Feynman Liang
STAT 205A
Lecture #21
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1 "Converge or Oscillate Infinitely"

Lemma 1.1. *Let (X_n) be a MG such that $|X_n - X_{n-1}| \leq K\ \forall n$. Then $P(C \cup D) = 1$ for the events*

$$C = \left\{\omega : \lim_{n\to\infty} X_n(\omega) \text{ exists and is finite}\right\} \tag{1.1}$$

$$D = \left\{\omega : \limsup_{n\to\infty} X_n(\omega) = +\infty \text{ and } \liminf_n X_n(\omega) = -\infty\right\} \tag{1.2}$$

Proof. WLOG assume $X_0 = 0$. Fix $L > 0$.

Define $T = \min\{n : X_n < -L\}$. The stopped process $(X_{T\wedge n}, n \geq 0)$ is a MG which is always $\geq -L - K$ (by def of T and assumption $|X_n - X_{n-1}| \leq K$).

By the (positive super-MG) convergence theorem, $X_{T\wedge n}$ converges to some finite limit a.s. as $n \to \infty$ (This is obvious for $T < \infty$ but still true for $T = \infty$).

This implies $\{\inf_n X_n > -L\} = \{T = \infty\} \subset C$.

As L was arbitrary, letting $L \to \infty$ yields

$$A_1 = \left\{\inf_n X_n > -\infty\right\} \subset C \tag{1.3}$$

Applying the same argument to $-X_n$ yields

$$A_2 = \left\{\sup_n X_n < \infty\right\} \subset C \tag{1.4}$$

Noting $D = (A_1 \cap A_2)^c$ completes the proof. □

2 Conditional Borel-Cantelli

Lemma 2.1 (Conditional Borel-Cantelli Lemma)**.** *Consider events (A_n) adapted to $(\mathcal{F}_n)$. Define $B_n = \cup_{m\geq n} A_m$ and $B = \cap_n B_n = \limsup_n A_n = \{A_n \text{ i.o.}\}$. Then*

(a) $\{A_n \text{ i.o.}\} \stackrel{a.s.}{=} \{\sum_{n=1}^\infty P(A_n \mid \mathcal{F}_{n-1}) = \infty\}$

(b) $P(B_{n+1} \mid \mathcal{F}_n) \stackrel{a.s.}{\to} 1_B$ *as* $n \to \infty$

$B_1 \stackrel{a.s.}{=} B_2$ *means* $P(B_1 \Delta B_2) = 0$.

Proof. **??** Consider $K < n$. Then $B \subset B_n \subset B_K$ and

$$P(B \mid \mathcal{F}_n) \leq P(B_{n+1} \mid \mathcal{F}_n) = P(B_{n+1} |_n) \leq P(B_K \mid \mathcal{F}_n) \tag{2.1}$$

Taking $n \to \infty$, by martingale convergence theorem

$$1_B \le \liminf_n P(B_{n+1} \mid \mathcal{F}_n) \le \limsup_n P(B_{n+1} \mid \mathcal{F}_n) \le 1_{B_K} \tag{2.2}$$

Let $K \uparrow \infty$. Then $1_{B_K} \downarrow 1_B \overset{\text{a.s.}}{=} \lim_n P(B_{n+1} \mid \mathcal{F}_n)$.

?? Consider $X_n = \sum_{m=1}^n (1_{A_m} - P(A_m \mid \mathcal{F}_{m-1}))$, which is a MG, and $|X_{n+1} - X_n| = |1_{A_{n+1}} - P(A_{n+1} \mid \mathcal{F}_n)| \le 1$. Then **??** implies that $P(C \cup D) = 1$. We want to show

$$\left\{ \sum_m 1_{A_m} = \infty \right\} \overset{\text{a.s.}}{=} \left\{ \sum_m P(A_m \mid \mathcal{F}_{m-1}) = \infty \right\} \tag{2.3}$$

Observe that $X_n = \sum_{m=1}^n 1_{A_m} - \sum_{m=1}^n P(A_m \mid \mathcal{F}_{m-1})$. On event $D = \{\omega : \limsup_n X_n = +\infty \text{ and } \liminf_n X_n = -\infty\}$, we have that both sums are infinite:

$$\infty = \limsup_n \sum_{m=1}^n 1_{A_m} - \sum_{m=1}^n P(A_m \mid \mathcal{F}_{m-1}) \le \sum_{m=1}^\infty 1_{A_m} \tag{2.4}$$

$$-\infty = \liminf_n \sum_{m=1}^n 1_{A_m} - \sum_{m=1}^n P(A_m \mid \mathcal{F}_{m-1}) \ge -\sum_{m=1}^\infty P(A_m \mid \mathcal{F}_{m-1}) \tag{2.5}$$

On event C, either both sums are finite or both sums equal ∞. □

3 "Product" martingales

Our discussion thus far has examined sums of MGs. In this section, we consider products of MGs.

3.1 Convergence for "Multiplicative" MGs

Theorem 3.1 (Kakutani's Theorem). *Take $(X_i, i \ge 1)$ to be independent, $X_i > 0$, $\mathbb{E}X_i = 1$. We know that $M_n = \prod_{i=1}^n X_i$ is a MG hence by MCT $M_n \overset{a.s.}{\to} M_\infty$ with $\mathbb{E}M_\infty \le 1$. Then TFAE:*

(a) $\mathbb{E}M_\infty = 1$

(b) $M_n \to M_\infty$ *in* L^1

(c) $(M_n, n \ge i)$ *is UI*

(d) Set $a_i = \mathbb{E}X_i^{1/2}$ *and note that* $0 \le a_i \le 1$, $\prod_{i=1}^\infty a_i > 0$.

(e) $\sum_i (1 - a_i) < \infty$

Proof. Conditions **??????** are equivalent by the L^1 MG convergence theorem.

Conditions **????** are equivalent by calculus: use $1 - x + x^2 \le e^{-x} \le 1 - x$ for small $x > 0$.

Suppose **??** holds. Consider

$$N_n = \frac{X_1^{1/2}}{a_1} \cdot \frac{X_2^{1/2}}{a_2} \cdot \dots \cdot \frac{X_n^{1/2}}{a_n}. \tag{3.1}$$

which is a MG. Note

$$\mathbb{E}[N_n^2] = \frac{\mathbb{E}M_n}{\prod_{i=1}^n a_i^2} \leq \frac{1}{\prod_{i=1}^\infty a_i^2} = K < \infty \tag{3.2}$$

Apply the Doob L^2 maximal inequality:

$$\mathbb{E}\left[\sup_n N_n\right] \leq 4K \tag{3.3}$$

Note that $M_n \leq N_n^2$ since $M_n = N_n^2 \prod_{i=1}^n a_i^2$. Therefore, $\mathbb{E}[\sup_n M_n] \leq (4K)^2 < \infty$. This implies that $(M_n, n \geq 1)$ is UI. If $Z \geq 0, \mathbb{E}Z < \infty$, then the family $\{X : 0 \leq X \leq Z\}$ is UI. This yields **??**.

Suppose that **??** is false, so $\prod_{i=1}^\infty a_i = 0$. For the MG (N_n), we have $N_n \overset{\text{a.s.}}{\to} N_\infty$, so we must have

$$N_\infty = \frac{\prod_{i=1}^\infty X_i^{1/2}}{\prod_{i=1}^\infty a_i} \tag{3.4}$$

Since the denominator is 0, then $\prod_{i=1}^\infty X_i^{1/2} = M_\infty^{1/2} = 0$ a.s., so **??** fails. $\square$

3.2 Likelihood ratios (absolute continuity of infinite product measures)

Given densities $(f_i, 1 \leq i < \infty)$ and $(g_i, 1 \leq i < \infty)$, assume $f_i > 0$ and $g_i > 0$. Take $\Omega = \mathbb{R}^\infty$ with $X_i(\vec{\omega}) = \omega_i$. Work with P, the product measure where the (X_i) are independent with densities f_i. Consider Q, where the (X_i) have densities g_i.

Definition 3.2. The *likelihood ratio*

$$L_n = \prod_{i=1}^n \frac{g_i(X_i)}{f_i(X_i)} \tag{3.5}$$

is the Radon-Nikodym density $\frac{dQ_n}{dP_n}$.

(Q_n is the probability measure with corresponding density $f_1 \otimes f_2 \otimes \cdots \otimes f_n$)

Know: $(L_n, n \geq 1)$ is a MG wrt P.

Suppose that $(L_n, n \geq 1)$ is UI. Then $L_n \to L_\infty$ in L^1 and $L_n = \mathbb{E}[L_\infty \mid \mathcal{F}_n]$. What this means, from the definition of R-N density, is

$$Q(A) = \mathbb{E}L_n 1_A \quad \forall A \in \mathcal{F}_n \tag{3.6}$$

$$= \mathbb{E}L_\infty 1_A \quad \forall A \in \bigcup_n \mathcal{F}_n \tag{3.7}$$

$$= \mathbb{E}L_\infty 1_A \quad \forall A \in \mathcal{F}_\infty \tag{3.8}$$

so L_∞ is the R-N density $\frac{dQ}{dP}$ on $\mathbb{R}^\infty$. Therefore, $Q \ll P$.

Similarly, if $Q \ll P$, then we can prove $(L_n, n \geq 1)$ is UI. So $Q \ll P \iff (L_n, n \geq 1)$ is UI $\underbrace{\iff}_{\text{Kakutani}} \sum_i (1 - a_i) < \infty$.

$$a_i = \mathbb{E}\left(\frac{g_i}{f_i}(X_i)\right)^{1/2} = \int g_i^{1/2}(x) f_i^{1/2}(x) dx \tag{3.9}$$

$$\text{algebra} \implies 1 - a_i = \frac{1}{2} \int \left(g_i^{1/2}(x) - f_i^{1/2}(x)\right)^2 dx \tag{3.10}$$

Our condition becomes $Q \ll P \iff$

$$\sum_{i=1}^{\infty} \int \left(g_i^{1/2}(x) - f_i^{1/2}(x)\right)^2 dx < \infty \tag{3.11}$$

"f_i and g_i become close for large i."

We know that if $f \neq g$, then Q and P are singular.