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STAT 210A
Lecture #16
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1 Review

$H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. LRT is MP.

$H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$ MLR in $T(X) \implies$ UMP test rejects when $T(x)$ large.

$H_0 : \theta = \theta_0$ vs $H_1 ; \theta \neq \theta_0$. Exponential families $\implies$ UMPU rejects when $T(x)$ extreme

2 Confidence sets/invertals

Model $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$

Definition 2.1. $C(X)$ is a $1-\alpha$ *confidence set* for $g(\theta)$ if $\forall\theta$

$$P_\theta(g(\theta) \in C(X)) \geq 1 - \alpha \tag{2.1}$$

3 Duality of tests and intervals

Suppose we have a level-α test $\phi_{\theta_0}(x)$ of $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$ for each $\theta_0 \in \Theta$. Assume non-randomized.

Let $C(x) = \{\theta : \phi_\theta(x) = 0\}$. $C(x)$ is a $1-\alpha$ confidence set for θ.

$$P_\theta(\theta \notin C(x)) = P_\theta(\underbrace{\phi_\theta(x)}_{\text{level-}\alpha\text{ test}} = 1) \leq \alpha \tag{3.1}$$

Alternatively, suppose $C(x)$ is a $1-\alpha$ confidence set for θ. To test $H_0 : \theta = \theta_0$ vs $H_1 ; \theta \neq \theta_0$

$$\phi(x) = 1_{\theta_0 \notin C(x)} \tag{3.2}$$

Example 3.1. $X \sim \text{Exp}(\theta) = \theta^{-1} e^{-x/\theta}$. CDF is $1 - \exp^{-x/\theta}$, $x > 0$.

Equal tailed test: $\phi_\theta(x)$ rejects θ unless

$$-\theta \log(1 - \alpha/2) \leq X \leq -\theta \log(\alpha/2) \tag{3.3}$$

$\iff$ reject θ unless

$$-x^{-}1 \log(1 - \alpha/2) \leq \theta^{-1} \leq -x^{-1} \log(\alpha/2) \tag{3.4}$$

Hence

$$C(x) = x \left(\frac{-1}{\log(\alpha/2)}, \frac{-1}{\log(1-\alpha/2)} \right) \tag{3.5}$$

4 Testing with nuisance parameters

$\mathcal{P} = \{P_{\theta,\zeta} : (\theta,\zeta) \in \Omega \subset \mathbb{R}^{r+s}\}$. $\theta \in \mathbb{R}^s$, $\zeta \in \mathbb{R}^r$.

$H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$.

Presence of ζ affects the distribution of the data, so power of test $\phi(\theta)$ depends on ζ. Goal is to control the dependence of the power of the test on the parameter ζ.

Example 4.1. $X_1, \cdots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$. $\sigma^2 > 0$ unknown.

$H_0 : \mu = 0$ vs $H_1 : \mu \neq= 0$. Nuisance parameter in this case is σ^2.

Example 4.2.

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2) \tag{4.1}$$

$$Y_1, \cdots, Y_n \overset{\text{iid}}{\sim} \mathcal{N}(\nu, \sigma^2) \tag{4.2}$$

$H_0 : \mu \leq \nu$ vs $H_1 : \mu > \nu$

$\theta = \mu - \nu$ ($\implies H_0 : \theta \leq 0$). $\zeta = (\mu + \nu, \sigma^2)$

Example 4.3. $X_i \overset{\text{indep}}{\sim} \text{Pois}(\lambda_i)$, $i = 1, 2$.

$H_0 :_1 \leq \lambda_2$ vs $H_1 : \lambda_1 > \lambda_2$.

$\theta = \lambda_1 / \lambda_2$. $H_0 : \theta \leq 1$. Nuisance λ_1 or $\lambda_1 \lambda_2$ or ...

Example 4.4. $X_1, \cdots, X_n \overset{\text{iid}}{\sim} P$. $Y_1, \cdots, Y_m \overset{\text{iid}}{\sim} Q$.

$H_0 : P = Q$ vs $H_1 : P \neq Q$.

4.1 Multi-parameter exponential families

Model:

$$p_{\theta,\zeta}(x) = e^{\theta T(X) + \zeta' U(X) - A(\theta,\zeta)} h(x)$$

. (4.3)

$H_0 : \theta = \theta_0$ or $H_1 : \theta \leq \theta_0$.

Under the null hypothesis, $e^{\theta T(X)}$ is constant so only $e^{\zeta' U(X)}$ is the complete sufficient statistic; the power of the test does not depend on ζ:

$$p_{\theta,\zeta}(x \mid U(x) = u) = \frac{e^{\theta T(x) + \cancel{\zeta' U(x)} - \cancel{A(\theta,\zeta)}} h(x) 1_{U(x)=u}}{\int_{U(z)=u} e^{\theta T(z) + \cancel{\zeta' u} - \cancel{A(\theta,\zeta)}} h(z) dz} \tag{4.4}$$

$$= e^{\theta T(x) - \tilde{A}_u(\theta)} h(x) 1_{U=u} \tag{4.5}$$

Example 4.5.

$$p_\lambda(x) = \lambda_1^{x_1}\lambda_2^{x_2}e^{-\lambda_1-\lambda_2}\frac{1}{x_1!x_2!} \tag{4.6}$$

$$= e^{x_1\log\lambda_1+x_2\log\lambda_2-\lambda_1-\lambda_2}\frac{1}{x_1!x_2!} \tag{4.7}$$

$$\propto \underbrace{e^{(X_1-X_2)}}_{T(X)}\underbrace{e^{\left(\frac{\log\lambda_1-\log\lambda_2}{2}\right)}}_{\theta}\underbrace{e^{(X_1+X_2)}}_{U(x)}\underbrace{e^{\left(\frac{\log\lambda_1+\log\lambda_2}{2}\right)}}_{\zeta}e^{\frac{1}{x_1!x_2!}} \tag{4.8}$$

Condition on $X_1 + X_2 = u$:

$$p_\theta(x \mid x_1 + x_2 = u) \propto e^{(x_1-x_2)\theta}\frac{u!}{x_1!x_2!} \tag{4.9}$$

$$p_\theta(x \mid x_2 = u - x_1) = e^{(2x_1-u)\theta}\binom{u}{x_1} \tag{4.10}$$

$$\propto e^{x_1\log\frac{\lambda_1}{\lambda_2}}\binom{u}{x_1} \tag{4.11}$$

$$= \mathrm{Binom}(u, \frac{\lambda_1}{\lambda_1+\lambda_2}) \tag{4.12}$$

Reject when $T(X) = X_1 - X_2$ large $\iff$ Reject when X_1 large (conditioned on U).
For $H_0 : \lambda_1 \leq \lambda_2$ reject if X_1 large compared to $\mathrm{Binom}(u, 1/2)$.

For a composite null, conditioning on a sufficient statistic can reduce composite null to a simple null.

4.2 UMPU Tests with nuisance parameters

Theorem 4.6. *Consider testing*

(a) $H_0 : \theta = \theta_0$ *vs* $\theta \neq \theta_0$

(b) $H_0 : \theta \leq \theta_0$ *vs* $\theta > \theta_0$

in model $\mathcal{P}$ of the form section 4.1.

Ω open, $\mathcal{P}$ full rank.

There is a UMPU test of the form $\phi^*(x) = \psi(T(x), U(x))$ where

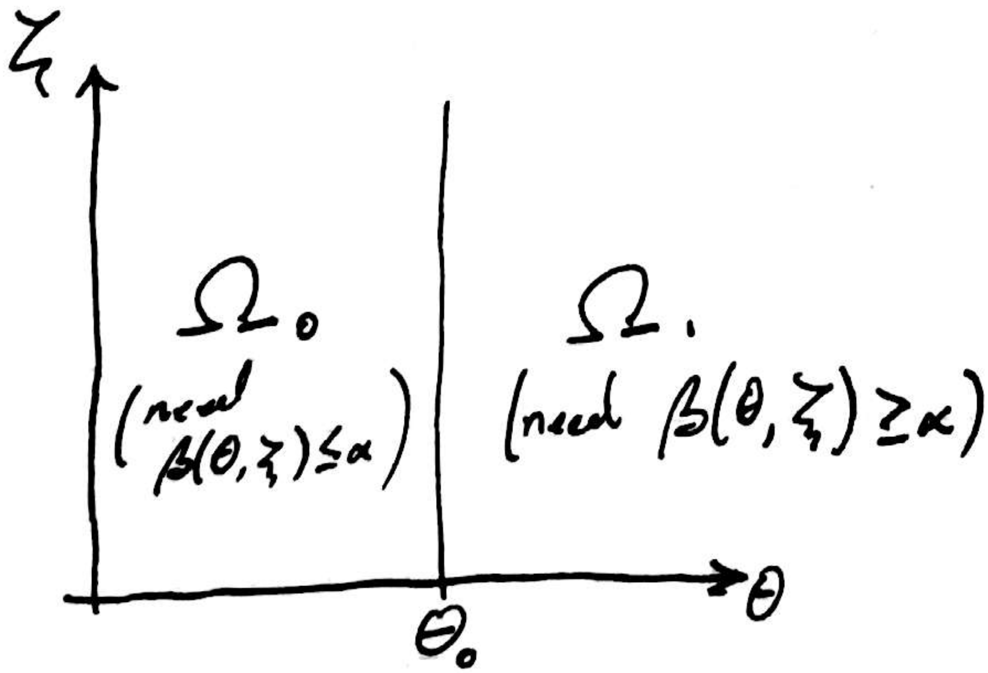
$$\psi(t,u) = \begin{cases} 1, & \text{if } t < c_1(u) \text{ or } t > c_2(u) \\ 0, & \text{if } t \in (c_1(u), c_2(u)) \\ \gamma_i, & \text{if } t = c_i(x) \end{cases} \tag{4.13}$$

for (a), and

$$\psi(t,u) = \begin{cases} 1, & \text{if } t > c(u) \\ 0, & \text{if } t < c(u) \\ \gamma_i, & \text{if } t = c(u) \end{cases} \tag{4.14}$$

for (b), where $c(\cdot)$, $\gamma(\cdot)$ chosen so

$$\mathbb{E}_{\theta_0}[\phi(x) \mid U(x) = u] = \alpha \quad \forall u \;\; \textit{for (a), (b)} \tag{4.15}$$
$$\mathbb{E}_{\theta_0}[T(x)(\phi(x) - \alpha) \mid U(x) = u] = 0 \quad \forall u \;\; \textit{for (a)} \tag{4.16}$$



Proof. (1) ϕ^* is unbiased, level α

(2) Any unbiased test has $\beta(\theta_0, \zeta) = \alpha \quad \forall \zeta$ (by continuity of β)

(3) Power $\equiv \alpha$ on boundary $\implies \mathbb{E}[\phi(X) \mid U(X) = u] \overset{\text{a.s.}}{=} \alpha$ (by completeness of $U(X)$)

(4) $\phi^*(X)$ optimal among all unbiased tests with $E_{\theta_0}[\phi \mid U] \overset{\text{a.s.}}{=} \alpha$ (by reduction to 1-param problem)

□