Feynman Liang
STAT 205A
Lecture #9
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1 Bounding maxima

Theorem 1.1 (Kolmogorov's Maximal Inequality)**.** *$(X_i)_{i=1}^n$ independent, $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 < \infty$.*
$S_m = \sum_{i=1}^m X_i$. $S_n^+ = \max_{1 \le m \le n} |S_m|$
Then $P(S_n^+ \ge x) \le \frac{\mathbb{E}S_n^2}{x^2}$, $x > 0$.

The proof uses a general trick related to martingales of considering stopping times.

Proof. Fix x. Event $\{S_n^+ \ge x\} = \cup_{k=1}^n A_k$ where $A_k = \{|S_k| \ge x, |S_i| < x, \text{ all } 1 \le i \le k\}$. Note (S_k, A_k) independent of $S_n - S_k$.

Notice $S_n = S_k + (S_n - S_k)$ so we can write

$$\mathbb{E}S_n^2 \ge \sum_{k=1}^n \mathbb{E}[S_n 1_{A_k}] \tag{1.1}$$

$$= \sum_{k=1}^n \left(\mathbb{E}(S_k^2 1_{A_k}) + \underbrace{2\mathbb{E}(\underbrace{S_k 1_{A_k}(S_n - S_k)}_{\mathbb{E}(S_n - S_k) = 0})}_{=0} + \underbrace{\mathbb{E}((S_n - S_k)^2 1_{A_k}}_{\ge 0} \right) \tag{1.2}$$

$$\ge \sum_{k=1}^n \mathbb{E}(S_k^2 1_{A_k}) \tag{1.3}$$

$$\ge \sum_{k=1}^n \mathbb{E}(x^2 1_{A_k}) \tag{1.4}$$

$$= x^2 P(\cup_{k=1}^n A_k) \tag{1.5}$$

$$= x^2 P(|S_n^+| \ge x) \tag{1.6}$$

where we have used independence of $S_k 1_{A_k}$ and $(S_n - S_k)$ in **??**, and $|S_k| \ge x$ on A_k in **??**. □

2 Almost sure convergence

"$\sum_{i=1}^\infty x_i$ converges" means $\lim_{n\to\infty} \sum_{i=1}^n x_i$ exists and is finite. This is equivalent to the Cauchy criterion:

$$\sup_{n \ge K} \left| \sum_{i=K+1}^n x_i \right| \to 0 \text{ as } K \to \infty \tag{2.1}$$

Thus, $\sum_{i=1}^{\infty} X_i$ converges a.s. means

$$P(\omega : \lim_{N\to\infty} \sum_{i=1}^{N} X_i(\omega) \text{ exists, finite}) = 1 \tag{2.2}$$

Theorem 2.1. *(X_i) independent, $\mathbb{E}X_i = 0$, $\sigma_i^2 = Var|X_i| < \infty$. If $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$, then $\sum_{i=1}^{\infty} X_i$ converges a.s.*

Comment:

$$\text{Var}(\sum_{i}^{n} X_i) = \sum_{=1}^{n} \sigma_i^2 \tag{2.3}$$

$$\text{Var} \sum_{i=1}^{\infty} = \sum_{i=1}^{\infty} \sigma_i^2 \leq \infty (*) \tag{2.4}$$

$$\implies \sum_{i=1}^{\infty} X_i \text{ is finite a.s.} \tag{2.5}$$

Exercise: Given theorem, show (*)

Proof. Define $M_k = \sup_{n\geq k} \left|\sum_{i=k+1}^{n} X_i\right|$. By Cauchy criterion, suffices to show $M_k \overset{\text{a.s.}}{\to} 0$ as $k \to \infty$.

$$P\left(\sup_{k<n\leq N} \left|\sum_{i=k+1}^{n} X_i\right| \geq \epsilon\right) \underset{??}{\leq} \epsilon^{-2}\text{Var}\left(\sum_{i=k+1}^{N} X_i\right) \tag{2.6}$$

$$= \epsilon^{-2} \sum_{i=k+1}^{N} \text{Var}\left(X_i\right) \tag{2.7}$$

$$\tag{2.8}$$

As $N \to \infty$

$$P(M_k > \epsilon) \leq \epsilon^{-2} \sum_{i=1}^{\infty} \sigma_i^2 \tag{2.9}$$

$$P(w_k > \epsilon) \leq P(M_k > \epsilon/2) \leq 4\epsilon^{-2} \sum_{1}^{\infty} \sigma_i^2 \tag{2.10}$$

where $w_k = \sup_{n_1 > n_1 > k} \left|\sum_{i=n_1+1}^{n_2} X_i\right|$. Note $M_k \leq w_k \leq 2M_k$ and $w_k \downarrow$ as $k \uparrow$. As $k \to \infty$, $w_k \downarrow_{\text{a.s.}} w_\infty$

$$P(w_\infty > \epsilon) = 0 \tag{2.11}$$

$$\implies w_\infty \overset{\text{a.s.}}{=} 0 \tag{2.12}$$

$$\implies w_k \downarrow_{\text{a.s.}} 0 \tag{2.13}$$

$$\implies M_k \overset{\text{a.s.}}{\to} 0 \tag{2.14}$$

□

Lemma 2.2 (Kronecker)**.** *$(x_n) \in \mathbb{R}^\omega$. $S_n = \sum_{i=1}^n x_i$. $0 < a_n \uparrow \infty$ as $n \uparrow \infty$. If $\sum_i \frac{x_i}{a_i}$ converges, then $\frac{s_n}{a_n} \to 0$.*

Proof. Exercise. □

Corollary 2.3. *(X_i) independent, $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 < \infty$. If $0 < a_n \uparrow \infty$ as $n \uparrow \infty$ and if $\sum_n \frac{\mathbb{E}X_n^2}{a_n^2} < \infty$, then $\frac{S_n}{a_n} \to 0$ a.s..*

Proof. Previous theorem implies $\sum_n \frac{X_n}{a_n}$ converges a.s.. Lemma implies $\frac{S_n}{a_n} \overset{\text{a.s.}}{\to} 0$. □

Specialization: Suppose also $\mathbb{E}X_n^2 \sim cn^{2\alpha}, \alpha > 0$. Take $a_n^2 = n^{1+2\alpha+\epsilon}$ ($\epsilon > 0$ small). Then corrolary implies $\frac{S_n}{n^{1/2+\alpha+\epsilon}} \overset{\text{a.s.}}{\to} 0$ TODO: Check the $1/2$.

Specialization: Suppose $\sup_n \mathbb{E}X_n^2 < \infty$. Take $a_n^2 = n(\log n)^{1+2\epsilon}$. The corollary implies $\frac{S_n}{\sqrt{n(\log n)^{1+\epsilon}}} \overset{\text{a.s.}}{\to} 0$.

Implicitly from CLT: If (X_i) i.i.d., then

$$\frac{S_n}{\sqrt{n}} \overset{\text{a.s.}}{\to} 0 \tag{2.15}$$

Law of iterated log.

Theorem 2.4 (Strong Law of Large Numbers (SLLN))**.** *Let (X_i) iid with $\mathbb{E}|X_i| < \infty$, $S_n := \sum_{i=1}^n X_i$. Then $\frac{S_n}{n} \overset{a.s.}{\to} \mathbb{E}X$ as $n \to \infty$.*

Proof. Truncate, center, apply corollary ($Z \geq 0$. $\mathbb{E}Z^k = \int_0^\infty kz^{k-1}P(Z \geq z)dz \approx \int_0^\infty x^k f(x)dx$)

(Truncate): Define $Y_k = X_k 1_{|X_k| \leq k}$, so Y_k are no longer iid. However

$$\sum_k P(Y_k \neq X_k) = \sum_{k=1}^\infty P(|X| > k) \leq \int_0^\infty P(|x| > x)dx = \mathbb{E}|X| \leq \infty \tag{2.16}$$

By Borel Cantelli 1, $P(Y_k = X_k \text{ e.v.}) = 1$. Thus, suffices to prove $\frac{1}{n}\sum_{k=1}^n Y_k \overset{\text{a.s.}}{\to} \mathbb{E}X$.

(Center): Define $X_k' = Y_k - \mathbb{E}Y_k$. **Claim**:

$$\sum_k \frac{\text{Var}(X_k')}{k^2} < \infty \tag{2.17}$$

To show the claim:

$$\mathbb{E}Y_k^2 = \int_0^\infty 2yP(|Y_k| > y)dy \tag{2.18}$$

$$= \int_0^\infty 2yP(k \geq |X_k| \geq y)1_{y\leq k}dy \tag{2.19}$$

$$\leq \int_0^\infty 2yP(|X_k| \geq y)1_{y\leq k}dy \tag{2.20}$$

$$\sum_k \frac{\mathrm{Var}X_n'}{k^2} \leq \sum_k \frac{\mathbb{E}Y_k^2}{k^2} \tag{2.21}$$

$$\leq \sum_k \frac{1}{k^2}\int_0^\infty 2yP(|X| \geq y)1_{y\leq k}dy \tag{2.22}$$

$$= \int_0^\infty \left(\underbrace{\sum_k \frac{1}{k^2}1_{y\leq k}\,2y}_{G(y)}\right) P(|X| \geq y)dy \tag{2.23}$$

Claim: $G(y) \leq 4$ for all $0 < y < \infty$. True for $y \leq 1$. Take $y > 1$

$$\frac{1}{k^2} \leq \int_{k-1}^k \frac{1}{x^2}dx \tag{2.24}$$

$$\sum_k \frac{1}{k^2}1_{y\leq k} = \sum_{k\geq\lceil y\rceil} \frac{1}{k^2} \leq \int_{\lceil y\rceil-1}^\infty \frac{1}{x^2}dx = \frac{1}{\lceil y\rceil - 1} \tag{2.25}$$

$$\implies G(y) \leq \frac{2y}{\lceil y\rceil - 1} \leq 4 \tag{2.26}$$

Hence

$$\sum_k \frac{\mathrm{Var}X_n'}{k^2} \leq 4\int_0^\infty P(|X| > y)dy = 4\mathbb{E}|X| \tag{2.27}$$

Apply corollary to X_n'

$$\frac{1}{n}\sum_{i=1}^n X_i' \overset{\text{a.s.}}{\to} 0 \tag{2.28}$$

$$\frac{1}{n}\sum_i^n (Y_i - \mathbb{E}Y_i) \overset{\text{a.s.}}{\to} 0 \tag{2.29}$$

Note $\mathbb{E}Y_i = \mathbb{E}X1_{|X|\leq ?} \to \mathbb{E}X$ by dominated convergence, so

$$\frac{1}{n}\sum_i^n (\mathbb{E}Y_i - \mathbb{E}X) \overset{\text{a.s.}}{\to} 0 \tag{2.30}$$

Adding **??** with **??** yields

$$\frac{1}{n}\sum_{i}^{n}(Y_i - \mathbb{E}X) \overset{\text{a.s.}}{\to} 0 \tag{2.31}$$

$$\frac{1}{n}\sum_{i}^{n}Y_i \overset{\text{a.s.}}{\to} \mathbb{E}X \tag{2.32}$$

$\square$