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STAT 205A
Lecture #7
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1 Polynomial Approximation Theorems

Main goal is to demonstrate proof techniques, results are not too important.

Theorem 1.1 (Bernstein). *Given continuous $f : [0,1] \to \mathbb{R}$, define*

$$f_n(x) := \sum_{n=0}^{n} \binom{n}{m} x^m (1-x)^{n-m} f(m/n), \quad 0 \leq x \leq 1 \tag{1.1}$$

(polynomial of degree n).

Then f_n converge uniformly to f i.e.

$$\sup_x |f_n(x) - f(x)| \to 0 \tag{1.2}$$

as $n \to \infty$.

Proof. Fix x. Take i.i.d. Bernoulli(x) r.v.s $(X_i, 1 \leq i < \infty)$. Let $S_n := \sum_{i=1}^n X_i \sim$ Binomial(n,x) and note $f_n(x) = \mathbb{E} f(S_n/n)$.

Want to bound

$$|f_n(x) - f(x)| = |\mathbb{E} f(S_n/n) - f(x)| \tag{1.3}$$
$$\leq \mathbb{E}|f(S_n/n) - f(x)| \tag{1.4}$$

Want to apply WLLN $S_n/n \xrightarrow{p} x$, so split

$$\mathbb{E}|f(S_n/n) - f(x)| = \mathbb{E}\left[|f(S_n/n) - f(x)| 1_{|S_n/n - x| \leq \delta} + |f(S_n/n) - f(x)| 1_{|S_n/n - x| > \delta}\right] \tag{1.5}$$

From analysis, we have

- $M := \sup |f(X)| \leq \infty$
- (Uniformly Continuous) $\forall \varepsilon > 0, \exists \delta > 0 : |y_1 - y_2| \leq \delta \implies |f(y_1) - f(y_2)| \leq \varepsilon$

Choosing $\varepsilon > 0$ and taking δ from uniform continuity

$$\mathbb{E}\left[|f(S_n/n) - f(x)|1_{|S_n/n-x|\leq\delta} + |f(S_n/n) - f(x)|1_{|S_n/n-x|>\delta}\right] \tag{1.6}$$
$$\leq \varepsilon + 2MP(|S_n/n - x| > \delta|) \tag{1.7}$$
$$\leq \varepsilon\frac{2M}{\delta^2}\mathrm{Var}(S_n/n) \tag{1.8}$$
$$= \varepsilon + \frac{2M}{\delta^2}\frac{x(-x)}{n} \tag{1.9}$$
$$\leq \varepsilon + \frac{M}{2\delta^2}\frac{1}{n} \tag{1.10}$$
$$\leq \varepsilon \quad \text{as } n \to \infty \tag{1.11}$$
$$= 0 \quad \text{holds } \forall\varepsilon > 0 \tag{1.12}$$

□

2 Backgrounnd for proving a.s. limits

2.1 Axioms

For events B_n, B

- $B_n \uparrow B \implies P(B_n) \uparrow P(B)$
- $B_n \downarrow B$ and $\exists n : P(B_n) < \infty \implies P(B_n) \downarrow P(B)$

Definition 2.1. Event A_n *infinitely often* (A_n i.o.) means $\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n$.
Event A_n *ultimately* (A_n ult.) means $\cup m = 1^{\infty} \cap n = m^{\infty} A_n$.

Proposition 2.2. *i.o. and ult. are opposites:* $(A_n \text{ i.o.})^c = (A_n^c \text{ ult.})$

Lemma 2.3 (Weak)**.**

(i) $P(A_n \textit{ i.o.}) \geq \limsup_n P(A_n)$

(ii) $P(A_n \textit{ ult.}) \geq \liminf_n P(A_n)$

Proof.

$$P(\cup_{n=m}^{Q} A_n) \geq \max_{m\leq n\leq Q} P(A_n) \tag{2.1}$$
$$P(\cup_{n=m}^{\infty} A_n) \geq \sup_{n\geq m} P(A_n) \qquad Q \to \infty \tag{2.2}$$
$$P(A_n \text{ i.o.}) \geq \limsup_{n} P(A_n) \qquad m \to \infty \tag{2.3}$$

□

Lemma 2.4 (First Borel-Cantelli)**.** *Events* $(A_n)_{n=1}^{\infty}$, *if* $\sum_n P(A_n) < \infty$ *then* $P(A_n \text{ i.o.}) = 0$.

Proof.

$$X_n = \sum_{i=1}^{n} 1_{A_i} = \text{(number of last } n \text{ events that occur)} \tag{2.4}$$

$$X_\infty = \sum_{i=1}^{\infty} 1_{A_i} = \text{(total number of events that occur)} \tag{2.5}$$

$$\mathbb{E}X_\infty = \sum_{i=1}^{\infty} P(A_i) \underset{\text{hyp}}{<} \infty \tag{2.6}$$

$$\implies P(X_\infty = \infty) = 0 \tag{2.7}$$

□

Lemma 2.5 (Second Borel-Cantelli)**.** *For* independent *events* $(A_i)_{i=1}^{\infty}$, *if* $\sum_i P(A_i) = \infty$ *then* $P(A_n \text{ i.o.}) = 1$.

(Many variants under alternative assumptions exist)

Proof. Fix m. We will prove $P(\cup_{n=m}^{\infty} A_n) = 1$ or prove $P(\cap_{n=m}^{\infty} A_n^c) = 0$.
Fact: $0 \leq x \leq 1 \implies 1 - x \leq e^{-x}$.
By independence

$$P(\cap_{n=m}^{q} A_n^c) = \prod_{n=m}^{q} P(A_n^c) \tag{2.8}$$

$$= \prod_{n=m}^{q} (1 - P(A_n)) \tag{2.9}$$

$$\leq \exp(-\sum_{n=m}^{q} P(A_n)) \tag{2.10}$$

Let $q \uparrow \infty$

$$P(\cap_{n=m}^{\infty} A_n^c) \leq \exp(-\sum_{n=m}^{\infty} P(A_n)) = 0 \tag{2.11}$$

□

Lemma 2.6. *Arbitrary* $\mathbb{R}$*-valued r.v.s* (Y_n) *and arbitrary* $-\infty < y < \infty$.
If $\sum_n P(Y_n \geq y + \varepsilon) < \infty$ *for all* $\varepsilon > 0$, *then* $\limsup_n Y_n \leq y$ *a.s..*

Corollary 2.7. *If* $\sum_n P(|Y_n| \geq \varepsilon) < \infty$, *each* $\varepsilon > 0$, *then* $Y_n \overset{a.s.}{\to} 0$.

Proposition 2.8 (Deterministic fact for real numbers)**.** *For reals* (y_n) *and* y, *"*$\limsup_n y_n \leq y$*" equivalent to "*$y_n \leq y + \varepsilon$*" ult., each* $\varepsilon > 0$, *equivalent to "*$y_n \leq y + 1/j$*" ult., each* $j \geq 1$.

Proof. By Hyp and B-C 1

$$\implies P(Y_n \leq y + 1/j \text{ ult.}) = 1) \tag{2.12}$$

By monotonicity $P(B_j) = 1 \forall j \implies P(B_j \text{ for all} j) = 1$ hence

$$\implies P(Y_n \leq y + 1/j \text{ ult., each } j \geq 1) = 1 \tag{2.13}$$

$$\implies P(\limsup_n Y_n \leq y) = 1 \tag{2.14}$$

$\square$

Theorem 2.9 (4th moment SLLN (strong law of large numbers))**.** *Let* $(X_i, 1 \leq i < \infty)$ *be i.i.d.,* $\forall i : \mathbb{E}X_i = 0$*,* $\mathbb{E}X^4 < \infty$*. Then*

(i) $\mathbb{E}S_n^4 \leq 3n^2\mathbb{E}X^4$

(ii) $S_n/n \overset{a.s.}{\to} 0$ *as* $n \to \infty$

Proof. (i) $\mathbb{E}S_n^4 = \sum_{i,j,k,l} \mathbb{E}[X_i X_j X_k X_l]$

Note that $\mathbb{E}[\cdot] = 0$ if some index j appears only once, e.g.

$$\mathbb{E}[X_4 X_6 X_6 X_6] = \overset{0}{\cancel{\mathbb{E}[X_4]}}\mathbb{E}[X_6^6] = 0 \tag{2.15}$$

Hence

$$\mathbb{E}S_n^4 = n\mathbb{E}X^4 + \binom{4}{2}\binom{n}{2}\mathbb{E}[X_1^2 X_2^2] \tag{2.16}$$

$$= n\mathbb{E}X^4 + 3n(n-1)\underbrace{(\mathbb{E}X^2)^2}_{\leq \mathbb{E}X^4} \tag{2.17}$$

$$\leq 3n^2\mathbb{E}X^4 \tag{2.18}$$

Fix $\varepsilon > 0$.

$$P(|S_n/n| \geq \varepsilon) \leq \mathbb{E}|S_n/n|^4/\varepsilon^4 \qquad \text{Markov ineq } \phi(x) = x^4 \tag{2.19}$$

$$\leq \varepsilon^{-4}n^{-4} \times 3n^2\mathbb{E}X^4 \tag{2.20}$$

$$\leq 3\varepsilon^{-4}\mathbb{E}X^4 n^{-2} \tag{2.21}$$

$$\implies \sum_n P(|S_n/n| \geq \varepsilon) \leq \sum_n 3\varepsilon^{-4}\mathbb{E}X^4 n^{-2} < \infty \tag{2.22}$$

Applying corollary 2.7, $S_n/n \overset{\text{a.s.}}{\to} 0$ $\square$

Corollary 2.10. *If* $(A_i)_{i=1}^{\infty}$ *independent Bernoulli*(p)*,* $S_n = \sum_{i=1}^n 1_{A_i}$*, then* $S_n/n \overset{a.s.}{\to} p$ *as* $n \to \infty$*.*

Definition 2.11. Given data real number $x_i, \cdots, x_n$, define:

Empirical distribution Uniform distribution on (x_1, x_n)

Empirical distribution function $G(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{x_i \le x}$

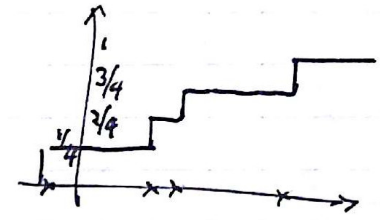

Figure 1: Example empirical distribution function

Theorem 2.12 (Glivenko-Cantelli). *$(X_i)_{i=1}^{\infty}$ i.i.d., arbitrary distribution function F, $G_n(\omega, x) = \frac{1}{n}\sum_{i=1}^{n} 1_{X_i(\omega)\le x}$ is empirical distribution of $(X_1(\omega), X_2(\omega), \cdots, X_n(\omega))$.*

For fixed x, events $\{X_i \le x\}$ are i.i.d. Bern$(f(x))$.

SLLN for events says

$$G_n(\omega, x) \to G(x) \text{ a.s. as } n \to \infty \tag{2.23}$$

for each x.