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STAT 210A
Lecture #17
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1 Review

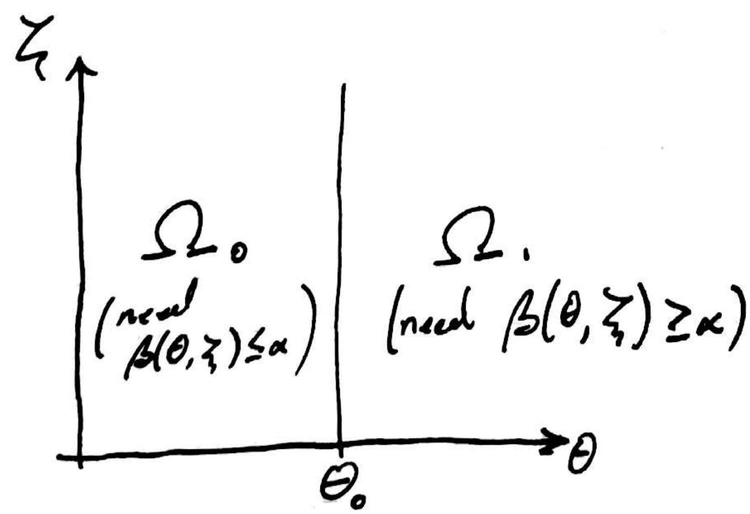
Testing with nuisance parameters $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$.

$$p_{\theta,\zeta}(x) = e^{\theta T(x) + \zeta' U(x) - A(\theta,\zeta)} h(x) \tag{1.1}$$

$$\phi^*(x) = \psi(T(x), U(x)) \tag{1.2}$$

$$\psi(t,u) = \begin{cases} 1, & \text{if } t > c(u) \\ \gamma(u), & \text{if } t = c(u) \\ 0, & \text{if } t < c(u) \end{cases} \tag{1.3}$$

$$\mathbb{E}_{\theta_0}[\phi \mid U] \overset{\text{a.s.}}{=} \alpha \tag{1.4}$$



2 Proof UMPU test for multi-param exp fam

(1) ϕ^* works, $\theta \leq \theta_0$

$$\mathbb{E}_{\theta,\zeta}[\phi^*(X)] = \mathbb{E}_{\theta,\zeta}[\mathbb{E}_\theta[\phi^* \mid U]] \leq \alpha \text{ if } \theta \leq \theta_0 \tag{2.1}$$

$$(\geq \alpha \text{ if } \theta \geq \theta_0) \tag{2.2}$$

(2) Recall Keener Theorem 2.4. Since $\mathbb{E}_{\theta,\zeta}[|\phi(X)|] < \infty$, $\beta(\theta,\zeta)$ is continuous, and all derivatives can be computed by differentiating under the integal.

$$\implies \beta_\phi(\theta_0,\zeta) = \alpha \quad \forall\zeta, \forall\phi \text{ level-}\alpha \text{ unbiased}$$

(3) Write $Q_{\theta_0} = \{q_\zeta(x) = p_{\theta_0,\zeta}(x)\}$. Then $q_\zeta(x) = e^{\zeta' U(x) - A(\theta_0,\zeta)} \underbrace{e^{\theta_0 T(x)} h(x)}$. $U(X)$ complete sufficient.

$$f(u) = \mathbb{E}_{\theta_0}[\phi(X) \mid U(X) = u] \tag{2.3}$$
$$\mathbb{E}_{\theta_0,\zeta}[f(U)] = \alpha \ \forall \zeta \tag{2.4}$$
$$\implies f(U) \overset{\text{a.s.}}{=} \alpha \tag{2.5}$$

(4) Suppose $\phi(X)$ is level-α given $U = u$

$$\mathbb{E}_{\theta,\zeta}[\phi(X)] = \mathbb{E}_{\theta,\zeta}[\mathbb{E}_\theta[\phi(X) \mid U]] \le \mathbb{E}_{\theta,\zeta}[\mathbb{E}_\theta[\phi^*(X) \mid U]] \le \mathbb{E}_{\theta,\zeta}[\phi^*(X)] \tag{2.6}$$

Step (2) says power is α on the boundary of $\Omega_0 = \{(\theta,\zeta) : \beta(\theta,\zeta) \le \alpha\}$.
Step (3) says conditional power is α on the boundary. U is complete because of some open set business wrt Ω_0.
Step (2)-(3) reduces to a 1-dimensional problem.
Step (4) shows that ϕ^* is uniformly most powerful.

3 Examples

Example 3.1 (Logistic Regression). $x_1, \ldots, x_n \in \mathbb{R}^d$.
$y_i \overset{\text{indep}}{\sim} \text{Bern}\left(\frac{e^{\beta' x}}{1+e^{\beta' x}}\right)$

$$p_\beta(y \mid x) = \prod_{i=1}^n e^{\beta' x_i y_i - A_i(\beta)} = e^{\beta'(x'y) - A(\beta)} \tag{3.1}$$

Test $H_0 : \beta_1 \le 0$ vs $\beta_1 > 0$
Condition on $X_2' y = u \iff$ Condition on y

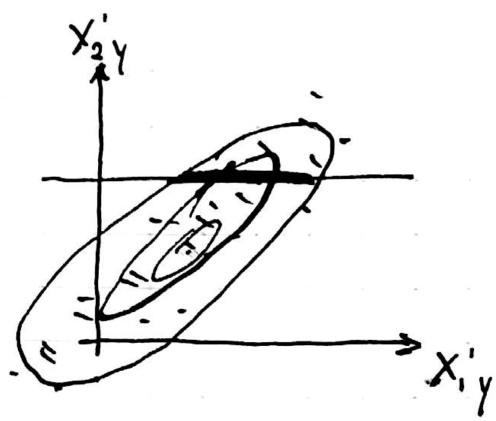


Figure 1: TODO: Does this belong here?

Example 3.2 (One-sample t-test). $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$, $\sigma^2 > 0$ unknown.

$$p_{\mu,\sigma^2}(x) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{\frac{\mu}{\sigma^2}\sum_{i=1}^n X_i - \frac{1}{2\sigma^2}\sum_{i=1}^n X_i^2 - \frac{n\mu^2}{2\sigma^2}} \tag{3.2}$$

$$\theta = \frac{\mu}{\sigma^2} \tag{3.3}$$

$$\zeta = \frac{-1}{2\sigma^2} \tag{3.4}$$

Test $H_0 : \mu \leq 0 \iff \theta \leq 0$ vs $H_1 : \mu > 0 \iff \theta > 0$
Condition on $\|X\|^2 = u$:

$$\implies p_{0,\sigma^2}(x \mid \|X\|^2 = u) \propto \cancelto{1}{e^{0\sum X_i}} 1_{\|X\|^2 = u}$$

($X \sim$ uniform on sphere of radius $\sqrt{u}$)

$$\iff \frac{X}{\|X\|} \mid \|X\| \sim \text{Unif(unit sphere)}$$

Optimal test rejects when

$$T(X) = n\bar{X} > c(u) \iff \text{reject when } \frac{\bar{X}}{\|X\|} > 0 \tag{3.5}$$

$$S^2 = \sum_i (X_i - \bar{X})^2 = \|X\|^2 - n\bar{X}^2 \iff \text{reject when } \frac{\bar{X}}{\sqrt{S^2/(n-1)}} \overset{\text{large}}{\sim} t_{n-1} \tag{3.6}$$

$$B(X) = \frac{\bar{X}}{\|X\|} \qquad D(X) = \frac{\bar{X}}{\sqrt{S}} \tag{3.7}$$

$$B^2 = \frac{X^2}{\|X\|^2} \qquad D^2 = \frac{X^2}{\|X\| - X^2} \tag{3.8}$$

$$\frac{1}{D^2} = \frac{1}{B^2} - 1 \tag{3.9}$$

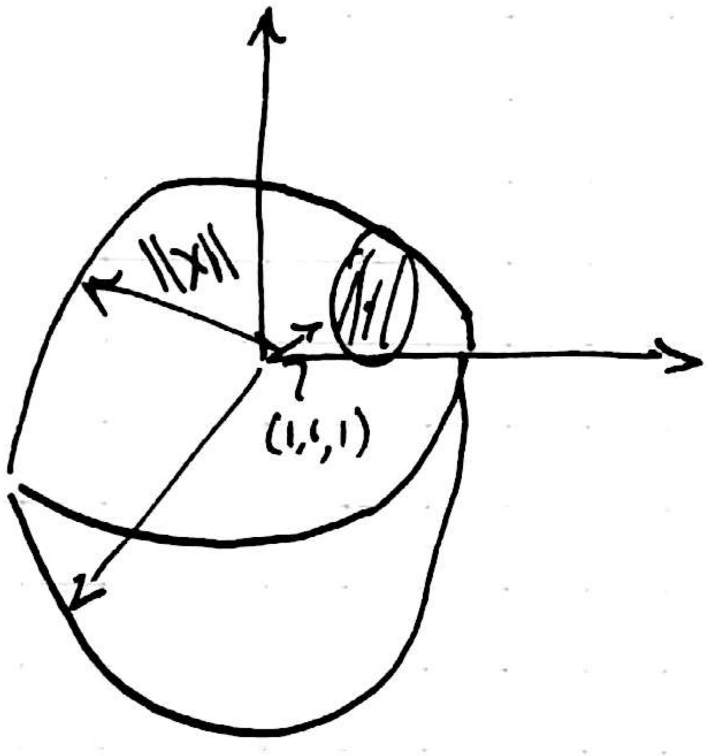


Figure 2: Rejection region is a cap on the sphere $\{\|X\| = \sqrt{u}\}$

Example 3.3 (Permutation Tests). $X_1, \dots, X_n \overset{\text{iid}}{\sim} P$.

$Y_1, \dots, Y_m \overset{\text{iid}}{\sim} Q$.

$H_0 : P = Q$, $H_1 : P \neq Q$.

Complete sufficient statistics under H_0 are the order statistics

$$Z = (X_1, \dots, X_n, Y_1, \dots, Y_m) \tag{3.10}$$
$$U(X,Y) = (Z_{(1)}, Z_{(2)}, \dots, Z_{(n)}) \tag{3.11}$$

Choose any test stat $T(X,Y)$. Under H_0 (not true under H_1)

$$T(X,Y) = \bar{X} - \bar{Y} \tag{3.12}$$
$$(X,Y) \mid U(X,Y) \sim \text{Unif over all permutations of } (Z_1, \dots, Z_{m+n}) \tag{3.13}$$

$$P_{H_0}(T(X,Y) > 0 \mid U(X,Y)) = \frac{1}{(n+m)!} \sum_{\pi \in S_{n+m}} 1_{T((Z_{\pi(1)}, \cdots, Z_{\pi(n)}), (Z_{\pi(n+1)}, \cdots, Z_{\pi(n+m)})) > c} \tag{3.14}$$

Reject when $T(X,Y)$ large.

Choose c such that under H_0 but not H_a: conditional power is $\alpha \implies$ marginal power is α.

TODO: WTF? This means that we should choose order statistics Z, not entire data, because the entire data statistic does not differ between null and alternative hypotheses.

4 Tests in linear models