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STAT 210A
Lecture #14
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# 1 Review

$$\mathcal{P} = \{P_\theta : \theta \in \Theta\} \tag{1.1}$$
$$H_0 : \theta \in \Theta_0 \subset \Theta \tag{1.2}$$
$$H_1 : \theta \in \Theta_1 = (\Theta \setminus \Theta_1) \tag{1.3}$$

*Critical function*: $\phi(x) \in [0,1]$, probability we reject if $X = x$.
*Power function*: $\beta_\phi(\theta) = \mathbb{E}_\theta(\phi(X))$.
**Goal**: Maximize $\beta(\theta)$ for $\theta \in \Theta_1$ while constraining significance level

$$\sup_{\theta \in \Theta_0} \beta(\theta) \leq \alpha \tag{1.4}$$

**Simple vs Simple**: $\Theta = \{\theta_0\}(= \{0\}), \Theta_1 = \{\theta_1\}(= \{1\})$.
**Likelihood Ratio Test**:

$$L(x) = \frac{p_1(x)}{p_0(x)} \quad \left(= \frac{\text{bang}}{\text{buck}}\right) \tag{1.5}$$
$$\text{LRT} : \phi^*(x) = \begin{cases} 0, & \text{if } L(x) < c \\ \gamma, & \text{if } L(x) = c \\ 1, & \text{if } L(x) > c \end{cases} \tag{1.6}$$

# 2 Neyman-Pearson

**Proposition 2.1** (Keener 12.1)**.** *Suppose $c \geq 0$, $\phi^*$ maximizes $\mathbb{E}_1\left[\phi(X)\right] - c\mathbb{E}_0\left[\phi(X)\right]$ among* all *critical functions.*

*If $\mathbb{E}_0\left[\phi^*(X)\right] = \alpha$, then $\phi^*$ maximizes $\mathbb{E}_1\left[\phi(X)\right]$ among all $\phi$ with level $\leq \alpha$.*

*Proof.* Suppose $\mathbb{E}_0[\phi(X)] \leq \alpha$. Then

$$\mathbb{E}_1[\phi(X)] \leq \mathbb{E}_1[\phi(X)] - c\mathbb{E}_0[\phi(X)] + c\alpha \tag{2.1}$$
$$\leq \mathbb{E}_1[\phi^*(X)] \cancelto{0}{- c\mathbb{E}_0[\phi^*(X)] + c\alpha} \tag{2.2}$$

□

**Lemma 2.2** (Neyman-Pearson)**.** *LRT with level $\alpha$ is optimal, among level $\leq \alpha$ tests (Simple vs Simple).*

*Proof.*

$$\begin{aligned}
&\mathbb{E}_1[\phi(X)] - c\mathbb{E}_0[\phi(X)] && (2.3)\\
&= \int (p_1(x) - cp_0(x))\phi(x)d\mu(x) && (2.4)\\
&= \int_{p_1(x)>cp_0(x)} |p_1 - cp_0(x)|\phi(x)d\mu(x) - \int_{p_1(x)<cp_0(x)} |p_1 - cp_0(x)|\phi(x)d\mu(x) && (2.5)
\end{aligned}$$

Any test maximizing eq. (2.5) has

$$\phi^*(x) = \begin{cases} 1, & \text{if } p_1 > cp_0 \\ 0, & \text{if } p_1 < cp_0 \end{cases} \tag{2.6}$$

Take $c$ s.t.

$$\begin{aligned}
\mathbb{P}_0[p_1(X) < cp_0(X)] \le 1 - \alpha && (2.7)\\
\mathbb{P}_1[p_1(X) > cp_0(X)] \le \alpha && (2.8)
\end{aligned}$$

□

**Example 2.3.** $X \sim \mathcal{N}(\theta, 1)$.
$H_0 : \theta = \theta_0$, $H_1 : \theta = \theta_1, \theta_1 > \theta_0)$

$$L(x) = \frac{p_{\theta_1}(x)}{p_{\theta_0}(x)} = \frac{e^{-(x-\theta_1)^2/2}}{e^{-(x-\theta_0)^2/2}} = \frac{e^{\theta_1 x - \theta_1^2/2}}{e^{\theta_0 x - \theta_0^2/2}} = e^{(\theta_1 - \theta_0)x + (\theta_1^2/2 - \theta_0^2/2)} \tag{2.9}$$

$L(x)$ is monotone in $X$

$$\implies \phi^*(X) = 1_{L(x)>c} = 1_{x>\tilde{c}} = 1_{x>\theta_0+z_\alpha} \tag{2.10}$$

This is true for all $\theta_1 > \theta_0$: *uniformly most powerful*.

**Example 2.4.** $X_1, \cdots, X_n \overset{\text{iid}}{\sim} p_\eta(x) = e^{\eta T(x) - A(\eta)}h(x)$.
$H_0 : \eta = \eta_0$ vs $H_1 : \eta = \eta_1$. $\eta \in \mathbb{R}$, $\eta_1 > \eta_0$.

$$L(x) = \frac{\prod_{i=1}^n p_{\eta_1}(x)}{\prod_{i=1}^n p_{\eta_0}(x)} = \frac{e^{\eta_1 \sum_i T(x_i) - nA(\eta_1)}}{e^{\eta_0 \sum_i T(x_i) - nA(\eta_0)}} = e^{(\eta_1 - \eta_0)\sum_i T(x_i) - n(A(\eta_1) - A(\eta_0))} \tag{2.11}$$

$$\phi^*(x) = \begin{cases} 1, & \text{if } \sum_i T(X_i) > c \\ \gamma, & \text{if } \sum_i T(X_i) = c \\ 1, & \text{if } \sum_i T(X_i) < c \end{cases} \tag{2.12}$$

Again $L(x)$ is monotone in $T(X)$.

# 3 Uniformly most powerful test

**Definition 3.1.** If $\phi^*(x)$ has level $\alpha$ and $\mathbb{E}_\theta[\phi^*(X)] \geq \mathbb{E}_\theta[\phi(X)]$ $\forall\theta \in \Theta_1$,$\forall\phi$ with level $\leq \alpha$. Then $\phi^*$ is *uniformly most powerful (UMP)*.

Typically exist only for 1-param families and 1-sided tests

$$\mathcal{P} = \{p_\theta : \theta \in \Theta \subset \mathbb{R}\} \tag{3.1}$$

$$H_0 : \theta \leq \theta_0 \text{ vs } H_1 : \theta > \theta_0 \tag{3.2}$$

**Definition 3.2.** Let $\mathcal{P} = \{p_\theta : \theta \in \Theta \subset \mathbb{R}\}$ be a dominated family. Then $\mathcal{P}$ has *monotone likelihood ratio (MLR)* if for some $T(X)$

$$\theta_1 < \theta_2 \implies \frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} \text{ is a non-decreasing function of } T(x) \tag{3.3}$$

(Also $p_{\theta_1} \overset{\text{a.s.}}{\neq} p_{\theta_2}$ for any $\theta_1 \neq \theta_2$)

**Corollary 3.3** (Keener cor. 12.4)**.** *If $p_0 \overset{a.s.}{\neq} p_1$, $\phi$ is LRT level $\alpha \in (0,1)$, $p_0$ vs $p_1$. Then $\mathbb{E}_1[\phi(X)] > \alpha$.*

**Theorem 3.4.** *Suppose $\mathcal{P}$ has MLR*

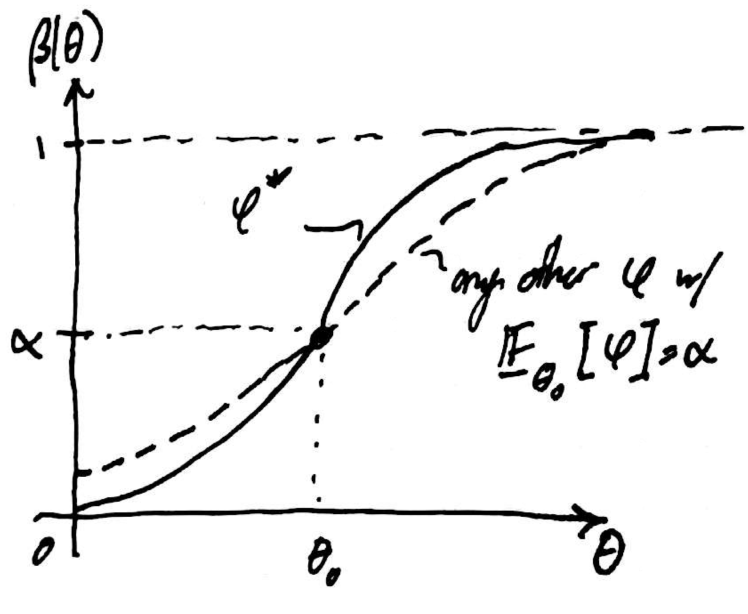
$$\phi^*(X) = \begin{cases} 0, & \text{if } T(x) < c \\ \gamma, & \text{if } T(x) = c \\ 1, & \text{if } T(x) > c \end{cases} \tag{3.4}$$

*(a) $\phi^*$ is UMP for testing $\theta \leq \theta_0$ vs $\theta > \theta_0$ amongst tests with level $\alpha = \mathbb{E}_{\theta_0}[\phi^*(X)]$*

*(b) Constants $c, \gamma$ can be adjusted to get any level*

*(c) $\beta_{\phi^*}(\theta) = \mathbb{E}_\theta[\phi^*(X)]$ is non-decreasing in $\theta$, strictly increasing when $\beta_{\phi^*}(\theta) \in (0,1)$.*

*(d) If $\theta_1 < \theta_0$, then $\phi^*$ minimizes $\mathbb{E}_{\theta_1}[\phi(X)]$ among all tests with $\mathbb{E}_{\theta_0}[\phi(X)] = \alpha$.*

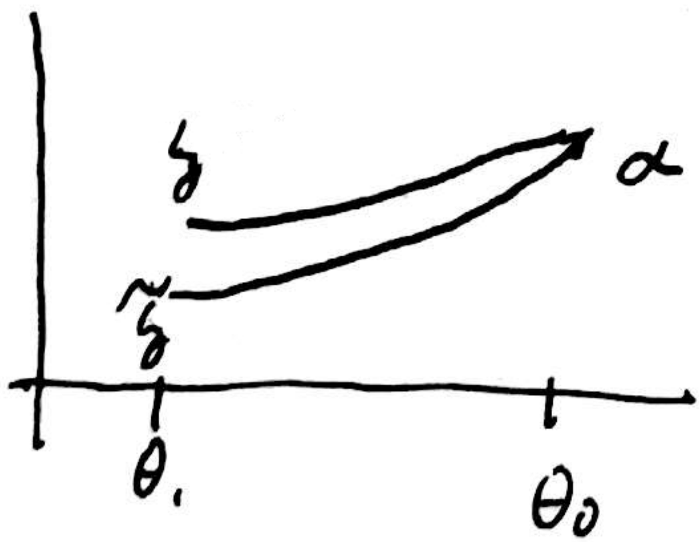


*Proof (c).* Suppose $\theta_1 < \theta_2$. Let $L(x) = \frac{p_{\theta_2}(x)}{p_{\theta_1}(x)}$. $L(x)$ non-decreasing function of $T(x)$, so $\phi^*(x)$ is a LRT for $\theta = \theta_1$ vs $\theta = \theta_2$ for some $\tilde{\alpha} = \mathbb{E}_{\theta_1}[\phi^*(X)]$.

By corollary 3.3, $\beta_{\theta^*}(\theta_2) > \tilde{\alpha}$. $\square$

*Proof (a).* Suppose $\theta_1 > \theta_0$ and $\tilde{\phi}$ has level $\leq \alpha$. Then $\mathbb{E}_{\theta_1}[\tilde{\phi}(X)] \leq \mathbb{E}_{\theta_1}[\phi^*(X)]$ $\square$

*Proof (d).* $\theta_1 < \theta_0$, $\mathbb{E}_{\theta_0}[\tilde{\phi}(X)] = \alpha$. Suppose $\tilde{\delta} = \mathbb{E}_{\theta_1}[\tilde{\phi}(X)] < \mathbb{E}_{\theta_1}[\phi^*(X)] = \delta^*$.




TODO: Figure out answer from Keener $\square$

# 4 Two-sided tests