Feynman Liang
STAT 205A
Lecture #4
9/6/2016

1 Abstract Integration (MT version) (sketchy details)

Setting:

- μ a measure (finite or σ-finite) on $(S, \mathcal{S})$
- Write $\mathcal{H}_+ :=$ set of measurable $h : S \to [0, \infty]$

Basic Theorem There exists a unique map $I : \mathcal{H}_+ \to [0, \infty]$ such that:

(a) $I(1_A) = \mu(A), \forall A \in \mathcal{S}$

(b) $I(h_1 + h_2) = I(h_1) + I(h_2), \forall h_i \in \mathcal{H}_+$

(c) $I(ch) = cI(h), \forall h \in \mathcal{H}_+, \forall c \geq 0$

(d) If $o \leq h_n \uparrow h \in \mathcal{H}_+$, then $I(h_n) \uparrow I(h) \leq \infty$

1.1 Background

$h \to \int_{-infty}^{\infty} h(x)dx$ will be the case $S = \mathbb{R}$, $\mu =$ Lebesgue measure.

In practice, write

$$I(h) := \int_S h d\mu = \int_S h(s)\mu(ds) \tag{1.1}$$

$$\int_A h d\mu := \int_S (h1_A) d\mu \quad \text{for } a \in \mathcal{S} \tag{1.2}$$

1.1.1 Definite integrals

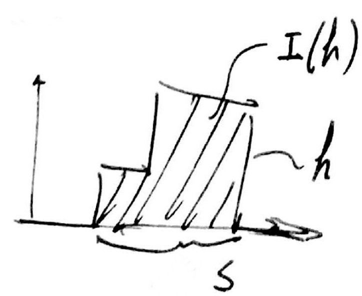

Figure 1: Area under curve interpretation of definite integral

1.1.2 Steps

(1) Define $I(1_A) := \mu(A)$

(2) For simple functions $h = \sum_i c_i 1_{A_i}$, define $I(h) = \sum_i c_i \mu(A_i)$

(3) For $0 \leq h \leq m$, constant, can write $h = \lim_n h_n$, h_n simple (old lemma) and define $I(h) := \lim_n I(h_n)$

(4) For general $h \in \mathcal{H}_+$, set $h_m = \min(h, m)$, so $h_m \uparrow h$. Define $I(h) = \lim_{m \uparrow \infty} I(h_m)$.

Note: Consider

$$h(s) = \begin{cases} \infty, & s \in A \text{where } \mu(A) = 0 \\ 0, & s \notin A \end{cases} \tag{1.3}$$

Here, $h_m(s) = \min(h(s), m) = m1_A$, $I(h_m) = m \cdot \mu(A) = 0$, $I(h) = \lim_{m\uparrow\infty} I(h_m) = 0$.

Notation (ALMOST EVERYWHERE): $h_1 = h_2$a.e. means $\{s : h_1(s) \neq h_2(s)\}$ has μ-measure $= 0$.

Notation: $x \in \mathbb{R}$, $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$, $h^+(s) = (h(s))^+ = \max(h(s), 0)$ $\implies x = x^+ - x^-$, $|x| = x^+ + x^-$, $|x - y| \leq |x| + |y|$

Definition: A measurable $h : S \to \bar{\mathbb{R}}$ is *integrable* (w.r.t. μ) if $\int_S |h| d\mu < \infty$. For integrable h, define $I(h) = I(h^+) - I(h^-)$ (but finite)

Lemma: Suppose h_1, h_2 integrable.

(1) (LINEARITY): For $c_1, c_2 \in \mathbb{R}$, $h := c_1 h_1 + c_2 h_2$, then h is integrable and $\int h d\mu = c_1 \int h_1 d\mu + c_2 \int h_2 d\mu$

(2) If $h_1 = 0$a.e., then $\int h_1 d\mu = 0$

(3) If $h_1 \geq 0$a.e., then $\int h_1 d\mu \geq 0$

(4) If $h_1 \leq h_2$a.e., then $\int h_1 d\mu \leq \int h_2 d\mu$

(5) $\left|\int h d\mu\right| \leq \int |h| d\mu$

Proof of (5):

$$\left|\int h d\mu\right| = \left|\int h^+ d\mu - \int h^- d\mu\right| \tag{1.4}$$

$$\leq \left|\int h^+ d\mu\right| + \left|\int h^- d\mu\right| \tag{1.5}$$

$$= \int (h^+ + h^-) d\mu \tag{1.6}$$

$$= \int |h| d\mu \tag{1.7}$$

2 Probability Theory (MT version)

Freshman: A r.v. X is a quantity with a range of possible values, the actual values determined somehow by chance. $P(X \leq 4)$ is the "chance it turns out that $X \leq 4$"

MT Version: Probability Space $(\Omega, \mathcal{F}, P)$:

- Ω : sample space, states of universe
- $\mathcal{F}$: events, σ-field on Ω
- P: probability measure

Events $A \in \mathcal{F}$ have probability $P(A)$.

Definition: A *random variable* (r.v.) is a measurable function $X : \Omega \to (S, \mathcal{S})$ or often $\mathbb{R}$.

So for measurable set $B \in \mathcal{S}$, $\{\omega : X(\omega) \in B\}$ is an event in $\mathcal{F}$ and so has a probability $P(\{\omega : X(\omega) \in B\}) = P(X \in B)$.

A given RV $X : \Omega \to (S, \mathcal{S})$ has a *distribution* (or *law*) μ defined by $\mu(B) = P(X \in B)$

Pushforward measure: The domain S of the RV has a PM obtained by pushing-forwards the PM P on Ω

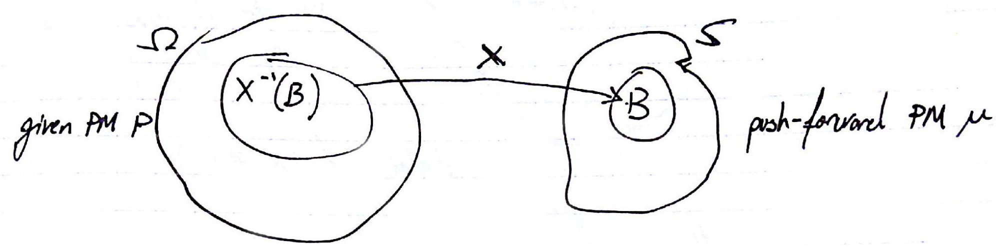

Figure 2: Pushforward of p.m. P on Ω through X to a p.m. U on $\mathcal{S}$.

Notation by example: $\mathbb{R}$-valued RVs X, Y, Z. $X^2 + Y^2 \leq 2 + 4$a.s. means $P(\{\omega : X^2(\omega) + Y^2(\omega) \leq Z(\omega) + 4\}) = P(X^2 + Y^2 \leq Z + 4) = 1$

Given $\mathbb{R}$-valued RVs X_n, X, $X_n \to X$a.s. means $P(\{\omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty\}) = 1$

Note also: Given arbitrary $\mathbb{R}$-valued X_n, $n \in \mathbb{N}$, can define $X^* := \limsup_n X_n$, $X^*(\omega) := \limsup_n X_n(\omega)$ and X^* is a RV (lim sup of measurable functions are measurable).

2.1 Expectation

The *expectation* of a RV $Y : (\Omega, \mathcal{F}, P) \to \mathbb{R}$ is

$$\mathbb{E}[Y] := \int_\Omega Y d\mathbb{P} \tag{2.1}$$

provided $\mathbb{E}|Y| := \int_\Omega |Y| d\mathbb{P} \leq \infty$ ("Y is integrable")

2.2 "Change of variable" lemmas

See Jim Pitman's notes for good explanation.

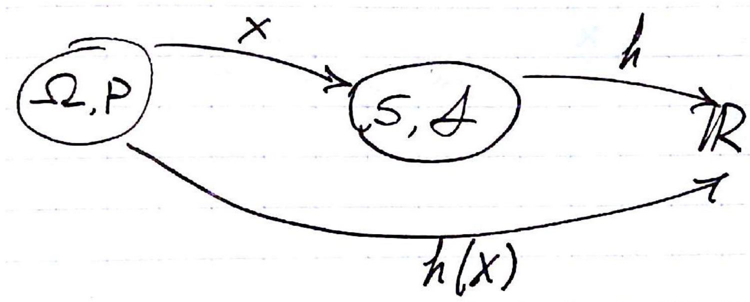

Figure 3: Functions of random variables $h(X)$ viewed as composition of measurable functions $h \circ X : \Omega \to \mathbb{R}$

Lemma 1: If $h(X)$ is integrable, then $\mathbb{E}h(X) = \int_S h d\mu$ for $\mu =$ distribution of X

Lemma 2: If ν is a PM on $\mathbb{R}$ with density f, then $\int_{\mathbb{R}} h d\nu = \int_{-\infty}^{\infty} h(x)f(x)dx$ provided h is ν-integrable

Proof: Consider the collection of h for which the stated $=$ is true.

- Consider $h = 1_B$, $B \in \mathcal{S}$.
  LHS $= \mathbb{E}h(X) = 1_{X \in B} = \mu(B) = \int 1_B d\mu =$ RHS
- Consider $h = 1_B, B \subset \mathbb{R}$
  LHS $= \int 1_B d\nu = \nu(B) = \int_B f(x)dx$ (definition of density $f(x)$ of ν) $=$ RHS

2.2.1 Steps of sketch proof of Basic Theorem

Both sides of $\cdot = \cdot$ are integrals, so:

- True for $1_B, \forall B \in \mathcal{S}$
- True for simple functions h
- True for bounded measurable h
- True for integrable h

Text: "Monotone class theorem"

Can combine Lemma 1 and Lemma 2:

Lemma: Suppose RV X is $\mathbb{R}$-valued, and its distribution has density f. Then $\mathbb{E}h(X) = \int h(x)f(x)dx$ provided $h(X)$ is integrable