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STAT 205A
Lecture #2
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1 Measure Theory Continued

Notation: $\mathcal{B} := \sigma\{\text{open sets of } S^{\text{topo}}\}$.

For $f : S_1 \to S_2$, have pullback $f^{-1} : S_2 \to S_1$

(a) f^{-1} commutes with finite Boolean operations and monotone limits, i.e.

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) \tag{1.1}$$

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2) \tag{1.2}$$

$$B_n \uparrow B \implies f^{-1}(B_n) \uparrow f^{-1}(B) \tag{1.3}$$

(b) Given $\mathcal{S}_2$, $\{f^{-1}(B) : B \in \mathcal{S}_2\}$ is a σ-field: "the pullback of a σ-field is a σ-field."

Definition 1.1. A function $f : S_1 \to S_2$ between two measurable spaces is *measurable* if $f^{-1}(B) \in \mathcal{S}_1$ for all $B \in \mathcal{S}_2$.

Lemma 1.2. *f is measurable if $f^{-1}(B) \in \mathcal{S}_1$ for all $B \in \mathcal{B}$ such that $\mathcal{S}_2 = \sigma(\mathcal{B})$.*

Proof. $\{B \in \mathcal{S}_2 : f^{-1}(B) \in \mathcal{S}_1\} \supset \sigma(\mathcal{B})$ is a σ-field by commutativity of f^{-1} wrt Boolean operations. It also $\supset \mathcal{B}$. □

Lemma 1.3. *$f^{cts} : S_1^{topo} \to S_2^{topo}$ is measurable (i.e. cts $\implies$ meas)*

Proof. cts $\implies$ $f^{-1}(G_2^{\text{open}}) \in S_1^{\text{open}} \supset \mathcal{S}_1$, where $\mathcal{S}_1$ is the Borel σ-algebra on S_1. The previous lemma implies f is measurable wrt $\sigma\{S_1^{\text{open}}\} = \mathcal{S}_1$. □

Lemma 1.4 (π-system sufficiency). *If $S_2 = \mathbb{R}$, it suffices to check $f^{-1}(-\infty, x] \in \mathcal{S}_1$ for all $x \in \mathbb{R}$.*

Proof. $\sigma\{(-\infty, x] : x \in \mathbb{R}\} = \sigma(\mathbb{R}) = \mathcal{S}_2$ □

$$\begin{tikzcd} S_1 \arrow[r, "h"] \arrow[rr, bend right, "f = g \circ h"'] & S_2 \arrow[r, "g"] & S_1 \end{tikzcd}$$

Lemma 1.5 (Composition). *If h and g are measurable, then $f = g \circ h$ is measurable.*

Lemma 1.6 (Multi-input composition). *Suppose* $\{f_i : (S, \mathcal{S}) \to \mathbb{R}\}_{i=1}^d$ *are measurable and* $g : \mathbb{R}^d \to \mathbb{R}$ *is measurable. Then* $g(f_1(s), f_2(s), \cdots, f_d(s))$ *is a measurable function* $\mathcal{S}_1 \to \mathbb{R}$.

Proof. Apply lemma 1.5 to $(S, \mathbb{R}^d, \mathbb{R})$ and $h(s_1) = \begin{bmatrix} f_1(s_1) & f_2(s_1) & \cdots & f_d(s_1) \end{bmatrix}$. Suffices to show $h : S\mathbb{R}^d$ measurable.

Use fact that $\mathcal{B}^d$ = Borel σ-field on $\mathbb{R}^d$ = σ-field generated by $\left\{\prod_{i=1}^d (-\infty, \hat{x}_i] : \hat{x} \in \mathbb{R}^d\right\}$.

Then

$$h^{-1}\left(\prod_{i=1}^d (-\infty, x_i]\right) = \bigcap_{i=1}^d \{s_1 : f_i(s_1) = x_i\} \in \mathcal{S}_1 \tag{1.4}$$

and by lemma 1.4 we are done. □

Corollary 1.7. $\{f_i : S \to \mathbb{R}\}$ *measurable, then* $f_1 + f_2$, $f_1 \cdot f_2$, *and* $\max\{f_1, f_2\}$ *are measurable.*

Proof. $g(x_1, x_2) = x_1 + x_2, x_1 \cdot x_2$, and $\max\{x_1, x_2\}$ are all continuous hence measurable. Applying lemma 1.6 with $\{f_i\}$ and g shows that the composition is measurable. □

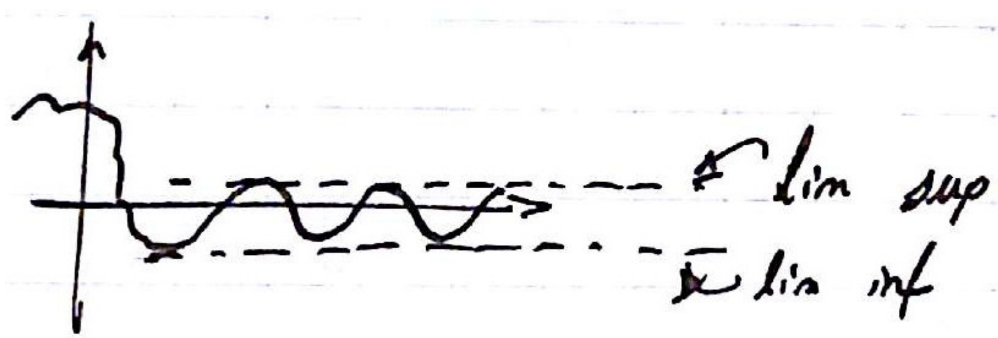
This is *very important*, make sure to grok the following definition:

Definition 1.8. For arbitrary $x_n \in \bar{\mathbb{R}}$, $n \in \mathbb{N}$, define

$$\limsup_n x_n := \lim_{N\uparrow\infty} \sup_{n\geq N} x_n = \inf N \uparrow \infty \sup_{n\geq N} x_n \in \bar{\mathbb{R}} \tag{1.5}$$

$$\liminf_n x_n := \lim_{N\uparrow\infty} \inf_{n\geq N} x_n = \sup N \uparrow \infty \inf_{n\geq N} x_n \in \bar{\mathbb{R}} \tag{1.6}$$

$$\tag{1.7}$$



Note that both lim sup and lim inf exist $\in \bar{\mathbb{R}}$, regardless of whether $\lim_n x_n$ does, and $\limsup \geq \lim \geq \liminf$.

These definitions may be generalized to ascending and descending sequences of sets, where sup is taken to be $\cup$ and inf as $\cap$.

Lemma 1.9. *Given measurable functions* $\{f_i : S \to \bar{\mathbb{R}}\}_{i=1}^\infty$, *define* $f^*(s) = \limsup_n f_n(s)$ *and* $f_*(s) = \liminf_n f_n(s)$. *Then* f^* *and* f_* *are measurable functions* $S \to \bar{\mathbb{R}}$.

Proof.

$$\{s : \limsup_n f_n(s) \le x\} = \{ss : f_n(s) \le x + 1/i \text{ ult. } \forall i \in \mathbb{N}\} \tag{1.8}$$

$$= \cap_{i=1}^{\infty} \{s : f_n(s) \le x + 1/i \text{ ult.}\} \tag{1.9}$$

$$= \cap_{i=1}^{\infty} \cup_{N=1}^{\infty} \{s : f_n(s) \le x + 1/i \; \forall n \ge N\} \tag{1.10}$$

$$= \cap_{i=1}^{\infty} \cup_{N=1}^{\infty} \cap_{n=N}^{\infty} \underbrace{\{s : f_n(s) \le x + 1/i\}}_{\in \mathcal{S}} \tag{1.11}$$

so f^* meaurable. □

2 On $\mathbb{R}$-valued measurable functions $(S, \mathcal{S}) \to \mathbb{R}$

Definition 2.1. For $A \in \mathcal{S}$, the *indicator function* $1_A(s) = \begin{cases} 1, & \text{if } s \in \mathcal{A} \\ 0, & \text{otherwise} \end{cases}$

Let $\vec{c} \in \mathbb{R}^n$ and $\{A_i\}_1^n$ be a partition of S into measurable sets. $f : (S, \mathcal{S}) \to \mathbb{R}$ is a *simple function* if $f(s) = \sum_i \underbrace{c_i 1_{A_i}}_{\text{step function on } A_i} \; (= c_i \text{ for } s \in A_i)$.

Lemma 2.2. *Let $h^{meas}; S \to [0, L]$. For $i \ge 1$, define*

$$h_i(s) = \max_{j \ge 0} \left\{ \frac{j}{2^i} : \frac{j}{2^i} \le h(s) \right\} = 2^{-i} \left\lfloor 2^i h(s) \right\rfloor \le h(s) \tag{2.1}$$

Then $h_i(s) \uparrow h(s)$ and each h_i is a simple function.

Exercise 2.3. Prove this.

3 Measures

$(S, \mathcal{S})$ a measurable space.

Definition 3.1. A *measure* is a function $\mu : \mathcal{S} \to [0, \infty]$ such that

(a) $\mu(\emptyset) = 0$

(b) (Countable additivity) For countable disjoint $A_i \in \mathcal{S}$, $\mu(\cup_i A_i) = \sum_i \mu(A_i) \le \infty$

Definition 3.2. μ is a *probability measure* if in addition $\mu(S) = 1$.
$\mu(S) < \infty$ is a *finite measure*.
If $\exists S_n \uparrow S$ s.t. $\mu(S_n) < \infty$ for all n, then μ is a *σ-finite measure*

3.1 Elementary Properties

- If $A \subset B$, then $\mu(A) \leq \mu(B)$
- If $(A \cup B) \leq \mu(A) + \mu(B)$, with equality if $A \cap B = \emptyset$
- For probability measures, $\mu(A^c) = 1 - \mu(A)$
- (Monotonicity) $A_n \uparrow A \implies \mu(A_n) \uparrow \mu(A)$.
  $A_n \downarrow A$ and some $\mu(A_n) < \infty \implies \mu(A_n) \downarrow \mu(A)$
- (Continuity) $A_n \downarrow \emptyset$, $\exists n : \mu(A_n) < \infty$, then $\mu(A_n) \downarrow 0$.