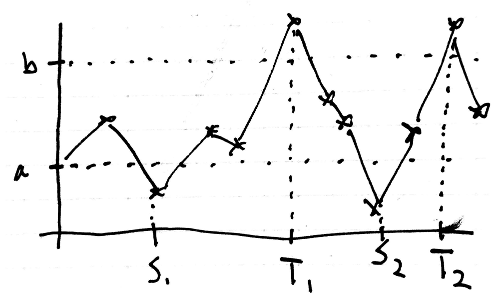
Feynman Liang
STAT 205A
Lecture #20
11/1/2016

# 1 Upcrossing Inequality

Take any $\mathbb{R}$-valued $(X_n, n \geq 0)$ and any $a < b \in \mathbb{R}$.

Let $S_1 = \min\{n : X_n \leq a\}$, $T_1 = \min\{n : X_n \geq b\}$, $S_2 = \min\{n > T_1 : X_n \leq a\}$, $T_2 = \min\{n > S_2 : X_n \geq b\}, \ldots$

**Definition 1.1.** Define $U_n = U_n[a,b] = \max\{k : T_k \leq n\}$ to be the *number of upcrossings* over $[a,b]$ completed by time $n$.




**Theorem 1.2** (Upcrossing Inequality)**.** *Suppose $(X_n)$ is a sub-MG. Then*

$$(b-a)\mathbb{E}U_n \leq \mathbb{E}(X_n - a)^+ - \mathbb{E}(X_0 - a)^+ \tag{1.1}$$

$$\leq \mathbb{E}X_n^+ + |a| \tag{1.2}$$

*Proof.* For the second inequality, note $(x-a)^+ \leq x^+ + |a|$ so $\mathbb{E}(X-a)^+ \leq \mathbb{E}X^+ + |a|$.

(*Trick*) When $X_n \geq a \; \forall n$, we will prove

$$(b-a)\mathbb{E}U_n \leq \mathbb{E}X_n^+ - \mathbb{E}X_0^+ \tag{1.3}$$

For general MG $(X_n)$, apply the result to $\max(X_n, a) - a$, which is a sub-MG.

Use the "buy low, sell high" strategy: buy 1 share at $S_i$ (low) and sell at $T_i$ (high). Consider $Y = H \cdot X$, where $H_n = \sum_i 1_{S_i < n \leq T_i}$. $(H_n)$ is predictable and $H_n \geq 0$, so (by Durett 2.7) $(Y_n)$ is a sub-MG.

$$Y_n = \sum_{i=1}^{U_n} \underbrace{(X_{T_i} - X_{S_i})}_{\text{profit}} + \underbrace{(X_n - X_{S_{U_n+1}})1_{n > S_{U_n+1}}}_{\substack{\text{value of stock if buy at } S_{U_n+1} \\ \text{and sell at time } n < T_{U_n+1}}} \tag{1.4}$$

$$\geq (b-a)U_n + 0 \tag{1.5}$$

$$\mathbb{E}Y_n \geq (b-a)\mathbb{E}U_n \tag{1.6}$$

Consider the opposite strategy $K$: $K_n = 1 - H_n$ ("Buy high, sell low"). $(K_n)$ is also predictable, $K_n \geq 0$, so

$$(X_n - Y_n) = X_n - (H \cdot X)_n = \sum_{i=1}^{n} \Delta_i^X + X_0 - \sum_{i=1}^{n} H_i \Delta_i^X = \sum_{i=1}^{n} \underbrace{(1 - H_i)}_{=K_i} \Delta_i^X + X_0 = (K \cdot X)_n + X_0 \tag{1.7}$$

is a sub-martingale and

$$\mathbb{E}[X_0 - \cancelto{0}{Y_0}] \leq \mathbb{E}[X_n - Y_n] \tag{1.8}$$
$$\mathbb{E}X_0 \leq \mathbb{E}X_n - \mathbb{E}Y_n \tag{1.9}$$
$$(b-a)\mathbb{E}U_n \leq \mathbb{E}X_n - \mathbb{E}X_0 \tag{1.10}$$

□

# 2 Martingale convergence

**Theorem 2.1** (Martingale Convergence Theorem (MCT)). *If $(X_n)$ is a sub-MG, $\sup_n \mathbb{E}X_n^+ < \infty$, then $X_n \xrightarrow{a.s.} X_\infty$ for some $X_\infty$ with $\mathbb{E}|X_\infty| < \infty$.*

*Proof.* $U_n[a,b] \uparrow U_\infty[a,b]$ so by monotone convergence theorem, upcrossing inequality, and the assumption $\mathbb{E}X_n^+ < \infty$

$$\mathbb{E}U_\infty[a,b] = \lim_n \mathbb{E}U_n[a,b] \leq \frac{\sup_n \mathbb{E}X_n^+ + |a|}{b-a} < \infty \tag{2.1}$$

This implies that $U_\infty[a,b] < \infty$ a.s., hence

$$P(U_n[a,b] < \infty \; \forall a, b \in \mathbb{Q}, a < b) = 1 \tag{2.2}$$

For reals $(x_n)$, if $\limsup_n x_n > \liminf_n x_n$, then $U_\infty[a,b] = \infty$ for some $a < b$.

Since $U_\infty[a,b] < \infty$ for all rational $a < b$, the contrapositive implies $\limsup x_n = \liminf x_n \in [-\infty, \infty]$. Therefore, $X_n \to X_\infty$ a.s..

We have so far $X_\infty \in [-\infty, \infty]$, but we would like $\mathbb{E}|X_\infty| < \infty$. To show this, recall Fatou's Lemma: If $Y_n \geq 0$, then $\mathbb{E} \liminf_n Y_n \leq \liminf_n \mathbb{E}Y_n$.

$X_n^+ \to X_\infty^+$ a.s. implies (by Fatou's Lemma) that $\mathbb{E}X_\infty^+ \leq \liminf_n \mathbb{E}X_n^+ < \infty$. Also

$$\mathbb{E}X_n^- = \mathbb{E}X_n^+ - \mathbb{E}X_n \leq \mathbb{E}X_n^+ - \mathbb{E}X_0 \tag{2.3}$$

since $(X_n)$ a sub-MG means $\mathbb{E}X_0 \leq \mathbb{E}X_n$. As $X_n^- \to X_\infty^-$ a.s., by Fatou's Lemma

$$\mathbb{E}X_\infty^- \leq \liminf_n \mathbb{E}X_n^- \leq \sup_n \mathbb{E}X_n^+ - \mathbb{E}X_0 < \infty \tag{2.4}$$

Since $\mathbb{E}X_\infty^+ < \infty$ and $\mathbb{E}X_\infty^- < \infty$, we have $\mathbb{E}|X_\infty| < \infty$. □

**Corollary 2.2.** *If $(X_n)$ is a super-MG, $X_n \geq 0$ a.s., then $X_n \overset{a.s.}{\to} X_\infty$ and $0 \leq \mathbb{E}X_\infty \leq \mathbb{E}X_0$.*

*Proof.* Apply MCT to $-X_n$, so $X_n \overset{\text{a.s.}}{\to} X_\infty$. Use Fatou's Lemma: $\mathbb{E}X_\infty \leq \liminf_n \mathbb{E}X_n \leq \mathbb{E}X_0$. $\square$

**Example 2.3** (WARNING: MCT does not imply $\mathbb{E}X_n \to \mathbb{E}X_\infty$)**.** Consider a simple random walk $X_0 = 1$, stopped at $T = \min\{n : X_n = 0\}$. Let $Y_n = X_{\min(T,n)}$. Then $Y_n \to 0 = Y_\infty$ a.s., but $\mathbb{E}Y_n = 1\ \forall n$ which differs from $\mathbb{E}Y_\infty = 0$.

This is similar to *uniform convergence*: if continuous $f_n$ converge uniformly to $f$, then $f$ is continuous. Not necessarily true for pointwise convergence.

For sequences of expectations to converge, we need *uniform integrability*.

# 3 Facts about Uniform (Equi-)Integrability

Consider $\mathbb{R}$-valued RVs $(Y_\alpha)$

**Definition 3.1.** A family $(Y_\alpha)$ is *uniformly integrable (UI)* if

$$\lim_{b\to\infty} \sup_\alpha \mathbb{E}\left[|Y_\alpha| 1_{|Y_\alpha|\geq b}\right] = 0 \tag{3.1}$$

"We have integrability *uniformly over all* RVs $Y_\alpha$ in the family"

If $\mathbb{E}|Y| < \infty$, then $\lim_{b\to\infty} \mathbb{E}\left[|Y| 1_{|Y|>b}\right] = 0$

Facts relating to UI (see Durrett or Bilingsley)

1. If $\sup_\alpha \mathbb{E}|Y_\alpha|^q < \infty$ for some $q > 1$, then $(Y_\alpha)$ is UI, which implies that $\sup_\alpha \mathbb{E}|Y_\alpha| < \infty$
2. If $Y_n \overset{\text{a.s.}}{\to} Y_\infty$ and $(Y_n)$ is UI, then $\mathbb{E}|Y_\infty| < \infty$ and $\mathbb{E}|Y_n - Y_\infty| \to 0$ (i.e. $Y_n \to Y_\infty$ in $L^1$)
3. If $Y_n \to Y_\infty$ in $L^1$, then $(Y_n)$ is UI.
4. If $\mathbb{E}|Y| < \infty$, the family $\{\mathbb{E}[Y \mid \mathcal{G}] : \forall \mathcal{G} \subset \mathcal{F}\}$ is UI.

**Theorem 3.2.** *For a MG (not sub-MG!) $(X_n)$, TFAE:*

*(a)* $(X_n)$ *is UI*

*(b)* $X_n$ *converges in* $L^1$

*(c) There exists a RV $X_\infty$ with $\mathbb{E}|X_\infty| < \infty$ such that $X_k = \mathbb{E}[X_\infty \mid \mathcal{F}_k]\ \forall k$*

*If any of these conditions hold, then $\exists X_\infty$ such that $X_n \to X_\infty$ both a.s. and in $L^1$.*

*Proof.* (c) $\implies$ (a), item 4..

(a) implies (by item 1.) that $\sup_n \mathbb{E}|X_n| < \infty$, which by MCT implies $X_n$ converges to some $X_\infty$ a.s., which implies by 2. that $X_n \to X_\infty$ in $L^1$ i.e. (b).

Given (b), $X_n \to X_\infty$ in $L^1$, which means that $\mathbb{E}|X_n - X_\infty| \to 0$ with $\mathbb{E}|X_\infty| < \infty$. We need to prove that $\mathbb{E}X_\infty 1_A = \mathbb{E}X_k 1_A$ for any $A \in \mathcal{F}_k$. Fix $A$ and $k$. By the MG property, for $n > k$, $\mathbb{E}[X_n \mid \mathcal{F}_k] = X_k$ so $\mathbb{E}X_n 1_A = \mathbb{E}X_k 1_A$. Hence

$$|\mathbb{E}X_\infty 1_A - \mathbb{E}X_n 1_A| \leq \mathbb{E}|X_\infty - X_n| \to 0 \tag{3.2}$$

as $n \to \infty$, so $|\mathbb{E}X_\infty 1_A - \mathbb{E}X_k 1_A| = 0$. □

Using UI with MCT leads to a convergence property for conditional expectations.

**Theorem 3.3** (Levy 0-1 Law). *Let $(Y_n)_{n \geq 0}$ be any process, $\mathcal{F}_n$ the natural filtration $\sigma(Y_k, 0 \leq k \leq n)$. Let $Z$ be any RV with $\mathbb{E}|Z| < \infty$ and $Z \in \mathcal{F}_\infty$.*

*Then $X_n = \mathbb{E}[Z \mid \mathcal{F}_n]$ is a UI martingale, so theorem 3.2 implies $X_n \to X_\infty$ both a.s. and in $L^1$. In terms of $Z$*

$$\mathbb{E}[Z \mid \mathcal{F}_n] \to \mathbb{E}[Z \mid \mathcal{F}_\infty] \quad \text{as } n \to \infty \tag{3.3}$$

*In fact, $X_\infty = Z$ because*

$$\mathbb{E}[X_\infty \mid \mathcal{F}_n] = X_n = \mathbb{E}[Z \mid \mathcal{F}_n] \qquad \textit{MG property} \tag{3.4}$$

$$0 = \mathbb{E}[X_\infty - Z \mid \mathcal{F}_n] = X_\infty - Z \qquad X_\infty - Z \textit{ is } \mathcal{F}_\infty\textit{-meas} \tag{3.5}$$

*Remark* 3.4. In particular, take $Z = 1_A$ for some event $A$. Then

$$P(A \mid \mathcal{F}_n)(\omega) \overset{\text{a.s.}}{\to} 1_A(\omega) \tag{3.6}$$

for all $A \in \sigma(Y_n, n \geq 0)$.

In English:

> If we are learning gradually all the information that determines the outcome of an event, then we will become gradually certain what the outcome will be.

For independent $(Y_n)$, let $A \in \mathcal{F}_\infty$ be some tail event. Then

$$P(A \mid \mathcal{F}_n)(\omega) = P(A) \overset{\text{a.s.}}{\to} 1_A \quad \text{as } n \to \infty \tag{3.7}$$

which implies that $1_A$ is a constant a.s. and hence $P(A) \in \{0, 1\}$. This is Kolmogorov's zero-one law.