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STAT 205A
Lecture #18
10/25/2016

1 Review

- General constructions of martingales
- Can define $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n) \subset \mathcal{F}$
- Usually **not** given a RV X_∞
- When we consider X_T for a stopping time T, care about $\{T = \infty\}$

Definition 1.1. A *filtration* $(\mathcal{F}_n, 0 \le n < \infty)$ on $(\Omega, \mathcal{F}, P)$ is a nested sequence of σ-fields, $F_i \subset F_{i+1}$.

Random variable $(X_n, 0 \le n < \infty$ *adapted* to filtration $\mathcal{F}_n$ means $X_n \in \mathcal{F}_n$, $0 \le < \infty$.

Example 1.2. Consider any X with $E|X| < \infty$. Then $X_n = E(X \mid \mathcal{F}_n)$, $0 \le n < \infty$ is a martingale.

$$E[X_n \mid \mathcal{F}_{n-1}] = E[E[X_n \mid \mathcal{F}_n] \mid \mathcal{F}_{n-1}] \tag{1.1}$$

$$\mathcal{F}_{n-1} \subset \mathcal{F}_n \underset{\text{tower}}{\Longrightarrow} = E[X_n \mid \mathcal{F}_n] \tag{1.2}$$

Similarly, for any event A, $Y_n = P(A \mid \mathcal{F}_n)$ is a martingale.

2 Doob Decomposition

Definition 2.1. For any $X = (X_n)$, define $\Delta_n^X = X_n - X_{n-1}$, $n \ge 1$. Then Δ_n^X is a (sub-)martingale $\iff$ $\Delta_n^X \in \mathcal{F}_n, n \ge 1$

$$E[\Delta_n^X \mid \mathcal{F}_{n-1}) = 0 \; (\ge 0 \text{ for sub-martingale a.s. } n \ge 1$$

$X_0 \in \mathcal{F}_n, E|X| \le \infty$. Call $(\Delta_n^X, n \ge 1)$ a **martingale difference sequence**.

Consider any $(X_n, n \ge 0)$ adapted to $(\mathcal{F}_n)$ and $E|X_n| < \infty \; \forall n$. Define (Y_n) by $Y_0 = X_0$, $\Delta_n^Y = \Delta_n^X - E[\Delta_n^X \mid \mathcal{F}_{n-1}]$ (shocks, martingale part). Define (Z_n) by $Z_0 = 0$, $\Delta_n^Z = E[\Delta_n^X \mid \mathcal{F}_{n-1}]$ (drift, predictable part). Then

1. $X_n = Y_n + Z_n$ a.s.
2. (Y_n) is a martingale
3. $Z_n \in \mathcal{F}_{n-1}$, $n \ge 1$ (Z_n is **predictable**) and $Z_0 = 0$ and $E|Z_n| < \infty$

This is the unique decomposition with these properties, called the **Doob decomposition**.

Uniqueness:

$$E[\Delta_n^X \mid \mathcal{F}_{n-1}) = E[\Delta_n^Y \mid \mathcal{F}_{n-1}) + E[\Delta_n^Z \mid \mathcal{F}_{n-1}) \tag{2.1}$$

$$= 0^{(martingale)} + \Delta_n^Z \tag{2.2}$$

3 Convexity Theorem

If (X_n) is a martingale then $(X_n - X_0, n \geq 0)$ is a martingale. Often say "WLOG assume $X_0 = 0$."

Theorem 3.1 (Convexity Theorem). *(X_n) adapted to $(\mathcal{F}_n)$, ϕ convex function, $E[\phi(X_n)] < \infty$*

1. *If (X_n) is a martingale then $(\phi(X_n))$ is a sub-martingale*
2. *If (X_n) is a sub-martingale and if ϕ is increasing, then $(\phi(X_n))$ is a sub-martingale.*

Proof.

$$E[\phi(X_{n+1}) \mid \mathcal{F}_n) \geq \phi(\underbrace{E[X_{n+1} \mid \mathcal{F}_n]}_{\geq X_n,\ \text{sub-martingale}}) \text{ by conditional Jensen's inequality.} \tag{3.1}$$

$$\geq \phi(X_n) \text{ because } \phi \text{ is increasing} \tag{3.2}$$

This checks $\phi(X_n)$ is a sub-martingale. □

Example 3.2. If (X_n) is a martingale, then (provided integrable)

(a) $|X_n|^p$ $(p \geq 1)$ is a sub-martingale, $x \to |x|^p$ is convex.

(b) X_n^2 is a sub-martingale.

(c) $\exp(\theta X_n)$ $(-\infty < \theta < \infty)$ is a sub-martingale.

(d) $\max(X_n, c)$ is a sub-martingale.

(e) $\min(X_n, c)$ is a super-martingale.

4 Stopping times in martingales

Definition 4.1. A RV $T : \Omega \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ is a *stopping time* if

$$\{T = n\} \in \mathcal{F}_n, 0 \leq n < \infty \tag{4.1}$$

Equivalent condition:

$$\{T \leq n\} \in \mathcal{F}_n, 0 \leq n < \infty \tag{4.2}$$

Definition 4.2. For a stopping time T, define $\mathcal{F}_T$ (the **pre-T σ-field**), as the collection of sets $A \in \mathcal{F}$ such that

(a) $A \cap \{T = n\} \in \mathcal{F}_n, 0 \leq n < \infty$

(b) $A \cap \{T \leq n\} \in \mathcal{F}_n, 0 \leq n < \infty$

Many "obvious" properties:

(a) If (X_n) is adapted, T is a stopping time, $T < \infty$, then X_T is $\mathcal{F}_T$-measurable.

Proof. Need to show $\{X_T \in B\} \in \mathcal{F}_T$ for all B. Equivalently

$$\{X_t \in B\} \cap \{T = n\} \in \mathcal{F}_n \tag{4.3}$$

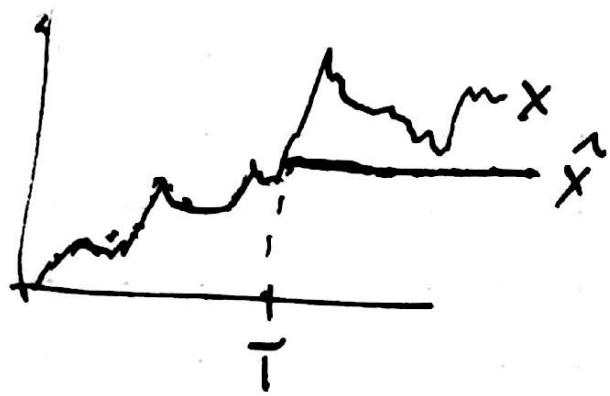
$$\underbrace{\{X_n \in B\}}_{\in \mathcal{F}_n,\text{ adapted}} \cap \underbrace{\{T = n\}}_{\in \mathcal{F}_n,\text{ stop time}} \in \mathcal{F}_n \tag{4.4}$$

□

(b) If $T_1 \subset T_2$ are stopping times, $\mathcal{F}_{T_1} \subset \mathcal{F}_{T_2}$.

(c) If S and T are stopping times, then $\{S = T\} \in \mathcal{F}_S \cap \mathcal{F}_T$ and for $A \subset \{S = T\}$, $A \in \mathcal{F}_S \iff A \in \mathcal{F}_T$.

(d) Given an adapted process (X_n) and a stopping time T, the process $\hat{X}_n = X_{\min(n,T)}$ is adapted. Call $\hat{X}$ the **stopped process**.



If want to sum $\sum_{n=1}^{T} X_n$, can use $\sum_{n=1}^{\infty} X_n 1_{T<n}$.

5 Story: stock market

You can buy a stock at end of any day n. X_n = the price of one share at the end of day n. H_n = the number of shares I hold during day n (bought day $n-1$ or earlier) Y_n = my accumulated profit at the end of day n

Question: What is the relation between X_n, H_n, and Y_n?

Answer: The relation is given by $\Delta_n^Y = H_n \Delta_n^X$, $Y_0 = 0$.

Definition 5.1. We write $Y = H \cdot X$ (*martingale transform* or *discrete time stochastic integral*)

Theorem 5.2 (2.7 Durett). *Suppose (X_n) is adapted and (H_n) is predictable. Consider $Y = H \cdot X$ (for simplicity assume H_n is bounded).*

1. *If (X_n) is a MG, then (Y_n) is a MG.*
2. *If (X_n) is a sub-MG and $H_n \geq 0$, then (Y_n) is a sub-MG.*

Proof. 2.

$$
\begin{aligned}
E(\Delta_n^Y \mid \mathcal{F}_{n-1}) &= E[H_n \Delta_n^X \mid \mathcal{F}_{n-1}) && (5.1) \\
&= \underbrace{H_n}_{\geq 0} \underbrace{E[\Delta_n^X \mid \mathcal{F}_{n-1})}_{\geq 0, \text{sub-MG}} && (5.2) \\
&\geq 0 && (5.3)
\end{aligned}
$$

Hence (Y_n) is a sub-MG. □

Corollary 5.3. *If (X_n) is a (sub)-MG, T a stopping time, then $\hat{X}_n = X_{\min(n,T)}$ is a (sub)-MG.*

Proof. Buy 1 share at end of day 0. Sell at end of day T. $H_n = 1_{0 \leq n \leq T}$. (H_n) is predictable because $\{n \leq T\} = \{T \leq n-1\}^c \in \mathcal{F}_{n-1}$. The process $Y = H \cdot X$ is explicitly $Y_n = X_{\min(n,T)} - X_0$. Apply Theorem. □