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STAT 205A
Lecture #27
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1 Aspects of Brownian motion

(a) Model or many processes fluctuating continuously (e.g. stock markets)

(b) Limit of RVs with small step size

(c) Gaussian process (multivariate normal over finite subsets)

(d) **Diffusion**: continuous-path Markov processes

(e) Martingale properties

Definition 1.1. *Brownian motion* $(B(t), 0 \leq t \leq \infty)$ has the proeprties

(a) For $s < t$, $B(t) - B(s) \stackrel{d}{=} N(0, t-s)$

(b) For $0 \leq t_1 < t_2 < \cdots < t_n$, the increments $(B(t_{i+1}) - B(t_i), 1 \leq i \leq n-1)$ are independent

(c) Sample paths $t \mapsto B(t)$ are continuous

1.1 Continuous-time martingales

Let $(\mathcal{F}_t, 0 \leq t < \infty)$ be a filtration. $(M_t, \mathcal{F}_t)$ is a *martingale (MG)* if

- $\mathbb{E}|M_t| < \infty$, $\forall t$
- M_t is adapted to $\mathcal{F}_t$
- For st, $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$ a.s.

All our MGs will have continuous paths — theory requires only right-continuity.

Definition 1.2. $T : \Omega \to [0, \infty]$ is a *stopping time* if $\{T \leq t\} \in \mathcal{F}_t$, all $0 \leq t < \infty$

Theorem 1.3. *If (M_t) is a MG, T a stopping time, $P(T \leq t_0) = 1$, then $\mathbb{E}M_T = \mathbb{E}M_0$.*

Proof. Fix m, look at times that are multiples of 2^{-m}.

Define $T_m = \inf\{2^{-m}i : 2^{-m}i > T\}$. This T_m is a stopping time for $(M_{2^{-m}i}, \mathcal{F}_{2^{-m}i}, i \geq 0)$ and $T_m \leq t_0 + 1$. Apply discrete-time OST $\implies$ $\mathbb{E}M_{T_m} = \mathbb{E}M_0$ and $M_{T_m} = \mathbb{E}[M_{t_0+1} \mid \mathcal{F}_{T_m}]$ (i.e .$(M_{T_m}, m \geq 1)$ is UI).

As $m \to \infty$, $T_m \downarrow T$, so right-continuity $\implies$ $M_{T_m} \stackrel{\text{a.s.}}{\to} M_T$.

Since $(M_{T_m}, m \geq 1)$ is UI, $\mathbb{E}M_{T_m} \to \mathbb{E}M_T$. $\square$

With BM we associate the natural filtration $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$ which makes all the B_s measurable.

Proposition 1.4. *The following are MGs*

(a) B_t

(b) $B_t^2 - t$

(c) $\exp(\theta B_t - \theta^2 t/2)$

(d) $B_t^3 - 3tB_t$

(e) $B_t^4 - 6tB_t^2 + 3t^2$

Proof. Fix $s < t$.
$B_t = B_s + (B_t - B_s)$.
$\mathbb{E}[B_t \mid \mathcal{F}_s] = B_s + \mathbb{E}[B_t - B_s \mid \mathcal{F}_s]$.
$B_t - B_s$ is independent of $(B_{s_1}, B_{s_2}, \cdots, B_{s_n}))$ for all $0 \le s_1 < s_2 < \cdots < s_n \le s$.
By measure theory on independence, independent increments on finite subsets $\implies$ $B_t - B_s$ independent of $\mathcal{F}_s \overset{\text{def}}{=} \sigma(B_u, 0 \le u \le s)$.
Hence

$$\begin{aligned} \mathbb{E}[B_t \mid \mathcal{F}_s] &= B_s + \mathbb{E}[B_t - B_s \mid \mathcal{F}_s] && (1.1) \\ &= B_s + \mathbb{E}[B_t - B_s] && (1.2) \\ &= B_s + 0 = B_s && (1.3) \end{aligned}$$

Write $Y_t = B_t^2 - t = (B_t + (B_t - B_s))^2 - t$.
$Y_t = Y_s + 2B_s(B_t - B_s) + (B_t - B_s)^2 - (t - s)$.
$\mathbb{E}[Y_t \mid \mathcal{F}_s] = Y_s + 2B_s \underbrace{\mathbb{E}[B_t - B_s \mid \mathcal{F}_s]}_{=0} + \underbrace{\mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s)}_{\substack{=\mathbb{E}(B_t - B_s)^2 \\ =t-s}} - (t - s) = Y_s$.

Remark 1.5. If $W \sim N(0, \sigma^2)$, then $\mathbb{E}\exp(\theta W) = \exp(\theta^2\sigma^2/2)$.
Write

$$\begin{aligned} Z_t &= \exp(\theta B_t - \theta^2 t/2) && (1.4) \\ &= Z_s \exp(\theta(B_t - B_s)) \exp\left(-\frac{\theta^2}{2}(t-s)\right) && (1.5) \end{aligned}$$

$$\mathbb{E}[Z_t \mid \mathcal{F}_s] = Z_s \exp\left(-\frac{\theta^2}{2}(t-s)\right) \underbrace{\mathbb{E}\exp(\theta(B_t - B_s))}_{=\exp(\theta^2(t-s)/2)} = Z_s \tag{1.6}$$

For the rest, informally: $(Z_t^\theta, 0 \le t < \infty)$ is a MG and since differentiation is linear

$$\left(\frac{d^k}{d\theta^k} Z_t^\theta, 0 \le t < \infty\right) \tag{1.7}$$

should be a MG.
If we differentiate k times, set $\theta = 0$, we get a sequence of polynomials in B_t. $\square$

Typical stopping time is $T_b = \inf\{t : B(t) = b\} = \inf\{t : B(t) \geq b > 0\}$.

Also, for $b > 0$, $t > 0$, $\{T_b \leq t\} = \{\sup_{s \leq t} B(t) \geq b\}$. Even though sup is an uncountable operation, this is measurable because

$$\sup_{s \leq t} B(t) = \sup_{\substack{u \leq t \\ u \in \mathbb{Q}}} B(u),\ \mathcal{F}_n\text{-meas.} \tag{1.8}$$

Lemma 1.6. *Fix $-a < 0 < b$. Consider $T = \min\{T_{-a}, T_b\}$ and $\mathbb{E}T = ab$. Then $P(B_T = b) = \frac{a}{a+b} = P(T_b < T_{-a}) = P(T_b < T_{-a})$ and $P(B_t = -a) = \frac{b}{a+b}$.*

Proof. Apply OST to 0 and $T \wedge t$.

$$0 = \mathbb{E}B_0 = \mathbb{E}B_{T\wedge t} \tag{1.9}$$

As $t \to \infty$, $B_{T\wedge t} \overset{\text{a.s.}}{\to} B_T$ and $|B_{T\wedge t}| \leq \max(a, b) \implies 0 = \mathbb{E}B_T$.

B_T takes values $(-a, b)$ only, so it must have the desired distribution.

Apply OST to $B_t^2 - t \implies \mathbb{E}B_{T\wedge t}^2 = \mathbb{E}[T \wedge t]$. Let $t \to \infty$, so $\mathbb{E}B_T^2 = \mathbb{E}T = b^2 \frac{a}{a+b} + (-a)^2 \frac{b}{a+b} = ab$.

Note that $P(T_b < \infty) \geq P(T_b < T_{-a}) \to 1$ as $a \to \infty$, so $T_b < \infty$ a.s. $\square$

Fix $c > 0$ and $-\infty < d < \infty$. Consider $T = \inf\{t : B_t = c + dt\} \leq \infty$.

TODO: Fig 27.1

Lemma 1.7. $\mathbb{E}\exp(-\lambda T) = \exp(-c(d + \sqrt{d^2 + 2\lambda}))$ *for $0 \leq \lambda < \infty$, the Laplace transform of T.*

Proof. Consider $\theta > \max(0, 2d)$. Apply OST to $\exp(\cdot)$ and $T \wedge t$.

$1 = \mathbb{E}\exp\left(\theta B_{T\wedge t} - \frac{\theta^2}{2}(T \wedge t)\right)$.

Case $d \leq 0, \theta > 0$. Here, $\theta B_{T\wedge t} - \frac{\theta^2}{2}(T \wedge t) \leq \theta c$, $T \leq T_c < \infty$.

Case $d > 0, \theta > 2d$. Then $\theta B_{T\wedge t} - \frac{\theta^2}{2}(T \wedge t) \leq \sup\left(\theta(c + ds) - \frac{\theta^2}{2}s\right) = \theta c$ and $\to -\infty$ as $t \to \infty$ on $\{T = \infty\}$.

Let $t \to \infty$. $1 = \mathbb{E}[\exp\left(\theta B_T - \frac{\theta^2}{2}T\right)]1_{T<\infty}$. But $B_t = c + dT$ on $\{T = \infty\}$.

$$1 = \mathbb{E}\exp\left(\theta c + \left(\theta d - \frac{\theta^2}{2}\right)T\right)1_{T<\infty} \tag{1.10}$$

Given $\lambda > 0$, define $\theta = \theta(x)$ as solution of $\theta d - \theta^2/2 = -\lambda$, so

$$\theta(\lambda) = d + \sqrt{d^2 + 2\lambda} \overset{?}{>} \max(0, 2d) \tag{1.11}$$
$$1 = \mathbb{E}\exp\left(c\theta(\lambda) - \lambda T\right) \tag{1.12}$$
$$\mathbb{E}\exp(-\lambda T) = \exp\left(-c\theta(\lambda)\right) \tag{1.13}$$

$\square$