Feynman Liang
STAT 205B
Lecture #3
1/24/2017

Homework due next Tuesday.
GSI office hours: Fridays 11-12, 426 Evans.
Optional lab ours: Mondays 3:00-4:00 ? TBA

1 Transforms — 3 variants of same idea

(a) X values in $\{0, 1, 2, \ldots\}$. Probability generating function

$$h_X(z) = \sum_{n=0}^{\infty} P(X = n) z^n = \mathbb{E} z^X, \quad 0 \le z \le 1 \tag{1.1}$$

(b) X values in $[0, \infty)$. Laplace transform

$$L_X(\theta) = \mathbb{E} e^{-\theta X} = \int_0^\infty e^{-\theta x} \underbrace{f_X(x) dx}_{\mu_X(dx)} \tag{1.2}$$

if X has density $f_X(x)$, or more generally if X has distribution μ_X.
Finite for $0 \le \theta < \infty$.

(c) X arbitrary $\mathbb{R}$-valued. Characteristic function (aka Fourier transform)

$$\phi_X(t) = \mathbb{E} e^{itX} = \mathbb{E} \cos(tX) + i \mathbb{E} \sin(tX) \tag{1.3}$$

$$= \int_{-\infty}^{\infty} e^{itx} f_X(x) dx \tag{1.4}$$

Proposition 1.1. *If $S = X_1 + X_2$, independent X_1, X_2, then*

$$h_S(z) = h_{X_1}(z) h_{X_2}(z) \tag{1.5}$$

$$L_S(z) = L_{X_1}(z) L_{X_2}(z) \tag{1.6}$$

$$\phi_S(z) = \phi_{X_1}(z) \phi_{X_2}(z) \tag{1.7}$$

Proof.

$$\mathbb{E} e^{itS} = \mathbb{E} \left(e^{itX_1} e^{itX_2} \right) \tag{1.8}$$

$$\text{Product rule} \implies = \left(\mathbb{E} e^{itX_1} \right) \left(\mathbb{E} e^{itX_2} \right) \tag{1.9}$$

□

Notation: $t, x, y \in \mathbb{R}$, $z \in \mathbb{C}$, $z = x + iy$. Then $|z| = \sqrt{x^2 + y^2}$, $|z_1 z_2| = |z_1||z_2|$.
For $\mathbb{C}$-valued RV $Z = X + iY$, $\mathbb{E}Z = \mathbb{E}X + i\mathbb{E}Y$. Jensen's gives $|\mathbb{E}Z| \leq \mathbb{E}|Z|$.
Consider the characteristic function

$$\phi : \mathbb{R} \to \mathbb{C} \tag{1.10}$$
$$t \mapsto \mathbb{E}e^{itX} \tag{1.11}$$

We have

$$\phi_X(t+h) - \phi_X(t) = \mathbb{E}\left(e^{i(t+h)X} - e^{itX}\right) = \mathbb{E}\left(e^{itX}(e^{ihX} - 1)\right) \tag{1.12}$$
$$|\phi_X(t+h) - \phi_X(t)| \leq \mathbb{E}\left(|e^{itX}||e^{ihX} - 1|\right) = \mathbb{E}|e^{ihX} - 1| =: \psi(h) \tag{1.13}$$

As $h \downarrow 0$, $e^{ihX} - 1 \to 0$. By bounded convergence $\phi(h) \to 0$ as $h \to 0$, so ϕ is uniformly continuous.

Proposition 1.2 (Inversion Formulas). *Let $\phi(t)$ be the characteristic function of a PM μ.*

(a) $\mu(a,b) + \frac{1}{2}(\mu(\{a\} + \mu\{b\}) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\phi(t)dt, \quad -\infty < a < b < \infty$

(b) If $\int_{-\infty}^{\infty}|\phi(t)|dt < \infty$ *then μ has bounded continuous density* $f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity}\phi(t)dt$

Proof.

$$I(T) := \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\phi(t)dt \tag{1.14}$$
$$S(T) := \int_{0}^{\infty} \frac{\sin x}{x}dx \to \frac{\pi}{2} \text{ as } T \to \infty \tag{1.15}$$
$$\frac{e^{-ita} - e^{-itb}}{it} = \int_a^b e^{-ity}dy \tag{1.16}$$

So $\left|\frac{e^{-ita} - e^{-itb}}{it}\right| \leq b - a$.
By Fubini's theorem

$$I(T) = \int_{-\infty}^{\infty}\left(\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}e^{itx}dt\right)\mu(dx) \tag{1.17}$$
$$\tag{1.18}$$

By symmetry

$$\int_{-T}^{T} \frac{e^{it(x-a)}}{it}dt = \int_{-T}^{T} \frac{\sin(t(x-a))}{t}dt + \frac{1}{i}\int_{-T}^{T} \frac{\cos(t(x-a))}{t}dt = 0 \tag{1.19}$$

and

$$\int_{-T}^{T} \frac{e^{it(x-a)}}{it}dt = 2\int_0^T \frac{\sin(\theta t)}{t}dt = 2S(\theta T), \quad \theta > 0 \tag{1.20}$$
$$= \underbrace{2\,\mathrm{sgn}(\theta)S(T|\theta|)}_{:=R(T,\theta)}, \quad -\infty < \theta < \infty \tag{1.21}$$
$$\to \pi\,\mathrm{sgn}(\theta) \text{ as } T \to \infty \tag{1.22}$$

Hence

$$I(T) = \int_{-\infty}^{\infty} (R(x-a,T) - R(x-b,T))\, \mu(dx) \tag{1.23}$$

$$\tag{1.24}$$

But since $R(x-a,T) - R(x-b,T) \leq 2\sup_{\theta,T} R(\theta,T) =: K < \infty$, letting $T \to \infty$ gives

$$\lim_{T\to\infty} I(T) = 2\pi \int_{-\infty}^{\infty} \chi_{a,b}(x)\mu(dx) \tag{1.25}$$

where $\chi_{a,b} := \begin{cases} 0, & \text{if } x < a \text{ or } x > b \\ 2\pi, & \text{if } a < x < b \\ \pi, & \text{if } x = a \text{ or } x = b \end{cases}$ THis shows TODO: reference (a).

In case TODO: reference (b), the integral

$$\int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} \phi(t) dt \tag{1.26}$$

is absolutely convergent, and $\left|\frac{e^{-ita}-e^{-itb}}{it}\phi(t)\right| \leq b-a$. Use TODO: ref (a) to get

$$\mu(a,b) + \frac{1}{2}\mu\{a\} + \frac{1}{2}\mu\{b\} \leq \frac{b-a}{2\pi}\int_{-\infty}^{\infty} |\phi(t)| dt \tag{1.27}$$

Note $(a',b') \downarrow \{x\} \implies \mu\{x\} = 0 \ \forall x$. TODO: ref (a) implies

$$\mu(a,b) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_a^b e^{-ity} dy\right)\phi(t)dt \tag{1.28}$$

$$\text{Fubini} \implies = \int_a^b \underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity}\phi(t)dt\right)}_{\text{the density function for } \mu} dy \tag{1.29}$$

□

Some comments:

(a) (Uniqueness) If $\phi_\mu(t) := \phi_\nu(t) \ \forall t$, then $\nu = \mu$

(b) In principle, we can calculate dist of $S_n = X_1 + X_2 + \cdots + X_n$ for independent X_i using $\phi_{S_n} = \prod_{i=1}^n \phi_{X_i}(t)$

Example 1.3. If $X \sim N(0,\sigma^2)$ then $\phi_X(t) = e^{-\sigma^2 t^2/2}$. So if X_1, X_2 are independent $N(0,\sigma_i^2)$, then $S = X_1 + X_2$ has

$$\begin{aligned} \phi_S(t) &= \phi_{X_1}(t)\phi_{X_2}(t) && (1.30) \\ &= \exp(-\sigma_1^2 t^2/2 - \sigma_2^2 t^2/2) && (1.31) \\ &= \exp(-(\sigma_1^2 + \sigma_2^2)t^2/2) && (1.32) \\ &= \text{CF of } N(0,\sigma_1^2+\sigma_2^2) && (1.33) \end{aligned}$$

Example 1.4. $X \sim$ Exponential(1), so $f(x) = e^{-x}$ for $x > 0$. THen

$$\phi_X(t) = \int_0^\infty e^{itx} e^{-x} dx = \int_0^\infty e^{-(1-it)x} dx = \frac{1}{1-it} \tag{1.34}$$

For $c > 0$, $\phi_{cX}(t) = \mathbb{E}e^{ictX} = \phi_X(ct)$.

Example 1.5 (Laplace RV)**.** Y has density $f_Y(y) = \frac{1}{2}e^{-|y|}$, $-\infty < y < \infty$.

$$\mu_Y = \frac{1}{2}\mu_X + \frac{1}{2}\mu_{-X} \tag{1.35}$$

$$\implies \phi_Y(t) = \frac{1}{2}\phi_X(t) + \frac{1}{2}\phi_{-X}(t) = \frac{1}{2}\left(\phi_X(t) + \phi_X(-t)\right) \tag{1.36}$$

$$= \frac{1}{2}\left(\frac{1}{1-it} + \frac{1}{1+it}\right) = \frac{1}{(1-it)(1+it)} = \frac{1}{1+t^2} \tag{1.37}$$

Proposition 1.6 (Parseval's Identity)**.** *Let μ and ν be PMs with CF ϕ_μ and ϕ_ν. Then*

$$\int_{-\infty}^\infty \phi_\nu(t)\mu(dt) = \int_{-\infty}^\infty \phi_\mu(t)\nu(dt) \tag{1.38}$$

Proof. Take X, Y independent, $\mathrm{dist}(X) = \mu$, $\mathrm{dist}(Y) = \nu$.

$$\mathbb{E}[e^{iXY} \mid Y = y] = \mathbb{E}e^{iyX} = \phi_\mu(y) \tag{1.39}$$

$$\implies \mathbb{E}e^{iXY} = \mathbb{E}\phi_\mu(Y) = \int_{-\infty}^\infty \phi_\mu(y)\nu(dy) = \text{Right side} \tag{1.40}$$

$$\mathbb{E}e^{iXY} = \mathbb{E}[\mathbb{E}[e^{iYX} \mid X]] = \text{Left side} \tag{1.41}$$

where we have applied Fubini's theorem. □

By choice of "simple" ν, get general identities between μ and ϕ_μ!

Example 1.7. ν uniform on $[-c, c]$. $\phi_\nu(t) = \frac{\sin(ct)}{cct}$. For any ν,

$$\frac{1}{2c}\int_{-c}^c \phi_\mu(t)dt = \int_{-\infty}^\infty \frac{\sin(ct)}{ct}\mu(dt) \tag{1.42}$$

Example 1.8. ν Normal$(0, \sigma^2)$. For any μ

$$\int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \phi_\mu(t)dt = \int_{-\infty}^\infty e^{-\frac{1}{2}\sigma^2 t^2}\mu(dt) \tag{1.43}$$