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STAT 205A
Lecture #25
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1 "Play Red"

Cards are shuffled and dealt out. Need to choose when to bet that next card is red (stopping time).

Finite S. $X_1, X_2, \dots, X_N$ uniform random ordering, clearly (finite) exchangeable sequence.

Lemma 1.1 (Last class). *If* $(Z_1, W) \overset{d}{=} Z(_2, W)$ *the* $\mathbb{E}[\phi(Z_1) \mid W] \overset{a.s.}{=} \mathbb{E}[\phi(Z_2) \mid W]$

Proposition 1.2. *If* $(X_1, \dots, X_N)$ *is exchangeable,* $0 \leq T \leq N-1$ *a stopping time wrt natural filtration of* (X_n)*, then* $X_{T+1} \overset{d}{=} X_1$.

Proof. By exchangeability

$$(X_{n+1}, X_1, \dots, X_n) \overset{d}{=} (X_N, X_1, \dots, X_n) \tag{1.1}$$

By **??**

$$P(X_{n+1} \in A \mid \mathcal{F}_n) \overset{\text{a.s.}}{=} P(X_N \in A \mid \mathcal{F}_n) \tag{1.2}$$

On $\{T = n\}$

$$P(X_{T+1} \in A \mid \mathcal{F}_T) \overset{\text{a.s.}}{=} P(X_N \in A \mid \mathcal{F}_T) \tag{1.3}$$

This holds for all n. Taking expectation

$$\mathbb{E}P(X_{T+1} \in A) = P(X_N \in A) \tag{1.4}$$

$$X_{T+1} \overset{d}{=} X_n \overset{d}{=} X_1 \tag{1.5}$$

□

2 de Finetti's Theorem

Parametric Bayes: Let A and $B > 0$ be RVs. Given $A = a$ and $B = b$, let $(X_i, 1 \leq i < \infty)$ be IID $\sim N(a, b)$.

Random Measure (more generally): Let $\mathcal{P}(\mathbb{R})$ be space of all PMs on $\mathbb{R}$, M a RV taking values in $\mathcal{P}(\mathbb{R})$. Given $M = \mu$, let $(X_i, i \geq 1)$ be IID $\sim \mu$.

This gives an infinite exchangeable sequence (X_i): de Finetti's is the converse.

Theorem 2.1 (de Finetti). *Let* $(X_i, 1 \leq i < \infty)$ *be exchangeable* $\mathbb{R}$*-valued. Let* τ *be tail* σ*-field. Then conditionally on* τ*, the* (X_i) *are IID. That is*

(a) $X_1, X_2, \dots$ *are conditionally independent given* τ

(b) $\exists$ *kernel* $Q(\omega, \cdot)$ *(random PM) such that* $Q(\omega, \cdot)$ *is a regular conditional distribution of* X_i *given* τ*, each* i

$$P(X_i \in A \mid \tau)(\omega) = Q(\omega, A) \quad \forall i \tag{2.1}$$

Proof of **??** *(Aldous' favorite, easier ones exist in textbooks.* Let $\phi : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. By exchangeability

$$(X_1, \dots, X_n) \stackrel{d}{=} (X_1, X_K, X_{K+1}, \dots, X_{n+K-1}) \tag{2.2}$$

Letting $n \to \infty$ and appealing to Kolmogorov's consistency theorem

$$(X_1, X_2, \dots) \stackrel{d}{=} (X_1, X_K, X_{K+1}, \dots) \tag{2.3}$$

The equality in distributions imply equality in expectation of the pushforward measures $\phi_* P_{X_1|X_2,X_2,\dots} = \phi_* P_{X_1|X_K,X_{K+1},\dots}$ hence

$$\mathbb{E}[\phi(X_1) \mid X_2, X_3, \dots] = \mathbb{E}[\phi(X_1) \mid X_K, X_{K+1}, \dots] \tag{2.4}$$

Applying reversed MG convergence

$$\mathbb{E}[\phi(X_1) \mid X_2, X_3, \dots] \stackrel{d}{=} \mathbb{E}[\phi(X_1) \mid \tau] \tag{2.5}$$

Exercise 2.2.

- If $\mathbb{E}[Z \mid \mathcal{G}] \stackrel{d}{=} Z$ then $\mathbb{E}[Z \mid \mathcal{G}] \stackrel{\text{a.s.}}{=} Z$.
- If $\mathcal{G} \subset \mathcal{H}$, $\mathbb{E}[Z \mid \mathcal{G}] \stackrel{d}{=} \mathbb{E}[Z \mid \mathcal{H}]$, then $\mathbb{E}[Z \mid \mathcal{G}] \stackrel{\text{a.s.}}{=} \mathbb{E}[Z \mid \mathcal{H}]$

By exercise

$$\mathbb{E}[\phi(X_1) \mid X_2, X_3, \dots] \stackrel{\text{a.s.}}{=} \mathbb{E}[\phi(X_1) \mid \tau] \tag{2.6}$$

The same argument applied $\forall k \geq 1$

$$\mathbb{E}[\phi(X_K) \mid X_{K+1}, X_{K+2}, \dots] \stackrel{\text{a.s.}}{=} \mathbb{E}[\phi(X_K) \mid \tau] \iff X_K \perp (X_{K+1}, X_{K+2}, \dots) \mid \tau \tag{2.7}$$

By exchangeability

$$(X_1, X_{i+1}, X_{i+2}, \dots) \stackrel{d}{=} (X_i, X_{i+1}, \dots) \tag{2.8}$$

Applying **??**

$$\mathbb{E}[\phi(X_1) \mid X_{i+1}, X_{i+2}, \dots] \stackrel{\text{a.s.}}{=} \mathbb{E}[\phi(X_i) \mid X_{i+1}, \dots] \tag{2.9}$$

Condition on τ

$$\mathbb{E}[\phi(X_1) \mid \tau] \stackrel{\text{a.s.}}{=} \mathbb{E}[\phi(X_i) \mid \tau] \tag{2.10}$$

Therefore, the conditional distributions of X_1 and X_i given τ coincide ($P_{X_1|\tau} = P_{X_i|\tau}$) $\square$

Proof of **??**. Recall the Glivenko-Cantelli theorem:

Theorem 2.3 (Glivenko-Cantelli). *Define $F(x_1, x_2, \ldots, x_n, t) = \frac{1}{n}\sum_{i=1}^{n} 1_{x_i \le t}$ the empirical distribution of $(x_1, \ldots x_n)$.*

If $(X_i, i \ge 1)$ are IID with distribution function F, then $F(X_1, \ldots, X_n, t) \overset{a.s.}{\to} F(t)$ for each t as $n \to \infty$.

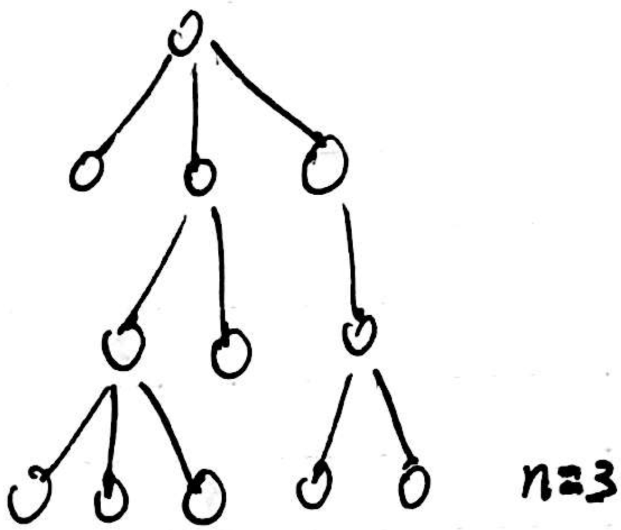
We can apply this theorem conditionally (given τ).
Given exchangeable $(X_i, 1 \le i < \infty)$, **??** implies

$$F(X_1, \ldots, X_n, t) \overset{\text{a.s.}}{\to} G(\omega, t) \tag{2.11}$$

which is the distribution function of $Q(\omega, \cdot)$. □

3 MGs in Galton-Watson processes

ξ takes values in $\{0, 1, 2, \ldots\}$.
Each individual in generation g has ξ offspring in generation $g+1$, ξ independent.
$Z_n = \#$ individuals in generation n
$Z_0 = 1$ as default



"extinction" = event $\{Z_n = 0 \text{ for some } n\}$
"survival" = event $\{Z_n \ge 1\ \forall n\}$.
Write $\mu = \mathbb{E}\xi < \infty$
Let $\mathcal{F} = \sigma(Z_1, Z_2, \ldots, Z_n)$.

$$\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = \mu Z_n \tag{3.1}$$
$$\implies \mathbb{E}Z_{n+1} = \mu \mathbb{E}Z_n \tag{3.2}$$
$$\implies \mathbb{E}Z_n = \mu^n \tag{3.3}$$

If $\mu < 1$, then $P(Z_n \ge 1) \le \mathbb{E}Z_n \le \mu^n \to 0$ so $P(\text{extinction}) = 1$.

Consider the case $\mu > 1$. **??** implies that $\left(\frac{Z_n}{\mu^n}, n \geq 0\right)$ is a MG. As $\mathbb{E}[\mu^{-n} Z_n] = 1 \leq \infty$, by MG convergence theorem

$$\mu^{-n} Z_n \overset{\text{a.s.}}{\to} \text{ some } W \geq 0, \quad \mathbb{E}W \leq 1 \tag{3.4}$$

Suppose $\mathbb{E}\xi^2 < \infty$. Will show $(\mu^{-n} Z_n, n \geq 1)$ is UI, which would mean that $\mu^{-n} Z_n \to W$ in L^1 and $\mathbb{E}W = 1$.

Clearly $\{\text{extinction}\} \subset \{W = 0\}$. Can prove $\{\text{extinction}\} \overset{\text{a.s.}}{=} \{W = 0\}$. So either extinction or Z_n grows exponentially fast.

Calculation $\mu^{-n} Z_n$ is UI:

$$\begin{aligned} \operatorname{Var}(Z_n) &= \mathbb{E}\operatorname{Var}(Z_n \mid \mathcal{F}_{n-1}) + \operatorname{Var}\mathbb{E}[Z_n \mid \mathcal{F}_{n-1}] && (3.5)\\ \implies \operatorname{Var}(\mu^{-n} Z_n) &= \mu^{-(n+1)}\operatorname{Var}(\xi) + \operatorname{Var}(\mu^{-(n-1)} Z_{n-1}) && (3.6)\\ \text{induction} \implies \operatorname{Var}(\mu^{-n} Z_n) &= \operatorname{Var}(\xi) \cdot \sum_{i=2}^{n+1} \mu^{-i} \leq K < \infty \ \forall n && (3.7)\\ \implies (\mu^{-n} Z_n, n \geq 1) &\text{ is UI} && (3.8) \end{aligned}$$

4 L^2 theory [see Durrett for more]

$(M_n, n \geq 0)$, $M_0 = 0$, $\Delta_n = M_n - M_{n-1}$. Suppose $\mathbb{E}M_n^2 < \infty$ for all n.

Orthogonality of increments: $\mathbb{E}[\Delta_i \Delta_j] = 0$, $i < j$, because

$$\mathbb{E}[\Delta_i \Delta_j \mid \mathcal{F}_{j-1}] = \Delta_i \mathbb{E}[\Delta_j \mid \mathcal{F}_{j-1}] = 0 \tag{4.1}$$

So $\mathbb{E}M_n^2 = \sum_{i=1}^n \mathbb{E}[\Delta_i^2]$.

Definition 4.1. $M - n$ is L^2 *bounded* if $\sup_n \mathbb{E}M_n^2 < \infty$, or equivalently $\sum_{i=1}^\infty \mathbb{E}[\Delta_i^2] < \infty$.

If (M_n) is L^2 bounded, then (L^1 convergence) $M_n \overset{\text{a.s.}}{\to} M_\infty$ and in L^1. In fact, also have $M_n \to M_\infty$ in L^2.

For $n_1 < n_2$, $\mathbb{E}[(M_{n_2} - M_{n_1})^2] = \sum_{i=n_1+1}^{n_2} \mathbb{E}[\Delta_i^2]$. If L^2-bounded

$$\lim_{n_1 \to \infty} \sup_{n_2 > n_1} \|M_{n_2} - M_{n_1}\|_2 = 0 \tag{4.2}$$

L^2 is a *complete* metric space, so

$$M_n \to \text{some } M_\infty \in L^2 \tag{4.3}$$