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STAT 205B
Lecture #1
1/17/2017

1 Logistics

https://www.stat.berkeley.edu/~aldous/205B/index.html

- 5 weeks — Convergence in Distribution
- 5 weeks — Markov Chains
- 2 weeks — Ergodic Theory
- 2 weeks — Brownian Motion

2 Background

Have definitions for

- Probability measure μ on $\mathbb{R}$.
  - Given F, $\exists$ a μ such that $F(x) = \mu[-\infty, x]$ holds.
  - This required first proving existence of Lebesgue measure on $[0, 1]$, then use inverse CDF.
- Distribution function F on $\mathbb{R}$.
  - Given μ, $F(x) = \mu[-\infty, x]$ is a distribution function.

x is a *continuity point* of F iff $F(x) = F(x-)$ iff $\mu(x) = 0$.

Theorem 2.1. *For PMs $(\mu_n, 1 \leq n < \infty)$ and μ on $\mathbb{R}$, the following are equivalent:*

(a) $F_{\mu_n}(x) \to F_\mu(x)$ as $n \to \infty$ for all continuity points x of F_μ

(b) $\int_{-\infty}^{\infty} g(x)\mu_n(dx) \to \int_{-\infty}^{\infty} g(x)\mu(dx)$ for all bounded continuous functions $g : \mathbb{R} \to \mathbb{R}$.

(c) There exists, on some probability space, random variables $(\hat{X}_n, 1 \leq n < \infty)$ and $(\hat{X})$ such that dist$(\hat{X}_n)$ = dist(X_n), $1 \leq n < \infty$, dist$(\hat{X})$ = dist(X), and $\hat{X}_n \xrightarrow{a.s.} X$ as $n \to \infty$.

Call this "weak convergence" $\mu_n \to \mu$.

Note: (b) and (c) make sense for PMs on metric space S and define *weak convergence* on S. In fact, (b) $\iff$ (c) on general S ("Skorohod representation theorem")

Theorem shows (a) not just arbitrary.

Write $X_n \xrightarrow{d} X$ "in distribution" to mean $\text{dist}(X_n) \to \text{dist}(X)$.

Proof. (c) $\implies$ (b).

$$\hat{X}_n \xrightarrow{\text{a.s.}} \hat{X} \implies g(\hat{X}_n) \xrightarrow{\text{a.s.}} g(\hat{X}) \; (g \text{ continuous}) \tag{2.1}$$
$$\implies \mathbb{E}g(\hat{X}_n) \xrightarrow{\text{a.s.}} \mathbb{E}g(\hat{X}) \; (g \text{ bounded}) \tag{2.2}$$
$$\implies \mathbb{E}g(\hat{X}_n) \xrightarrow{\text{a.s.}} \mathbb{E}g(X) \tag{2.3}$$

(b) $\implies$ (a). TODO: Fig 1.1

Fix x_0. Define bounded continuous $f_j(x)$ by

$$F_{\mu_n}(x_0) = \int_{-\infty}^{\infty} 1_{x \le x_0} \mu_n(dx) \le \int_{-\infty}^{\infty} f_j(x)\mu_n(dx) \tag{2.4}$$
$$\limsup_n F_{\mu_n}(x_0) \le \lim_n \int_{-\infty}^{\infty} f_j(x)\mu_n(dx) \overset{(b)}{=} \int_{-\infty}^{\infty} f_j(x)\mu(dx) \le_\mu (x_0 + 1/j) \tag{2.5}$$

But this holds for every j, so letting $j \to \infty$

$$\limsup_n F_{\mu_n}(x_0) \le F_\mu(x_0) \tag{2.6}$$

Symmetrically, define $g_j(x)$ by TODO: Fig 1.2

$$\liminf_n F_n(x_0) \ge \lim_n \int_{-\infty}^{\infty} g_j(x)\mu_n(dx) = \int_{-\infty}^{\infty} g_j(x)\mu(dx) \ge F_\mu(x_0 - 1/j) \tag{2.7}$$

Letting $j \to \infty \implies \liminf_n F_n(x_0) \ge F_\mu(x_0-)$.

If x_0 is a continuity point, then $\liminf = \limsup$.

(a) $\implies$ (c). Recall inverse function of F_μ

$$F_\mu^{-1}(y) := \sup\{x : F(x) < y\} = \inf\{x : F(x) \ge y\} \tag{2.8}$$

TODO: Fig 1.3

Exercise 2.2. (a) implies $F_{\mu_n}^{-1}(y) \to F_\mu^{-1}(y)$ for all y such that $\{x : F_\mu(x) = y\}$ is either empty or a single point x.

The other case is $\{x : F_\mu(x) = y\}$ is a non-trivial interval (i.e. when $F(x)$ has a plateau). This can only happen for countably many y.

$$F_{\mu_n}^{-1}(u) \xrightarrow{\text{a.s.}} F_\mu^{-1}(u) \text{ (all } U \text{ outside countable set)} \tag{2.9}$$

This is (c). $\square$

Elementary examples where we show (a) by calculation.

(a) X_n uniform on $\{1, 2, \ldots, k\}$. Then $\frac{X_n}{n} \xrightarrow{d} U[0,1]$.

(b) X_θ has Geometric(θ) distribution $P(X > i) = (1-\theta)^i, i = 0, 1, 2, \ldots$, then $\theta X_\theta \xrightarrow{d} Y$ with Exponential(1) distributions $P(Y > y) = e^{-y}, 0 \le y < \infty$.

(c) B_n ="birthday RV"$= \min\{j : \xi_j = \xi_i, 1 \le i < j\}$ for IID $\xi_i \sim U\{1, 2, \ldots, n\}$, then $n^[-1/2]B_n \xrightarrow{d} R$ with Rayleigh distribution $P(R > x) = \exp(-x^2/2)$.