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STAT 210A
Lecture #26
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(a) Testing in high dimensions

(b) Asymptotic performance of two tests

Reference: Stats 300C notes, Emmanuel Candès

1 Testing in high dimensions

Setup:
$X \sim N_d(\theta, I_d)$, $\theta\ in \mathbb{R}^d$
Test $H_0 : \theta = 0$ vs $H_1 : \theta \neq 0$.
For convenience, write $\varepsilon \sim N(0, I_d)$ so $X = \theta + \varepsilon$.

1.1 Two competing tests

(a) χ^2**-test**: reject for large $\|X\|_2^2 \overset{H_0}{\sim} \chi_d^2$.
(Reject if $\|X\|_2^2 > \chi_d^2(\alpha)$)

(b) **Max test**: Reject when $\|X\|_\infty = \max_i |X_i|$ large

$$P_0(\|X\|_\infty \le c) = P_0(|X_1| \le c)^d \tag{1.1}$$
$$(1-\alpha)^{1/d} = P_0(|X_1| \le c) \tag{1.2}$$
$$c = z_{\tilde{\alpha}_d/2} \tag{1.3}$$
$$\tilde{\alpha}_d = 1 - (1-\alpha)^{1/d} \tag{1.4}$$

For large d, small α

$$1 - (1-\alpha)^{1/d} \approx \alpha/d \tag{1.5}$$

e.g. $\alpha = 0.05, d = 10 \implies \tilde{\alpha} = 0.00512$.

χ^2 test vs max test
Note $l(\theta; X) = \frac{-1}{2}\|\theta - X\|_2^2 - \frac{n}{2}\log(2\pi)$. GLRT statistic for testing

$$H_0 : \theta = 0 \quad \text{vs } H_1 : \theta \neq 0 \tag{1.6}$$
$$2(l(X;X) - l(0;X)) = \|X\|_2^2 \tag{1.7}$$

What about GLRT for

$$H_0 : \theta = 0 \quad \text{vs } H_1 : \theta = \mu e_i \tag{1.8}$$

where $\mu \in \mathbb{R} \setminus \{0\}, i \in \{1, \dots, d\}$.

$\Theta_0 \cup \Theta_1 = \{\mu e_i : \mu \in \mathbb{R}, i \in \{1, \dots, d\}\}$.

$\hat{\theta} = \mathrm{proj}_\Theta(X) = (0, 0, \dots, 0, X_{i^*}, 0, \dots, 0)$ where $i^* = \arg\max |X_i|$.

GRLT statistic is

$$2(\ell(\hat{\theta}; X) - \ell(0; X)) = |X_{i^*}|^2 = \max_{i=1}^{d} |X_i|^2 = \|X\|_\infty^2 \tag{1.9}$$

Sparse regime: We will consider a sparse regime where

$$\theta_1 = \cdots = \theta_k = \mu > 0 \tag{1.10}$$

$$\theta_{k+1} = \cdots = \theta_d = 0 \tag{1.11}$$

$$k_d = d^\beta \quad \beta \in (0, 1) \tag{1.12}$$

Question: When is the max-test powerful?

Lemma 1.1. *As $d \to \infty$, $\frac{\max_{i=1}^d |\varepsilon_i|}{\sqrt{2 \log d}} \xrightarrow{p} 1$*

Proof. It is equivalent to show

(a) $P(\max |\varepsilon_i| > c\sqrt{2 \log d}) \to 0$ if $c > 1$

(b) $P(\max |\varepsilon_i| > c\sqrt{2 \log d}) \to 1$ if $c < 1$

Write $z(d) = c\sqrt{2 \log d}$. Recall useful inequality

$$2(1 - z^{-2}) \frac{\phi(z)}{z} \leq P(|\varepsilon_1| > z) \leq 2\frac{\phi(z)}{z} \tag{1.13}$$

This is true $\forall z > 0$

If $c > 1$ (a), then

$$P(\max |\varepsilon_i| > z) \leq dP(|\varepsilon_1 \geq \varepsilon) \tag{1.14}$$

$$\leq 2d\frac{\phi(z)}{z} \tag{1.15}$$

$$= \frac{2}{\sqrt{2\pi}} d \frac{e^{-z^2/\alpha}}{z} \tag{1.16}$$

$$= \frac{2}{\sqrt{2\pi}} d \frac{d^{-c^2}}{c\sqrt{2 \log d}} \to 0 \tag{1.17}$$

Otherwise if $c < 1$ (b), then

$$P(\max |\varepsilon_i| < z) = P(|\varepsilon_1| < z)^d \tag{1.18}$$

$$\leq \left(1 - 2(1 - z^{-2}) \frac{\phi(z)}{z}\right)^d \tag{1.19}$$

$$\leq \left(1 - \frac{1}{10} \frac{d^{-c^2}}{\sqrt{2 \log d}}\right)^d \tag{1.20}$$

$$= \left(1 - \frac{1}{d}\left(\frac{d^{1-c^2}}{10\sqrt{2 \log d}}\right)\right)^d \to 0 \tag{1.21}$$

Note this lemma implies max-test threshold is $\sqrt{2\log d}(1+o(1))$.
(α fixed, $d \to \infty$)
Similar argument shows $\frac{\max \varepsilon_i}{\sqrt{2\log d}} \xrightarrow{\text{P}} 1$ □

Theorem 1.2. *Suppose $\mu(d) = \sqrt{2r\log d}$.*

(a) If $r > (1-\sqrt{\beta})^2$, then Power $\to 1$

(b) If $r < (1-\sqrt{\beta})^2$, then Power $\to \alpha$

Proof.

$$\frac{\max_{i=1}^k |X_i|}{\sqrt{2\log d}} \geq \frac{\sqrt{2r\log d} + \sqrt{2\log k}\,\overbrace{\frac{\max_{i\leq k}\varepsilon_i}{\sqrt{2\log k}}}^{\xrightarrow{\text{P}} 1}}{\sqrt{2\log d}} \tag{1.22}$$

$$\xrightarrow{\text{P}} \sqrt{r} + \sqrt{\beta} \tag{1.23}$$

(a) Threshold is $\frac{z_{\tilde{\alpha}/2}}{\sqrt{2\log d}} \to 1$.
$\implies P(\max_{i\leq k}|X_i| > z_{\tilde{\alpha}/2}) \to 1$ if $\sqrt{r}+\sqrt{\beta} > 1 \iff r \geq (1-\sqrt{b})^2$
(b) $P(\|X\|_\infty \geq z_{\alpha/2}) \leq P(\max_{i=1}^k |X_i| \geq z_{\alpha/2}) + P(\max_{i>k}|X_i| \geq z_{\alpha/2})$
If $\sqrt{r}+\sqrt{\beta} < 1$, then $P(\max_{i=1}^k |X_i| \geq z_{\alpha/2}) \to 0$.
We know the second term $\leq \alpha$, so overall $\lim P(\|X\|_\infty \geq z_{\alpha/2}) \leq \alpha$. So $\lim$ Power $\leq \alpha$.
Power $\geq \alpha$ because test unbiased. □

Question: When is χ^2 test powerful?
Under H_0, $\|X\|_2^2 \sim \chi_d^2$

$$\|X\|_2^2 = \sum_{i=1}^d X_i^2 \quad X_i^2 \sim (1,2) \tag{1.24}$$

$$\frac{1}{\sqrt{d}}(\|X\|_2^2 - d) \Rightarrow N(0,2) \tag{1.25}$$

$$\chi_d^2(\alpha) = d + \sqrt{2d}z_\alpha + o(\sqrt{d}) \tag{1.26}$$

Under H_1, everything is rotationaly invariant so distribution only depends on $\|\theta\|_2^2$ (non-central χ^2)

$$\mathbb{E}X_i^2 = 1+\theta_i^2 \tag{1.27}$$

$$\mathrm{Var}(X_i^2) = 4\theta_i^2 + 2 \tag{1.28}$$

$$\|X\|_2^2 \approx N(d+\|\theta\|^2, 4\|\theta\|^2 + 2d) \tag{1.29}$$

So if we normalize the test statistic

$$\frac{\|X\|_2^2 - d}{\sqrt{d}} \approx N(\frac{\|\theta\|^2}{d}, 2 + 4\frac{\|\theta\|^2}{d}) \tag{1.30}$$

So

- $\frac{\|\theta\|^2}{\sqrt{2d}} \gg 1$, power very high
- $\frac{\|\theta\|^2}{\sqrt{2d}} \ll 1$, power very low

$$\frac{\|\theta\|^2}{\sqrt{2d}} = \frac{k\mu^2}{\sqrt{2d}} \tag{1.31}$$

1.2 Power comparison

k	χ^2 needs	max needs
$d^{1/2}$	$\boxed{\mu > 3}$	$\mu > 0.29\sqrt{2\log d}$
$d^{1/4}$	$\mu > 3d^{1/8}$	$\boxed{\mu > 0.5\sqrt{2\log d}}$
$d^{3/4}$	$\boxed{\mu > 3d^{-1/8}}$	$\mu > 0.13\sqrt{2\log d}$