Feynman Liang
STAT 210A
Lecture #18
10/25/2016

1 Outline

(a) Testing in General Linear Models

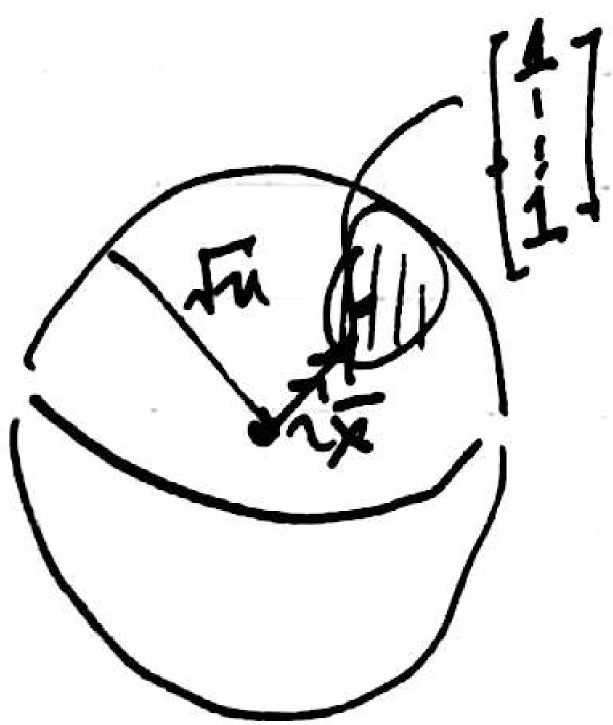
(b) t, Z, F, χ^2 tests

(c) Aymptotics: Conv. in Prob.

2 Review

Example 2.1 (1-sample t-test). $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, $\sigma^2 > 0$ unknown. Test $H_0 : \mu \leq 0$ vs $H_1 : \mu > 0$ ($\iff \frac{\mu}{\sigma^2} \leq 0$ vs $\frac{\mu}{\sigma^2} > 0$).

Condition on $\|X\|^2 = u$ ($\implies X \overset{H_0}{\sim} \text{Unif}(\{z : |z| = \sqrt{u}\})$. Reject when $\frac{\bar{X}}{\|X\|} = c \iff$ reject when $\frac{\bar{X}}{\sqrt{\frac{\|X\|^2 - n\bar{X}^2}{n-1}}} > c$



Under H_0, X is uniformly distributed on sphere with radius $\sqrt{u}$. $\bar{X}$ measures the magnitude of X in the $\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ direction.

3 Basic Setup

Observe $Y \sim N(\theta, \sigma^2 I_n)$. $\theta \in \mathbb{R}^n$, $\sigma^2 > 0$ possibly unknown. Models/hypotheses framed in terms of linear constraints on θ. $\mathcal{P}$ constrains $\theta \in \Theta$, d-dimensional affine space. $H_0 : \theta \in \Theta_0 \subset \Theta$ d_0-dimensional.

Example 3.1. One-sample t-test. $\mathcal{P}$ constrains $\theta = \mu 1_n$, $H_0 : \theta = 0_n$.

Example 3.2. Two-sample testing $H_0 : \mu = \nu$ vs $H_1 : \mu \neq \nu$ $Y_{1,1}, \dots, Y_{1,n} \overset{iid}{\sim} N(\mu, \sigma^2)$ $Y_{1,1}, \dots, Y_{1,n} \overset{iid}{\sim} N(\nu, \sigma^2)$ $Y = (Y_{1,1}, \dots, Y_{1,n}, Y_{2,1}, \dots, Y_{2,n}) \sim N(\theta, \sigma^2 I_{2n})$ $\Theta = \mu(1_n, 0_n) + \nu(0_n, 1_n)$ $\Theta_0 = \mu 1_{2n}$

Example 3.3. One-way ANOVA $H_0 : \mu_1 = \cdots = \mu_k$ $Y_{k,i} \overset{indeed}{\sim} N(\mu_k, \sigma^2)$ $\dim(\Theta) = k$ $\dim(\Theta_0) = 1$

Example 3.4. Linear regression $X \in \mathbb{R}^{n \times d}$, $d < n$, X full column rank. $Y \sim N(X\beta, \sigma^2 I_n)$, $\beta \in \mathbb{R}^d$. $H_0 : \beta_1 = \cdots = \beta_s = 0$, $s \leq d$. $d = d$, $d_0 = d - s$.

4 General Strategy

Rotate Y by some orthogonal matrix Q

$$Q = \left[\underbrace{\overset{d_0}{Q_0}}_{\text{o.n. basis for } \Theta_0}, \quad \underbrace{\overset{d-s}{Q_1}}_{\text{o.n. basis for } \Theta \cap \Theta_0^\perp} \quad \underbrace{\overset{n-d}{Q_r}}_{\text{o.n. basis for } \mathbb{R}^d \cap \Theta^\perp} \right]$$

$$Z = Q'Y \sim N\left(\begin{pmatrix} Q_0'\theta \\ Q_1'\theta \\ Q_r'\theta \end{pmatrix}, \sigma^2 I_n \right) \; \mathrm{Var}(Q'Y) = Q'\mathrm{Var}(Y)Q = \sigma^2 I_n \begin{pmatrix} Z_0 \\ Z_1 \\ Z_r \end{pmatrix} \sim N\left(\begin{pmatrix} \nu_0 \\ \nu_1 \\ \nu_r \end{pmatrix}, \sigma^2 I_n \right)$$

Model $\implies$ $\nu_r = 0_{n-d}$. $H_0 : \nu_1 = 0$.

Example 4.1. $\Theta_0 = \mathrm{span}(X_{s+1}, \dots, X_d)$. $\Theta = \mathrm{span}(X_1, \dots, X_d)$.

$$Q_0 = \left\{ \frac{X_{s+1}}{\|X_{s+1}\|}, \frac{\prod_{X_{s+1}^\perp} X_{s+2}}{\| \cdot \|}, \dots, \frac{\prod_{(X_{s+1}, \dots, X_d)^\perp} X_d}{\| \cdot \|} \right\}$$

Use Gram-Schmidt to get the successive o.n. bases.

We now have the problem in this canonical form:

$$Z_0 \sim N(\nu_0, \sigma^2 I_{d_0})$$
$$Z_1 \sim N(\nu_1, \sigma^2 I_s)$$
$$Z_r \sim N(0, \sigma^2 I_{n-d})$$

All 3 independent normal random vectors with different means, $H_0 : \nu_1 = 0$.

How to test H_0?

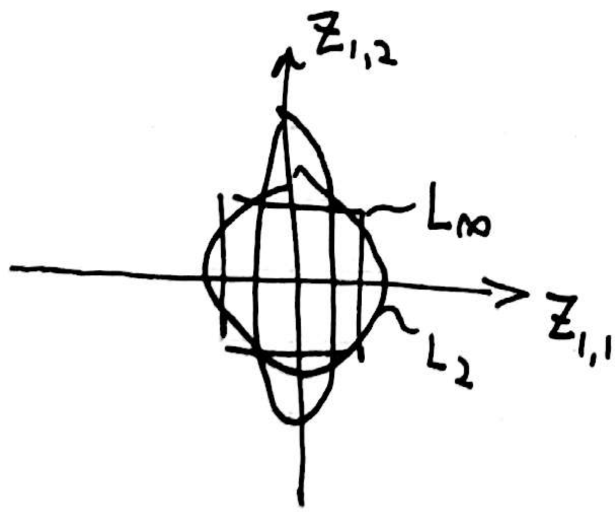
5 χ^2, t, F distributions

$Z_1, \dots, Z_d \overset{iid}{\sim} N(0,1) \implies \|Z\|_2^2 = \sum_{i=1}^n Z_i^2 \sim \chi_d^2$.

$Y \sim N(0,1)$ independent of $Z \implies \frac{Z_1}{\sqrt{\|Z\|^2/d}} \sim t_d$.

$Y_1, \dots, Y_{d_1} \overset{iid}{\sim} N(0,1)$ and $Z_1, \dots, Z_{d_2} \overset{iid}{\sim} N(0,1)$ independent $\implies \frac{\|Y\|^2/d_1}{\|Z\|^2/d_2} \sim F_{d_1,d_2}$.

Case 1: σ^2 known. **s=1**: reject for $\frac{Z_1}{\sigma}(\overset{H_0}{\sim} N(0,1))$ large or extreme. **s > 1**: reject for $\frac{\|Z_1\|^2}{\sigma^2}2(\overset{H_0}{\sim} \chi_S^2))$ large.



Case 2: σ^2 unknown, $\mathrm{Var}(Z_1)$ unknown, $\frac{\|Z_r\|^2}{\sigma^2} \sim \chi_r^2$.

s=1: reject for $\frac{Z_1}{\sqrt{\|Z_r\|^2/(d-s)}}$ $Z_1 \overset{H_0}{\sim} N(0,\sigma^2)$, $\frac{Z_1}{\sigma} \sim N(0,1)$ $V = \sqrt{\frac{\|Z_1\|^2}{\sigma^2}} \sim \sqrt{\chi_{n-d}^2}$ $\frac{Z_r/\sigma}{\sqrt{\frac{\|Z_r\|^2}{\sigma^2}/(d-s)}} = \frac{Z_1}{\sqrt{\frac{\|Z_r\|^2}{n-d}}} \sim t_{n-d}$

$\hat\sigma^2 = \frac{\|Z_r\|^2}{n-d}$ $E\|Z_r\|^2 = (n-d)E(Z_{r,1}^2)$ $E[\hat\sigma^2] = \sigma^2$

$\mathrm{Var}(\|Z_r\|^2) = (n-d)\mathrm{Var}(Z_{r,1}^2) = (n-d)2\sigma^4$

The point is that $\mathrm{Var}(\hat\sigma^2) = \frac{2\sigma^4}{(n-d)^2} \to 0$ so $\hat\sigma^2$ is a consistent estimator of σ^2.

s > 1: reject for $\frac{\|Z_1\|^2/s}{\hat\sigma^2} = \frac{\|Z_1\|^2/s}{\|Z_r\|^2/(n-d)} \sim F_{s,n-d}$

6 Summary

	$s=1$	$s>1$
σ^2 known	$\frac{Z_1}{\sigma}$ (Z-test)	$\frac{\|Z_1\|^2}{\sigma}$ (χ_s^2-test)
σ^2 unknown	$\frac{Z_1}{\hat\sigma}$ (t_{n-d}-test)	$\frac{\|Z_1\|_2^2/s}{\hat\sigma^2}$ ($F_{s,n-d}$-test)

Example: Regression $X \in \mathbb{R}^{n \times d}$. $Y \sim N(X\beta, \sigma^2 I_n)$ $H_0 : \beta_d = 0$

$$Q_0 = \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_{2\perp}}{\|X_{2\perp}\|} & \cdots & \frac{X_{d-1\perp}}{\|X_{d-1\perp}\|} \end{bmatrix}$$

$$X_{k\perp} = \prod_{\text{span}(X_1, \ldots, X_{k-1})^\perp} (X_k)$$

$$Q_1 = \frac{X_{d\perp}}{\|X_{d\perp}\|} = \frac{(I - Q_0 Q_0')X_d}{\| \cdot \|}$$

$$Q_r = \text{completion of basis}$$

$$Z_1 = \frac{X_{d\perp}' Y}{\|X_{d\perp}\|} = \frac{\hat{\beta}_d}{\text{s.e.}(\hat{\beta}_d)}$$

$$\|Z_r\|^2 = \|(I - \prod_{\text{span}(x)})Y\|^2 = \sum_i (Y_i - \hat{Y}_i)^2 = \text{RSS}$$