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STAT 205A
Lecture #8
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1 Glivenko-Cantelli Theorem

Theorem 1.1. *Let $(X_i)_{i=1}^N$ be i.i.d. with distribution function F and $G_n(\omega, x) = \frac{1}{n}\sum_{i=1}^n 1_{X_i(\omega) \leq x}$ be its empirical distribution function. Then*

$$\sup_x |G_n(\omega, x) - F(x)| \overset{a.s.}{\to} 0 \quad \text{as } n \to \infty \tag{1.1}$$

(i.e. $\|G_n(\omega, x) - F(x)\|_\infty \to 0$ or $G_n \overset{L_\infty}{\to} F$

To prove theorem 1.1, we need a lemma proved in a later homework.

Definition 1.2. For a CDF $F : \mathbb{R} \to [0, 1]$, $x \in \mathbb{R}$ is an *atom* if

$$F(x) - F(x^-) = P(X = x) > 0 \tag{1.2}$$

Lemma 1.3. *Let F_n, F be distribution functions. If*

(a) For all $x \in \mathbb{Q}$: $F_n(x) \to F(x)$

(b) For each atom x of F: $F_n(x) \to F(x)$ and $F_n(x^-) \to F(x^-)$

then $\sup_x |F_n(x) - F(x)| \to 0$ almost surely.

Proof of theorem 1.1. Fix $x \in \mathbb{R}$. The events $\{X_i \leq x\}_{i=1}^n$ are i.i.d. with probability $= F(x)$. By SLLN, $G_n(\omega, x) \overset{\text{a.s.}}{\to} F(x)$ as $n \to \infty$. Consider $S = \mathbb{Q} \cup \{\text{atoms of } F\}$, which is countable (TODO: why?). Notice $P(G_n(\omega, x) \to F(x) \quad \forall x \in S) = 1$ so by lemma 1.3 $P(\sup_x |G_n(\omega, x) - F(x)| \to 0) = 1$. □

2 Gambling on a favorable game

Suppose we are playing a game where we stake an amount $s \in \mathbb{R}$ and recieve payoff

$$\begin{cases} +s, & \text{w.p. } \frac{1}{2} + \alpha \\ -s, & \text{w.p. } \frac{1}{2} - \alpha \end{cases} \tag{2.1}$$

Consider a strategy where at every time the stake s is equal to some proportion $q \in$

$[0,1]$ of your currrent total. Let X_n denote your total fortune after n bets. Then

$$X_{n+1} = (1-q)X_n + \begin{cases} 2qX_n, & \text{if win} \\ 0, & \text{if loose} \end{cases} \tag{2.2}$$

$$= (1-q)X_n + 2qX_n \underbrace{1_{A_{n+1}}}_{\text{win } (n+1)\text{st bet}} \tag{2.3}$$

$$= (1 - q + 2q1_{A_{n+1}})X_n \tag{2.4}$$

$$\implies X_n = x_0 \prod_{i=1}^{n} (1 - q + 2q1_{A_i}) \tag{2.5}$$

$$\frac{\log X_n}{n} = \frac{\log x_0}{n} + \frac{1}{n} \sum_{i=1}^{n} \underbrace{\log(1 - q + 2q1_{A_i})}_{Y_i} \tag{2.6}$$

By SLLN, as $n \to \infty$

$$\frac{\log Y_n}{n} \overset{\text{a.s.}}{\to} \mathbb{E}Y \tag{2.7}$$

(Note: $\frac{1}{n} \log X_n \to c$ is slightly weaker than $x_n \approx e^{cn}$, $c =$"asymptotic growth rate")
The optimal choice of q should maximize $\mathbb{E}Y$

$$\mathbb{E}Y = \left(\frac{1}{2} + \alpha\right) \log(1+q) + \left(\frac{1}{2} - \alpha\right) \log(1-q) \tag{2.8}$$

$$\approx 2\alpha q - \frac{1}{2}q^2 \quad \text{for } \alpha, q \text{ small} \tag{2.9}$$

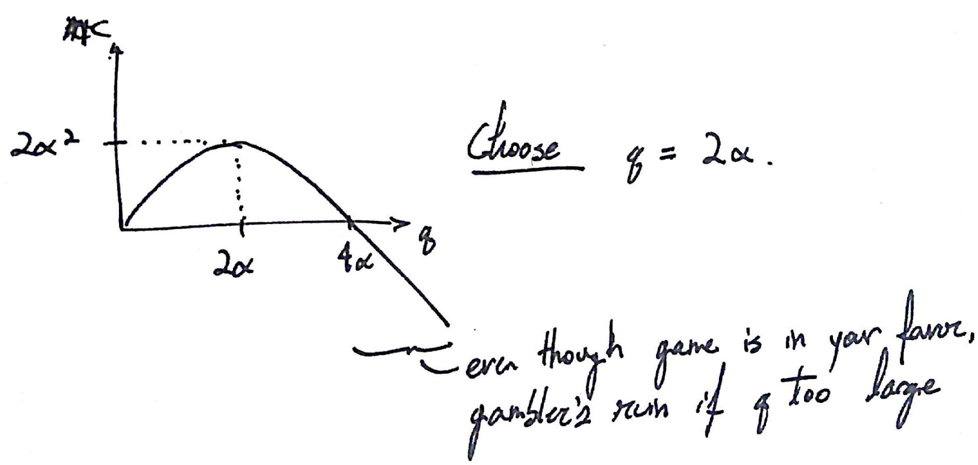

Figure 1: Asymptotic growth rate for different bet proportions q

$\therefore$ the optimal choice is $q = 2\alpha$
However, notice that

$$\mathbb{E}X_n = x_0(1 + 2q\alpha)^n \to \infty \qquad \text{but } X_n \overset{\text{a.s.}}{\to} ifq \geq q_{crit} \approx 4\alpha \tag{2.10}$$

3 Almost-sure limits for maxima

Example 3.1. $(X_i)_1^n \overset{\text{iid}}{\sim} \text{Exp}(1)$, $P(X > x) = e^{-x}$.
Write $M_n = \sup_{1 \le i \le n} X_i$. Then

$$\limsup_n \frac{X_n}{\log n} \overset{\text{a.s.}}{=} 1 \tag{3.1}$$

and

$$\frac{M_n}{\log n} \overset{\text{a.s.}}{\to} 1 \tag{3.2}$$

Proof. Fix $\varepsilon > 0$. $P(X_n / \log n > 1 + \varepsilon) = \exp(-(1+\varepsilon)\log n) = n^{-(1+\varepsilon)}$

$$\sum_{n \in \mathbb{N}} n^{-(1+\varepsilon)} < \infty \underset{\text{(BC 1)}}{\Longrightarrow} P\left(\frac{X_n}{\log n} \le 1 + \varepsilon \text{ ult.}\right) = 1 \tag{3.3}$$

$$\iff \limsup_n \frac{X_n}{\log n} \overset{\text{a.s.}}{\le} 1 + \varepsilon \tag{3.4}$$

$$\underset{\varepsilon \to 0}{\Longrightarrow} \limsup_n \frac{X_n}{\log n} \overset{\text{a.s.}}{\le} 1 \tag{3.5}$$

To obtain a lower bound, $P(\underbrace{X_n / \log n \ge 1 - \varepsilon}_{\text{indep.}}) = n^{-(1-\varepsilon)}$ so

$$\sum_{n \in \mathbb{N}} n^{-(1-\varepsilon)} = \infty \underset{\text{(BC 2)}}{\Longrightarrow} P\left(\frac{X_n}{\log n} \ge 1 - \varepsilon \text{ i.o.}\right) = 1 \tag{3.6}$$

$$\underset{\varepsilon \to 0}{\Longrightarrow} \limsup_n \frac{X_n}{\log n} \overset{\text{a.s.}}{\ge} 1 \tag{3.7}$$

Together, we have $\limsup_n X_n / \log n \overset{\text{a.s.}}{=} 1$.
To prove the second part, we first prove a deterministic lemma:

Lemma 3.2 (Deterministic Lemma)**.** *If* $X_n \ge 0$ *and* $0 < b_n \uparrow \infty$, *then* $\limsup_n \frac{\max_{1 \le i \le n} X_i}{b_n} = \limsup_n \frac{X_n}{b_n}$.

Proof. "$\ge$" is obvious. Fix j.

$$\limsup_n \frac{\max_{1 \le i \le n} X_i}{b_n} = \limsup_n \frac{\max_{j \le i \le n} X_i}{b_n} \qquad b_n \uparrow \infty,\ x_j \text{ fixed} \tag{3.8}$$

$$\le \lim_n \max_{j \le i \le n} \frac{x_i}{b_i} \qquad \frac{x_i}{b_i} \ge \frac{x_i}{b_n} \tag{3.9}$$

$$= \sup_{i \ge j} \frac{x_i}{b_i} \tag{3.10}$$

Letting $j \to \infty$ shows "$\le$". $\square$

Combining eq. (3.1) and lemma 3.2 imply $\limsup_n \frac{M_n}{\log n} \overset{\text{a.s.}}{=} 1$.

It remains to show $\liminf_n \frac{M_n}{logn} \overset{\text{a.s.}}{=} 1$. But since $1 \overset{\text{a.s.}}{=} \limsup_n M_n / \log n \geq \liminf_n M_n / \log n$, it suffices to show $\liminf_n M_n / \log n \overset{\text{a.s.}}{\geq} 1$

Fix $\varepsilon > 0$.

$$\begin{aligned} P(M_n \leq (1-\varepsilon)\log n) &= P(X \leq (1-\varepsilon)\log n)^n && (3.11) \\ &= (1 - n^{-(1-\varepsilon)})^n && (3.12) \\ &\leq \exp\left(-n^{-(1-\varepsilon)}\right)^n \qquad 1 - x \leq e^{-x} && (3.13) \\ &= \exp(-n^{\varepsilon}) && (3.14) \\ & && (3.15) \end{aligned}$$

$$\begin{aligned} \sum_n P(M_n \leq (1-\varepsilon)\log n) \leq 1 - \frac{1}{1-e^{\varepsilon}} = \frac{1}{1-e^{\varepsilon}} < \infty && (3.16) \\ \underset{(\text{BC } 1)}{\Longrightarrow} M_n \geq (1-\varepsilon)\log n \text{ ult. a.s.} && (3.17) \\ \Longrightarrow \liminf_n \frac{M_n}{\log n} \overset{\text{a.s.}}{\geq} 1 - \varepsilon && (3.18) \\ \Longleftrightarrow \liminf_n \frac{M_n}{\log n} \overset{\text{a.s.}}{\geq} 1 - \varepsilon && (3.19) \\ \underset{\varepsilon \to 0}{\Longrightarrow} \liminf_n \frac{M_n}{\log n} \overset{\text{a.s.}}{\geq} 1 && (3.20) \end{aligned}$$

$\square$

Remark 3.3. Here $X_n / \log n \to 0$ in probability (i.e. $P(X_n / \log n \geq \varepsilon) = n^{-\varepsilon} \to 0$, but not a.s. (which requires showing $P(\lim_n X_n / \log n = 0) = 1$).

4 Second moment Strong Law of Large Numbers

Theorem 4.1 (Second moment SLLN). *Given* $(X_i)_{i=1}^n$ *with:*

(a) $\mathbb{E}X_i = 0$ *for all* i

(b) $\sup_i \mathbb{E}X_i^2 = B < \infty$

(c) ***Orthogonal****:* $\mathbb{E}[X_i X_j] = 0$ *for all* $i \neq j$

Write $S_n = \sum_{i=1}^n X_i$. *Then* $S_n / n \overset{a.s.}{\to} 0$.

We first show a deterministic lemma used in the proof.

Lemma 4.2 (Deterministic Lemma). *Given* $S_n \in \mathbb{R}$, *to show* $S_n / n \to 0$ *it suffices to show* $\exists$ *subsequence* $n(j) \uparrow \infty$ *such that:*

(a) $S_{n(j)} / n(j) \to 0$ *as* $j \to \infty$

(b) $D_j/n(j) \to 0$ *where* $D_j := \max_{n(j)\le n\le n(j+1)} |S_n - S_{n(j)}|$

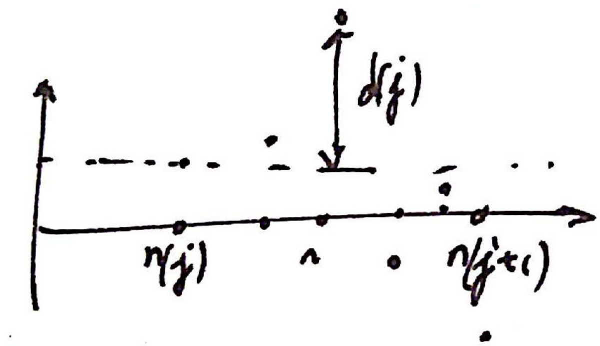

Figure 2: $d(j)$ and $n(j)$ as defined for lemma 4.2

Proof of lemma 4.2. Given n, for some j where $n(j) \le n < n(j+1)$

$$\left|\frac{S_n}{n}\right| \le \left|\frac{S_n}{n(j)}\right| \le \frac{|S_{n(j)}| + D_j}{n(j)} \overset{\text{a.s.}}{\to} 0 \tag{4.1}$$

□

Proof of theorem 4.1. $\mathrm{Var}(S_n) \le nB$ so by Chebyshev's inequality

$$P\left(\frac{|S_n|}{n} \ge \varepsilon\right) \le \frac{nB}{n^2\varepsilon^2} = \frac{B}{n\varepsilon^2} \tag{4.2}$$

Take $a(j) = j^2$.

$$P\left(\frac{|S_{n(j)}|}{n(j)} \ge \varepsilon\right) \le \frac{B}{\varepsilon^2}\frac{1}{j^2} \tag{4.3}$$

$\sum_j P\left(\frac{|S_{n(j)}|}{n(j)} \ge \varepsilon\right) \le \infty$ so by Borel-Cantelli 1 $\limsup_n S_{n(j)}/n(j) \overset{\text{a.s.}}{\le} \epsilon$. Taking $\epsilon \to 0$ yields $\limsup_n S_{n(j)}/n(j) \overset{\text{a.s.}}{=} 0$, and since $S_{n(j)}/n(j) \ge 0$ for all j, $\liminf_n S_{n(j)}/n(j) \ge 0$ hence $S_{n(j)}/n(j) \overset{\text{a.s.}}{\to} 0$.

By lemma 4.2 it suffices to show $D_j/j^2 \overset{\text{a.s.}}{\to} 0$ for $D_j = \max_{j^2\le n\le (j+1)^2}|S_n - S_{j^2}|$.

$$D_j^2 = \max_{j^2\le n\le (j+1)^2}(S_n - S_{j^2})^2 \le \sum_{n=j^2}^{(j+1)^2}(S_n - S_{j^2})^2 \tag{4.4}$$

$$\mathbb{E}D_j^2 \le \sum_{n=j^2}^{(j+1)^2}\mathbb{E}(S_n - S_{j^2})^2 = \sum_{n=j^2}^{(j+1)^2}\mathrm{Var}\left(S_n - S_{j^2}\right) = \sum_{n=j^2}^{(j+1)^2}\mathrm{Var}\left(\sum_{i=j^2+1}^{n} X_i\right) \tag{4.5}$$

$$\le \sum_{n=j^2}^{(j+1)^2} B(n - j^2) = B\sum_{i=1}^{2j+1} i = \frac{1}{2}(2j+1)(2j+2)B \tag{4.6}$$

By Chebyshev bound

$$P\left(\frac{D_j}{j} \geq \varepsilon\right) \leq \frac{\mathbb{E}D_j^2}{\varepsilon^2 j^4} \in O(j^{-2}) \tag{4.7}$$

By Borel-Cantelli 1, $D_j/j^2 \xrightarrow{\text{a.s.}} 0$. □

Remark 4.3. Theorem 4.1 does not rely on independence, only bounded variance and orthogonality.

5 Misc. MT

Definition 5.1. For a RV X and a non-negative integrable RV Y ($Y \geq 0$, $\mathbb{E}Y \leq \infty$), X is *dominated by* Y (written $X \ll Y$) means $|X_n| \overset{\text{a.s.}}{\leq} Y$.

Theorem 5.2 (Dominated Convergence Theorem). *If $X_n \xrightarrow{a.s.} X$ and $X_n \ll Y$, then:*

(a) $\mathbb{E}X_n \to \mathbb{E}X$

(b) $\mathbb{E}|X_n - X| \to 0$

(c) $\mathbb{E}|X| < \infty$

Proof. Fix $\epsilon > 0$. Define $A_N = \{|X_n - X| \leq \epsilon \text{ for all } n \geq N\}$. Then $A_N \uparrow A$, $P(A) = 1$ implies $A_n^c \downarrow A^c$, $P(A^c) = 0$.

$$\mathbb{E}|X_N - X| = \mathbb{E}|X_N - X|1_{A_N} + \mathbb{E}|X_N - X|1_{A_N^c} \tag{5.1}$$

$$\leq \epsilon + \mathbb{E}|X_N - X|1_{A_N^c} \tag{5.2}$$

TODO: Bernstein's theorem from pre

$$\limsup_N \mathbb{E}|X_N - X| \leq \epsilon + 0A_n^c \to A^c \text{ by monotone convergence} \tag{5.3}$$

True $\forall \epsilon$ so $\mathbb{E}|X_N - X| \to 0$. □

This proof may need Fatou's lemma.