# Project 1

Nico O'Neill

September 2025

## 1 Introduction

A Fabry-Perot laser cavity is defined by two mirrors which can build up optical power when in a resonant state. In optical experimentation, reaching this state of resonance requires fine alignment of a laser beam onto the input mirror and proper tuning of the laser to fit the resonance conditions. This code simulates how fabry-perot cavity is effected by changing of these parameters.

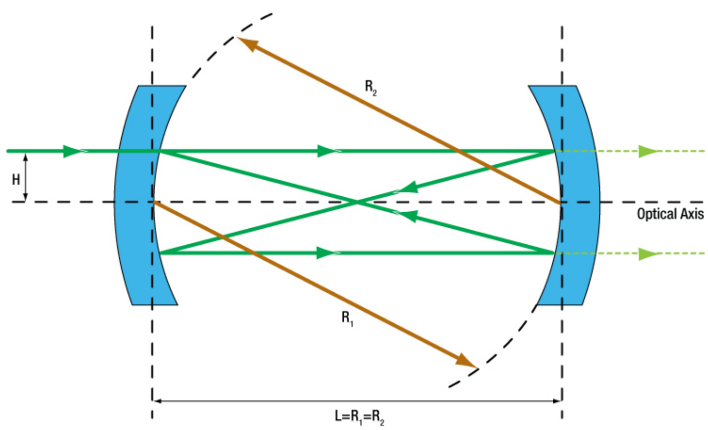


Figure 1: A diagram of a Fabry-Perot Cavity.

## 2 Intracavity Power

### 2.1 Mathematical Background

When light enters this cavity, the light circulates around and builds up power, while leaking out both mirrors in some amount. The laser frequency will have a required $\omega_C$, which is the resonant frequency.

$$\omega_C = \frac{n\pi c}{L}$$
where,
$c$ is the speed of light
$L$ is the length of the cavity

To find the ODE for the intracavity power, we must first define some properties of the cavity.

$R_n$ is the reflectivity of each mirror.
$T_n$ is the transmissivity of each mirror.
$\tau_{RT} = \frac{2L}{c}$ is the time for the laser to pass through the cavity.
$\Delta = \omega - \omega_C$ is how far the laser frequency is from resonance.
$\phi = \Delta\tau_{RT}$ is the phase shift accumulated through one trip.

Due to the transmissive properties of the mirrors, we lose a small amount of power each time we circulate the cavity.

$$R_{RT} = (1 - T_1)(1 - T_2)(1 - L_{RT})$$
where,
$L_{RT}$ represents the round trip loss.

The real amplitude scales by $\sqrt{R_{RT}}$ and can be simplified down to

$$r_{RT} = 1 - \frac{1}{2}(T_1 + T_2 + L_{rt})$$

using a Taylor expansion.
Now that we understand the losses, we can start to calculate the round-trip amplitude:

$$a(t + \tau_{RT}) = r_{RT}e^{i\phi}a(t) + \sqrt{T_1}s_{in}(t)$$
where,
$e^{i\phi}$ is the detuning phase,
$sqrtT_1 s_{in}(t)$ is the transmitted input field.

Through Taylor expanding the first term into $a(t) + \tau_{RT}\frac{da}{dt}$ and substitutions, we can obtain the equation:

$$\frac{da}{dt} = (i(\omega - \omega_c) - \frac{c}{4L_{cav}}(T_1 + T_2 + L))a + \sqrt{\frac{c}{2L_{cav}}T_1}s_{in}(t)$$

where,
$\omega - \omega_c$ is the detuning factor
$L_{cav}$ is the cavity length
$T_x$ is the transmissivity of each mirror
$s_{in}$ is the incident field.

The three terms here can be roughly interpreted as the sum of the intercavity phase rotation the cavity experiences when the laser is off resonance, the amplitude decay due to losses, and the input laser driving the cavity.

For this, we used 3 different numerical integration methods. The first was the explicit Euler method, which is a first-order method that comes from the Taylor expansion.

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0,$$
$$y_{n+1} = y_n + \Delta t\, f(t_n, y_n)$$

The second method is the Semi-Implicit Euler method, which is similar to the explicit Euler, but has a staggered update method which allows it to deal with oscillations better.

$$y^* = y_n + \Delta t\, f(t_n, y_n),$$
$$y_{n+1} = y_n + \Delta t\, f(t_{n+1}, y^*)$$

Finally, we used the fourth-order Runge-Kutta method, which combines four intermediate slopes through a weighted averaging.

$$k_1 = f(t_n, y_n),$$
$$k_2 = f\left(t_n + \tfrac{\Delta t}{2},\, y_n + \tfrac{\Delta t}{2} k_1\right),$$
$$k_3 = f\left(t_n + \tfrac{\Delta t}{2},\, y_n + \tfrac{\Delta t}{2} k_2\right),$$
$$k_4 = f(t_n + \Delta t,\, y_n + \Delta t\, k_3),$$
$$y_{n+1} = y_n + \tfrac{\Delta t}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)$$

We anticipate the RK4 will simulate our system the best, as it is the highest order method, followed by the Semi-Implicit due to its ability to handle oscillations.

## 2.2 Modeling Goals

There are two results we aim to express with the model. The first is the ringdown effect. We can imagine a two mirror cavity filled with circulating light, which then has the input laser turned off. In this case we see the power inside the cavity ring down as the light leaks out through the mirrors and are "absorbed" through the internal losses. From the general equation, we can set $s_{in}$ to be 0, so our equation simplifies to:

$$\frac{da}{dt} = (i(\omega - \omega_c) - \tfrac{c}{4L_{cav}}(T_1 + T_2 + L))a$$

Analytically we can solve this ODE to find that:

$$a(t) = a_0 e^{(i(w - w_C) - \frac{c}{2L}(T_1 + T_2 + L_{RT}))t}$$

Here, we can understand that the circulating power decays with the storage time of the cavity, defined as:

$$\tau_{store} = \frac{1}{\kappa}$$
where,
$$\kappa = \frac{c}{2L}(T_1 + T_2 + L_{RT})$$

In the case that the cavity is on resonance before it is turned off, we expect to see an exponential decay–however, if there is some detuning factor, we can expect to see an exponential decay with some phase rotation.
The next result we want to model out of this code is the step-on resonant response. Now we assume that the laser is off and then we turn it on to some set input power. Here we expect to see the cavity power build up as it circulates throughout the cavity. For a cavity on resonance, we expect to see an exponential rise in power. For an off-resonance cavity, we expect to see some oscillations in power at low time steps as the cavity begins to resonate, and then loses power through losses.

## 2.3 Results

Our ringdown results follow our assumptions. We see that for a cavity on resonance there is an exponential decay.

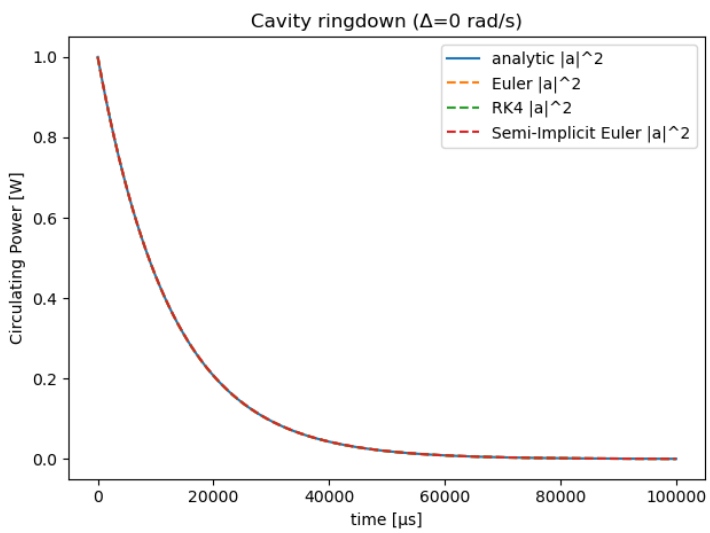


Figure 2: A comparison of the ringdown method without detuning.

For a cavity with a detuning factor, we have:

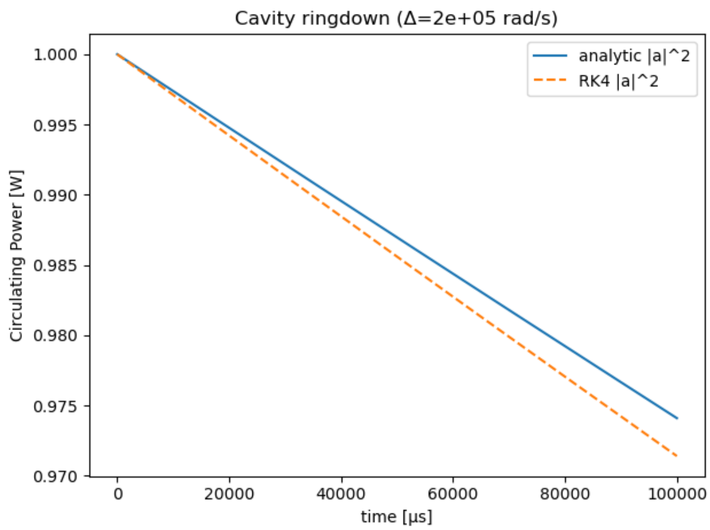


Figure 3: A comparison of the ringdown method a phase rotation.

We see here that all 3 methods of integration are extremely accurate. This is due to the fact that we are working with a linear, first-order ODE. As we increase the complexity of the ODE, shown in the second figure with the detuning factor, we add in some complex phase rotation and has an oscillatory component that is difficult for the numerical methods to solve depending on the time step.

Our step-up results also fit our predictions well. Our resonant cavity shows an exponential rise, and our 3 methods of numerical integration fit our analytic well.

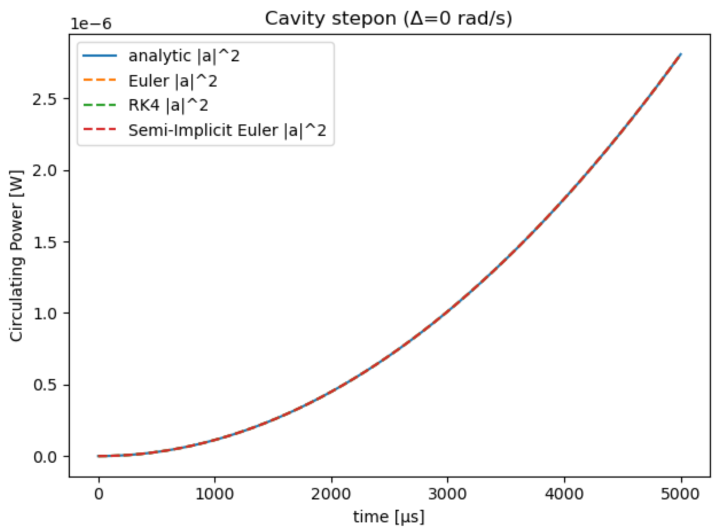


Figure 4: A comparison of the step-on method without detuning.

For a cavity with a detuning factor, we have:

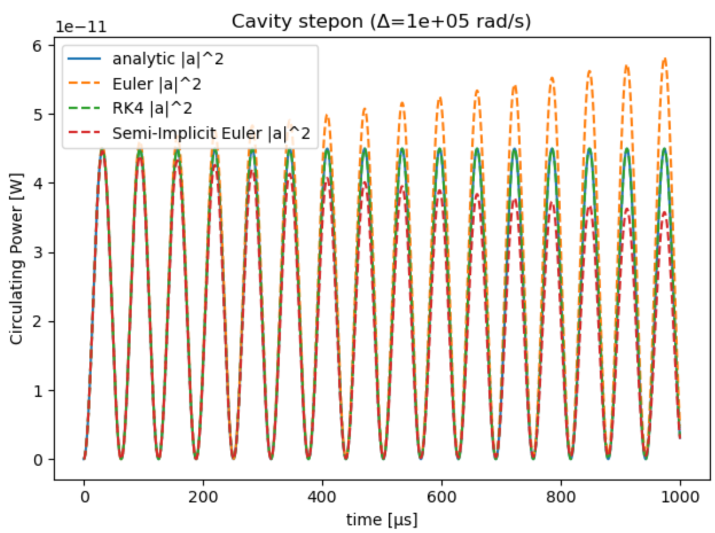


Figure 5: A comparison of the step-on method a phase rotation.

Here we can see the oscillatory behavior caused by the complex phasing term. Specifically, we see a beating between the laser and the cavity resonances, caus-

ing these peaks. Our methods of integration are behaving like we expect them to with complicated oscillations. We are seeing the Euler method over-estimate our magnitude, while the Semi-Implicit Euler method slowly underestimates. These results are validated by physical response. We expect the cavity power decay in the ring down scene(because power must decrease if the input decreases) and we expect the ramp up when the laser is turned on.
We can also use the Scipy Library to compare, which uses a complex RK45 method:

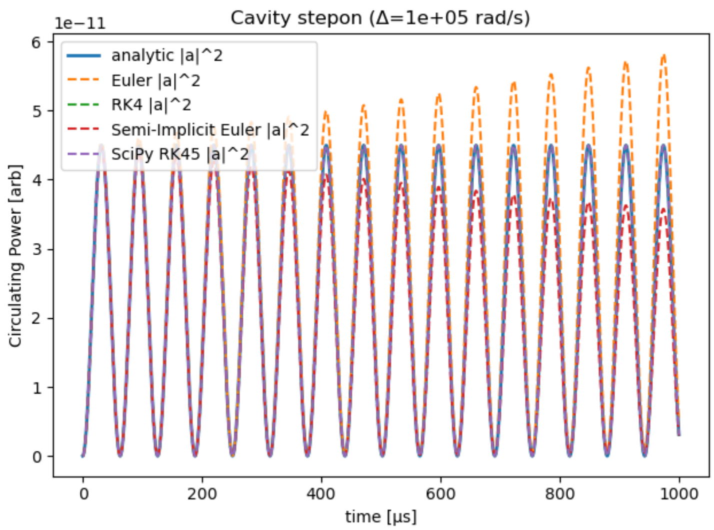


Figure 6: A comparison of the step-on method a phase rotation, using RK45.

This result shows the same as the RK4 method, due to the simplicity of the function. Using higher and higher order methods will not improve accuracy after a certain point.

# 3 Clipping Loss

## 3.1 Mathematical Background

When attempting to align a beam into a cavity, you must first align onto the front facing mirror. Precision in doing this is important because if the full radius of your beam is not contained within the radius of your mirror, you end up with clipping losses. If we assume a Gaussian beam and a circular mirror, the fraction of power transmitted is found by:

$$F(a) = \frac{\int_0^a 2\pi r e^{\frac{-2r^2}{g^2}} dr}{\int_0^\infty 2\pi r e^{\frac{-2r^2}{g^2}} dr}$$

We used three integration methods to model this effect:
The Riemann Midpoint rule subdivides the integeral into equal parts with equal widths and then approximates by:

$$I \approx \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \Delta x, \qquad x_i = a + i\Delta x.$$

This can be understood as summing the area of the rectangles of equal width and heights set by the integral function.
The second is the trapezodial rule, which acts similarly but assumes the area under the curve is a trapezoid rather than a rectangle:

$$I \approx \frac{\Delta x}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right],$$

Finally, we used Simpson's rule, which assumes a parabolic shape for each subinterval defined:

$$I \approx \frac{\Delta x}{3}\left[f(x_0) + 4\sum_{i=1}^{n-1} f(x_i) + 2\sum_{i=2}^{n-2} f(x_i) + f(x_n)\right].$$

### 3.2 Modeling Goals

The plan is to model a centered Gaussian beam, but modulate the size of the optic. This shows that how the clipping effects the overall power by changing how much of the beam is incident on the mirror. We expect to see that for small mirrors, we lose most of our power but this increases as we capture more of the beam.

### 3.3 Results

We can compare the results from our different methods:

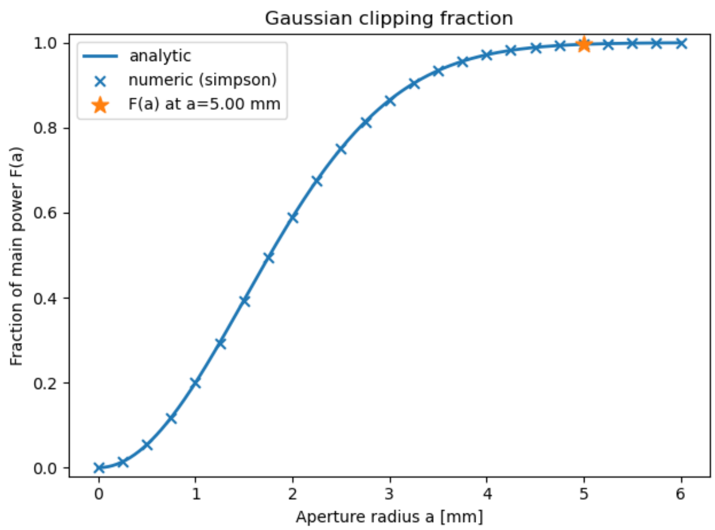


Figure 7: A Gaussian Beam clipping using Simpson's method.

Simpson's method resulted in an error of $2.148 * 10^{-11}$.

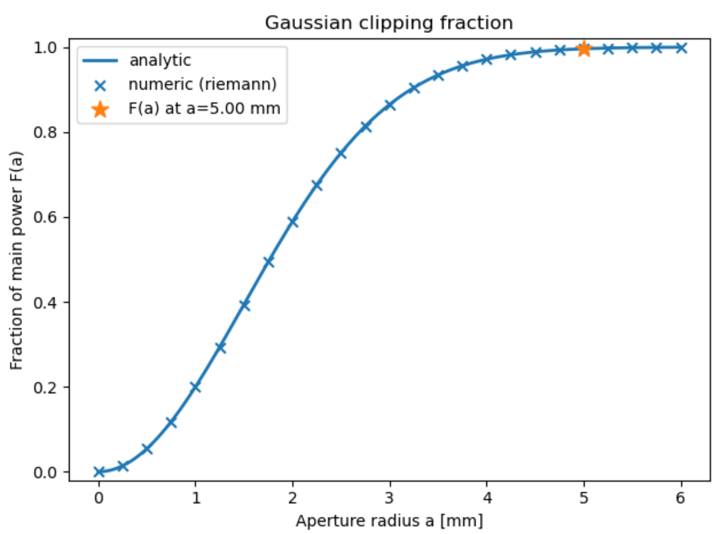


Figure 8: A Gaussian Beam clipping using Riemann sum.

The Riemann sum resulted in an error of $1.379 * 10^{-6}$.

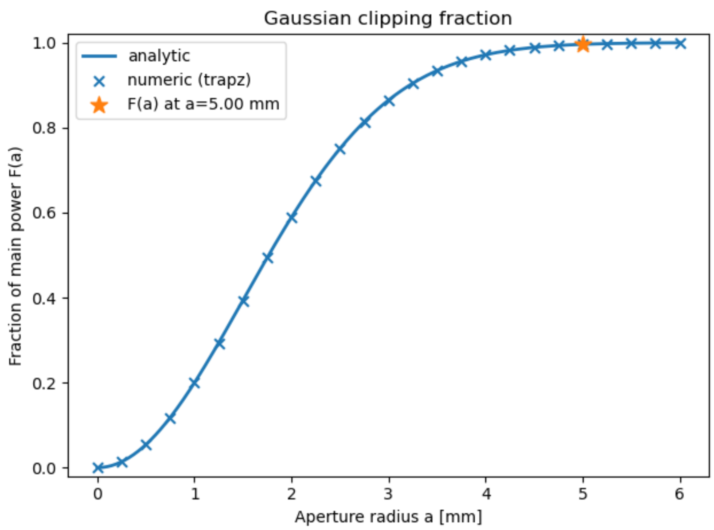


Figure 9: A Gaussian Beam clipping using trapezoidal method.

The Trapezoidal method resulted in an error of $2.759 * 10^{-6}$.
Here, we see that the trapezoidal and Riemann sum methods are pretty similar in terms of accuracy, while Simpson's method is the most accurate at estimating an integral.
These results validate our physical knowledge, as we can only expect that as more of the input laser is incident on a mirror our fractional power will reach 1 as the aperture reaches the size of the laser.

# 4 Truncation Error

In both of these simulations, we are able to see the effect of truncation errors. For our first simulation, the truncation error of both Euler's methods is $O(t^2)$ and the RK4 is $O(t^4)$.

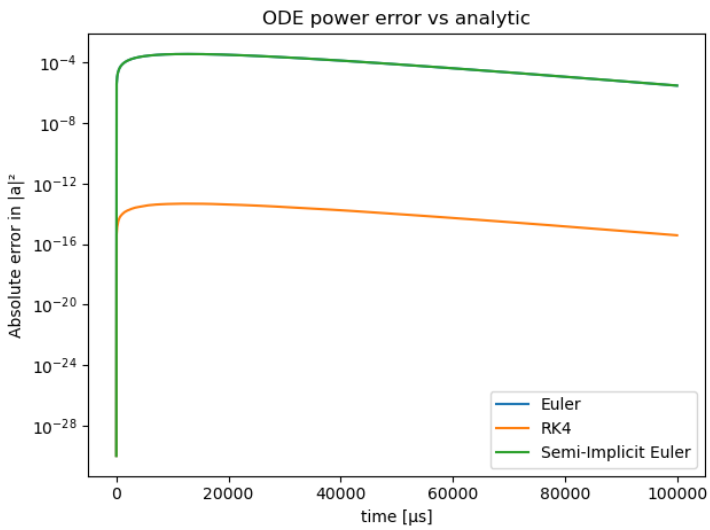


Figure 10: The Error in a ringdown simulation with no phasing.

As mentioned before, the ringdown error is very low with no phase offset, due to the simplicity of the function. As a contrast:

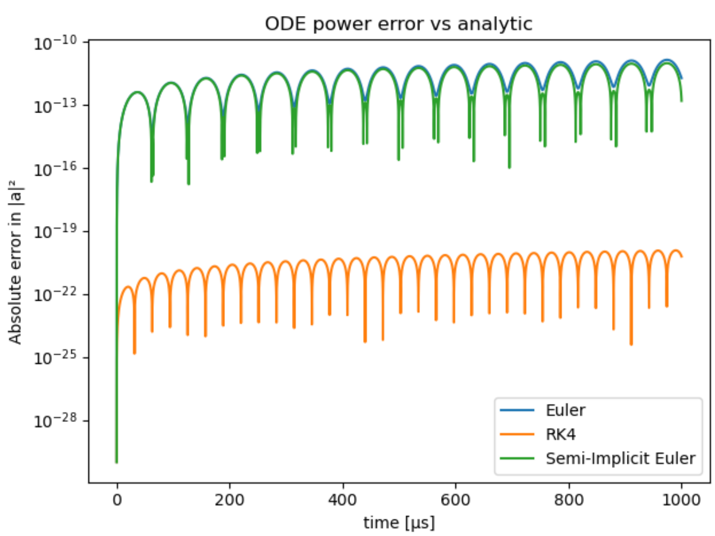


Figure 11: The Error in a stepup simulation with phasing.

We can see the rising error for a stepon function with some phase offset,

which is due to the complicated oscillatory behavior.
The trunctation error for Simpson's method is $O(t^4)$, for Riemann sum is $O(t)$, and for trapezoidal is $O(t^3)$.

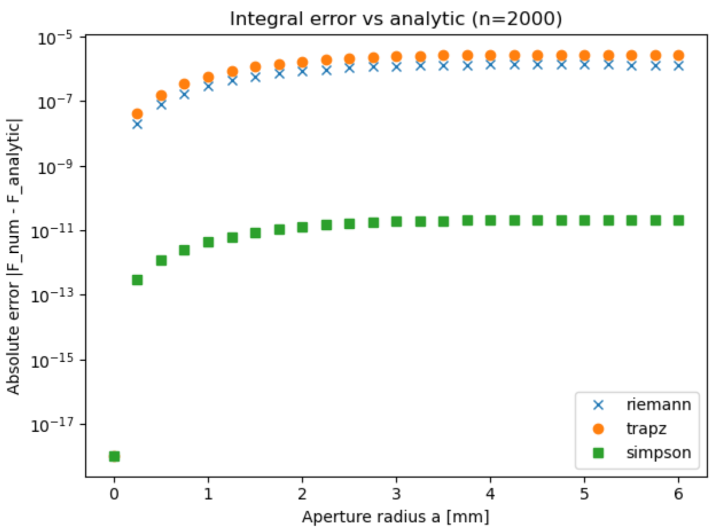


Figure 12: The Error in an integral simulation.

These make sense and we see as the truncation exponent decreases, our error increases. The error for riemann and trapezodial are very close here due to the time step used. We can expect as the time step increases we will see the trapezodial get more and more accurate.

# 5 Sources


16-Scipy. https://www.southampton.ac.uk/ fangohr/teaching/python/book/html/16-scipy.html. Accessed 30 Sep. 2025.

"Fabry–Pérot Interferometer." Wikipedia, 22 Sep. 2025. Wikipedia, https://en.wikipedia.org/w/index.php?
RK45 — SciPy v1.16.2 Manual. https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.RK45
Accessed 30 Sep. 2025.

"Runge-Kutta 4th Order Method to Solve Differential Equation." GeeksforGeeks, 31 Jan. 2016, https://www.geeksforgeeks.org/dsa/runge-kutta-4th-order-method-solve-differential-equation/.

"Truncation Error (Numerical Integration)." Wikipedia, 14 Jun. 2025. Wikipedia, https://en.wikipedia.org/w/index.php?title=Truncation$_{e}rror_{(}numerical_{i}ntegration)oldid =$

1295503829.