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STAT 205A
Lecture #1
8/25/2016

1 Abstract Measure Theory

Let S be a set. Capital letters $A, B, C \cdots \subset S$ denote *subsets*. Lowercase letters $s \in S$ denote *elements*. Calligraphic leters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \cdots, \mathcal{F}, \cdots, \mathcal{S}, \cdots \subset 2^S$ denote *collections of subsets*.

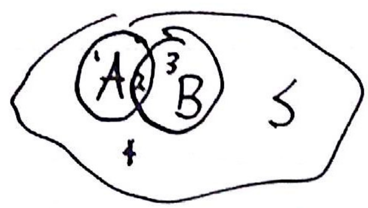
Definition 1.1. $\mathcal{S}$ is a *field* (or *algebra*) if $\mathcal{S}$ is closed under Boolean operations. i.e. if $A, B \in \mathcal{S}$, then:

(a) $A \cup B \in \mathcal{S}$

(b) $A \cap B \in \mathcal{S}$

(c) $A \setminus B \in \mathcal{S}$

Also $\mathcal{S} \neq \emptyset$.



There are 16 total possible Boolean operations, each a disjoint union of sets from $\{A \setminus B, A \cap B, B \setminus A, S \setminus (A \cup B)\}$ hence isomorphic to a binary string of length 4.

Example 1.2. $\mathcal{F} = \{\emptyset, \mathcal{S}\}$ is a field.
$\mathcal{F} = \{\emptyset, A, A^c, \mathcal{S}\}$.

Exercise 1.3. Show that to show $\mathcal{S}$ is a field, it suffices to check $\forall A, B \in \mathcal{S}$, $A^c, A \cup B \in \mathcal{S}$.

Lemma 1.4. *Let $\mathcal{S}_1, \mathcal{S}_2$ be fields. Then $\mathcal{S}_1 \cap \mathcal{S}_2$ is a field (*Not true *for $\mathcal{S}_1 \cup \mathcal{S}_2$).*
More generally, if $(\mathcal{S}_\theta)_{\theta \in \Theta}$ is any collection of fields on S, $\cap_{\theta \in \Theta} \mathcal{S}_\theta$ is a field.

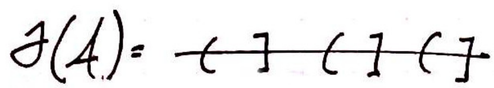
Definition 1.5. Let $\mathcal{A}$ be any collection of subsets of $\mathcal{S}$. Then

$$\mathcal{F}(\mathcal{A}) := \bigcap_{\substack{\mathcal{F} \text{ a field} \\ \mathcal{F} \supset \mathcal{A}}} \mathcal{F} \tag{1.1}$$

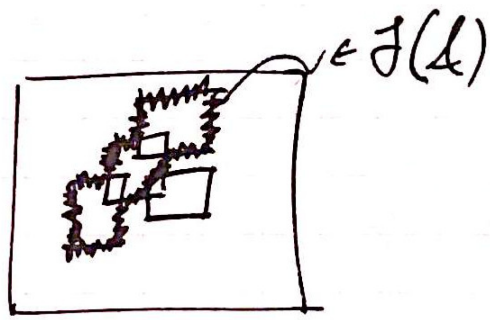
is a field, called the *field generated by* $\mathcal{A}$.

Exercise 1.6. (HW 1) Show that $\mathcal{F}(\mathcal{A})$ is the collection of subsets that can be obtained from sets in $\mathcal{A}$ via a finite number of Boolean operations.

Example 1.7. $S = \mathbb{R}$, $\mathcal{A}$ collection of $(-\infty, x]$ for $x \in \mathbb{R}$. $\mathcal{F}(\mathcal{A})$ = union of disjoint half-open intervals



Example 1.8. $S = [0,1]^2$. $\mathcal{A}$ = rectangles $(x_1, x_2] \times (y_1, y_2]$



Definition 1.9. $\mathcal{S}$ is a *σ-field* (*σ-algebra*) if:

(a) $\mathcal{S}$ is a field

(b) $\mathcal{S}$ is closed under *countable* unions and intersections

Exercise 1.10. Suffices to show closed under increasing unions i.e. $A_i \in \mathcal{S}, A_1 \subset A_2 \subset \cdots$, then $\cup_i A_i \in \mathcal{S}$.

Definition 1.11. Let $\mathcal{A} \subset 2^S$, then

$$\sigma(\mathcal{A}) := \bigcap_{\substack{\mathcal{G}\ \sigma\text{-field} \\ \mathcal{G} \supset \mathcal{A}}} \mathcal{G} \tag{1.2}$$

is a σ-fied, called the *σ-field generated by $\mathcal{A}$*.

Definition 1.12. A *measurable space* is $(S, \mathcal{S})$ where S is a set, $\mathcal{S}$ a σ-field on S.

If S is a topological space and $\mathcal{G}$ the collection of open sets, then $\sigma(\mathcal{G})$ is called the *Borel σ-field* on S.

Example 1.13. On $\mathbb{R}^d$, the Borel σ-field is the same as the σ-field generated by the d-dimensional cubes $\prod_{i=1}^d (x_i, y_i)$.