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STAT 205A
Lecture #10
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1 Truncation

Corollary 1.1. *(of SLLN). Take i.i.d.* (X_i) *where* $\mathbb{E}X^+ = \infty$, $\mathbb{E}X^- < \infty$. *Let* $S_n = \sum_1^n X_i$. *Then* $\frac{S_n}{n} \overset{a.s.}{\to} \infty$.

Proof. Fix large $B < \infty$. Define $Y_i = X_i 1_{X_i \le B}$, $Y_i \le X_i$. (Y_i) iid, $\mathbb{E}|Y_i| < \infty$, so can apply SLLN to (Y_i).

$\implies \frac{1}{n}\sum_1^n Y_i \overset{\text{a.s.}}{\to} \mathbb{E}Y = \mathbb{E}X1_{X \le B}$

$\implies \liminf_n \frac{1}{n} S_n \ge \liminf \frac{1}{n}\sum_1^n Y_i \overset{\text{a.s.}}{=} \mathbb{E}X1_{X \le B}$

True for each B, so letting $B \uparrow \infty$

$$\mathbb{E}[X1_{X \le B}] \uparrow -\mathbb{E}X^- + \mathbb{E}X^+ = +\infty \tag{1.1}$$

$\implies \liminf_n \frac{1}{n} S_n \ge \infty$ □

Lemma 1.2. *(Deterministic) Reals* $s_0 = 0$, $\frac{S_n}{n} \to a \in (0, \infty)$. *Let* $h(t) = \min\{n : s_n \ge t\}$, $m(t) = \max\{n : S_n \le t\}$. *Note* $m(t) \ge h(t) - 1$. *Then* $\frac{h(t)}{t} \to \frac{1}{a}$ *and* $\frac{m(t)}{t} \to \frac{1}{a}$ *as* $t \to \infty$.

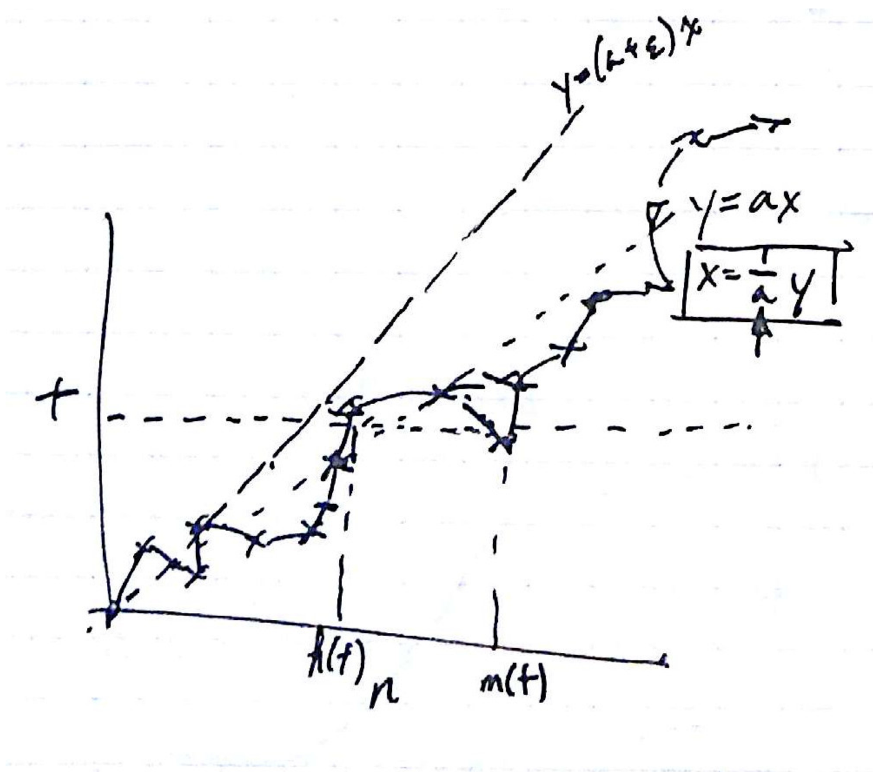

Figure 1: Depiction of $h(t)$ and $m(t)$

Proof. Fix $\epsilon > 0$. $S_n \le (a+\epsilon)n$e.v.

$\implies h(t) \ge \frac{t}{a=\epsilon}$e.v.

$\implies \liminf_t \frac{h(t)}{t} \ge \frac{1}{a+\epsilon}$

$\underset{\epsilon\downarrow 0}{\implies} \liminf_t \frac{h(t)}{t} \ge \frac{1}{a}$

Similarly $m(t) \le \frac{t}{a-\epsilon}$e.v.

$\implies \limsup_t \frac{m(t)}{t} \le \frac{1}{a}$

Altogether

$$\frac{1}{a} \le \liminf_t \frac{h(t)}{t} \le \liminf_t \frac{m(t)}{t} \le \limsup_t \frac{m(t)}{t} \le \frac{1}{a} \tag{1.2}$$

□

Corollary 1.3. *(renewal SLLN) (X_i) iid. $\mathbb{E}X = \mu \in (0,\infty)$, $S_n = \sum_1^n X_i$.*
Define $N_t = \max\{n : S_n \le t\}$, $H_t = \min\{n : S_n \ge t\}$.
Then $\frac{N(t)}{t} \to \frac{1}{\mu}$ and $\frac{H(t)}{t} \to \frac{1}{\mu}$ a.s. as $t \to \infty$.

Proof. Use SLLN and Deterministic Lemma. □

Consider light bulbs have IID lifetimes $X_1, X_2, \cdots > 0$. New bulb at time 0, N_t = # bulbs replaced by time t. Renewal SLLN says if average lifetime is 1/2 year, then you should replace at 2 per year. TODO: ???

2 Martingales

Random variables $X_i = (\Omega, \mathcal{F}, P) \to \mathbb{R}$ are measurable functions. Givne $X_0, X_1, \cdots, X_n$, define $\mathcal{F}_n = \sigma(X_0, \cdots, X_n)$ which is comprised of events of the form $\{\omega : (X_0(\omega), \cdots, X_n(\omega)) \in B\}$ for some measurable $B \subset \mathbb{R}^{n+1}$, $\mathcal{F}_n \subset \mathcal{F}$ for every n. We sometimes interpret $\mathcal{F}_n$ as "information at time n."

Definition 2.1. A *stopping time* is a r.v. $T : (\Omega, \mathcal{F}, P) \to \{0,1,2,\cdots\} \cup \{\infty\}$ such that

$$\{T = n\} \in \mathcal{F}_n, 0 \le n < \infty \tag{2.1}$$

Proposition 2.2. *This is equivalent to*

$$\{T \le n\} \in \mathcal{F}_n, 0 \le n < \infty \tag{2.2}$$

Proof. Given eq. (2.1),

$$\{T \le n\} = \{T = 0\} \cup \{T = 1\} \cup \cdots \cup \{T = n\} \in \mathcal{F}_n \tag{2.3}$$

so all are $\in \mathcal{F}_n$.

Conversely, given eq. (2.2)

$$\{T = n\} = \{T \le n\} \setminus \{T \le n-1\} \in \mathcal{F}_n \tag{2.4}$$

are also all $\in \mathcal{F}_n$. □

Example 2.3. $T = \min\{n : X_n \in B\}$ for measurable $B \subset \mathbb{R}^1$ is a stopping time, because $\{T \le n\} = \cup_{i=0}^n \{X_i \in B\} \in \mathcal{F}_n$.

Example 2.4. Arbitrary $X_1, X_2, \cdots, X_n$, define $S_0 = 0, S_m = \sum_{i=1}^m X_i$. Then $\sigma(X_1, \cdots, X_n) = \sigma(S_0, S_1, \cdots, S_n) = \mathcal{F}_n$ ($(X_1, \cdots, X_i) \to S_i$ through summing, $(S_i, S_{i-1}) \to X_i$ through differences) and so $T = \min\{n : S_n \ge b\}$ is a stopping time.

Example 2.5. Given $X_1, \cdots, X_N$, given N, $T = \max\{n : n \le N, X_n \ge a\}$ is *not* a stopping time.

Proposition 2.6 (Wald's equation/identity/formula)**.** *(X_i) iid, $\mathbb{E}X = \mu < \infty$, $S_n = \sum_{i=1}^n X_i$, T a stopping time with $\mathbb{E}T < \infty$. Then $\mathbb{E}S_t = \mu \mathbb{E}T$*

Note: Undergraduate result assumed T is independent of (X).

Remark 2.7. $\mathbb{E}\sum_i^\infty Y_i = \sum_i^\infty \mathbb{E}Y_i$ holds provided $\sum_i^\infty \mathbb{E}|Y_i| < \infty$

Proof. $\sum_i^n Y_i \to \sum_i^\infty Y_i$ and dominated by RV $\sum_i^\infty |Y_i|$. Use dominated convergence. ☐

Proof of Wald's.

$$S_n = \sum_1^\infty X_i 1_{i \le n} \implies S_T = \sum_1^\infty X_i 1_{i \le T} \tag{2.5}$$

$$\underbrace{\{i \le T\}^c}_{\in \mathcal{F}_{i-1}} = \{T \le i-1\} \in \mathcal{F}_{i-1} \tag{2.6}$$

$$\implies \{i \le T\} \text{ independent of } X_i \tag{2.7}$$

$$\mathbb{E}[X_i 1_{i \le T}] = \mu P(T \ge i) \tag{2.8}$$

$$\implies \sum_i^\infty \mathbb{E}[X_i 1_{1 \le T}] = \mu \mathbb{E}T \tag{2.9}$$

By earlier fact, to show

$$\mathbb{E}[S_t] = \mathbb{E}\sum_1^\infty X_i 1_{i \le T} \stackrel{?}{=} \sum_1^\infty \mathbb{E}[X_i 1_{1 \le T}] = \mu \mathbb{E}T \tag{2.10}$$

suffices to show $\sum_1^\infty \mathbb{E}|X_i| 1_{i \le T} < \infty$

By applying eq. (2.9) $|X_i|$

$\implies \sum_{i=1}^\infty \mathbb{E}[|X_i| 1_{i \le T}] = (\mathbb{E}|X|)\mathbb{E}T < \infty$ ☐

Lemma 2.8 (Fatou's lemma)**.** *Arbitrary $X_n \ge 0$. Then $\mathbb{E}[\liminf_n X_n] \le \liminf_n \mathbb{E}X_n \le \infty$.*

Corollary 2.9. *Arbitrary $X_n \ge 0$. If $X_n \stackrel{a.s.}{\to} X_\infty$, then $\mathbb{E}X_\infty \le \liminf_n \mathbb{E}X_n \le \infty$.*

TODO: Fatou's automatically gives us a lower bound.

Recall overaggrssive "gambling favorable game" example $X_n > 0$, $X_n \stackrel{\text{a.s.}}{\to} 0$ and $\mathbb{E}X_n \to \infty$.

Proof. Define $Y_N = \inf_{n \ge N} X_n$. Then $0 \le Y_N \uparrow \liminf X_n$. By monotone convergence, $0 \le \mathbb{E}Y_n \uparrow \mathbb{E}(\liminf X_n)$. Since $Y_n \le X_n$, $\implies \mathbb{E}(\liminf X_n) = \liminf_N \mathbb{E}Y_n \le \liminf_N \mathbb{E}X_n$ ☐

3 Back to renewal theory

Under assumptions of corollary 1.3, we also assume $X \geq 0$ a.s.. Then $\mathbb{E}\left[\frac{N(t)}{t}\right] \to \frac{1}{\mu}$ as $t \to \infty$

Proof. Fatou's lemma:

$$\frac{1}{\mu} \leq \liminf_{\substack{t\to\infty \\ t\in\mathbb{N}}} \mathbb{E}\left[\frac{N(t)}{t}\right] \tag{3.1}$$

$$= \liminf_{t\to\infty} \mathbb{E}\left[\frac{N(t)}{t}\right] \tag{3.2}$$

So enough to show upper bound

$$\limsup_{t} \mathbb{E}\left[\frac{N(t)}{t}\right] \leq \frac{1}{\mu} \tag{3.3}$$

$X > 0 \implies N(t) + 1 = \min\{n : S_n > t\}$ is a stopping time.

(Truncation) Consider $\min\{N(t) + 1, m)$, can check a stopping time. Applying Wald's identity

$$\mathbb{E}S_{\min\{N(t)+1,m)} = \mu\mathbb{E}\min\{N(t) + 1, m) \tag{3.4}$$

(Untruncate) Letting $m \uparrow \infty$ yields

$$\mathbb{E}S_{N(t)+1} = \mu\mathbb{E}(N(t) + 1) \leq \infty \tag{3.5}$$

Fix k. Let $\hat{X}_i = \min(X_i, k)$, $\hat{S}_n = \hat{N}(t)$.

$\implies \hat{S}_n \leq S_n \implies \hat{N}(t) \geq N(t)$.

We can apply eq. (3.5) to $(\hat{x}_i)$

$$\mathbb{E}(\hat{N}(t) + 1) \tag{3.6}$$

$$\mathbb{E}\min(X, k) = \mathbb{E}\hat{S}_{N(t)+1} \leq t + k < \infty \tag{3.7}$$

$$\implies \frac{\mathbb{E}(N(t) + 1)}{t} \leq \frac{(t + k)}{t} \times \frac{1}{\mathbb{E}\min(X, k)} \tag{3.8}$$

$$\implies \limsup_{t} \frac{\mathbb{E}N(t)}{t} \leq \frac{1}{\mathbb{E}\min(X, k)} \qquad \text{true } \forall k \tag{3.9}$$

Let $k \uparrow \infty$ shows $\leq \frac{1}{\mathbb{E}X} = \frac{1}{\mu}$. $\square$