CAB203 Discrete Structures

Lecture Notes

Jack Girard 2023

Contents

1 What is Mathematics?

1.1 What is Mathematics?

1.1.1 Abstraction

Abstraction can be used to simplify a problem by ignoring all the information that is not needed. They capture the relevant properties of a situation, and the relationships between them. Those properties can then be used to work out the solution.

- You have x apples
- You have y friends
- Are there enough apples for all your friends?

The usual abstraction for problems involving counting is *natural numbers* (non-negative integers). By abstracting away irrelevant properties such as size, shape, colour etc. the problem can be simplified to:

$$x \geq y?$$

Limitations of abstractions can include:

- Not enough information
- Too much information
- Incorrect information

1.1.2 Mathematical Theories

An abstraction without reference to a particular problem is called a *mathematical theory*, usually consisting of:

- Mathematical objects (numbers, operations, etc.)
- Axioms (statements about how objects relate to each other)

1.1.3 Axioms

Some examples of axioms for the natural numbers include:

1. 0 is a natural number
2. $x = x$
3. if $x = y$ then $y = x$
4. if $x = y$ and $y = z$ then $x = z$
5. if $x = w$ then w is a natural number
6. $S(x)$ is a natural number
7. if $S(x) = S(y)$ then $x = y$
8. $S(x) = 0$ is always false

Axioms can be combined to create new true statements. For example, axiom 6 in the list above states $S(x)$ is a natural number, so by combining it with itself it can be deduced that $S(S(x))$ is also a natural number.

Note

S refers to the *successor function* that increments a natural number.

$$S(x) = x + 1$$

1.1.4 Mathematical Objects

Mathematical objects are abstract objects that can correspond to concrete objects. They can only be defined in how they relate to other objects - this is done through axioms. Formally, objects are just symbols. They have no meaning, and are just names.

Relationships between objects are given by *propositions*. Propositions are statements that can be true or false. Axioms are propositions that assert to be true for objects in the mathematical theory.

1.1.5 Models

A mathematical theory can apply to a real situation if:

- Every object in the theory matches up to something in the real situation (at least hypothetically)
- All axioms in the theory remain true in the real situation

If this can be done, the real situation can be defined as a *model* for the theory.

Another instance of a model is a *mathematical model*. This type of model is an abstraction of a particular system to be studied and analysed in a mathematical way, and is the primary focus of this course when referring to ”model”.

1.1.6 Truth in Mathematics

Statements in mathematics are always relative to a particular mathematical theory. A statement may be true in one theory and false in another.

- For example, $ab = ba$ is true for real numbers, but not for matrices.

A true statement in a theory is irrelevant to a real situation, **unless** it is a model for that theory. The rules of logic guarantee that true statements in a theory are also true in every model of the theory.

> **Note**
>
> It is possible for every statement in a theory to be true **and** false. This is called an *inconsistent* theory and cannot have any models.

1.2 Modular Arithmetic

1.3 Exponents and Logarithms

2 Data Representation