Feynman Liang
STAT 205A
Lecture #24
11/15/2016

1 Examples using method of bounded differences

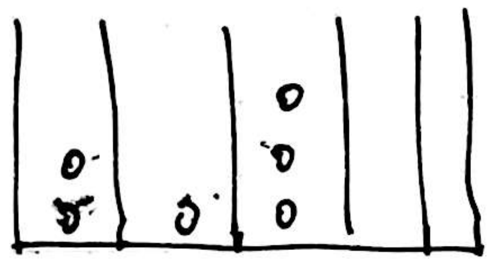
Theorem 1.1 (Method of bounded differences, from last class)**.** *$\xi_1, \xi_2, \ldots, \xi_n$ independent, $Z = f(\xi_1, \ldots, \xi_n)$ where f has the property*

$$|\{i : x_i \neq y_i\}| = 1 \implies |f(\tilde{x}) - f(\tilde{y})| \leq 1 \tag{1.1}$$

Then

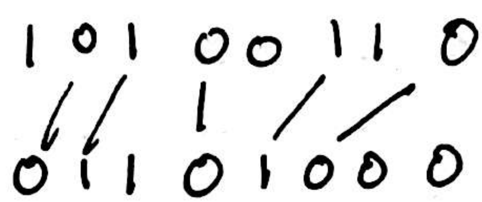
$$P(|Z - \mathbb{E}Z| \geq \lambda\sqrt{n}) \leq 2e^{-\lambda^2/2}, \quad \lambda > 0 \tag{1.2}$$

Example 1.2. Put n balls "at random" into m boxes.



Consider $Z(m, n)$ = # empty boxes.
From combinatorics, $\mathbb{E}Z(n, m) = m(1 - 1/m)^n$.
Apply to ξ_i = box containing ball i, $1 \leq i \leq n$. Equation (1.1) holds.

Example 1.3 (Simple open (unsolvable?) problem)**.** Take 2 independent Bernoulli$(1/2)$ sequences of length n.



Want to match digits between two strings such that no lines are crossing (i.e. longest common subsequence). In example 1.3, this is 01010.

Let Z_n = length of longest common subsequence.

Fact: $n^{-1}Z_n \overset{\text{a.s.}}{\to} c$ as $n \to \infty$, no formula for c.

Take $\xi_i = \binom{\cdot}{\cdot}$ at position i.

Any change $\tilde{x} \to \tilde{y}$ has

$$
\begin{aligned}
& f(\tilde{y}) - f(\tilde{x}) \geq -2 && (1.3)\\
\iff & f(\tilde{x}) - f(\tilde{y}) \leq 2 && (1.4)
\end{aligned}
$$

So $Z_n/2$ satisfies eq. (1.1).

Definition 1.4. A *c-coloring of a graph* G is a function color : $V(G) \to \{1, 2, \ldots, c\}$ such that $(v_1, v_2) \in E(G) \implies \text{color}(v_1) \neq \text{color}(v_2)$.

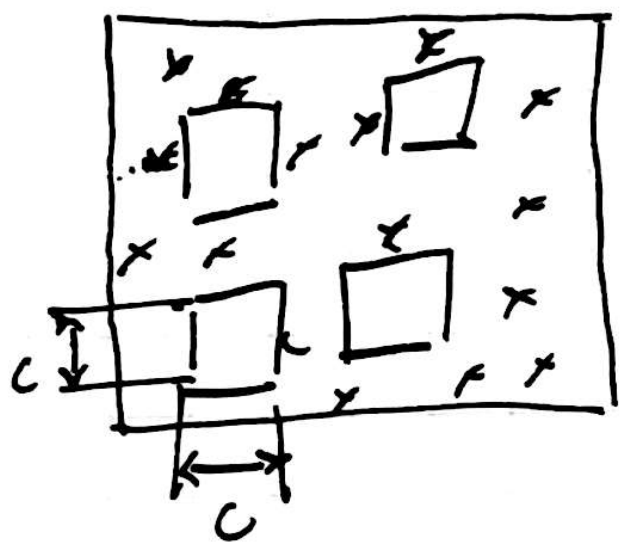
The *chromatic number* $\chi(G) = \min\{c : \exists c\text{-coloring of } G\}$

Definition 1.5. The *Erdös-Renyi random graph model* $\mathcal{G}(n, p)$ has n vertices with each of the $\binom{n}{2}$ possible edges present with probability p.

Let $Z = \chi(\mathcal{G}(n, p))$. Order vertices arbitrarily $1, 2, \ldots, n$.

For $i \geq 2$, let $\xi_i = (1_{(i,1) \in E(G)}, \ldots, 1_{(i,i-1) \in E(G)})$. Since we can always just add a color when we increment i, eq. (1.1) holds for $Z = f(\xi_1, \ldots, \xi_n)$.

Example 1.6. Put n points IID uniform in unit square. Fix $0 < c < 1$. Let $Z(n, c)$ = max number of disjoint $c \times c$ squares containing 0 points.



Moving any single point can only reduce $Z(n, c)$ by 1, so eq. (1.1) holds.

2 Reversed martingales

Consider sub-σ-fields, $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \mathcal{G}_2 \supset \ldots$, $\mathcal{G}_\infty = \cap_n \mathcal{G}_n$. In Durrett, $\mathcal{G}_n = \mathcal{F}_{-n}$.

Definition 2.1. $(X_n, \mathcal{G}_n)$ is a *reversed martingale* if

(a) $\mathbb{E}|X_n| < \infty$

(b) $\mathbb{E}(X_m \mid \mathcal{G}_n) = X_n$, $m \leq n$

(c) (X_n) adapted to $(\mathcal{G}_n)$
Together, these imply $X_n = \mathbb{E}[X_0 \mid \mathcal{G}_n]$

Reversed martingales are in some sense easier, since we are given that the limit X_0 always exists

Theorem 2.2. *For a reversed MG, $X_n \to \mathbb{E}[X_0 \mid \mathcal{G}_0]$ a.s. and in L^1.*

Proof. $(X_N, X_{N-1}, \ldots, X_0)$ is a MG. Let U_N = # upcrossings of $(X_N, \ldots, X_0)$ over $[a, b]$. By the upcrossing inequality

$$\mathbb{E}U_n \leq \frac{\mathbb{E}|X_0| + |a|}{b-a} \tag{2.1}$$

[As in proof for MGs] $U_N \uparrow U_\infty$, so

$$\mathbb{E}U_{\leq}\frac{\mathbb{E}|X_0| + a}{b-a} \implies U_\infty < \infty \text{ a.s.} \tag{2.2}$$

So $X_n \overset{\text{a.s.}}{\to} X_\infty$ for some $X_\infty \in [-\infty, \infty]$.

But $X_n = \mathbb{E}[X_0 \mid \mathcal{G}]$ implies (X_n) is UI, which implies $X_n \overset{L^1}{\to} X_\infty$, $\mathbb{E}|X_\infty| < \infty$.
Need to show $X_\infty = \mathbb{E}[X_0 \mid \mathcal{G}_\infty]$.

$$X_n \in \mathcal{G}_n \subset \mathcal{G}_k \qquad n > k \tag{2.3}$$
$$n \to \infty \qquad X_\infty \in \mathcal{G}_k \tag{2.4}$$
$$k \to \infty \qquad X_\infty \in \mathcal{G}_\infty \tag{2.5}$$

So X_∞ is $\mathcal{G}_\infty$-measurable.
Need to show $\mathbb{E}X_\infty 1_G = \mathbb{E}X_0 1_G$, $G \in \mathcal{G}_\infty$.

$$X_n = \mathbb{E}[X_0 \mid \mathcal{G}_n] \implies \mathbb{E}X_n 1_G = \mathbb{E}X_0 1_G \qquad G \in \mathcal{G}_\infty \tag{2.6}$$
$$X_n \overset{L^1}{\to} X_\infty \implies \mathbb{E}X_n 1_G \to \mathbb{E}X_\infty 1_G \tag{2.7}$$
$$\implies \mathbb{E}X_0 1_G = \mathbb{E}X_\infty 1_G \tag{2.8}$$
$$\tag{2.9}$$

□

3 Exchangeable sequences

Definition 3.1. A sequence of RVs $(X_1, X_2, \ldots)$ is called *exchangeable* if

$$(X_1, X_2, \ldots, X_n) \overset{d}{=} (X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}) \tag{3.1}$$

for all n and all permutations $\pi \in S_n$.

By Kolmogorov's consistency theorem, if this holds for all finite sequences, then it holds for infinite sequences as well.

Theorem 3.2. *Suppose $(X_i)_{i=1}^{\infty}$ are exchangeable, $\mathbb{R}$-valued, and $\mathbb{E}|X_1| < \infty$. Write $S_n = \sum_{i=1}^{n} X_i$.*

Then $n^{-1}S_n \to \mathbb{E}[X_1 \mid \tau]$ a.s. and in L^1, where $\tau = tail(X_i, i \geq 1)$.

Corollary 3.3. *If (X_i) IID, $\mathbb{E}|X_1| < \infty$, then τ is trivial.*

$\implies \mathbb{E}[X_1 \mid \tau] = \mathbb{E}X_1$ and Theorem $\implies n^{-1}S_n \to \mathbb{E}X_1$. This shows WLLN.

Lemma 3.4. *If $(Z_1, W) \stackrel{d}{=} (Z_2, W)$ and $\mathbb{E}|Z_1| < \infty$, then $\mathbb{E}[Z_1 \mid W] = \mathbb{E}[Z_2 \mid W]$.*

Proof. Q is a kernel assoc with $\text{dist}(Z_1, W)$.

$$\mathbb{E}[Z_1 \mid W] = \phi(W) \quad \text{where } \phi(W) = \int Z Q(W, dz) \tag{3.2}$$

$$\mathbb{E}[Z_2 \mid W] = \phi(W) \tag{3.3}$$

□

Exercise 3.5. Let $\mathbb{E}|X| < \infty$. If $X \stackrel{d}{=} \mathbb{E}[X \mid \mathcal{G}]$, then $X = \mathbb{E}[X \mid \mathcal{G}]$ a.s.

Comment: easy if $\mathbb{E}X^2 < \infty$, more work if only integrable.

Proof of theorem 3.2. Define $\mathcal{G}_n = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots) = \sigma(S_n, S_{n+1}, S_{n+2}, \ldots)$.

Lemma 3.6. $\mathbb{E}[X_i \mid \mathcal{G}_n] \stackrel{a.s.}{=} \mathbb{E}[X_i \mid \mathcal{G}_n], 1 \leq i \leq n.$

Proof. Take permutation π of $(1, \ldots, n)$.

$$(X_{\pi(1)}, \ldots, X_{\pi(n)}, X_{n+1}, X_{n+2}, \ldots) \stackrel{d}{=} (X_1, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots) \tag{3.4}$$

Set $W = (S_n, X_{n+1}, X_{n+2}, \ldots)$.

$$\implies \quad (X_{\pi(1)}, \ldots, X_{\pi(n)}, W) \stackrel{d}{=} (X_1, \ldots, X_n, W) \tag{3.5}$$

$$\implies \quad (X_{\pi(1)}, W) \stackrel{d}{=} (X_1, W) \tag{3.6}$$

$$\implies \quad (X_i, W) \stackrel{d}{=} (X_1, W) \tag{3.7}$$

$$\text{Lemma 3.4} \implies \quad \mathbb{E}[X_i \mid W] = \mathbb{E}[X_1 \mid W] \tag{3.8}$$

Can extend with Kolmogorov consistenty to infinite sequences. □

$\mathcal{G}_n \supset \mathcal{G}_{n+1} \ldots$ decreasing.

$$S_n = \mathbb{E}[S_n \mid \mathcal{G}_n] = \sum_{i=1}^{n} \mathbb{E}[X_i \mid \mathcal{G}_n] = n\mathbb{E}[X_1 \mid \mathcal{G}_n] \tag{3.9}$$

$$\implies n^{-1}S_n = \mathbb{E}[X_1 \mid \mathcal{G}_n] \to \mathbb{E}[X_1 \mid \mathcal{G}_\infty] \text{ a.s., } L^1 \tag{3.10}$$

Note $\mathcal{G}_\infty \supset \tau$. But $\lim n^{-1}S_n$ is τ-measurable, so $\mathbb{E}[X_1 \mid \mathcal{G}_\infty]$ is τ-measurable. Combined with the tower property, we have

$$\mathbb{E}[X_1 \mid \tau] = \mathbb{E}[\mathbb{E}[X_1 \mid \mathcal{G}_\infty] \mid \tau] = \mathbb{E}[X_1 \mid \mathcal{G}_\infty] \tag{3.11}$$

□