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STAT 205A
Lecture #3
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1 Measures continued

Example 1.1. $S = \mathbb{N}, \mathcal{S} = 2^{\mathbb{N}}$.

- Given $p_0, p_2, \cdots \geq 0, \sum_i p_i = 1$, define $\mu(A) := \sum_{i \in A} p_i$ for $A \subset S$. μ is a prob measure.
- Given p.m. μ on S, define $p_i := \mu(\{i\})$ and $\sum_i p_i = 1$ holds.

Definition 1.2. A class of subsets of S, $\mathcal{A}$, is a *π-class* (or *π-system*) if $A_1, A_2 \in \mathcal{A} \implies A_1 \cap A_2 \in \mathcal{A}$.

Definition 1.3. A class of subsets of $S, \mathcal{C}$, is a *λ-class* if:
(a) $S \in \mathcal{C}$
(b) $A, B \in \mathcal{S}, A \subset B \implies B \setminus A \in \mathcal{C}$
(c) $A_n \in \mathcal{C}, A_n \uparrow A \implies A \in \mathcal{C}$

Lemma 1.4 (Dynkin). *If $\mathcal{C}$ is a λ-class, $\mathcal{A}$ is a π-class, $\mathcal{C} \supset \mathcal{A}$, then $\mathcal{C} \supset \sigma(\mathcal{A})$.*

Proof. See Durrett. □

Lemma 1.5 (Identification of PMs). *μ_1, μ_2 prob. meaures on $(S, \mathcal{S})$. If $\mu_1(A) = \mu_2(A)$ for all $A \in \mathcal{A}$, $\mathcal{A}$ is a π-class such that $\mathcal{S} = \sigma(\mathcal{A})$, then $\mu_1 = \mu_2$ (i.e. $\mu_1(B) = \mu_2(B)$ for all $B \in \mathcal{S}$).*

Proof. Consider $\mathcal{C} = \{A \in \sigma(\mathcal{A}) : \mu_1(A) = \mu_2(A)\}$, so $\mathcal{C} \supset \mathcal{A}$. To apply Dynkin's, it suffices to check $\mathcal{C}$ is a λ-class (clear from definition of PM). □

Theorem 1.6 (Existance of Lebesgue measure). *$\exists$ σ-finite measure λ on $(\mathbb{R}^1, \mathcal{B}^1)$ such that $\lambda([a, b]) = b - a$ for all $a, b \in \mathbb{R}$.*

$\exists$ PM λ_1 on $[0, 1]$, called the uniform distribution *on $[0, 1]$, such that $\lambda_1([a, b]) = b - a$.*

Proof. See Durrett □

Proposition 1.7. *Given $f^{meas} : S_1 \to S_2$, PM μ on $(S_1, \mathcal{S}_1)$, can define PM $\hat{\mu}$ on S_2 by*

$$\hat{\mu}(B) = \mu(f^{-1}(B)) \tag{1.1}$$

for all $B \in \S_2$

Proof. $\hat{\mu}$ is PM because f^{-1} commutes with Boolean operations. □

2 Probability measures on $\mathbb{R}$

Given PM μ on $\mathbb{R}$, define $F(x) = \mu((-\infty, x])$. F is

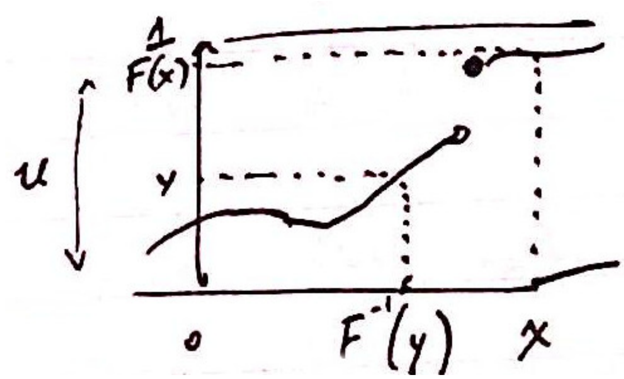
- (Increasing) $x_1 < x_2 \implies F(x_1) < F(x_2)$
- (Right-Continuous) $x_n \downarrow x \implies F(x_n) \downarrow F(x)$
- $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$

Definition 2.1. A function satisfying the above is called a *distribution function*.

Theorem 2.2. *Given a distribution function F, exists unique probability measure $\mu : F(x) = \mu((-\infty, x])$ for all x.*

2.1 Pullback of random variables

(Undergrad) $U \sim \text{Unif}[0, 1]$. Then $F^{-1}(U)$ is a RV with distribution function F.

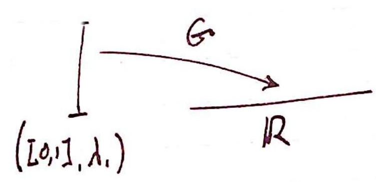


Define G (version of F^{-1}) on $0 < y < 1$ as

$$G(y) := \sup\{x : F(x) < y\} \tag{2.1}$$
$$= \inf\{x : F(x) \geq y\} \tag{2.2}$$

G is increasing $\implies$ G is measurable.

For each x, $\{y : G(y) \leq x\} = \{y : y \leq F(x)\}$



Lemma 2.3 (Push-forward). *$\exists$ PM $\hat{\mu}$ on $\mathbb{R}$ such that*

$$\lambda_1(\underbrace{[0, F(x)]}_{=G^{-1}(-\infty,x]}) = \hat{\mu}((-\infty, x]) \tag{2.3}$$

Proof. Needs right-cts $\square$

3 Coin-tossing space

2-element set $B = \{H, T\}$.

Sequence space $B^{\mathbb{N}} = \{\vec{b} = (b_1, b_2, \cdots)) : b_i \in B\}$.

Given finite sequence $\pi = (\pi_i)_{i=1}^{n}, \pi_i \in B$, let $A_\pi = \{\vec{b} : b_i = \pi_i \quad 1 \leq 1 \leq n\} \subset B^{\mathbb{N}}$ be all sequences starting out as π.

Define σ-field $\mathcal{B}^{\mathbb{N}}$ on $B^{\mathbb{N}}$ as σ(all A_π such that π is finite string).

Theorem 3.1. *$\exists$ PM μ on $(B^{\mathbb{N}}, \mathcal{B}^{\mathbb{N}})$ such that $\mu(A_\pi) = \frac{1}{2^{|\pi|}} \quad \forall \pi$*

Proof idea. Existance is ensured by theorem 1.6.

Consider the binary expansion of real $x \in (0, 1)$, e.g.

$$x = 0.1101101 \cdots \tag{3.1}$$

$$= 0.b_1(x)b_2(x)b_3(x) \cdots \tag{3.2}$$

In general, $b_i = 1_{2^i x \text{ is odd}}$. b_i is measurable.

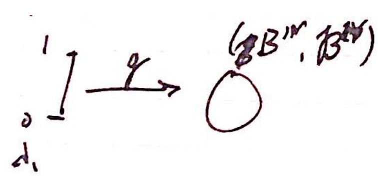
Define $g : [0, 1] \to B^{\mathbb{N}}$ by $x \mapsto (b_1(x), b_2(x), \cdots)$. g is measurable.

Use lemma 2.3 to get PM $\mu : \mathcal{B}^\infty \to \mathbb{R}^+$ mapping

$$A_\pi \mapsto \lambda\{x : g(X) \in A_\pi\} \tag{3.3}$$

$$= \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \text{ for some } k \text{ if } |\pi| = n \tag{3.4}$$

$$= \frac{1}{2^n} \tag{3.5}$$



$\square$