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STAT 205B
Lecture #4
1/26/2017

Optional lab Monday 1:00 to 2:00 in 344 Evans

1 More on characteristic functions

Recall from last lecture

Theorem 1.1 (Inversion formula)**.** *If a PM μ has CF ϕ such that $\int_{-\infty}^{\infty} |phi(t)| dt < \infty$, then μ has a bounded continuous density*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t) dt \tag{1.1}$$

Corollary 1.2. *Given a PM μ with CF ϕ. Suppose $\phi(\cdot)$ is $\mathbb{R}$-valued, $\phi \geq 0$, and $\int_{-\infty}^{\infty} \phi(t) dt < \infty$. Then*

$$g(x) := \frac{\phi(x)}{2\pi f(0)} \tag{1.2}$$

is a density function, with CF $f(t)/f(0)$.

Here, f and g are called dual pairs.

Proof. By inversion formula

$$\frac{f(y)}{f(0)} = \int_{-\infty}^{\infty} e^{-ity} \underbrace{\frac{\phi(t)}{2\pi f(0)}}_{=g(t)} dt \tag{1.3}$$

holds for all y, so in particular for $y = 0$

$$1 = \int_{-\infty}^{\infty} \underbrace{\frac{\phi(t)}{2\pi f(0)}}_{=g(t)} dt \tag{1.4}$$

so g integrates to one. Since $\phi \geq 0$, $g \geq 0$, hence g is a density function.

Equation (1.3) also shows that the CF of g is $f(y)/f(0)$. □

Example 1.3 (Last class)**.** If $f(x) = \frac{1}{2} e^{-|x|}$, then $\phi(t) = \frac{1}{1+t^2}$. The dual is $g(x) = \frac{\phi(x)}{\pi} = \frac{1}{\pi(1+x^2)}$, the standard Cauchy distribution, and this has CF $\frac{f(t)}{f(0)} = e^{-|t|}$, $-\infty < t < \infty$.

Write W for a RV with standard Cauchy distribution. Take iid copies $W_1, W_2, \ldots$.

$$\phi_{W_1 + W_2 + \cdots + W_n}(t) = (e^{-|t|})^n = e^{-n|t|} = \phi_{nW}(t) \tag{1.5}$$

Uniqueness of CF implies $\sum_{i=1}^{n} \overset{d}{=} nW$, or $\frac{1}{n} \sum_{i=1}^{n} W_i \overset{d}{=} W$.

LLN doesn't hold here, because $\mathbb{E}|W| = \infty$, so this is a good example of where calculations using CF ("in transform land") are easier.

General facts: $\phi_{aW}(t) = \phi_W(at)$ and $\phi_{X-x}(t) = e^{-itx}\phi_X(t)$

Exercise 1.4. If $Y_n \xrightarrow{\text{d}} c$, then $Y_n \xrightarrow{\text{p}} c$.

Exercise 1.5. If $Y_n \xrightarrow{\text{d}} c$, then $X_n + Y_n \xrightarrow{\text{d}} X + c$ for any X.

A second proof of the inversion formula:

Proof. Take X with $\text{dist}(X) = \mu$.
Take $Z_\sigma \stackrel{d}{=} N(0, \sigma^2)$ independent of X.
$X + Z_\sigma \xrightarrow{\text{d}} X$ as $\sigma 0$.
Note $X + Z_\sigma$ has density defined by the convolution

$$f_{X+Z_\sigma}(0) = \int_{-\infty}^{\infty} f_{2\sigma}(-t)\mu(dt) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} \mu(dt) \tag{1.6}$$

By Parseval's identity for Normals

$$f_{X+Z_\sigma}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t^2\sigma^2/2} \phi(t) dt \tag{1.7}$$

Apply to $X - x$ instead of X to get

$$f_{X+Z_\sigma}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t^2\sigma^2/2} \phi(t) dt \tag{1.8}$$

Let $\sigma \downarrow 0$ and appeal to bounded convergence to get

$$\lim_{\sigma \downarrow 0} f_{X+Z_\sigma}(x) = \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t) dt}_{=: f(x)} \tag{1.9}$$

Final detail: $P(a \leq X \leq b) = \lim_{\sigma \downarrow 0} P(a \leq X + Z_\sigma \leq b) = \int_a^b f(x)dx$ at continuity points a, b of X, which is enough to prove f is the density of X (TODO: why?). □

Theorem 1.6 (Continuity Theorem)**.** *Let X_n have CF ϕ_n.*

(a) If $X_n \xrightarrow{d} X_\infty$, then $\phi_n(t) \to \phi_\infty(t)$ for each t.

(b) Suppose $\lim_{n\to\infty} \phi_n(t)$ exists (say $= \phi(t)$) for each t. If any of the following are true:

(i) ϕ is a CF

(ii) $\phi(t) \to 1$ as $t \to 0$

(iii) $(X_n, n \geq 1)$ are tight

then $X_n \xrightarrow{d} X_\infty$ and X_∞ has CF ϕ.

Proof. (a): $X_n \xrightarrow{\text{d}}_\infty$ implies $\mathbb{E}g(X_n) \to Eg(X_\infty)$ for bounded continuous g. Take $g(x) = e^{itx}$ to get $\phi_n(t) \to \phi_\infty(t)$ as $n \to \infty$, t fixed.

(b): Suppose (iii). Helly's selection theorem implies there exists subsequence $X_{n_s} \xrightarrow{\text{d}}$ some $\hat{X}$. Then (a) and hypothesis of (b), $\hat{X}$ has CF ϕ. By previous lemma, because every convergent subsequence has same limiting distribution we have that the whole sequence $X_n \xrightarrow{\text{d}} \hat{X}$ with CF ϕ. This proves (b).

Claim: (i) $\implies$ (ii), because a CF ϕ is continuous and $\phi(0) = 1$.

Need to prove (ii) and hypothesis of (b) imply (iii).

Fix K, put $c = \frac{2}{K}$. Trick: bound

$$\begin{aligned}
P(|X_n| \geq K) &\leq \mathbb{E}\left[2\left(1 - \frac{1}{c|X_n|}\right) 1_{|X_n| \geq K}\right] && (1.10)\\
&\leq 2\mathbb{E}\left[\left(1 - \frac{\sin(c|X_n|)}{c|X_n|}\right) 1_{|X_n| \geq K}\right] && (1.11)\\
&\leq 2\mathbb{E}\left[1 - \frac{\sin(c|X_n|)}{c|X_n|}\right] && (1.12)\\
&= 2\left(1 - \frac{1}{2c}\int_{-c}^{c} \phi_n(t)dt\right) = \frac{1}{c}\int_{-c}^{c}(1 - \phi_n(t))dt && (1.13)
\end{aligned}$$

where the last line applies Parseval's identity for $U[-c, c]$. Bounded convergence as $n \to \infty$ implies

$$\begin{aligned}
\limsup_n P(|X_n| \geq K) &\leq \frac{1}{c}\int_{-c}^{c}(1 - \phi(t))dt && (1.14)\\
\lim_{K\uparrow\infty} \limsup_n P(|X_n| \geq K) &\leq \lim_{c\downarrow 0} \frac{1}{c}\int_{-c}^{c}(1 - \phi(t))dt = 0 && (1.15)
\end{aligned}$$

by (ii), which implies tightness. □

2 CFs and moments

$$e^{itx} = \sum_{m=0}^{\infty} \frac{(itx)^m}{m!} \tag{2.1}$$

This suggests that CF ϕ of X is

$$\phi_X(t) = \sum_{m=0}^{\infty} \frac{\mathbb{E}(itX)^m}{m!} = 1 + it\mathbb{E}X - \frac{t^2}{2}\mathbb{E}X^2 \cdots \tag{2.2}$$

Lemma 2.1 (Durrett 3.3.7)**.**

$$\left| e^{iy} - \sum_{m=0}^{n} \frac{(iy)^m}{m!} \right| \leq \min\left(\frac{|y|^{n+1}}{(n+1)!}, \frac{2|y|^n}{n!}\right) \tag{2.3}$$

Applying the lemma to $y = tX$ gives

$$\left| \phi_X(t) - \sum_{m=0}^{n} \frac{\mathbb{E}(itX)^m}{m!} \right| \leq \mathbb{E} \min \left(\frac{|y|^{n+1}}{(n+1)!}, \frac{2|y|^n}{n!} \right) \tag{2.4}$$

$$= \frac{|t|^n}{n!} \mathbb{E} \min \left(\frac{|t||X|^{n+1}}{n+1}, 2|X|^n \right) \tag{2.5}$$

Corollary 2.2. *Suppose* $\mathbb{E}|X|^n < \infty$. *Then* $\phi_X(t) = \sum_{m=0}^{n} \frac{\mathbb{E}(itX)^m}{m!} + o(|t|^n)$ *as* $t \to \infty$.

Proof. Define the RV $Z_t := \min\left(\frac{|t||X|^{n+1}}{n+1}, 2|X|^n\right)$. $Z_t \xrightarrow{\text{a.s.}} 0$ as $t \to \infty$, dominated by $2|X|^n$ integrable. This implies $\mathbb{E}Z_t \to 0$ as $t \to 0$. $\square$