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STAT 210A
Lecture #4
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1 Minimal Sufficiency

Definition 1.1. T is *minimal sufficient* if

- T is sufficient
- For any other sufficient statistic S, $\exists f$ such that $T = f(S)$a.e.

Minimal sufficiency expresses the notion of the "most compressed sufficient statistic."

Definition 1.2. $\mathbb{P}_\theta(x) \propto_\theta \mathbb{P}_\theta(y)$ means that $\exists$ function $c(x, y)$ *independent of* θ such that $c(x, y)\mathbb{P}_\theta(x) = \mathbb{P}_\theta(y)$.

Theorem 1.3 (3.11)**.** *Let $\mathcal{P} = \{\mathbb{P}_\theta : \theta \in \Theta\}$, $\mathbb{P}_\theta << \mu$, T suff.*
If $\mathbb{P}_\theta(x) \propto_\theta \mathbb{P}_\theta(y) \implies T(x) = T(y)$, then T is minimal sufficient.

Proof. Consider another sufficient statistic $\tilde{T}$. We need to show $\exists f : f(\tilde{T}) = T$. Will proceed by contradiction i.e. assuming $\tilde{T}(x) = \tilde{T}(y)$ but $T(x) \neq T(y)$, (f cannot be well-defined).

By factorization theorem, $\mathbb{P}_\theta(x) = \tilde{g}_\theta(\tilde{T}(x))\tilde{h}(x)$. By (contradiction) assumption $\tilde{g}_\theta(\tilde{T}(x)) = \tilde{g}_\theta(\tilde{T}(y))$ hence $\mathbb{P}_\theta(y) = \tilde{g}_\theta(\tilde{T}(y))\tilde{h}(y)$.

$\tilde{h}(y)/\tilde{h}(x)$ serves as the proportionality constant showing $\mathbb{P}_\theta(x) \propto_\theta \mathbb{P}_\theta(y)$, so by theorem premise $\implies T(x) = T(y)$, a contradiction! □

Example 1.4 (Exponential family)**.** $\mathbb{P}_\theta(x) = e^{\eta(\theta)^\intercal T(x) - B(\theta)}h(x)$. Under what circumstances will $T(x)$ be minimal sufficient?

To apply theorem, need to check:

- Is $T(x)$ sufficient? Yes (apply factorization theorem)
- $\mathbb{P}_\theta(x) \propto_\theta \mathbb{P}_\theta(y) \implies T(x) = T(y)$? Considering the ratio of $\mathbb{P}_\theta(x)$ and $\mathbb{P}_\theta(y)$, this is true iff $\forall \theta \in \Theta$

$$e^{\eta(\theta)^\intercal T(x)} \propto_\theta e^{\eta(\theta)^\intercal T(y)} \tag{1.1}$$

$$\eta(\theta)^\intercal T(x) = \eta(\theta)^\intercal T(y) + c(x, y) \tag{1.2}$$

For a particular θ_1:

$$\eta(\theta_1)^\intercal T(x) = \eta(\theta_1)^\intercal T(y) + c(x, y) \tag{1.3}$$

Subtracting:

$$[\eta(\theta) - \eta(\theta_1)]^\intercal T(x) = [\eta(\theta) - \eta(\theta_1)]^\intercal T(y) \tag{1.4}$$

$$[\eta(\theta) - \eta(\theta_1)]^\intercal [T(x) - T(y)] = 0 \quad \forall \theta, \theta_1 \in \Theta \tag{1.5}$$

$$T(x) - T(y) \in \text{span}\{\eta(\theta) - \eta(\theta_1) : \theta, \theta_1 \in \Theta\}^\perp \tag{1.6}$$

- Since we need to show $T(x) - T(y) = 0$, we are done if $\text{span}\{\eta(\theta) - \eta(\theta_1) : \theta, \theta_1 \in \Theta\}^\perp = \{0\}$ i.e. $\text{span}\{\eta(\theta) - \eta(\theta_1) : \theta, \theta_1 \in \Theta\} = \mathbb{R}^s$

Definition 1.5. An exponential family is *full rank* if $\text{span}\{\eta(\theta) - \eta(\theta_1) : \theta, \theta_1 \in \Theta\} = \mathbb{R}^s$

Example 1.6 (Full rank)**.** Normal(θ, 1), $\theta \in \mathbb{R}$, $\mathbb{P}_\theta(x) = e^{-\frac{(x-\theta)^2}{2}}$, $\eta(\theta) = \theta$. Then $\{\eta(\theta) - \eta(\theta_1)\} \simeq \mathbb{R}$ so the exponential family is full rank.

Example 1.7. title $x_1, x_2, \cdots, x_n \overset{\text{iid}}{\sim} \mathbb{P}_\theta(x) = \frac{1}{2}e^{-|x-\theta|}$

The order statistics $X_{(1)} \leq \cdots \leq X_{(n)}$ are minimal sufficient.

2 Completeness

Definition 2.1. $T(X)$ is *complete* if $\forall$ functions f, $\exists$ constant c s.t.

$$[\forall \theta : \mathbb{E}_\theta f(T) = c] \implies f(\cdot) = c \text{a.e.} \tag{2.1}$$

Example 2.2. $X_i \overset{\text{iid}}{\sim} Unif(0, \theta), \theta \in (0, \infty)$

$T(X) = \max_i X_i$

$\mathbb{P}_\theta(T \leq t) = \left(\frac{t}{\theta}\right)^n \implies \mathbb{P}_\theta(t) = \frac{nt^{n-1}}{\theta^n}$

$\mathbb{E}_\theta f(T) = 0 \iff \int_0^\infty f(t)\frac{nt^{n-1}}{\theta^n}dt = 0$

"differentiate wrt θ" will show $f(\theta)\frac{n\theta^{n-1}}{\theta^n} = 0, \implies f(\theta) = 0$

This means that if f has $\mathbb{E}_\theta f(T) = 0$, then $f = 0$, so f is complete.

Proposition 2.3. *The sufficient statistic for a full rank exponential family is complete.*

Theorem 2.4. *If T is sufficient and complete, then T is minimal.*

Proof. Consider a minimal sufficient statistic $\tilde{T}$. Since any sufficient statistic can be mapped to a minimal $\tilde{T}$, it suffices to show $\exists g : g(\tilde{T}) = T$.

Define $g(\tilde{T}) := \mathbb{E}_\theta[T|\tilde{T}]$. Note that g is independent of θ because $\tilde{T}$ is sufficient. Applying the tower property for expectations

$$\mathbb{E}_\theta g(\tilde{T}) = \mathbb{E}_\theta T \tag{2.2}$$

Since $\tilde{T}$ is minimal, $\exists f : \tilde{T} = f(T)$ so

$$\mathbb{E}_\theta g(f(T)) = \mathbb{E}_\theta T \quad \forall \theta \tag{2.3}$$

$$\mathbb{E}_\theta\left[g(f(T)) - T\right] = 0 \quad \forall \theta \tag{2.4}$$

By completeness of T, we have $g(f(T)) = T$. $\square$