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STAT 205A
Lecture #19
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1 Last class

(X_n) sub-MG wrt $(\mathcal{F}_n)$, i.e. $\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] \geq X_{n-1}$.

(H_n) predictable process, bounded. Interpret $H_n = \#$ shares held on day n.

Define $Y = H \cdot X$ by $Y_0 = 0$, $\Delta_n^Y = H_n \Delta_n^X$

Then (Y_n) is a sub-MG provided $H_n \geq 0$

Remark 1.1. If $\mathbb{E}[Z 1_A] \geq 0$ for all $A \in \mathcal{G}$, then $\mathbb{E}[Z \mid \mathcal{G}] \geq 0$ a.s.

2 Today

Corollary 2.1. *(X_n) a sub-MG. $0 \leq T_1 \leq T_2 \leq t_0$ stopping times. Then $\mathbb{E}[X_{T_2} \mid \mathcal{F}_{T_1}] \geq X_{T_1}$*

Proof. Fix event $A \in \mathcal{F}_{T_1}$. Strategy:

> If A happens, buy 1 share at T_1, sell at T_2.
>
> If A doesn't happen, do nothing.

In math: $H_n = 1_A 1_{T_1 < n \leq T_2}$.

Want to check H is predictable, that is

$$A \cap \{T_1 < n \leq T_2\} \in \mathcal{F}_{n-1} \tag{2.1}$$

$$= A \cap \underbrace{\{T_1 \leq n-1\}}_{\substack{\in \mathcal{F}_{n-1} \\ \text{def of } A \in \mathcal{F}_{T_1}}} \setminus A \cap \underbrace{\{T_2 \leq n-1\}}_{\in \mathcal{F}_{n-1}} \text{ because } T_2 \geq T_1 \tag{2.2}$$

So H is predictable.

So (Y_n) is a sub-MG, $Y_n = (X_{T_2 \wedge n} - X_{T_1 \wedge n}) 1_A$

$$\implies \mathbb{E} Y_{t_0} \geq \mathbb{E} Y_0 = 0 \tag{2.3}$$

$$\implies \mathbb{E}[(X_{T_2} - X_{T_1}) 1_A] \geq 0 \qquad \forall A \in \mathcal{F}_{T_1} \tag{2.4}$$

$$\implies \mathbb{E}[X_{T_2} - X_{T_1} \mid \mathcal{F}_{T_1}] \geq 0 \text{ a.s.} \qquad \textbf{??} \tag{2.5}$$

□

3 Optional Sampling Theorem (OST)

Theorem 3.1 (Basic version)**.** *If* (X_n) *is a (sub-)MG,*
$0 = T_0 \le T_1 \le T_2 \le \cdots$ *are stopping times,*
$T_i \le t_i$ *(constant) for every* i*,*
then $(X_{T_i}, i = 0, 1, 2, \ldots)$ *is a (sub-)MG wrt* $(\mathcal{F}_{T_i}, i = 0, 1, 2 \ldots)$.

In particular, $\mathbb{E}X_T \ge \mathbb{E}X_0$ for sub-MG, $= \mathbb{E}X_0$ for MG.
Many other versions without restriction $T \le t_0$ exist.

4 Maximal inequalities

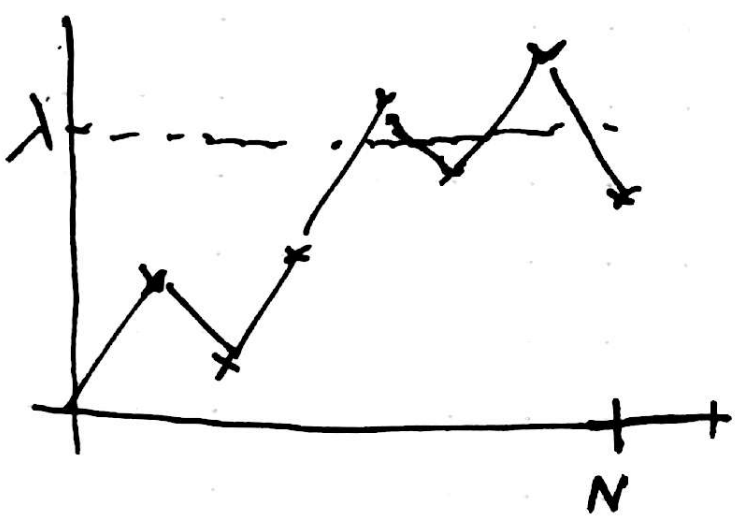
$X_N^* = \max(X_0, X_1, \ldots, X_N)$

always $P(X_N^* \ge x) \le \sum_{n=0}^{N} P(X_n \ge x)$

MG get better than above

if independent $P(X_N^* \ge x) = 1 - \prod_{n=0}^{N} P(X_n \le x)$

Lemma 4.1. (X_n) *a super-MG,* $X_n \ge 0$ *a.s.*
Write $X^* = \sup_n X_n$*, so* $X_N^* \uparrow X^*$ *as* $N \to \infty$*.*
Then $P(X^* \ge \lambda) \le \frac{\mathbb{E}X_0}{\lambda}$*, all* $\lambda > 0$



Proof. Define $T = \min\{n : X_n \ge \lambda\}$. Apply OST to 0 and $T \wedge N$.

$$\implies \mathbb{E}X_0 \geq \mathbb{E}X_{T\wedge N} = \mathbb{E}X_T 1_{T\leq N} + \mathbb{E}X_n 1_{T>N} \tag{4.1}$$
$$\geq \lambda P(T \leq N) + 0 \tag{4.2}$$
$$\implies P(T \leq N) \leq \lambda^{-1}\mathbb{E}X_0 \tag{4.3}$$
$$\implies P(X_N^* \geq \lambda) \leq \lambda^{-1}\mathbb{E}X_0 \tag{4.4}$$
$$N \to \infty \implies P(X^* > \lambda) \leq \lambda^{-1}\mathbb{E}X_0 \tag{4.5}$$
$$\text{Apply to } \lambda_j \uparrow \lambda \text{ (check)} \implies P(X^* \geq \lambda) \leq \lambda^{-1}\mathbb{E}X_0 \tag{4.6}$$

□

Lemma 4.2 (Doob's L_1 maximal inequality)**.** *(X_n) sub-MG. For $\lambda > 0$*

$$\lambda P(X_N^*\lambda) \leq \mathbb{E}[X_n 1_{X_N^*\geq\lambda}] \leq \mathbb{E}X_N^+ = \mathbb{E}\max(X,0) \tag{4.7}$$

Proof. $T = \min\{n : X_n \geq \lambda\}$.
Apply OST to $T\wedge N$ and $N \implies \mathbb{E}X_{T\wedge N} \leq \mathbb{E}X_N$.
$\implies \mathbb{E}X_T 1_{T\leq N} + \cancel{\mathbb{E}X_n 1_{T>N}} \leq \mathbb{E}X_N 1_{TN} + \cancel{\mathbb{E}X_N 1_{T>N}}$
$X_T \geq \lambda \implies \lambda P(T \leq N) \leq \mathbb{E}X_N 1_{T\leq N} = \mathbb{E}X_n 1_{X_N^*\geq\lambda}$ □

Corollary 4.3. *If (X_n) is a MG then (because $Y_n = |X_n|$ is a sub-MG)*

$$\lambda P(\max_{0\leq n\leq N} |X_n| \geq) \leq \mathbb{E}|X_n|/\lambda \tag{4.8}$$

Also, $Z_n = X_n^2$ is a sub-MG (provided $\mathbb{E}X_n^2 < \infty$).
Apply Lemma to (Z_n)

$$\lambda P(\max_{0\leq n\leq N} X_n^2 \geq \lambda) \leq \mathbb{E}X_N^2 \tag{4.9}$$
$$\lambda^2 P(\max_{0\leq n\leq N} X_n^2 \geq \lambda^2) \leq \mathbb{E}X_N^2 \tag{4.10}$$
$$P(\max_{0\leq n\leq N} |X_n| \geq \lambda) \leq \lambda^{-2}\mathbb{E}X_N^2 \tag{4.11}$$

Different bounds for same quantity (c.f. Markov/Chebyshev)

Lemma 4.4 (Doob's L_2 maximal inequality)**.** *(X_n) sub-MG.*

$$\mathbb{E}(0 \vee X_N^*)^2 \leq 4\mathbb{E}(X_N^+)^2 \tag{4.12}$$

Proof.

$$\underbrace{\mathbb{E}(0 \vee Z)^2}_{a} = 2\int_0^\infty \lambda P(Z \geq \lambda) d\lambda \tag{4.13}$$

$$\implies \mathbb{E}(0 \vee X_N^*)^2 = 2\int_0^\infty \lambda P(X_N^* \geq \lambda) d\lambda \tag{4.14}$$

$$\leq 2\int_0^\infty \mathbb{E}[X_N 1_{X_N^* \geq \lambda}] d\lambda \tag{4.15}$$

$$\leq 2\int_0^\infty \mathbb{E}[X_N^+ 1_{X_N^* \geq \lambda}] d\lambda \tag{4.16}$$

$$= 2\mathbb{E}\left[X_N^+ \int_0^\infty 1_{X_N^* \geq \lambda} d\lambda\right] \tag{4.17}$$

$$= 2\mathbb{E}[X_N^+(0 \vee X_N^*)] \tag{4.18}$$

$$\text{Cauchy-Schwarz} \leq 2\sqrt{\underbrace{\mathbb{E}[(X_N^+)^2]}_{b} \times \underbrace{\mathbb{E}[(0 \vee X_N^*)^2]}_{a}} \tag{4.19}$$

$a \leq 2\sqrt{ba} \implies a \leq 4b.$ □

If we use Hölder instead of Cauchy-Schwarz

$$\mathbb{E}[(0 \vee X_N^*)^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[(X_N^+)p] \quad 1 < p < \infty \tag{4.20}$$

Example 4.5 (*not* true for $p = 1$). $X_0 = 1$, simple symmetric random walk on $\mathbb{Z}$, stop at $T = \min\{n \geq 1 : X_n = 0\}$.

(X_n) is a MG. $\mathbb{E}X_n = 1\ \forall n$.
$X_N^* \uparrow X^* = \sup_n X_n$. Elementary $P(X^* \geq m) = m^{-1}$.
$\implies \mathbb{E}X^+ = \infty \implies \mathbb{E}X_N^* \uparrow \infty$.
$\therefore \mathbb{E}X_n = 1\ \forall N$, so cannot bound the ratio for $p = 1$.