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STAT 205A
Lecture #11
9/29/2016

1 Miscellaneous measure-theory related topics

Theorem 1.1 (Kolmogorov 0-1 Law). $(X_1, X_2, \cdots)$ *with any range space.*

Define $\tau_n = \sigma(X_n, X_{n+1}, X_{n+2}, \cdots)$.

Define the tail σ-field

$$\tau = \bigcap_{n \geq 1} \tau_n \tag{1.1}$$

If $(X_1, X_2, \cdots)$ *independent, then* τ *is* trivial *i.e.*

$$\forall A \in \tau, P(A) \in \{0, 1\} \tag{1.2}$$

Proof. Define $\mathcal{F}_{n-1} = \sigma(X_1, \cdots, X_{n-1})$.

$$\begin{aligned}
& \mathcal{F}_{n-1} \text{ is independent of } \tau_n && (1.3)\\
& \implies \mathcal{F}_{n-1} \text{ independent of } \tau && (1.4)\\
& \implies \text{ field } \cup_n \mathcal{F}_n \text{ independent of } \tau && (1.5)\\
\text{TODO: Dynkin? } \pi - \lambda \text{ Lemma } & \implies \sigma(\cup_n \mathcal{F}_n) = \sigma(X_1, X_2, \cdots) \text{ independent of } \tau && (1.6)\\
& \implies \tau \text{ is independent of } \tau && (1.7)\\
A \in \tau & \implies P(A \cap A) = P(A)P(A) = P(A) && (1.8)\\
x^2 = x & \implies x = 0 \text{ or } 1 && (1.9)
\end{aligned}$$

□

Lemma 1.2. *If* $\mathcal{A}$ *is a trivial* σ*-field,* X *an* $\mathcal{A}$*-measurable RV with values in* $[-\infty, \infty]$*, then* $\exists x_0$ *such that* $P(X = x_0) = 1$.

Proof. Define $x_0 = \inf\{x : P(X \leq x)\}$. TODO: Finish □

1.1 Modes of convergence for $\mathbb{R}$-valued RVs

Almost-sure convergence $X_n \xrightarrow{\text{a.s.}} X$ means $P(\omega : X_n(\omega) \to X(\omega)\} = 1$

Converges in L^p space $X_n \xrightarrow{L^p} X$ means $\mathbb{E}|X_n - X|^p \to 0$ and $\sup_n \mathbb{E}|X_n|^p < \infty$ $(1 \leq p < \infty)$

Converges in probability $X_n \xrightarrow{\text{p}} X$ means $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$, $\forall \epsilon > 0$

(a) $\xrightarrow{L^p}$ implies $\xrightarrow{p}$, not conversely

Example 1.3. U uniform on $[0,1]$. $X_n = n1_{U \le \frac{1}{n}}$. $X_n \xrightarrow{p} 0$, $\mathbb{E}X_n = 1$, but $X_n \not\to 0$

(b) $\xrightarrow{\text{a.s.}}$ implies $\xrightarrow{p}$, not conversely

Proof. $X_n \xrightarrow{\text{a.s.}} X$ means

$$0 = P(|X_n - X| \ge \epsilon \text{ i.o.}) \ge \limsup_n P(|X_n - X| \ge \epsilon) = 0 \tag{1.10}$$

$$\implies X_n \xrightarrow{p} X \tag{1.11}$$

□

Example 1.4. Take independent events (A_n) with $P(A_n) \to 0 \implies 1_{A_n} \xrightarrow{p} 0$.

$$\sum_n P(A_n) = \infty \underset{\text{(BC 2)}}{\implies} P(A_n \text{ i.o.}) = 1 \tag{1.12}$$

$$\implies 1_{A_n} \overset{\text{a.s.}}{\not\to} 0 \tag{1.13}$$

Recall the dominated convergence theorem, restated here

Theorem 1.5. *If $X_n \xrightarrow{a.s.} X$, if $\exists Y \ge 0$ with $\mathbb{E}Y < \infty$ and $|X_n| \le Y$ for all n, then $\mathbb{E}|X_n - X| \to 0$ and $\mathbb{E}X_n \to \mathbb{E}X$.*

Lemma 1.6. *If $X_n \xrightarrow{p} X$ then $\exists$ subsequence $n_1 < n_2 < n_3 < \cdots$ such that $X_{n_j} \xrightarrow{a.s.} X$ as $j \to \infty$.*

Proof. Choose n_j inductively: $n_j = \min\left\{n > n_{j-1} : P(|X_n - X| \ge 2^{-j}) \le 2^{-j}\right\}$.

$$\sum_j P(|X_{n_j} - X| \ge 2^{-j}) \le \frac{1}{2} < \infty \tag{1.14}$$

$$\underset{\text{(BC 1)}}{\implies} |X_{n_j} - X| \le 2^{-j} \text{ ult. in } j \text{ a.s.} \tag{1.15}$$

$$\iff X_{n_j} \xrightarrow{\text{a.s.}} X \tag{1.16}$$

□

Remark 1.7. Related to fact "a.s. convergence" not convergence in a metric.

Corollary 1.8. *The dominated convergence theorem (DCT) remains true under assumption $X_n \xrightarrow{p} X$.*

Proof. Suppose false: $\exists \varepsilon > 0$ and a subsequence $m_1 < m_2 < \cdots$ such that $\mathbb{E}|X_{m_j} - X| \ge \varepsilon$ for all j.

Now $X_{m_j} \xrightarrow{p} X$ so lemma implies $\exists$ subsequence n_j of m_j such that $X_{n_j} \xrightarrow{\text{a.s.}} X$ and $\mathbb{E}|X_{n_j} - X| \ge \epsilon$ for all j, contradicting DCT. □

1.2 2 views of integration calculus

(1) Given f, a, b, $\int_a^b f(x)dx =$ a number

(2) $F(x) = \int_0^x f(y)dy \iff f(x) = \frac{dF(x)}{dx}$
An operator $f \mapsto F$, opposite of $F \mapsto F'$.

The analogue of $\frac{dF(x)}{dx}$ involves measures, not functions.

Consider a measurable space $(S, \mathcal{S})$. Fix a σ-finite μ. Consider measurable $h : S \to [0, \infty)$. For $A \in \mathcal{S}$, define $\nu(A) = \int_A h d\mu \leq \infty$.

Proposition 1.9. *ν is a σ-finite measure on $(S, \mathcal{S})$.*

Proof. μ σ-finite $\implies \exists A_n \uparrow S, \mu(A_n) < \infty$.
Define $B_n = A_n \cap \{s : h(s) \leq n\}$. Then $B_n \uparrow S$ and $\nu(B_n) \leq n, \mu(A_n) \leq \infty$. □

The two measures ν and μ have a relationship:

Definition 1.10. ν is *absolutely continuous* wrt μ, written $\nu \ll \mu$, if

$$\forall A \in \mathcal{S} : \mu(A) = 0 \implies \nu(A) = 0 \tag{1.17}$$

Theorem 1.11 (Radon-Nikodym)**.** *If μ and ν are σ-finite measures on $(S, \mathcal{S})$, if $\nu \ll \mu$, then $\exists$ measurable $h : S \to [0, \infty]$ such that*

$$\forall A \in \mathcal{S} : \nu(A) = \int_A h d\mu \tag{1.18}$$

Proof. Two ways: (1) See MT Text, (2) Via martingales, later □

Definition 1.12. We write h from theorem 1.11 as $h = \frac{d\nu}{d\mu}$ and call it the *Radon-Nikodym density* of ν with respect to μ

In particular, if μ is a probability measure on $\mathbb{R}^1$, $\mu \ll$ Leb, then $h = \frac{d\mu}{d\text{Leb}}$ exists: the *density function*.

1.3 Probability measures on $\mathbb{R}$

Know 1-1 correspondence between probability measures μ and distribution functions F

$$F(x) = \mu(-\infty, x] \tag{1.19}$$

There are three basic types of PMs μ

(1) $\mu \ll$ Leb, so can be described by density f

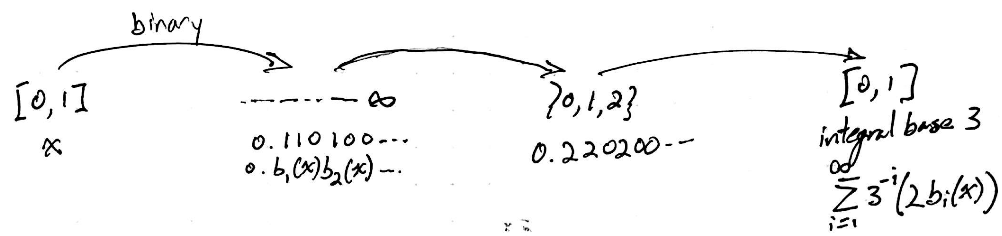
$$F(x) = \int_{-\infty}^x f(y)dy \tag{1.20}$$

Here, f can be any measurable function with $f \geq 0$ and $\int_{-\infty}^\infty f(x)dx = 1$

(2) μ is *purely atomic* (discrete):

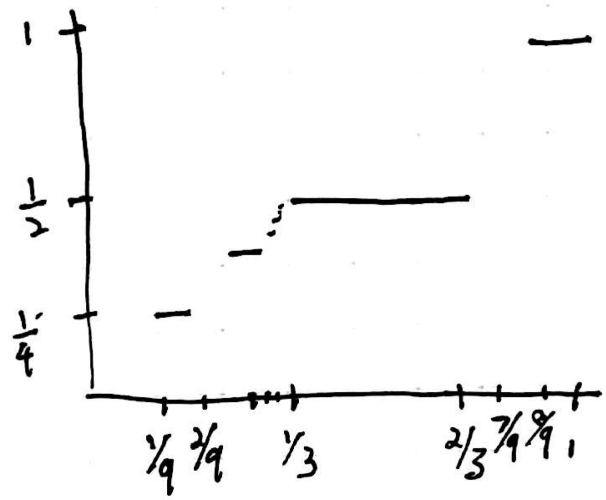
$\exists$ countable set of atoms $x_1, x_2, \cdots$ and $\sum_i \mu(\{x_i\}) = 1, \implies \mu(\mathbb{R} \setminus \cup_i \{x_i\}) = 0$

Example 1.13 (Uniform distribution on Cantor set)**.** $x \in [0,1]$, write out binary expansion $x = 0.10110100\ldots = 0.b_1(x)b_2(x)b_3(x)\ldots$.



Put together gives measurable map $H : [0,1] \to [0,1]$. Take U uniform $[0,1]$. What is the distribution of $H(U)$?

$$F(x) = P(H(U) \leq u) \tag{1.21}$$



The set of possible values of H = "base-3 expansion has no 1" = cantor set = $\mathcal{C}$ and $\text{Leb}(\mathcal{C}) = 0$ while $P(H(U) \in C) = 1$.

(3) μ is a *singular measure*:

$\exists A$ such that $\text{Leb}(A) = 0$, $\mu(A) = 1$ but no atoms.

Proposition 1.14. *Any PM μ on $\mathbb{R}^1$ has a unique decomposition*

$$\mu = a_1\mu_1 + a_2\mu_2 + a_3\mu_3 \tag{1.22}$$

where μ_1 admits a density, μ_2 is purely atomic, and μ_3 is singular, $a_i \geq 0$, $\sum_1^3 a_i = 1$.