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STAT 210A
Lecture #15
10/13/2016

1 Review

Simple vs Simple: Reject for large $\frac{p_1(X)}{p_0(X)}$

One-sided: $\Theta \subset \mathbb{R}$.
$H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$
If $\mathcal{P}$ MLR in $T(X)$, UMP test rej for large $T(X)$.
$= \{P_\theta : \theta \in \Theta \subset \mathbb{R}\}$.
Test $\theta \leq \theta_0$ vs $\theta > \theta_0$.
$T(X) \in \mathbb{R}$ stochastically increasing in θ.
$P_\theta((x) \leq t)$ non-increasing in θ.
$\beta(\theta) = P_\theta(T(x) > c)$ non-decreasing in θ.

Example 1.1. $X_i \overset{\text{iid}}{\sim} p_\theta(x) = p_0(x - \theta), i \leq n$.
$T(X) =$ sample mean/median.

Example 1.2. $X_i \overset{\text{iid}}{\sim} p_\theta(x) = \frac{1}{\theta} p\left(\frac{x}{\theta}\right)$
$T(X) = \sum_i X_i^2, \sum |X_i|$

Example 1.3. $X_i \overset{\text{iid}}{\sim} N(\theta, 1)$. $T(X) = \bar{X}$.

$$\tilde{T}(X) = \mathbb{E}[\tilde{T}(X) \mid \bar{X}] + \text{random noise} \tag{1.1}$$
$$= \bar{X} + \varepsilon \approx N(\theta, \frac{1+\delta}{n}) \tag{1.2}$$

2 Two-sided tests, unbiasedness

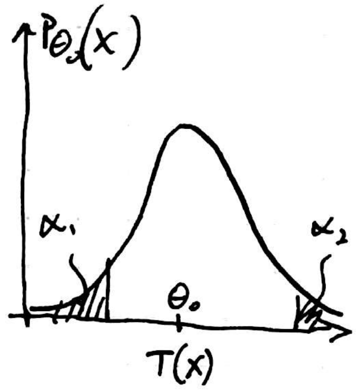
$H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$

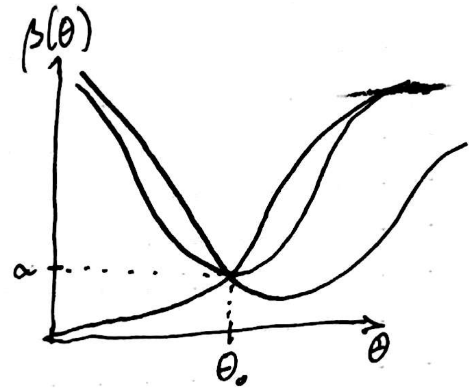
Example 2.1. $X \sim N(\theta, 1)$, $H_0 : \theta = 0$

(a) Reject when $|X| > z_{n/2}$

(b) Reject when $X > z_\alpha$

(c) Reject when $X < z_{1-\alpha_1}$ or $X > z_{\alpha_2}$ such that $\alpha_1 + \alpha_2 = \alpha$.





2.1 Equal-tailed test

Reject for extreme $T(X)$

$$(*)\phi(X) = \begin{cases} 1, & \text{if } T > c_1, T < c_1 \\ 0, & \text{if } T \in (c_1, c_2) \\ \gamma, & \text{if } T = c_i \end{cases} \tag{2.1}$$

Equal-tailed: choose c_i, γ_i so $\alpha_1 = \alpha_2 = \frac{\alpha}{2}$.

2.2 Unbiased test

A test is *unbiased* if

$$\inf_{\theta \in \Theta_1} \mathbb{E}_\theta[\phi(X)]\alpha \tag{2.2}$$

(for exp fam $\theta_0 \in \Theta^\circ, \beta'(\theta_0) = 0$)

Suppose $\theta_0 \in \Theta^\circ$. There is a two-sided UMP unbiased level-α test of the form (*) with c_i, γ_i, chosen to satisfy

$$\mathbb{E}_{\theta_0}\phi(X) = \alpha \tag{2.3}$$

$$\beta'(\theta_0) = \mathbb{E}_{\theta_0}(T(X)(\phi(X) - \alpha)) = 0 \tag{2.4}$$

$$p_0(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, & \text{if } x > 0 \\ \frac{1}{2} e^{-|x|}, & \text{if } x < 0 \end{cases} \tag{2.5}$$

3 p-Values

Example 3.1. $X \sim N(\theta, 1)$. $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$.

$$p(X) = P_{\theta_0}(|X - \theta_0| > |x - \theta_0|) \tag{3.1}$$

$$p(x) = P_{\theta_0}(X > x) \tag{3.2}$$

Setup: Testing problem $\mathcal{P}$, H_0, H_1
Non-random Test $\phi_\alpha(x)$ for each significance level $\alpha \in (0,1)$.

$$\phi_\alpha(x) = 1_{x \in R_\alpha} \tag{3.3}$$

Example 3.2. test $X \sim N(\theta, 1)$, $H_0 : \theta = 0$. $R_\alpha = (-\infty, z_{\alpha/2}] \cup [z_{\alpha/2}, \infty))$

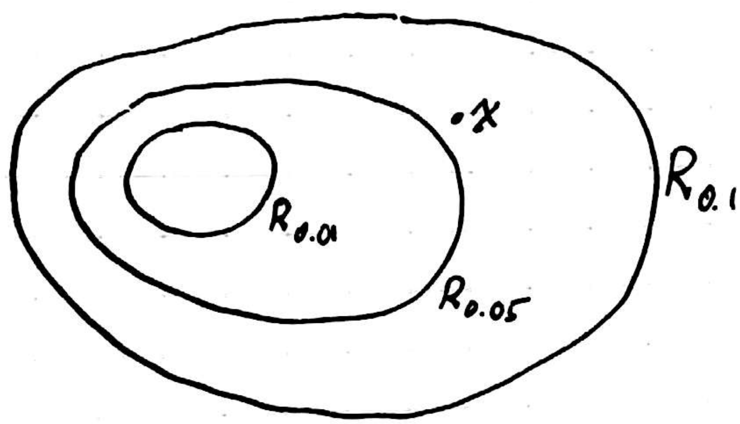
Assume rejection regions are *nested*

$$\alpha_1 \leq \alpha_2 \implies R_{\alpha_1} \subset R_{\alpha_2} \tag{3.4}$$
$$(\phi_{\alpha_1}(X) \leq \phi_{\alpha_2}(X)) \tag{3.5}$$

Define *p-value*

$$p(x) = \inf\{\alpha : \phi_\alpha(x) = 1\} \tag{3.6}$$
$$= \inf\{\alpha : x \in R_\alpha\} \tag{3.7}$$

Picture:

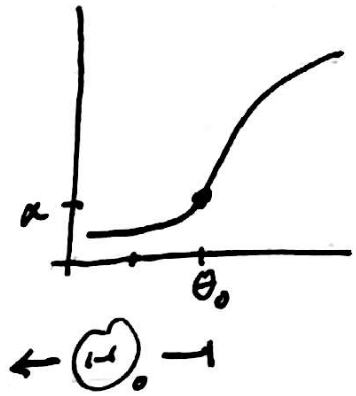


$\forall \theta \in \Theta_0$

$$P_\theta(p(X) \leq \alpha) = \sup_{\tilde{\alpha} < \alpha} P_\theta(x \in R_{\tilde{\alpha}}) \leq \sup_{\tilde{\alpha} < \alpha} \tilde{\alpha} = \alpha \tag{3.8}$$

If $\phi_\alpha(x)$ reject for large $T(X)$

$$p(x) = \sup_{\theta \in \Theta_0} P_\theta(T(X) > T(x)) \tag{3.9}$$



4 Confidence intervals/regions

Accept/reject decision only so informative

Definition 4.1. Model $\mathcal{P}_{1-\alpha} = \{P_\theta : \theta \in \Theta\}$.

$C(X)$ is a *confidence set* for $g(\theta)$ if $P_\theta(g(\theta) \in C(X)) \geq 1 - \alpha$ for all θ.