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STAT 205A
Lecture #22
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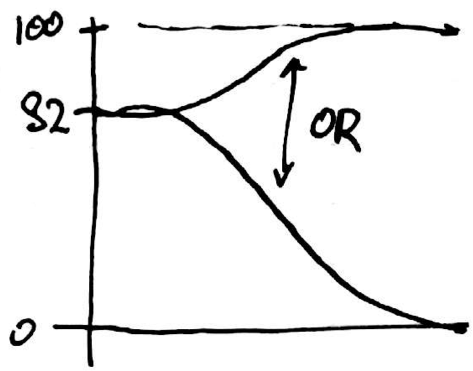
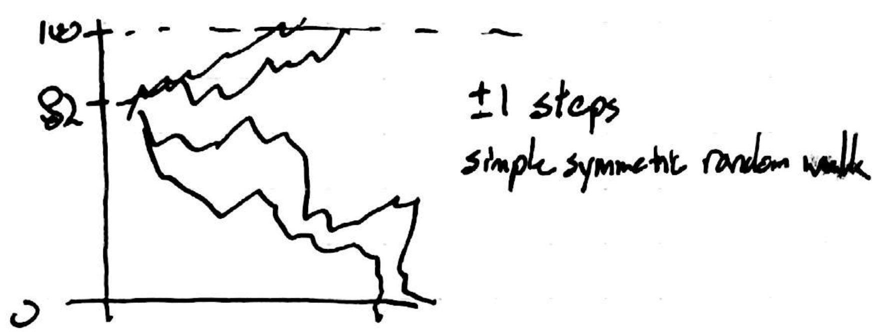
1 Optional Sampling Theorem

Intuition:



Math/data:



Setup for OST. Let $(X_n, n \geq 0)$ be a sub-MG. To conclude $\mathbb{E}X_0 \leq \mathbb{E}X_T$, what *extra assumptions* do we need?

Know: Sufficient $T \leq t_0 < \infty$ a.s. So, sufficient that

$$\mathbb{E}|X_T - X_{T \wedge n}| \to 0 \tag{1.1}$$

Theorem 1.1 (Optional Sampling Theorem (OST))**.** *If*

(a) $\mathbb{E}|X_n| 1_{T>n} \to 0$ *as* $n \to \infty$

(b) $\mathbb{E}|X_T| < \infty$

then $\mathbb{E}X_0 \leq a\mathbb{E}X_T$.

Proof. See Durrett. $\square$

Theorem 1.2 (Useful version of OST)**.** *Suppose (X_n) is a sub-MG, T a stopping time, $\mathbb{E}T < \infty$. Write $\Delta_{n=X_n-X_{n-1}}$. If $\exists b > 0$ such that*

$$\mathbb{E}[|\Delta_n| \mid \mathcal{F}_{n-1}] \le b \quad on\ \{n \le T\} \tag{1.2}$$

then $\mathbb{E}X_0 \le \mathbb{E}X_T$.

Proof. Note $X_T = X_0 + \sum_{n=1}^{T} \Delta_n$, and $|X_T| \le Y$ hence $|X_{T\wedge n}| \le Y$.
Consider

$$Y = |X_0| + \sum_{n=1}^{T} |\Delta_n| \tag{1.3}$$

$$\mathbb{E}Y = \mathbb{E}|X_0| + \sum_{n=1}^{T} |\Delta_n| \tag{1.4}$$

$$\mathbb{E}[|\Delta_n| 1_{T\ge n} \mid \mathcal{F}_{n-1}] = 1_{T\ge n}\mathbb{E}[|\Delta_n| \mid \mathcal{F}_n] \le b1_{T\ge n} \tag{1.5}$$

$$\mathbb{E}[|\Delta_n| 1_{T\ge n}] = \mathbb{E}\mathbb{E}[|\Delta_n| 1_{T\ge n} \mid \mathcal{F}_{n-1}] \le bP(T \ge n) \tag{1.6}$$

$$\mathbb{E}Y \le \mathbb{E}_0 + \sum_{n=1}^{\infty} bP(T \ge n) = \mathbb{E}X_0 + b\mathbb{E}T < \infty \tag{1.7}$$

Recall $\mathbb{E}|W| < \infty$ and $P(A_n) \to 0 \implies \mathbb{E}(W1_{A_n})) \to 0$, so taking $W = Y$ and $A_n = \{T > n\}$ shows (a).
$\mathbb{E}|X_T| \le \mathbb{E}Y < \infty$ shows (b). □

2 Equalities from inequalities using martingales

Principle: Given a MG proof of exact formula, one can often get equality conclusions out of inequality assumptions.

Corollary 2.1 (Inequality version of Wald identity)**.** *Suppose (ξ_i) independent, $\mu_1 \le \mathbb{E}\xi_i \le \mu_2$, and $\sup_i \mathbb{E}|\xi_i| < \infty$.*
Let $S_n = \sum_{i=1}^{n} \xi_i$. Then for any stopping time T where $\mathbb{E}T < \infty$

$$\mu_1\mathbb{E}T \le \mathbb{E}S_T \le \mu_2\mathbb{E}T \tag{2.1}$$

Proof. Apply theorem 1.2 to $X_n = S_n - n\mu_1$ (i.e. $\Delta_n = \xi_n - \mu_1$).

$$\mathbb{E}[\Delta_n \mid \mathcal{F}_{n-1}] = \mathbb{E}S_n - \mu_1 \ge 0 \implies (X_n) \text{ is a sub-MG} \tag{2.2}$$

$$\mathbb{E}[|\Delta_n| \mid \mathcal{F}_{n-1}] = \mathbb{E}|\Delta_n| \le \mathbb{E}|S_n| + |\mu_1| \le b \qquad \text{by hypothesis} \tag{2.3}$$

□

Wald: if (ξ_i) i.i.d.

$$\mathbb{E}S_t = (\mathbb{E}\xi) \cdot (\mathbb{E}T) \tag{2.4}$$

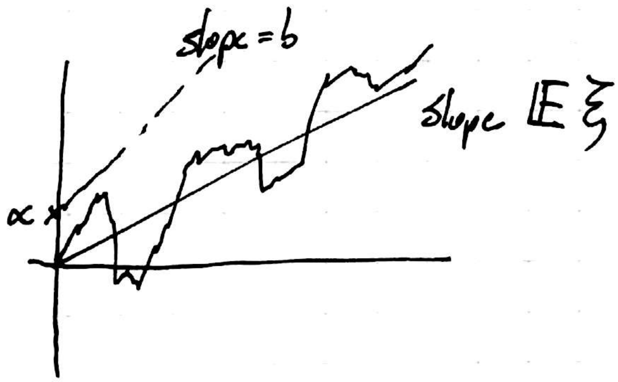
$$\text{Theorem} \implies \mathbb{E}X_0 \le \mathbb{E}X_T \tag{2.5}$$

$$\implies 0 \le \mathbb{E}S_t - \mu_1\mathbb{E}T \tag{2.6}$$

$$\implies \mathbb{E}S_t \ge \mu_1\mathbb{E}T \tag{2.7}$$

Lemma 2.2. *Take (ξ_i) i.i.d., $S_n = \sum_{i=1}^{n} \xi_i$. Fix $a > 0$ and $b > \mathbb{E}\xi$. Suppose $\exists \theta > 0$ such that $\mathbb{E}\exp(\theta\xi) = \mathbb{E}^{\theta b}$. Then*

$$P(S_n \geq a + nb \text{ for some } n \geq 0) \leq e^{-\theta a} \tag{2.8}$$



Proof. Set $\hat{\xi}_i = \xi_i - b$, so $\hat{S}_n = S_n - nb$.
$\mathbb{E}\exp(\theta\hat{\xi}) = 1$ by definition, so $(\exp(\theta\hat{S_n}), n \geq 0)$ is a MG.
Apply L^1 maximal inequality

$$P(\sup_n \exp(\theta\hat{S_n}) \geq \lambda) \leq \lambda^{-1} \tag{2.9}$$

Set $\lambda = e^{\theta a} \implies P(\sup_n \hat{S_n} \geq a) \leq e^{-\theta a}$ □

Lemma 2.3. *Suppose (ξ_i) i.i.d., $S_n = \sum_{i=1}^{n} \xi_i$, $\exists \theta > 0$ such that moment generating function $\phi(\theta) = \mathbb{E}\exp(\theta\xi) = 1$, T is a stopping time with $\mathbb{E}T < \infty$, and $\forall n : S_n \leq B$ on $\{n < T\}$.*
Then $\mathbb{E}\exp(\theta S_T) = 1$.

Proof. $X_n := \exp(\theta S_n)$ is a MG. Need to check eq. (1.2) from theorem 1.2.

$$\Delta_n = X_n - X_{n-1} = X_{n-1}(\exp(\theta S_n) - 1) \tag{2.10}$$
$$|\Delta_n| \leq X_{n-1}|\exp(\theta\xi_n) - 1| \tag{2.11}$$
$$\mathbb{E}[|\Delta_n| \mid \mathcal{F}_{n-1}] \leq X_{n-1}\mathbb{E}|\exp(\theta\xi) - 1| \tag{2.12}$$

as $\{n \leq T\} = \{n - 1 < T\}$ we have

$$S_{n-1} \leq B \tag{2.13}$$
$$X_{n-1} \leq e^{\theta B} \tag{2.14}$$

This verifies theorem 1.2 so apply MG version to conclude $\mathbb{E}X_0 = \mathbb{E}X_T$. TODO: Follow up □

2.1 Boundary crossing inequalities

Setting:

- (ξ_i) i.id.
- $S_n = \sum_{i=1}^{n} \xi_i$
- $|\xi_i| \leq L$
- $\mathbb{E}\xi < 0$
- $P(\xi > 0) > 0$

Fix $a < 0 < b$, consider $T = \min\{n : S_n \geq b \text{ or } S_n \leq a\}$

Exercise 2.4. Check $\mathbb{E}T < \infty$

Let $P(\underbrace{S_T \geq b}_{\iff S_T \leq b-L}) = x$, so $P(\underbrace{S_t \leq a}_{\iff S_T \geq a+L}) = 1 - x$.
Consider $\phi(\theta) = \mathbb{E}\exp(\theta\xi) < \infty$, $\phi(a) = 1$, $\phi'(0) = \mathbb{E}\xi < 0$, $\phi(\theta) \to \infty$ as $\theta \to \infty$.
$\implies \exists\theta > 0 : \phi(\theta) = 1$.
Apply Lemma $\implies \mathbb{E}\exp(\theta S_T) = 1$.

$$xe^{\theta b} + (1-x)e^{\theta(a-L)} \leq 1 \leq xe^{\theta(b+L)} + (1-x)e^{\theta a} \tag{2.15}$$

Rearranging

$$\frac{1 - e^{\theta a}}{e^{\theta(b+L)} - e^{\theta a}} \leq x \leq \frac{1 - e^{\theta(a-L)}}{e^{\theta b} - e^{\theta(a-L)}} \tag{2.16}$$

Special case for simple random walk:

$$P(\xi = 1) = p < \frac{1}{2} \tag{2.17}$$
$$P(\xi = -1) = q = 1 - p \tag{2.18}$$
$$a < 0 < b \in \mathbb{Z} \tag{2.19}$$

Here, the upper bound is an equality so

$$x = \frac{1 - e^{\theta a}}{e^{\theta b} - e^{\theta a}} \tag{2.20}$$

$\phi(\theta) = pe^{\theta} + qe^{-\theta} = 1$, so solving yields $e^{\theta} = q/p$ and

$$x = \frac{1 - (q/p)^a}{(q/p)^b - (q/p)^a} \tag{2.21}$$

which is the undergraduate result from directly solving simple random walk.