

Functions of Random Variables

Functions of Random Variables

X - income, $Y = 0.3X$ - income tax

X - driving speed, $Y = 2^x$ - speeding ticket

$X \sim f_X$ f_X known distribution

$Y = g(X)$ g known deterministic function

f_Y ?

What is the distribution of Y?

Power-law pdf's

$$a > -1$$

$$\int_0^1 \underbrace{(a+1)x^a dx}_{\geq 0} = x^{a+1} \Big|_0^1 = 1$$

$$a < -1 \text{ later}$$

$$f(x) = \begin{cases} (a+1)x^a & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

pdf!

$$F(x) = \int_0^x (a+1)u^a du = u^{a+1} \Big|_0^x = x^{a+1}$$

$$a \leq x \leq 1$$

Use $f(x) = 3x^2$

$$F(x) = x^3$$

$$0 \leq x \leq 1$$

g

$$Y = g(X) \triangleq X^{\frac{3}{2}}$$

$$F_Y(y) = P(Y \leq y)$$

$$0 \leq y \leq 1$$

$$= P(X^{\frac{3}{2}} \leq y)$$

$$= P(X \leq y^{\frac{2}{3}})$$

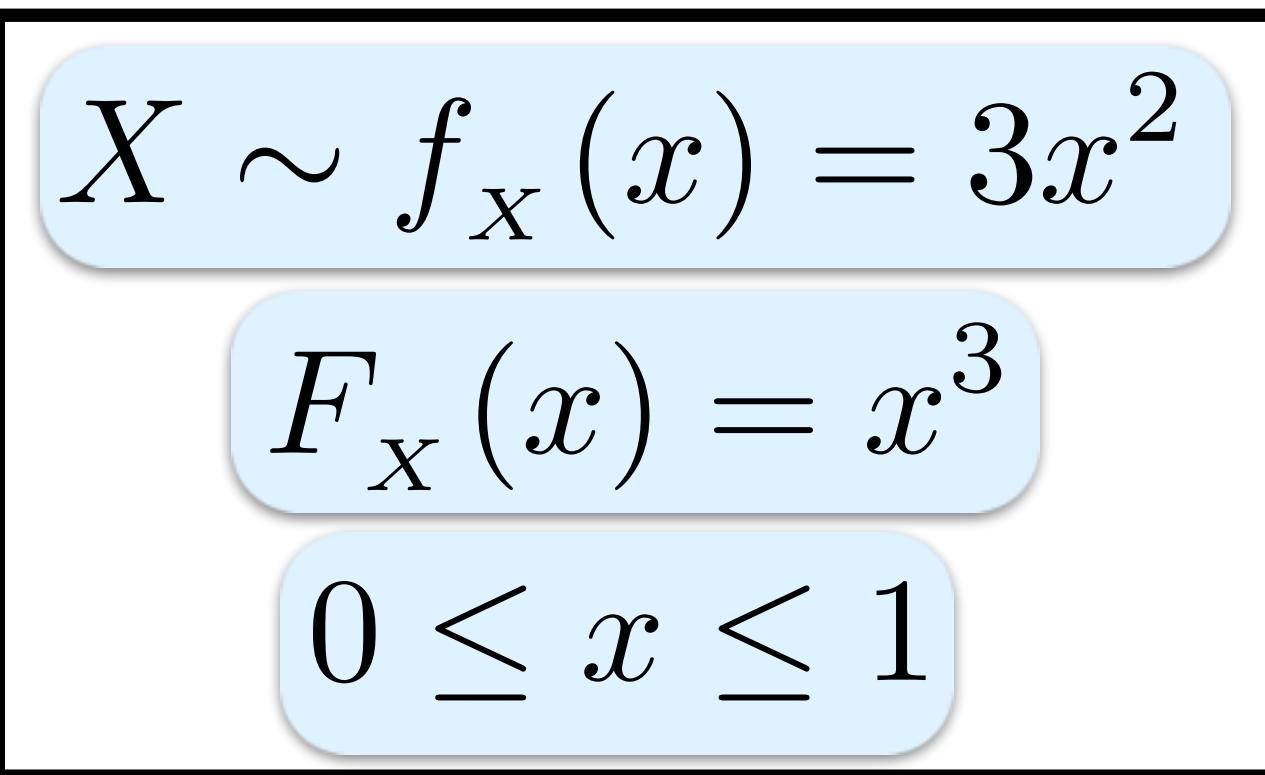
$$= F_X(y^{\frac{2}{3}})$$

$$= y^2$$

$$F_Y(0) = 0$$

$$F_Y(1) = 1$$
✓

$$f_Y(y) = F'_Y(y) = 2y$$



g

$$Y = g(X) \triangleq X^{-3}$$

$$F_Y(y) = P(Y \leq y) \quad y \geq 1$$

$$= P(X^{-3} \leq y)$$

$$= P(X \geq y^{-\frac{1}{3}})$$

$$= 1 - P(X \leq y^{-\frac{1}{3}})$$

$$F_Y(1) = 0$$

$$= 1 - F_X(y^{-\frac{1}{3}})$$

✓

$$F_Y(\infty) = 1$$

$$= 1 - y^{-1}$$

$$f_Y(y) = F'_Y(y) = y^{-2}$$

g ↗

$$f_X(x) = 3x^2$$

$$F_X(x) = x^3$$

$$g(x) = x^{\frac{3}{2}}$$

$$h(y) = y^{\frac{2}{3}}$$

$$f_Y(y) = 2y$$

$$F_Y(y) \triangleq P(Y \leq y)$$

$$f_Y(y) = F'_Y(y)$$

$$= P(g(X) \leq y)$$

$$P(X^{\frac{3}{2}} \leq y)$$

$$= [F_X(h(y))]'$$

$$[(y^{\frac{2}{3}})^3]'$$

$$= P(X \leq g^{-1}(y))$$

$$P(X \leq y^{\frac{2}{3}})$$

$$= F'_X(h(y)) \cdot h'(y)$$

$$3(y^{\frac{2}{3}})^2 \cdot \frac{2}{3}y^{-\frac{1}{3}}$$

$$= F_X(g^{-1}(y))$$

$$F_X(y^{\frac{2}{3}})$$

$$= f_X(h(y)) \cdot h'(y)$$

$$2y$$

$$= F_X(h(y))$$

$$h(y) \stackrel{\text{def}}{=} g^{-1}(y)$$

g ↘

$$F_Y(y) \triangleq P(Y \leq y)$$

$$f_Y(y) = F'_Y(y)$$

$$= P(g(X) \leq y)$$

$$= [1 - [F_X(h(y))]'$$

$$= P(X \geq g^{-1}(y))$$

$$= -F'_X(h(y)) \cdot h'(y)$$

$$= 1 - P(X \leq g^{-1}(y))$$

$$= -f_X(h(y)) \cdot h'(y)$$

$$= 1 - F_X(g^{-1}(y))$$

$$= 1 - F_X(h(y))$$

Combining

$g \nearrow$

$$f_Y(y) = f_X(h(y)) \cdot h'(y)$$

$g \searrow$

$$f_Y(y) = -f_X(h(y)) \cdot h'(y)$$

For both

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$

Alternative formulation

$$f_Y(y) = \frac{1}{|g'(x)|} f_X(x) \Big|_{y=g(x)}$$

Functions of Random Variables

Uniform Distributions



A wide, flat-topped mesa rises above a desert landscape under a clear sky. The mesa's surface is a uniform reddish-brown color, contrasting with the dry, yellowish-brown ground in the foreground. The horizon is flat, emphasizing the uniformity of the distribution.

Uniform distribution

Definintuition

For $a < b$, the **uniform** distribution $U_{[a,b]}$ is constant inside $[a,b]$ and 0 outside

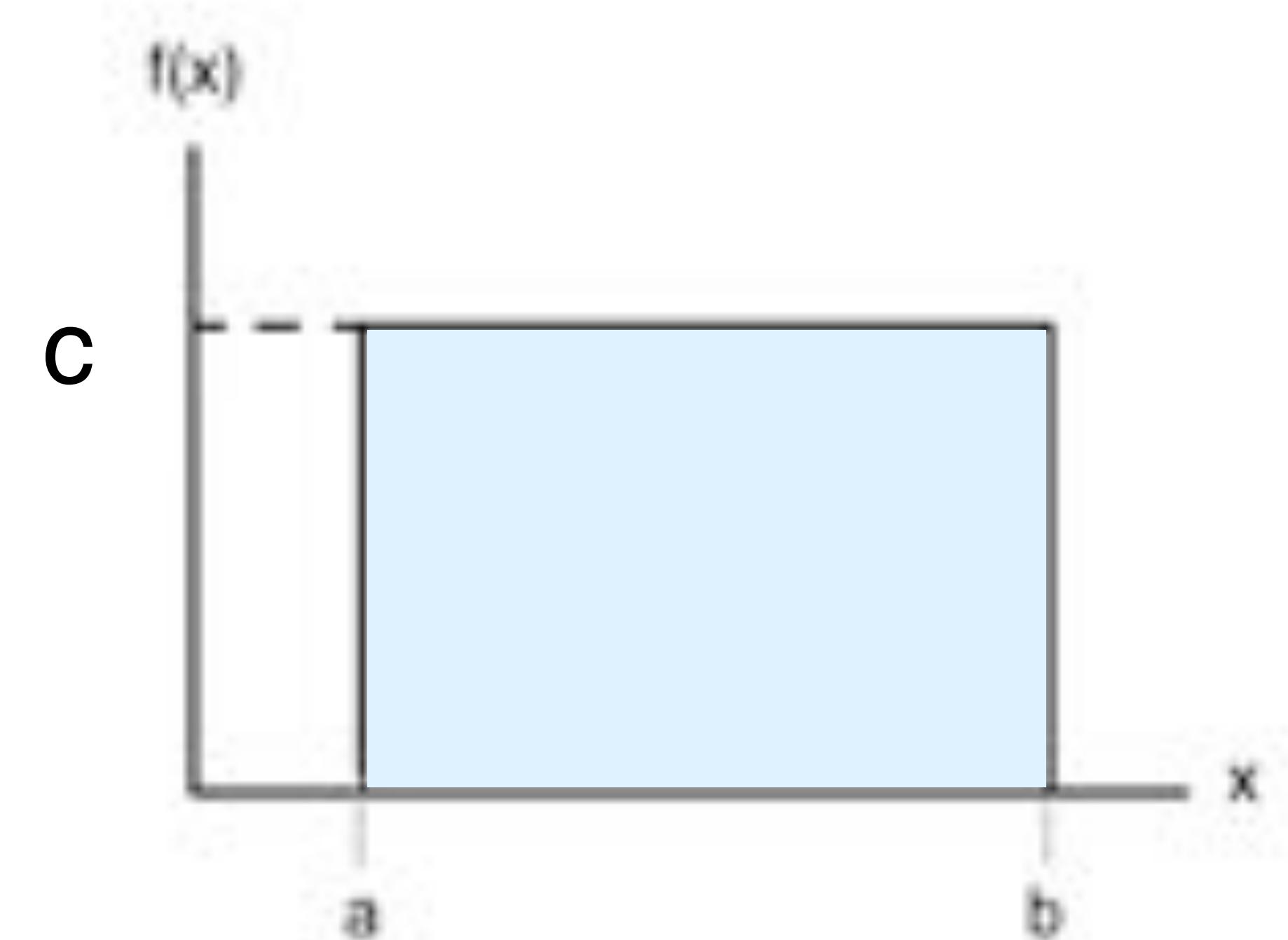
$$f(x) = \begin{cases} c & x \in [a,b] \text{ equally likely} \\ 0 & x \notin [a,b] \text{ never happen} \end{cases}$$

For $(\alpha, \beta) \subseteq [a,b]$

Probability determined by, and \propto , length $\beta - \alpha$

Area under curve is always a rectangle

Integrals are just height \times width



c?

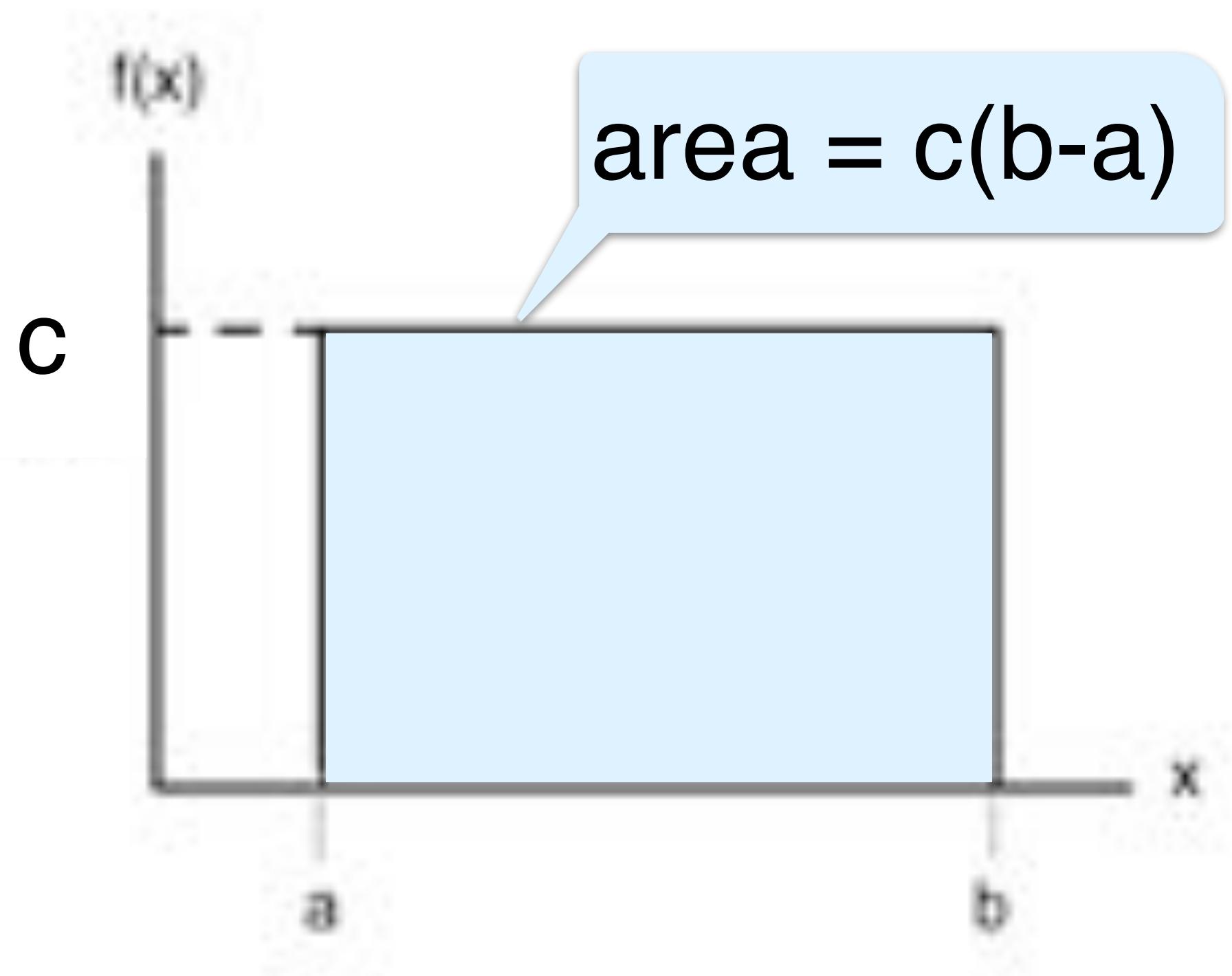
$$f(x) = \begin{cases} c = \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f(x)dx = c(b - a)$$

$$c = \frac{1}{b-a}$$

Σ WILL IT ADD?

YES IT
ADDS!



Who's Uniform

Departure times

Wait time for a bus

Location of chip defect

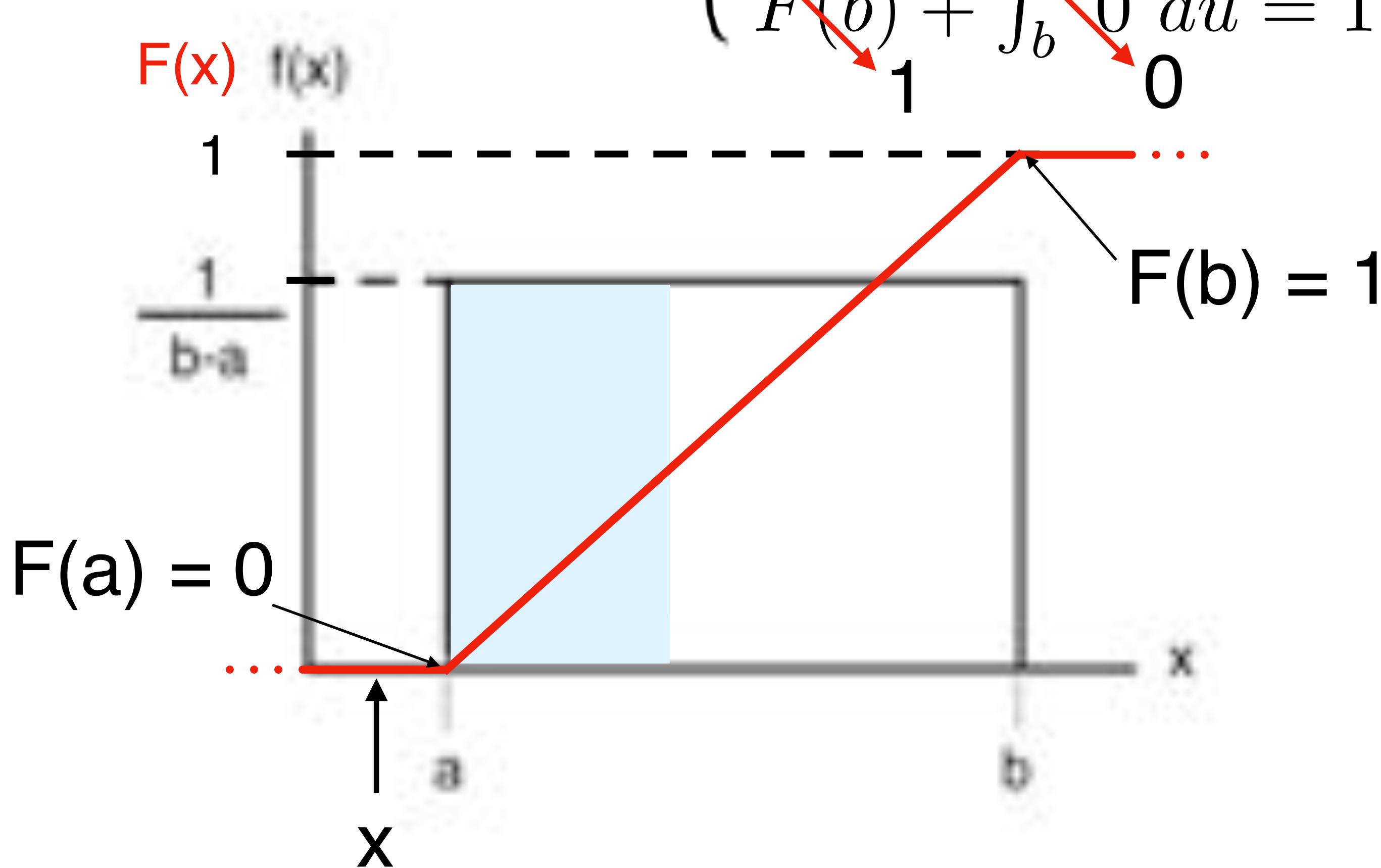
Location of a molecule in space

Considering a small area in time or space

Not so many...

CDF

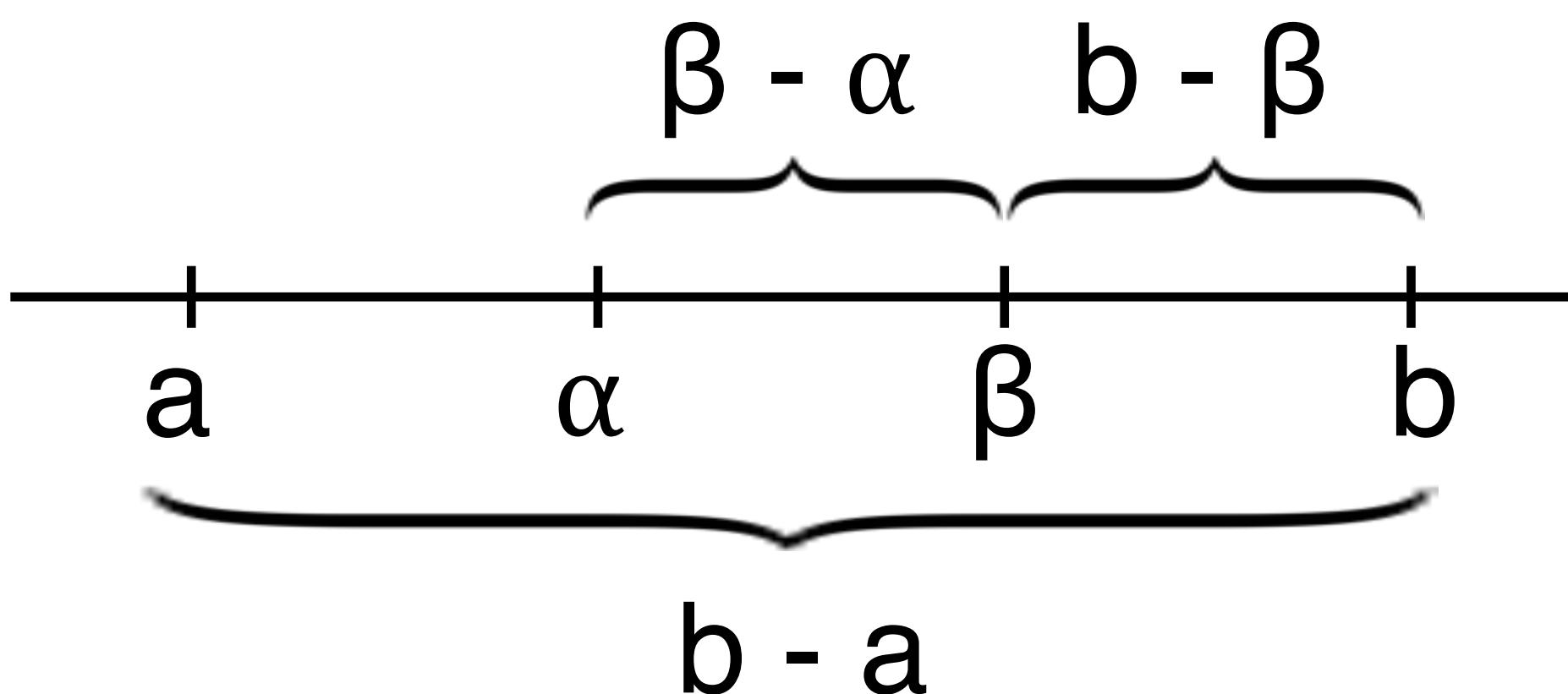
$$F(x) = \int_{-\infty}^x f(u)du = \begin{cases} \int_{-\infty}^x 0 du = 0 & x \leq a \\ F(a) + \int_a^x \frac{1}{b-a} du = \frac{x-a}{b-a} & a \leq x \leq b \\ F(b) + \int_b^x 0 du = 1 & x \geq b \end{cases}$$



Interval Probabilities

For $a \leq \alpha \leq \beta \leq b$

Interval	Probability
$(\alpha, \beta]$	$F(\beta) - F(\alpha) = \frac{\beta-a}{b-a} - \frac{\alpha-a}{b-a} = \frac{\beta-\alpha}{b-a}$
$[\beta, \infty)$	$F(\infty) - F(\beta) = 1 - \frac{\beta-a}{b-a} = \frac{b-\beta}{b-a}$
$\{\alpha\}$	$F(\alpha) - F(\alpha) = 0$



μ & σ

$X \sim U_{[0, 1]}$ first

$$f(x) = 1, \quad 0 \leq x \leq 1$$

$$EX = \int_0^1 x \, dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$$

$$EX^2 = \int_0^1 x^2 \, dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}$$

$$V(X) = EX^2 - (EX)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\sigma = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}} \approx 0.29$$

Translation and Scaling

Uniformity preserved under translation and scaling

$$X \sim U[0, 1]$$

wolog

$$(a \neq 0)$$

For any constants $a > 0$ and b , $Y \stackrel{\text{def}}{=} aX + b \sim U[b, a+b]$

Range:

$$0 \rightarrow b$$

$$1 \rightarrow a+b$$

PDF

$$Y = aX + b \stackrel{\text{def}}{=} g(X)$$

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|} \Big|_{x=g^{-1}(y)} = \frac{1}{a}$$

Or: equal-length interval map to $=$ -length intervals

General μ & σ

$$Y \sim U[a, b]$$

$$Y = (b - a)X + a$$

$$EX = \frac{1}{2}$$

$$EY = (b - a)EX + a = \frac{b-a}{2} + a = \frac{a+b}{2}$$

$$V(X) = \frac{1}{12}$$

$$V(Y) = V[(b - a)X + a] = (b - a)^2 V(X) = \frac{(b-a)^2}{12}$$

$$\sigma = \frac{b-a}{2\sqrt{3}} \approx 0.29(b - a)$$

Uniform Distributions

$U_{[a,b]}$ $a < b$

PDF $f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$

CDF $F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x \geq b \end{cases}$

$$\mu = \frac{a+b}{2}$$

$$V = \frac{(b-a)^2}{12}$$

$$\sigma = \frac{b-a}{2\sqrt{3}}$$



Exponential Distribution



Definition

Extends geometric distribution to continuous values

$$\lambda > 0$$

$$f_{\lambda}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\geq 0$$



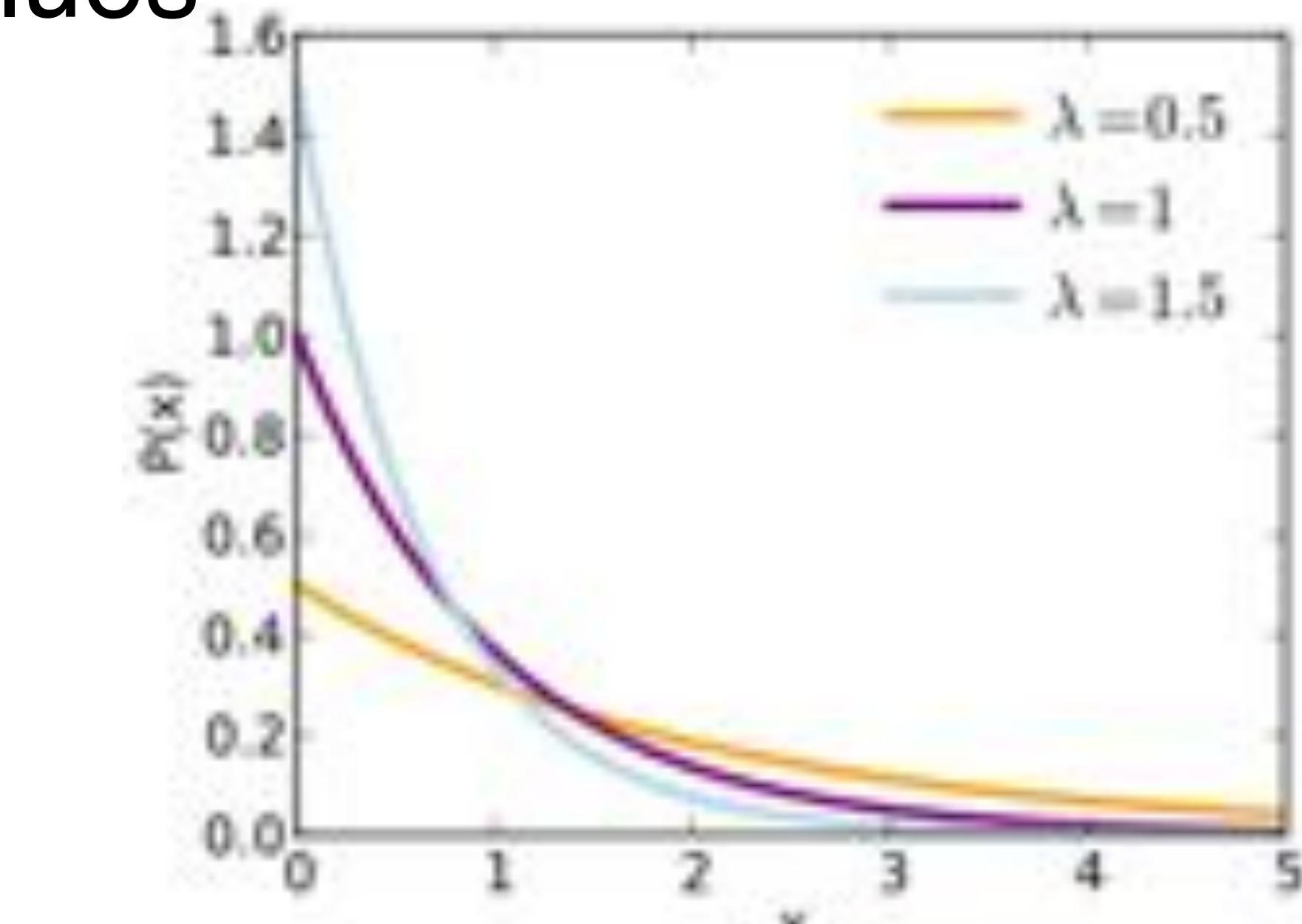
SUM WILL IT ADD?

$$\int f(x)dx = \int_0^{\infty} \lambda e^{-\lambda x} dx$$

$$= -e^{-\lambda x} \Big|_0^{\infty}$$

$$= 0 - (-1)$$

$$= 1$$



YES IT
ADDS!

Who's Exponential

Duration of a phone call

Wait time when you call an airline

Lifetime of a car

Time between accidents

CDF

$$x \geq 0$$

$$P(X > x) = \int_x^{\infty} \lambda e^{-\lambda u} du$$

$$= -e^{-\lambda u} \Big|_x^{\infty}$$

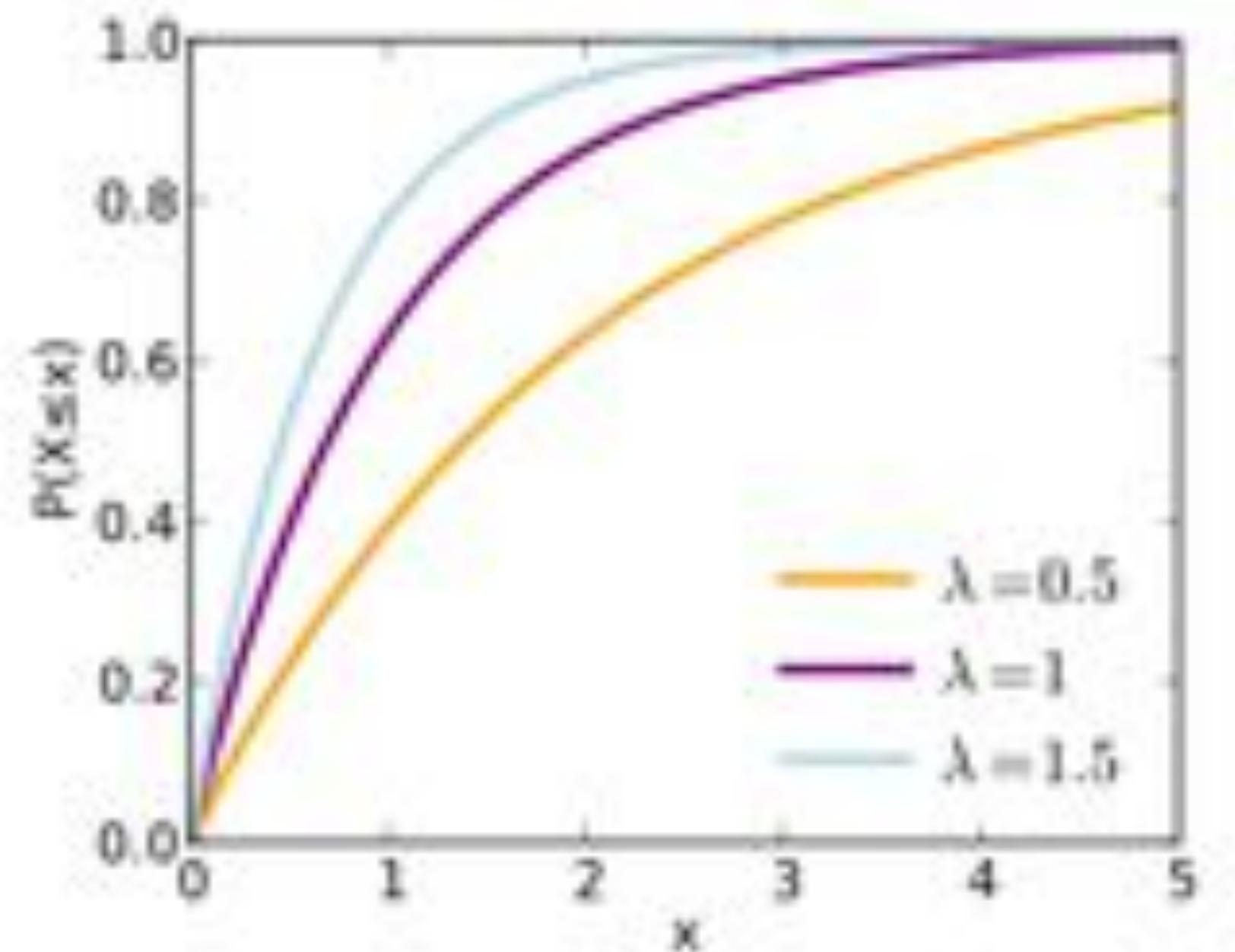
$$= e^{-\lambda x}$$

$$F(x) = P(X \leq x) = 1 - P(X > x) = 1 - e^{-\lambda x}$$

$$x \geq 0$$

$$F(x) = 0$$

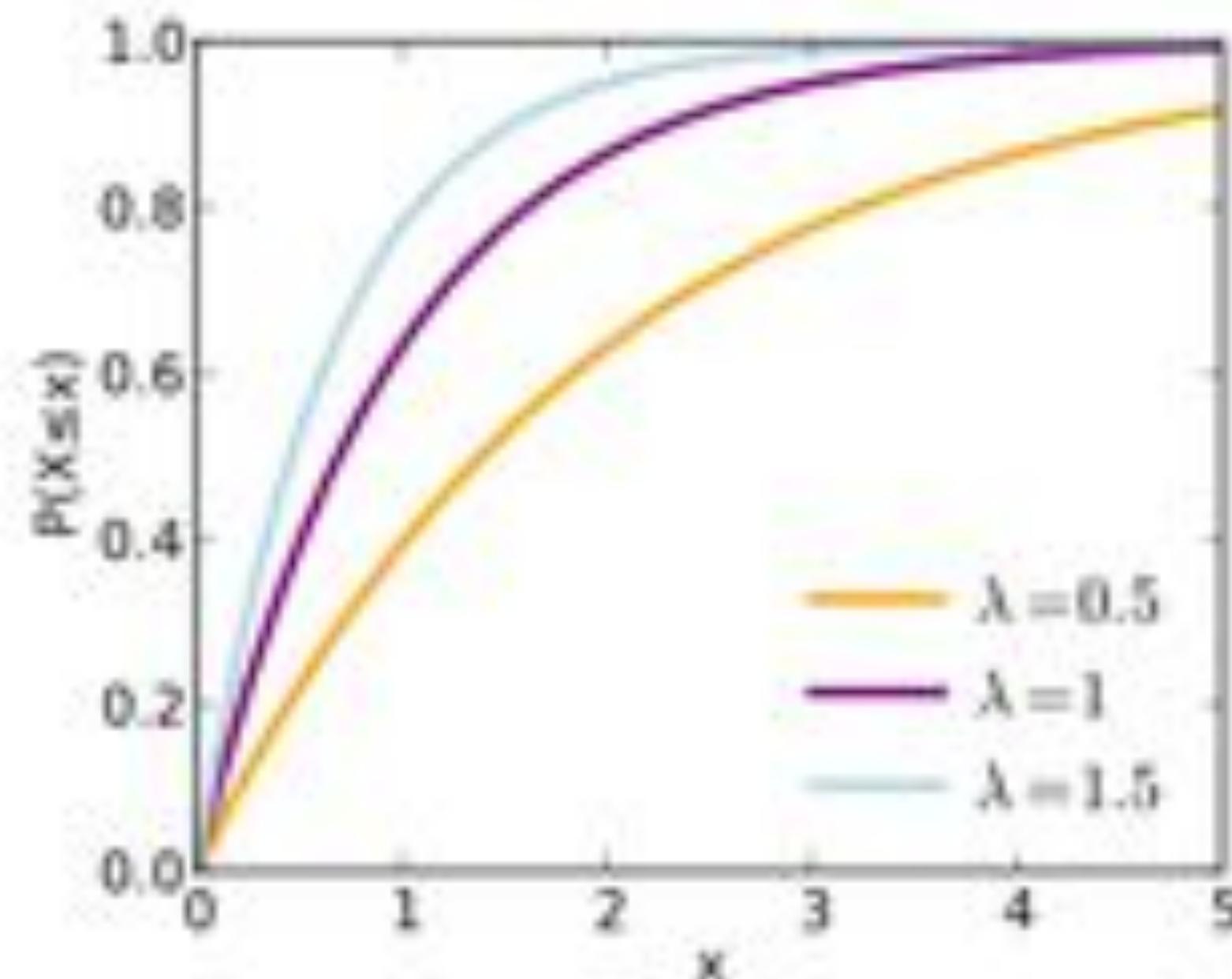
$$x \leq 0$$



CDF

$$P(X > x) = \begin{cases} \int_x^{\infty} \lambda e^{-\lambda u} du = -e^{-\lambda u} \Big|_x^{\infty} = e^{-\lambda x} & x \geq 0 \\ 1 & x \leq 0 \end{cases}$$

$$F(x) = P(X \leq x) = \begin{cases} 1 - P(X > x) = 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$



Example

$$0 \leq a \leq b$$

$$P(a \leq X \leq b) = P(a < X < b)$$

$$= F(b) - F(a)$$

$$= (1 - e^{-\lambda b}) - (1 - e^{-\lambda a})$$

$$= e^{-\lambda a} - e^{-\lambda b}$$

Expectation

$$EX = \int_0^\infty x \lambda e^{-\lambda x} dx$$

$$u = x$$

$$dv = \lambda e^{-\lambda x} dx$$

$$\int u \ dv = uv - \int v \ du$$

$$du = 1$$

$$v = -e^{-\lambda x}$$

$$= -xe^{-\lambda x} \Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx$$

$$= 0 - \frac{1}{\lambda} e^{-\lambda x} \Big|_0^\infty$$

$$= \frac{1}{\lambda}$$

Variance

$$EX^2 = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$u = x^2$$

$$dv = \lambda e^{-\lambda x} dx$$

$$du = 2x \ dx$$

$$v = -e^{-\lambda x}$$

$$\int u \ dv = uv - \int v \ du$$

$$= -x^2 e^{-\lambda x} \Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x} dx$$

$$EX = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$= 0 + \frac{2}{\lambda} EX = \frac{2}{\lambda^2}$$

$$V(X) = EX^2 - (EX)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\sigma = \frac{1}{\lambda}$$

Memoryless

$$X \sim f_\lambda$$

$$a, b \geq 0$$

$$P(X \geq a + b | X \geq a) = \frac{P(X \geq a + b, X \geq a)}{P(X \geq a)}$$

$$= \frac{P(X \geq a + b)}{P(X \geq a)}$$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$$

$$= e^{-\lambda b}$$

$$= P(X \geq b)$$

$$P(X < a + b | X \geq a) = 1 - P(X \geq a + b | X \geq a) = 1 - P(X \geq b) = P(X < b)$$

$$f(X = a + b | X \geq a) = f(X = b)$$

While Waiting in Line

DMV has 2 clerks, each with exponential service time

When you arrive, one person is in line 😕

While you wait, someone cuts in front of you 😠

At some point a clerk becomes available and starts serving the first person

Before first person finishes, other clerk starts serving second person

If all three of you served randomly, $P(\text{you finish last}) = \frac{1}{3}$

$P(\text{you finish last now})?$

Evaluation

A - time first person finishes

B - time second person finishes

C - time you finish

Service	$P(A < B < C)$
Fixed	1
Exponential	?

Orders	Probability
$A < B < C$	$\frac{1}{4}$
$A < C < B$	$\frac{1}{4}$
$B < A < C$	$\frac{1}{4}$
$B < C < A$	$\frac{1}{4}$
$C < A < B$	0
$C < B < A$	0

?

$$P(A < B < C) = \underbrace{P(A < B)}_{\frac{1}{2}} \cdot \underbrace{P(B < C | A < B)}_{\frac{1}{2}} = \frac{1}{4}$$

$$P(B < C < A) = \underbrace{P(B < A)}_{\frac{1}{2}} \cdot \underbrace{P(C < A | B < A)}_{\frac{1}{2}} = \frac{1}{4}$$

Conclusion

All three of you served randomly, $P(\text{you finish last}) = \frac{1}{3}$

Fixed service time, $P(\text{you finish last}) = 1$

Exponential (memoryless) service time

You won't finish first

All 4 other orders equally likely

$P(\text{you finish last}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Only slightly larger than $\frac{1}{3}$

Orders	P
$A < B < C$	$\frac{1}{4}$
$A < C < B$	$\frac{1}{4}$
$B < A < C$	$\frac{1}{4}$
$B < C < A$	$\frac{1}{4}$
$C < A < B$	0
$C < B < A$	0



Summary

Exponential

$$\text{PDF } f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

$$\text{CDF } F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

$$EX = \frac{1}{\lambda} \quad V(X) = \frac{1}{\lambda^2} \quad \sigma = \frac{1}{\lambda}$$

Memoryless



Normal Distribution