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STAT 210A
Lecture #6
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1 Review

1.1 Sufficiency

$T(X)$ is *sufficient* if $P_\theta(X|T)$ does not depend on θ i.e. $(p_\theta(X) = g_\theta(T(x))h(x))$

$T(X)$ is *minimal sufficient* if:

- T is sufficient
- $T(X) \overset{\text{a.s.}}{=} f(S(X))$ for any S sufficient

Theorem: T is minimal sufficient if $p_\theta(x) \propto_\theta p_\theta(y) \implies T(x) = T(y)$

$T(X)$ is *complete sufficient* if:

- T is sufficient
- $\mathbb{E}_\theta[g(T(X))] = 0 \quad \forall\theta \implies g(T) \overset{\text{a.s.}}{=} 0$

Proposition 1.1. *If T is complete, then it is independent of all ancillary statistics.*

1.2 Exponential familites

$p_\theta(x) = e^{\eta(\theta)' T(x) - B(\theta)} h(x)$ for $\theta \in \Theta$, $\eta(\theta) \in \Xi \subset \mathbb{R}^s$.

If $\text{span}(\{\eta(\theta_1) - \eta(\theta) : \theta_1, \theta \in \Theta\}) = \mathbb{R}^s$, then T is minimal sufficient.

If $\eta(\Theta)$ contains an open set, we say η is *full-rank*. $\implies T(X)$ is complete sufficient.

A *curved* exponential family is one which is not full rank.

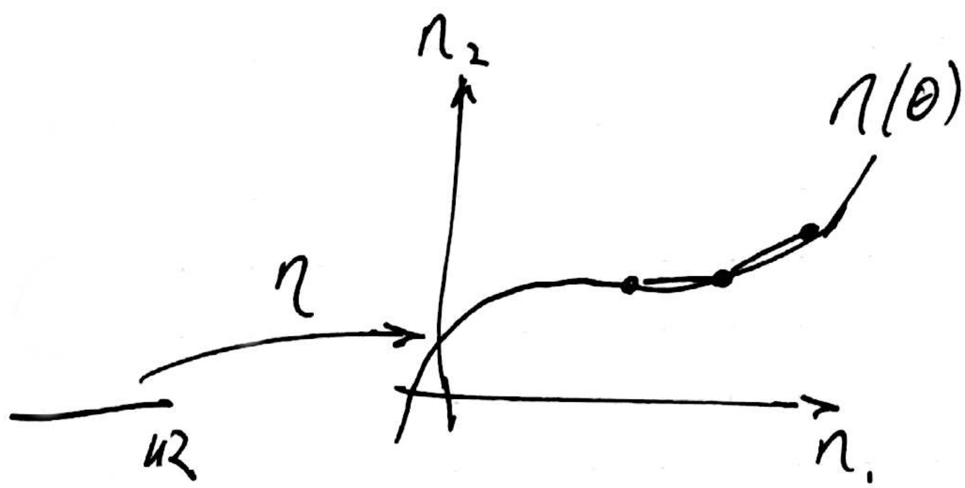

Figure 1: Example of curved exponential family, not full rank because closed in $\mathbb{R}^s = \mathbb{R}^2$

1.3 Rao-Blackwell

Theorem 1.2. *If T is sufficient, $\tilde{\delta}(X)$ some estimator, $L(\theta, \cdot)$ convex loss, then the estimator*

$$\delta(X) = \mathbb{E}[\tilde{\delta}(X)|T] \tag{1.1}$$

is better i.e. $L(\theta, \delta) \leq L(\theta, \tilde{\delta})$

2 Unbiased estimators

Example 2.1 (Bias-Variance Tradeoff)**.**

$$\begin{aligned}
\mathrm{MSE}(\theta;\delta) &= \mathbb{E}_\theta[(\delta(X) - g(\theta))^2] && (2.1)\\
&= \mathbb{E}_\theta[(\delta(X) - \mathbb{E}_\theta\delta + \mathbb{E}_\theta\delta - g(\theta))^2] && (2.2)\\
&= \mathbb{E}_\theta[(\delta - \mathbb{E}_\theta\delta)^2] + (\mathbb{E}_\theta(\delta) - g(\theta))^2 + 2(\mathbb{E}_\theta(\delta) - g(\theta))\mathbb{E}_\theta[\cancelto{0}{(\delta(X) - \mathbb{E}_\theta[\delta])}] && (2.3)\\
&= \mathrm{Var}_\theta(\delta) + \mathrm{Bias}_\theta(\delta)^2 && (2.4)
\end{aligned}$$

Definition 2.2. $\delta(X)$ is an *unbiased* estimator of g if $\mathrm{Bias}_\theta(\delta) = \mathbb{E}_\theta[g(\theta) - \mathbb{E}_\theta\delta] = 0$.

Definition 2.3. A statistic g is U-estimable if there exists an unbiased estimator.

Definition 2.4. $\delta(X)$ is *UMVU* (uniform minimum variance unbiased estimator) if δ is unbiased and $\mathrm{Var}_\theta(\delta) \leq \mathrm{Var}_\theta(\tilde{\delta})$ for all $\theta \in \Theta$ and any other unbiased $\tilde{\theta}$.

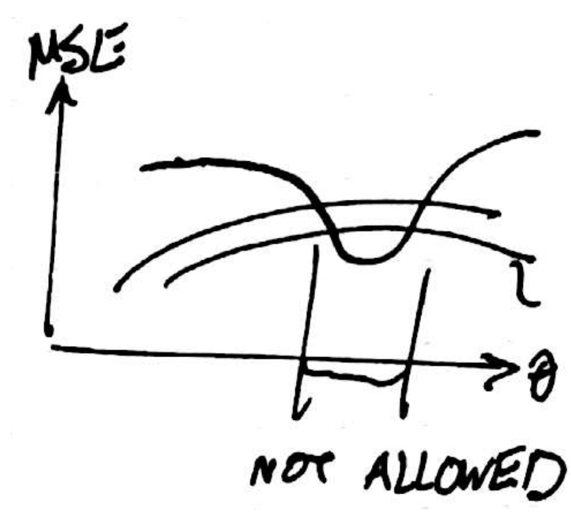

Figure 2: No estimator in this figure is UMVU; the bold estimator is not minimum variance in the indicated region

Theorem 2.5. *If $g(\theta)$ is U-estimable and T is complete sufficient, $\exists!$ UMVU estimator $\delta(T(X))$.*

Proof. Let $\delta_0(X)$ be unbiased ($\exists$ since g is U-estimable) and consider the Rao-Blackwellized estimator $\delta(T) = \mathbb{E}_\theta[\delta_0|T]$.

We first show δ is unbiased:

$$\begin{aligned}
\mathbb{E}_\theta \delta(T) &= \mathbb{E}_\theta \mathbb{E}_\theta[\delta_0|T] && (2.5)\\
&= \mathbb{E}_\theta[\delta_0] && \text{Tower property} \quad (2.6)\\
&= g(\theta) && \delta_0 \text{ unbiased} \quad (2.7)
\end{aligned}$$

We next show essential uniqueness of δ. Suppose $\tilde{\delta}(T)$ unbiased. As both are unbiased, $\mathbb{E}_\theta[\delta - \tilde{\delta}] = 0$ for all θ so by completeness $\delta \overset{\text{a.s.}}{=} \tilde{\delta}$.

By Rao-Blackwell, no other $\delta^*(X)$ could be better. $\square$

2 ways to find UMVUE:

(1) Directly

(2) Find any unbiased $\delta(T)$, Rao-Blackwellize

Example 2.6. $X_1, \cdots, X_n \overset{\text{iid}}{\sim} \text{Pois}(\theta)$. $p_\theta(x) = \frac{\theta^x e^{-\theta}}{x!}$ for $\theta \geq 0$, $x \in \mathbb{N}^*$. Complete sufficient statistic $T = \sum X_i \sim \text{Pois}(n\theta)$. $p_\theta^T(t) = \frac{(n\theta)^t e^{-n\theta}}{t!}$. Estimate θ^2.

One way:

$$\begin{aligned}
\theta^2 &= \mathbb{E}[X_1]^2 && (2.8)\\
&= \mathbb{E}[X_1 X_2] = \delta_0 && (2.9)
\end{aligned}$$

and Rao-Blackwellize.

Another way: Any unbiased $\delta(T)$ must satisfy

$$\sum_{t=1}^{\infty} \delta(t) p_\theta^T(t) = \theta^2 \quad \forall \theta \tag{2.10}$$

$$\sum_{t=1}^{\infty} \delta(t) \frac{n^t \theta^t}{t!} = e^{n\theta}\theta^2 \quad \forall \theta \tag{2.11}$$

$$= \sum_{k \geq 0} \frac{n^k}{k!} \theta^{k+2} \tag{2.12}$$

Matching coefficients implies

$$\delta(T) = \left(\frac{T(T-1)}{n^2} \right)_+ \approx (T/n)^2 \tag{2.13}$$

(Note: this shows that $\mathbb{E}[X_1 X_2 | X_1 + X_2 = T] = (T(T-1))/n^2$ TODO: really?)

Example 2.7. In Keener. $X_1, \cdots, X_n \overset{\text{iid}}{\sim} U[0, \theta]$. $T = X_{(n)}$ complete sufficient.

$$p_\theta^T(t) = \frac{\partial}{\partial t} P_\theta[T = t] \tag{2.14}$$

$$= \frac{\partial}{\partial t} \left(\frac{t}{\theta} \right)^n = \frac{n}{\theta^n} t^{n-1} \tag{2.15}$$

$$\mathbb{E}_\theta T = \frac{n}{n+1}\theta \implies \frac{n+1}{n} T \text{ is UMVU} \tag{2.16}$$

Example 2.8 (Nonparamtric Problem). $X_1, \cdots, X_n \overset{\text{iid}}{\sim} p(X)$ wrt Leb. s.t. $\mathbb{E}_p[|X|] < \infty$. Estimate $\mu(p) = \int_{-\infty}^{\infty} xp(x)dx = \mathbb{E}_p X$.

Know $T = (X_{(1)}, \cdots, X_{(n)})$ is sufficient (because iid), can show it is also complete.

X_1 is unbiased, so Rao-Blackwellizing

$$\begin{aligned} \mathbb{E}_p[X_1 | X_{(1)}, \cdots, X_{(n)}] &= \frac{1}{n} \sum_{i=1}^{n} X_{(i)} && \text{Data exchangeable} && (2.17) \\ &= \bar{X} && && (2.18) \end{aligned}$$

So $\bar{X}$ is UMVU.

2.1 U-statistics (U for Unbiased)

Suppose we want to etimate

$$g(p) = \mathbb{E}_p[|X_1 - X|2|] \tag{2.19}$$

(Gini's mean difference)

Or more generally

$$g(p) = \mathbb{E}_p[h(X_1, \cdots, X_m)] \quad m \leq n \tag{2.20}$$

First of all

$$\delta_0 = h(X_1, \cdots, X_m) \tag{2.21}$$

is an unnbiased estimator obtained by just taking the first m observations. But we can do better by Rao-Blackwellizing

$$\delta(X) = \frac{n!}{(n-m)!} \sum_{\substack{i_1, \cdots, i_m \\ \text{distinct}}} h(X_{i_1}, \cdots, X_{i_m}) \tag{2.22}$$

This is called a *U-statistic*.