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STAT 205B
Lecture #2
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1 Wrapping up Scheffe's theorem

Theorem 1.1 (Scheffe's Theorem). *$\theta(\cdot)$ σ-finite measure on S.*

If h_n and $h : S \to [0, \infty]$ satisfy

$$\int_S h_n d\theta = 1, \quad \int_S h d\theta = 1, \quad h_n(s) \to h(s)\ \theta - a.e. \tag{1.1}$$

then $\int_s |h_n(s) - h(s)| \theta(ds) \to 0$

Proposition 1.2. *Suppose $(X_n, 1 \leq n < \infty)$ and X are integer-valued. The following are equivalent:*

(a) $X_n \xrightarrow{d} X$

(b) $P(X_n = i) \underset{n\to\infty}{\to} P(X = i), \forall i$

(c) $\sum_i |P(X_n = 1) - P(X = i)| \to 0$

Proof. (a) $\implies$ (b): $P(X_n \leq i + \frac{1}{2}) \to P(X \leq i + \frac{1}{2})$.

$P(X_n = i) = P(X_n \leq i + \frac{1}{2}) - P(X_n \leq i - \frac{1}{2}) \to P(X \leq i + \frac{1}{2}) - P(X \leq i - \frac{1}{2}) = P(X = i)$

(b) $\implies$ (c): Scheffe's theorem for $\theta(i) = 1\ \forall i$, $h_n(i) = P(X_n = i)$.

(c) $\implies$ (a):

$$|P(X_n \leq x) - P(X \leq x)| = |\sum_{i \leq x} (P(X_n = i) - P(X = i))| \tag{1.2}$$

$$\leq \sum_i |P(X_n = i) - P(X = i)| \tag{1.3}$$

□

Proposition 1.3. *If X_n and X have probability densities $f_n(x)$ and $f(x)$, $f_n(x) \to f(x)$ for almost all x, then $X_n \xrightarrow{d} X$.*

Proof. Scheffe's theorem. $|P(X_n \leq x) - P(X \leq x)| \leq \int |f_n(x) - f(x)| dx \to 0$ □

2 Compactness/metric related theory

Let $(X_n, 1 \leq n < \infty)$ be $\mathbb{R}$-valued.

Definition 2.1. Say (X_n) is *tight* if $\lim_{B\uparrow\infty}\sup_n P(|X_n|\geq B)=0$.
Say (X_n) is *uniformly integrable* (UI) if $\lim_{B\uparrow\infty}\sup_n \mathbb{E}[|X_n|1_{|X_n|\geq B}]=0$.

These are actually properties of $\mu_n=\text{dist}(X_n)$.

Lemma 2.2. *(a) If $\sup_n \mathbb{E}|X_n|<\infty$, or more generally if $\sup_n \mathbb{E}\phi(|X_n|)<\infty$ for some $0\leq\phi(x)\uparrow\infty$ as $x\uparrow\infty$, then (X_n) is tight.*

(b) If $\sup_n \mathbb{E}X_n^2<\infty$, or more generally if $\sup_n \mathbb{E}\phi(|X_n|)<\infty$ for some $0\leq\phi(x)\uparrow\infty$ such that $\frac{\phi(x)}{x}\to\infty$ as $x\to\infty$, then (X_n) is UI.

Proof. (a): Markov's inequality $P(|X_n|>B)\leq\frac{\mathbb{E}\phi(|X_n|)}{\phi(B)}$ □

Lemma 2.3 (From 205A)**.** *If $\hat{X}_n\xrightarrow{a.s.}\hat{X}$, if $(\hat{X}_n, 1\leq n<\infty)$ is UI, then $\mathbb{E}|\hat{X}|<\infty$ and $\mathbb{E}\hat{X}_n\to\mathbb{E}\hat{X}$.*

Corollary 2.4. *If $X_n\xrightarrow{d}X$, if $(X_n, 1\leq n<\infty)$ is UI, then $\mathbb{E}|X|<\infty$ and $\mathbb{E}X_n\to\mathbb{E}X$.*

Definition 2.5. A *distribution function* F satisfies

- $0\leq F(x)\leq 1, \forall x\in(-\infty,\infty)$
- $x\mapsto F(x)$ increasing
- $F(x+)=F(x)$ i.e. right-continuity
- $\lim_{x\uparrow\infty}F(x)=1$, $\lim_{x\downarrow-\infty}F(x)=0$

Definition 2.6. The *extended distribution function* (EDF) has the first three properties of a distribution function, and

$$\lim_{x\uparrow\infty}F(x)=\text{“}F(\infty)\text{''}\leq 1 \tag{2.1}$$
$$\lim_{x\downarrow-\infty}F(x)=\text{“}F(-\infty)\text{''}\geq 0 \tag{2.2}$$

1-1 correspondence between PMs μ on $[-\infty,\infty]$ and EDFs.
Think of RVs X with values in $[-\infty,\infty]$.

Theorem 2.7 (Helly's selection theorem)**.** *Let $F_1, F_2, \dots$ be distribution functions on $(-\infty,\infty)$.*

(a) There exist $n_j\to\infty$ and an EDF G such that $F_{n_j}(x)\to G(x)$ for all continuity points x of G.

(b) If $(F_n, 1\leq n<\infty)$ is tight, then G is a distirbution function on $(-\infty,\infty)$

Example 2.8. Z standard normal, $\Phi(z)$.
J uniform on $\{1,2,3\}$.

$$X_n=\begin{cases}-n & \text{if } J=1\\ Z & \text{if } J=2\\ +n & \text{if } J=3\end{cases} \tag{2.3}$$

TODO: Fig 2.1

Proof. (a) Let $q_1, q_2, q_3, \ldots$ be the rationals. The sequence $f_1(q_1), F_2(q_2), F_3(q_1), \ldots$ is a $[0,1]$. By compactness of $[0,1]$, exists convergent subsequence $m(1,1)$, $m(1,2)$, $m(1,3)$, such that

$$F_{m(1,1)}(q_1) \underset{i\to\infty}{\rightarrow} \text{ some limit } G_0(q_1) \tag{2.4}$$

Diagonal Argument: $F_{m(1,i)}(q_2)$, $i = 1, 2, \ldots$ is a sequence in $[0,1]$; exists convergent subsequence $m(2,1)$, $m(2,2)$, $m(2,3)$, ..., such that $F_{m(2,i)}(q_2) \rightarrow$ some limit $G_0(q_2)$.

Repeat for eack $k \geq 1$, find subsequence $(m(k,i), i \geq 1)$ of $(m(k-1,i), i > 1)$ such that $F_{m(k,i)}(q_k) \underset{i\to\infty}{\rightarrow}$ some $G_0(q_k)$.

Consider $m(i,i)$ "diagonal". This has property $F_{m(i,i)}(q_k) \underset{i\to\infty}{\rightarrow} G_0(q_k)$ for all k.

Now define an EDF G by $G(x) = \inf_{\substack{q\in\mathbb{Q}\\ q>x}} G_0(q)$.

Check that G is an EDF.

Fix x. For any $q > x$, $\limsup_i F_{m(i,i)}(x) \leq \limsup_n F_{m(i,i)}(q) = G_0(q) \leq G(x)$. Let $q \downarrow x$. By same argument, $\liminf_i F_{m(i,i)}(x) \geq G(x)$. So if $G(x) = G(x-)$ (i.e. continuity point), then $F_{m(i,i)}(x) \rightarrow G(x)$.

(b) tight $\implies \exists k(B)$ such that ${}_nP(X_n \leq B) \geq 1 - k(B)$, $K(B) \downarrow 0$ as $B \uparrow \infty$. Consider $F_{m(i,i)}(q) \rightarrow G(q)\ \forall q \implies G(B) \geq 1 - k(B) \implies G$ puts 0 mass on $+\infty$. ☐

Corollary 2.9. *Given $(X_n, 1 \leq n < \infty)$ and X ($\mathbb{R}$-valued RVs).*

Suppose (X_n) is tight.

Suppose that, whenever $X_{n_j} \xrightarrow{d}$ some Y as $j \to \infty$ for some (n_j) we have $Y \overset{d}{=} X$.

Then $X_n \xrightarrow{d} X$ as $n \to \infty$.

Proof. By contradiction. If $X_n \overset{\mathrm{d}}{\not\rightarrow} X$, then $\exists x_0$ continuity point of X such that $P(X_n \leq x_0)$ $P(X_n \leq x_0) \rightarrow P(X \leq x_0)$.

$\implies \exists \varepsilon > 0$ and $m_j \to \infty$ such that $P(X_{m_j} \leq x_0) - P(X \leq x_0)| \geq \varepsilon$, $\forall j$.

Apply Helly to (X_{m_j}): $\exists$ subsequence $X_{n_j} \xrightarrow{\mathrm{d}}$ some Y. But $Y \overset{d}{=} X$ by hypothesis. $|P(X_{n_j} \leq x_0) - P(X \leq x_0)| \to 0$, which is not $\geq \varepsilon$. Contradiction. ☐

Lemma 2.10. *Suppose $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $\mathbb{E}X^4 \leq K$. Then there exists $c(K) \geq 0$, depending only on K, such that $P(X > 0) \leq c(K)$.*

Proof. By contradiction. Suppose $\exists K$ such that the assertion is false. So $\exists X_n$ such that $\mathbb{E}X_n = 0$, $\mathbb{E}X_n^2 = 1$, $\mathbb{E}X_n^4 \leq K$, but $P(X_n > 0) \leq n^{-1}$. These X_n are tight, so by Helly $\exists$ subsequence $X_{n_j} \xrightarrow{\mathrm{d}}$ some X.

So $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $P(X > 0) = 0$. But this is impossible. ☐