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STAT 205B
Lecture #5
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1 Review

Theorem 1.1 (Convergence Theorem). *Let X_n have characteristic function ϕ_n.*

If $\phi_n(t) \to \phi_\infty(t)$ as $n \to \infty$, each t, and if $\phi_\infty(t)$ is the characteristic function of some X_∞, then $X_n \xrightarrow{d} X_\infty$.

Lemma 1.2 (Technical bound using moments). *Suppose $\mathbb{E}|X|^n < \infty$. Then*

$$\left| \phi_X(t) - \sum_{m=0}^{n} \frac{\mathbb{E}[itX]^m}{m!} \right| = o(|t|^n) \text{ as } t \to \infty \tag{1.1}$$

$$n = 1 \implies |\phi_X(s) - (1 + is\theta)| = o(|s|) \tag{1.2}$$

$$n = 2 \implies \left| \phi_X(s) - \left(1 + \frac{s^2\sigma^2}{2}\right) \right| = o(s^2) \tag{1.3}$$

2 Using CFs

To demonstrate the technique we will use in more elaborate forms later on.

Theorem 2.1 (Weak law of large numbers). *Let $X_1, X_2, \ldots$ be IID with $\mathbb{E}X = \theta < \infty$, $S_n = \sum_{i=1}^n X_i$. Then $n^{-1}S_n \xrightarrow{d} \theta$., and hence (because θ is a constant) $n^{-1}S_n \xrightarrow{p} \theta$.*

Proof. Prob. meas. δ_θ has CF $e^{i\theta t}$ so by theorem 1.1 suffices to show $\phi_{n^{-1}S_n}(t) \to e^{i\theta t}$ as $n \to \infty$, t fixed.

Motivated by $z_n \to z \in \mathbb{C} \implies (1 + n^{-1}z_n)^n \to e^z$, write

$$\phi_{n^{-1}S_n}(t) = \left(\phi_X(n^{-1}t)\right)^n \tag{2.1}$$

$$= \left(1 + \frac{n(\phi_X(n^{-1}t) - 1)}{n}\right)^n \tag{2.2}$$

So it suffices to show $n(\phi_X(n^{-1}t) - 1) \to i\theta t$. Applying lemma 1.2 with $s = t/n$

$$n(\phi_X(n^{-1}t) - 1) = n\left(i\frac{t}{n}\theta + o\left(\frac{|t|}{n}\right)\right) \tag{2.3}$$

$$= it\theta + no\left(\frac{|t|}{n}\right) \to i\theta t \tag{2.4}$$

□

Remark 2.2. Proof shows that $\phi_X'(0) = \theta$ is sufficient for WLLN. In fact (not obvious), this is also necessary.

Remark 2.3. $\mathbb{E}X = \theta \implies \phi_X'(0) = \theta$, but not conversely.

Theorem 2.4 (IID Central Limit Theorem)**.** *$(X_i, i \geq 1)$ IID, $\mathbb{E}X = \mu$, $Var(X) = \sigma^2 < \infty$. Then*

$$\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2) \tag{2.5}$$

Proof. WLOG take $\mu = 0$. Suffices to show $\phi_{n^{-1/2}S_n}(t) \to \exp(-\sigma^2 t^2/2)$.

$$\phi_{n^{-1/2}S_n}(t) = \left(\phi_X(n^{-1/2}t)\right)^n = \left(1 + \frac{n(\phi_X(n^{-1/2}t) - 1)}{n}\right)^n \tag{2.6}$$

So by definition of e suffices to show $n(\phi_X(n^{-1/2}t) - 1) \to \sigma^2 t^2/2$. Taking $s = n^{-1/2}t$ in lemma 1.2

$$n(\phi_X(n^{-1/2}t) - 1) = n\left(\frac{t^2}{n}\frac{\sigma^2}{2} + o\left(\frac{t^2}{n}\right)\right) \tag{2.7}$$

$$= t^2\sigma^2/2 + no\left(\frac{t^2}{n}\right) \to t^2\sigma^2/2 \tag{2.8}$$

□

There are many variations of CLT, one modification removes the identical distribution assumption.

Theorem 2.5 (Lindeberg's Theorem)**.** *For each n let $X_{n,1}, X_{n,2}, \dots, X_{n,n}$ be independent, $\mathbb{E}X_{n,m} = 0$, $VarX_{n,m} = \sigma_{n,m}^2 < \infty$.*

Write $S_n = \sum_{m=1}^n X_{n,m}$, so $\mathbb{E}S_n = 0$ and $\sigma_n^2 = \sum_{m=1}^n \sigma_{n,m}^2 = Var(S_n)$.

Suppose

(a) $\sigma_n^2 \to \sigma^2 < \infty$ as $n \to \infty$

(b) (Lindenberg's condition or uniformly asymptotically negligible, UAN)

$$\lim_{n\to\infty} \sum_{m=1}^n \mathbb{E}[X_{n,m}^2 1_{|X_{n,m}|>\varepsilon}] = 0, \quad \forall \varepsilon > 0 \tag{2.9}$$

Then $S_n \xrightarrow{d} N(0, \sigma^2)$.

Proof. Let $\phi_{n,m}(t)$ be the CF of $X_{n,m}$.

From last time (Durrett 3.3.7), we know

$$\left|\phi_{n,m}(t) - \left(1 - \frac{t^2\sigma_{n,m}^2}{2}\right)\right| \leq \mathbb{E}\min\left(\frac{|tX_{n,m}|^3}{6}, |tX_{n,m}|^2\right) \tag{2.10}$$

Cheap trick: if $|X| \leq \varepsilon$, then $|X|^3 \leq \varepsilon X^2$. Splitting $X_{n,m}$ into two parts, one where $X_{n,m} < \varepsilon$ and another where $X_{n,m} \geq \varepsilon$

$$\left|\phi_{n,m}(t) - \left(1 - \frac{t^2\sigma_{n,m}^2}{2}\right)\right| \leq \frac{\varepsilon|t|^3}{6}\mathbb{E}[X_{n,m}^2] + |t|^2\mathbb{E}[X_{n,m}^2 1_{|X_{n,m}|\geq\varepsilon}] \tag{2.11}$$

$$\limsup_n \underbrace{\sum_{m=1}^n \left|\phi_{n,m}(t) - \left(1 - \frac{t^2\sigma_{n,m}^2}{2}\right)\right|}_{B_n(t)} \leq \frac{\varepsilon|t|^3}{6}\sigma^2 + 0 \tag{2.12}$$

where the $+0$ comes from 2.9. After letting $\varepsilon \downarrow 0$, we have $B_n(t) \to 0$ as $n \to \infty$.

Claim. *(i)* $\max_m \sigma_{n,m}^2 \to 0$ *as* $n \to \infty$

(ii) $\sum_m \sigma_{n,m}^4 \to 0$ *as* $n \to \infty$

Proof. (i):

$$\sigma_{n,m}^2 = \mathbb{E}X_{n,m}^2 1_{|X_{n,m}|\geq\varepsilon} + \mathbb{E}X_{n,m}^2 1_{|X_{n,m}|\leq\varepsilon} \tag{2.13}$$

$$\leq \sum_m [\cdot] + \varepsilon^2 \tag{2.14}$$

$$\limsup_n \max_m \sigma_{n,m}^2 \leq \underbrace{0}_{\text{by 2.9}} + \varepsilon^2 \tag{2.15}$$

Let $\varepsilon \downarrow 0$

(ii):

$$\sum_m \sigma_{n,m}^4 \leq \left(\max_m \sigma_{n,m}^2\right) \sum_m \sigma_{n,m}^2 \to 0 \tag{2.16}$$

by (a). □

Lemma 2.6. *If $w_i, z_i \in \mathbb{C}$, $|w_i| \leq 1$, $|z_i| \leq 1$, then*

$$\left|\prod_{i=1}^n z_i - \prod_{i=1}^n w_i\right| \leq \sum_{i=1}^n |w_i - z_i| \tag{2.17}$$

Proof.

$$|z_1 z_2 \ldots z_i w_{i+1} \ldots w_n - z_1 \ldots z_{i+1} w_{i+2} \ldots w_n| = |(z_{i+1} - w_{i+1}) \cdot A| \tag{2.18}$$

$$\leq |z_{i+1} - w_{i+1}| \tag{2.19}$$

where $|A| \leq 1$. □

Consider $\phi_{S_n}(t) = \prod_{m=1}^n \phi_{n,m}(t)$. Lemma 2.6 implies

$$\left|\phi_{S_n}(t) - \prod_{m=1}^n \left(1 - \frac{t^2\sigma_{n,m}^2}{2}\right)\right| \leq B_n(t) \to 0 \tag{2.20}$$

using (i).

The goal is to prove $\phi_{S_n}(t) \to \exp(-t^2\sigma^2/2)$, so it's enough to prove

$$\prod_{m=1}^{n}\left(1-\frac{t^2\sigma_{n,m}^2}{2}\right) \to \exp(-t^2\sigma^2/2) \tag{2.21}$$

Lemma 2.7. *Let $a_{n,m} \in \mathbb{R}$. If*

(1) $\sum_m a_{n,m} \to a$ as $n \to \infty$

(2) $\sum_m a_{n,m}^2 \to 0$

Then $\prod_{m=1}^{n}(1-a_{n,m}) \to e^{-a}$.

Proof. We know $\max_m |a_{n,m}| \to 0$ by (2), so

$$|\log(1-x)+x| \leq Cx^2 \quad \text{for } |x| \leq \frac{1}{2} \tag{2.22}$$

$$\implies \left|\sum_{m=1}^{n}\log(1-a_{n,m}) + \sum_{m=1}^{n} a_{n,m}\right| \leq C\sum_m a_{n,m}^2 \quad \text{for large } n \tag{2.23}$$

$$\to 0 \text{ as } n \to \infty \tag{2.24}$$

$$\implies \log\prod_{m=1}^{n}(1-a_{n,m}) \to -a \tag{2.25}$$

$$\implies \prod_{m=1}^{n}(1-a_{n,m}) \to e^{-a} \tag{2.26}$$

□

(1) of lemma is assumption (a) of theorem and (2) of lemma is (ii) of claim, so applying the lemma to $a_{n,m} = -t^2\sigma_{n,m}^2/2$ yields the desired result. □

Theorem 2.8 (Equivalent form of Lindeberg CLT)**.** *For each n, let $X_{n,m}$, $1 \leq m \leq n$, be independent, $\mathbb{E}X_{n,m} = 0$. Let $S_n = \sum_{m=1}^{n} X_{n,m}$ and $s_n^2 = Var(S_n) = \sum_{m=1}^{n} Var(X_{n,m})$.*

Suppose

$$\sum_{m=1}^{n}\mathbb{E}\left(\frac{X_{n,m}^2}{S_n^2}1_{|X_{n,m}|\geq \varepsilon S_n}\right) \to 0 \quad \text{as } n \to \infty \tag{2.27}$$

Then $\frac{S_n}{s_n} \xrightarrow{d} N(0,1)$.

This follow from previous theorem applied with $\hat{X}_{n,m} = \frac{X_{n,m}}{s_n}$.

Looks more like IID version.