Feynman Liang
STAT 210A
Lecture #13
10/06/2016

1 Review

Let $X \sim \mathcal{N}_d(\theta, I_d)$, $\theta \in \mathbb{R}^d$.

Bayes: Assume $\theta \sim \mathcal{N}_d(0, \tau I_d)$

$$\implies \delta(X) = (1-\zeta)X \tag{1.1}$$
$$\zeta = \frac{1}{1+\tau^2} \tag{1.2}$$

H-Bayes: Assume $\tau \sim \Lambda(t)$

$$\implies \delta(X) = (1-\hat{\zeta})X \tag{1.3}$$
$$\hat{\zeta} = \mathbb{E}\left[\frac{1}{1+\tau^2} \mid x\right] \tag{1.4}$$

E-Bayes: No assumptions on τ^2

$$\delta_{JS}(X) = \left(1 - \frac{d-2}{\|X\|^2}\right)X \tag{1.5}$$

2 Wrap up James-Stein

2.1 (Frequentist) Risk

Use $\hat{R} = d + \|\delta(X) - X\|^2 + \text{tr}(D(\delta(X) - X))$

$$\delta(X) = X \to \text{MSE} = d \tag{2.1}$$
$$\delta(X) = (1-\zeta)X \to \text{MSE} = (1-\zeta)^2 d + \zeta^2\|X\|^2 \tag{2.2}$$
$$\text{MSE}(\delta_{JS}) = d - \mathbb{E}_\theta\left[\frac{d-2}{\|X\|^2}\right] \qquad \underbrace{< d}_{\text{Shocker: "Stein's paradox"}} \tag{2.3}$$
$$= 2 \text{ for } \theta = 0 \tag{2.4}$$

Shrinkage towards sample mean:

$$\delta_{JS+1}(X)_i = \bar{X} + \left(1 - \frac{d-2}{\|X - \bar{X}\|^2}\right)(X_i - \bar{X}) \tag{2.5}$$

Can be derived by empirical Bayes argument.

A little bit of shrinkage is good:

$$\mathrm{MSE}((1-\zeta)X) = (1-\zeta)^2 d + \zeta^2 \|\theta\|^2 \tag{2.6}$$

never min at 0. Minimizer is:

$$\zeta^* = \frac{d}{\underbrace{d + \|\theta\|^2}_{\mathbb{E}\|X\|^2}} \tag{2.7}$$

3 Hypothesis testing basics

Statistical model $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ (possibly non-parameteric).

We test whether

$$H_0 : \theta \in \Theta_0 \qquad \text{null hyp} \tag{3.1}$$

$$\text{vs } H_1 : \theta \in \Theta_1 \qquad \text{alt. hyp} \tag{3.2}$$

Default decision is to "accept" H_0 or given strong evidence against H_0, reject H_0.

Example 3.1. $X \sim \mathcal{N}(\theta, 1)$. Test $H_0 : \theta \leq 0$ vs $H_1 : \theta > 0$.
Or $H_0 : \theta = 0$ vs $H_1 : \theta \neq 0$.
Or $H_0 : \theta \in [-\delta, \delta]$ vs H_1 : not

Example 3.2. $X_{1:n} \overset{\text{iid}}{\sim} P$, $Y_{1:m} \overset{\text{iid}}{\sim} Q$.
$H_0 : P = Q$ vs $H_1 : P \neq Q$.

Test *procedure* fully described by *critical function* (*test function*)

$$\phi(X) = \begin{cases} 0, & \text{accept} \\ \gamma \in (0,1), & \text{reject w.p. } \gamma \\ 1, & \text{reject} \end{cases} \tag{3.3}$$

For a non-randomized test, can describe in terms of *rejection region* $R = \{x : \phi(x) = 1\}$. $\mathcal{X} \setminus R$ is the *acceptance region*.

Performance of ϕ described by its *power function*

$$\beta_\phi(\theta) = \mathbb{E}_\theta[\phi(x)] = \mathbb{P}_\theta[\text{Reject} H_0] \tag{3.4}$$

Significance level (*size*)

$$\alpha = \sup_{\theta \in \Theta_0} \beta(\theta) \tag{3.5}$$

$$\alpha = 0.05 \text{ ubiquitous} \tag{3.6}$$

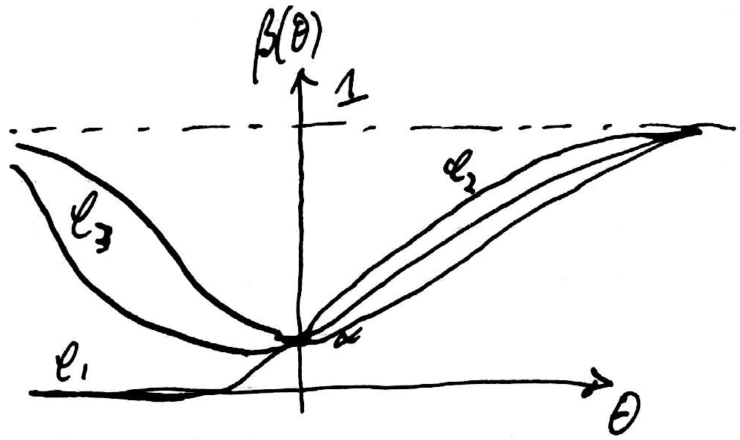
Example 3.3. $X \sim \mathcal{N}(\theta, 1)$, $H_0 : \theta = 0$ (vs $\theta \neq 0$).

$$\phi_1(x) = 1_{|X| > z_{\alpha/2}} \tag{3.7}$$

$$\phi_2(x) = 1_{X > z_\alpha} \tag{3.8}$$

$$\phi_3(x) = 1_{X < z_{\alpha/5} \text{ or } X > z_{2\alpha/3}} \tag{3.9}$$

$z_\alpha = \Phi^{-1}(1 - \alpha)$



Example 3.4. $X \sim \text{Binom}(n, \theta)$, $H_0 : \theta \leq \frac{1}{2}$.

$$\mathbb{P}_{\theta = \frac{1}{2}}(X \in R) = \frac{1}{2^n} \sum_{x \in R} \binom{n}{x} \tag{3.10}$$

$\alpha = 0.05$ no level$-\alpha$ non-rand test exists.

3.1 Likelihood Ratio Test (LRT)

$$\phi^*(x) = \begin{cases} 0, & L(x) < c \\ \gamma, & L(x) = c \\ 1, & L(x) > c \end{cases} \tag{3.11}$$

Intuition

$$\int_R p_0(x) d\mu(x) \leftrightarrow \qquad \text{“Buck”, Budget} \tag{3.12}$$

$$\int_R p_1(x) d\mu(x) \leftrightarrow \qquad \text{“Bang”, Value} \tag{3.13}$$

4 Neyman-Pearson Lemma