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STAT 205A
Lecture #26
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1 Brownian motion

Definition 1.1. A $\mathbb{R}^1$-valued process $(B(t), 0 \leq t < \infty)$ is *(standard) Brownian motion (Wiener process)* if $B(0) = 0$ and

(a) ("independent increments") $B(t_0)$, $B(t_1) - B(t_0), \ldots, B(t_n) - B(t_{n-1})$ are independent, any $0 \leq t_0 < t_1 < \cdots < t_n$

(b) $B(t) - B(s) \sim N(0, \underbrace{t-s}_{\text{variance}})$ distribution

(c) Sample path $t \mapsto B(t)$ are continuous.

Need proof of existence.

Proof. Write Q_2 = dyadic rationals $= \{2^{-j}i : i, j \geq 0\}$. Suffices to consider time interval $[0,1]$. Enumerate Q_2 as $q_1, q_2, \ldots$. For each n, items (a) and (b) specify specify a joint distribution of $(B(q_1), B(q_2), \ldots, B(q_n))$. These are *consistent* as n increases.

TODO: Fig 26.1 Check $N(0, s-t_1) \overset{\text{ind}}{+} (0, t_2 - s) = N(0, t_2 - t_1)$

Use Kolmogorov extension theorem to show there exists a process $(B(q), q \in Q_2 \cap [0,1])$.

For $f : Q_2 \cap [0,1] \to \mathbb{R}$, and $\delta > 0$, define

$$w(f, \delta) = \sup_{\substack{0 \leq q_1 < q_2 \leq 1 \\ q_2 \in Q_2 \\ q_2 - q_1 \leq \delta}} |f(q_1) - f(q_2)| \tag{1.1}$$

Lemma 1.2. *(exists continuous extension from Q_2 to all of $\mathbb{R}$)*

If $w(f,\delta) \to 0$ as $\delta \to 0$, then $\exists$ cts $\tilde{f} : [0,1] \to \mathbb{R}$ such that $\tilde{f}(q) = f(q)\ \forall q \in Q_2 \cap [0,1]$.

Proof. Define $\tilde{f}(t) = \limsup_{\substack{q \downarrow t \\ q \in Q_2}} f(q)$ (lim sup guaranteed well-defined). If $|t-s| < \delta$, then $|\tilde{f}(t) - \tilde{f}(s)| \leq w(f, \delta)$. The lemma premise implies f is continuous. □

It is sufficient to show $P(w(B(\cdot), \delta) \geq \varepsilon) \to 0$ as $\delta \downarrow 0$, $\varepsilon > 0$ fixed, because then $w(B(\cdot), \delta) \to 0$ a.s. as $\delta \downarrow 0$ and by lemma 1.2 $\exists \tilde{B}$ such that $t \mapsto \tilde{B}(\omega, t)$ is continuous a.s (item (c)). Easy to check (using property of normals) items (a) and (b) remains true for $t \in \mathbb{R}$. Redefine $B(t, \omega) \equiv 0\ \forall t$ on null set.

Define

$$\bar{w}(f, 2^{-m}) = \max_{0 \leq j \leq 2^m - 1} \sup_{2^{-m}j \leq q \leq 2^{-m}(j+1)} |f(q) - f(2^{-m})| \tag{1.2}$$

Consider $0 \leq q_1 < q_2 \leq 1$ with $q_2 - q_1 \leq 2^{-m}$.

TODO: Fig 26.2 They must be in either the same or adjacent intervals, so by the triangle inequality (see TODO: Fig 26.2 ref)

$$|f(q_2) - f(q_1)| \leq 3\bar{w}(f, 2^{-m}) \tag{1.3}$$

So it suffices to show $P(\bar{w}(B(\cdot), 2^{-m}) \geq \varepsilon) \to 0$ as $m \to \infty$.

Define $S_m = \sup_{0 \leq q \leq 2^{-m}} |B(q)|$. $\bar{w}(B(\cdot), 2^{-m})$ is the max of 2^m identically distributed RVs, so $P(\bar{w}(B(\cdot), 2^{-m}) \geq \varepsilon) \leq 2^m P(S_m \geq \varepsilon)$.

Fix m, take $n > m$. Consider $B(2^{-n}i, 0 \leq i \leq 2^{n-m})$. This is a martingale, so by convexity theorem $B^4(2^{-n}, i \geq 0)$ is a sub-martingale. Applying L^1 maximal inequality

$$P\left(\max_{2^{-n} \leq 2^{-m}} B^4(2^{-m}) \geq \varepsilon^4\right) \leq \varepsilon^{-4} \mathbb{E} B^4(2^{-m}) \tag{1.4}$$

Let $Z \sim N(0,1)$, so $B(t) \overset{d}{=} t^{1/2} Z$ and $P(S_m \geq \varepsilon) \leq \varepsilon^{-4} 2^{-2m} \mathbb{E} Z^4$.

$$P\left(\max_{2^{-n} \leq 2^{-m}} B^4(2^{-m}) \geq \varepsilon^4\right) \leq \varepsilon^4 2^{-2m} \mathbb{E} Z^4 \tag{1.5}$$

Let $n \to \infty$, so

$$P(\bar{w}(B(\cdot), 2^{-m}) \geq \varepsilon) \leq 2^m P(S_m \geq \varepsilon) \leq 2^{-m} \varepsilon^{-4} \mathbb{E} Z^4 \tag{1.6}$$

Taking $m \to 0$, this quantity $\to 0$ showing continuity. □

Theorem 1.3. *For almost all ω, the sample path $t \mapsto B(\omega, t)$ is nowhere differentiable.*

Proof. **From analysis**: Consider $f : [0,1] \to \mathbb{R}$. Fix $C < \infty$. Suppose $\exists s$ such that $f'(s)$ exists and $|f'(s)| \leq C/2$. Then $\exists n_0$ such that for $n \geq n_0$,

$$|f(t) - f(s)| \leq C|t - s| \text{ for all } t \text{ such that } |t - s| \leq \frac{3}{n} \tag{1.7}$$

Rewrite: define $A_n = \{f : \text{above property holds for some } s\}$. As $n \to \infty$, $A_n \uparrow A \supset \{f : |f'(s)| \leq C/2 \ \text{ for some } s\}$

For $0 \leq K \leq n - 1$, define

$$Y(f, k, n) = \max\left(|f(\frac{k+3}{n}) - f(\frac{k+2}{n})|, |f(\frac{k+2}{n}) - f(\frac{k+1}{n})|, |f(\frac{k+1}{n}) - f(\frac{k}{n})|\right) \tag{1.8}$$

TODO: Fig 26.3

Given $f \in A_n$, suppose the s for which the property holds satisfies $k/n \leq s < (k+1)/n$. Then

$$\implies Y(f, k, n) \leq \frac{sC}{n} \tag{1.9}$$

$$\implies YA_n \subset D_n \overset{\text{def}}{=} \{f : Y(f, k, n) \leq \frac{sC}{n} \text{ for some } K \leq n - 1\} \tag{1.10}$$

Computing the probability

$$P\left(\left|\underbrace{B\left(\frac{k+1}{n}\right)-B\left(\frac{k}{n}\right)}_{N(0,n^{-1})=n^{-1/2}Z}\right|\leq\frac{sC}{n}\right)=P(|Z\leq sC/n^{1/2})\leq(2\pi)^{-1/2}\times 10C/n^{1/2} \tag{1.11}$$

Regard $B(\cdot)$ as random

$$P\left(Y(B,k,n)\leq\frac{5C}{n}\right)=P\left(\left|B\left(\frac{k+1}{n}\right)-B\left(\frac{k}{n}\right)\right|\leq\frac{sC}{n}\right)^3\leq 100C^3/n^{3/2} \tag{1.12}$$

So

$$P(B(\cdot)\in D_n)\leq n\times P\left(Y(B,k,n)\leq\frac{5C}{n}\right)\leq 100C^3/n^{3/2} \tag{1.13}$$

$$P(B(\cdot)\in A_n)\leq 100C^3/n^{1/2} \tag{1.14}$$

Let $n\to\infty$, $P(B(\cdot)\in A)=0$. $\square$