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STAT 205A
Lecture #14
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1 Last Class

Given PM μ on $S_1 \times S_2$:

- Exists PM μ_1 on S_1
- if S_2 Borel space, then there exists kernel Q from S_1 to S_2 such that (BR1)-(BR3) hold.

Interpretation: If $\mu = \text{dist}(X, Y)$ then $\mu_1 = \text{dist}(X)$, $Q(x, B) = P(Y \in B \mid X = x)$.

2 Product Measure

Given PM μ_1 on $(S_1, \mathcal{S}_1)$, μ_2 on $(S_2, \mathcal{S}_2)$, there exists a *product measure* $\mu = \mu_1 \otimes \mu_2$ on $S_1 \times S_2$ such that

(a) $\mu(A \times B) = \mu_1(A) \times \mu_2(B)$ for all $A \in \mathcal{S}_1, B \in \mathcal{S}_2$

(b) $D \in \mathcal{S}_1 \otimes \mathcal{S}_2$, $\mu(D) = \int \mu_2(D_{s_1})\mu_1(ds_1)$

Product measures also satisfy:

Theorem 2.1 (Fubini). *For measurable $h : S_1 \times S_2 \to \mathbb{R}$*

$$\int h(s_1, s_2)\mu(ds) = \int_{S_1} \int_{S_2} h(s_1, s_2)\mu_2(ds_2)\mu_1(ds_1) \tag{2.1}$$

provided $h \geq 0$ or $|h|$ is μ-integrable.

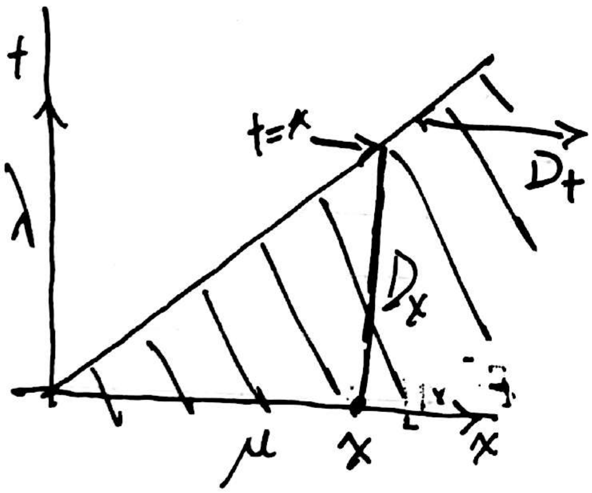
$\text{dist}(X, Y) = \mu_1 \otimes \mu_2 \iff X$ and Y are independent, $\text{dist}(X) = \mu_1$ and $\text{dist}(Y) = \mu_2$.

Remark 2.2. Fubini's theorem works for σ-finite measures. If $\lambda =$ Lebesgue measure on $\mathbb{R}^1$, then Fubini's theorem reads

$$\mathbb{E}h(X_1, X_2) = \mathbb{E}h_1(X_1) \text{ where } h_1(x_1) = \mathbb{E}h(x_1, X_2) \tag{2.2}$$

The general identity is (usually) best viewed as calculating the same quantity in 2 different ways.

Example 2.3. If $X \geq 0$ then $\mathbb{E}X = \int_0^\infty P(X \geq t)dt$.
$D = \{(x, t) : x \geq t\}$, $\mu = \text{dist}(X)$



$\lambda(D_x) = x$. $D_t = (t, \infty)$.

By Fubini

$$(\mu \times \lambda)(D) = \int \underbrace{\lambda(D_x)}_{=x} \mu(dx) = \mathbb{E}X \tag{2.3}$$

$$(\mu \times \lambda)(D) = \int \underbrace{\mu(t, \infty)}_{=P(X \geq t)} \lambda(dt) \tag{2.4}$$

Example 2.4. $j = 1, 2$. X_1, X_2 independent. $\mu_j = \text{dist}(X_j)$.

$\phi_j(t) = \mathbb{E}\exp(itX_j)$ for $t \in \mathbb{R}$ the *characteristic function* (probabilists) or *Fourier transform* (everyone else).

Parseval's identity refers to

$$\int \phi_2(t)\mu_1(dt) = \int \phi_1(t)\mu_2(dt) \tag{2.5}$$

To show this

$$\mathbb{E}\exp(iX_1X_2) = \mathbb{E}h_1(X_1) \tag{2.6}$$

$$h_1(x_1) = \mathbb{E}\exp(ix_1X_2) = \phi_2(x_1) \tag{2.7}$$

$$\implies \mathbb{E}\exp(iX_1X_2) = \mathbb{E}\phi_2(X_1) = \int \phi_2(t)\mu_1(dt) \tag{2.8}$$

Similarly

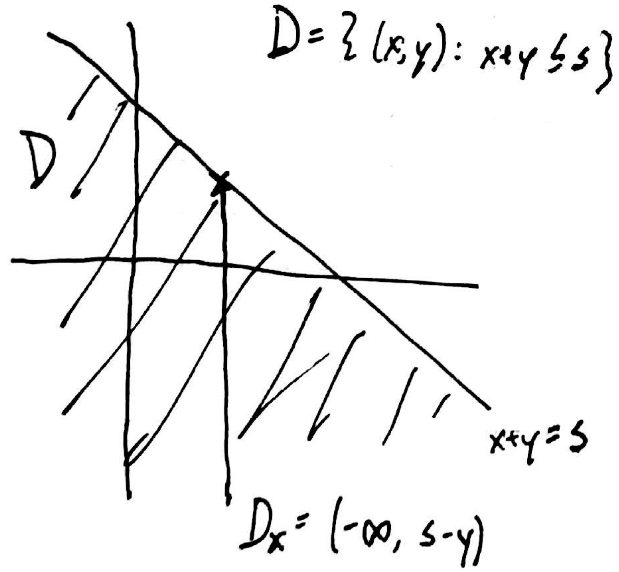
$$\mathbb{E}\exp(iX_1X_2) = \mathbb{E}h_2(X_2) \tag{2.9}$$

$$h_2(x_2) = \mathbb{E}\exp(iX_1x_2) = \phi_1(x_2) \tag{2.10}$$

$$\implies \mathbb{E}\exp(iX_1X_2) = \mathbb{E}\phi_1(X_2) = \int \phi_1(t)\mu_2(dt) \tag{2.11}$$

Example 2.5 (Convolution formula (Undergrad))**.** Suppose X and Y independent densities f_X, f_Y, distribution functions F_X, F_Y.

Then $S = X + Y$ has density $f(s) = \int_{-\infty}^{\infty} f_Y(s-x) f_X(x) dx$.



Example 2.6. No regularity assumptions.

$$P(S \le s) = \mu_x \otimes \mu_y(A) = \int \underbrace{F_Y(s-x)}_{=\mu_y(D_x)} \mu_x(d\mu) \tag{2.12}$$

Suppose μ_x has density $f_X \implies P(S \le s) = \int F_Y(s-x) f_X(x) dx$. Formally, $\frac{d}{dx}$ provided μ_Y has a density f_Y.

"change of variable" $\int (\cdot) \mu_x(dx) = \int (\cdot) f_X(x) dx$

2.1 Justifying identities involving differentiation by checking integral form

How to justify (?):

$$f_S(s) \overset{(?)}{=} \int f_Y(s-x) f_X(x) dx \tag{2.13}$$

Need to show

$$\int_{-\infty}^{s_0} \left(\int_{-\infty}^{\infty} f_Y(s-x) f_X(x) dx \right) ds = P(S \le s_0) \tag{2.14}$$

$$= \int \left(\int_{-\infty}^{s_0} f_Y(s-x) ds \right) \mu_X(dx) \tag{2.15}$$

$$= \int F_Y(s_0 - x) \mu(dx) \overset{(**)}{=} P(S \le s_0) \tag{2.16}$$

Example 2.7. Suppose (X,Y) has joint density $f(x,y)$, marginal $f_1(x)$.

Define $f(y \mid x) = f(x,y)/f_1(x)$.

Define a kernel Q by $Q(x,\cdot)$ is the PM with density $y \mapsto f(y \mid x)$.

Then this Q is the kernel in general theorem about $\mu = \mathrm{dist}(X,Y)$.

Need to verify (BR1):

$$P(X \in A, Y \in B) = \int_A Q(x,B)\mu_X(dx) \tag{2.17}$$

$$\text{Left} = \int\int 1_{x\in A}1_{y\in B} f(x,y)dxdy \tag{2.18}$$

$$\overset{f(y|x)=f(x,y)/f_1(x)}{=} \int\int 1_{x\in A}1_{y\in B} f(y \mid x)f_1(x)dxdy \tag{2.19}$$

$$\overset{\text{Fubini}}{=} \int\int 1_{x\in A}\left(\int 1_{y\in B} f(y \mid x)dy\right) f_1(x)dx \tag{2.20}$$

$$= \text{Right} \tag{2.21}$$

3 RVs and PMs

Know: $X = (\Omega, \mathcal{F}, P) \to (S, \mathcal{S})$ has distribution $\mu = \mathrm{dist}(X)$ a PM on $(S,\mathcal{S})$. "given μ_1, is there an X with $\mathrm{dist}(X) = \mu$?" Non-trivial "yes" answer.

Know: $\exists$ RV U with uniform distribution $[0,1]$

Know: For any PM μ on $\mathbb{R}$, the RV $X = F_\mu^{-1}(U)$ has $\mathrm{dist}(X) = \mu$

Know: Binary expansion $U = 0.b_1(U)b_2(U)b_3(u)\cdots$ gives infinite sequence of RBs $(b_i(U))_i$ independent $P(b_1(U)=0) = 1/2, P(b_1(U)=1) = 1/2$.

Definition 3.1. $(S,\mathcal{S})$ is a *Borel space* if there exists a Borel-measurable $A \subset \mathbb{R}$ and a bijection $\phi : A \to S$ such that ϕ and ϕ^{-1} are measurable.

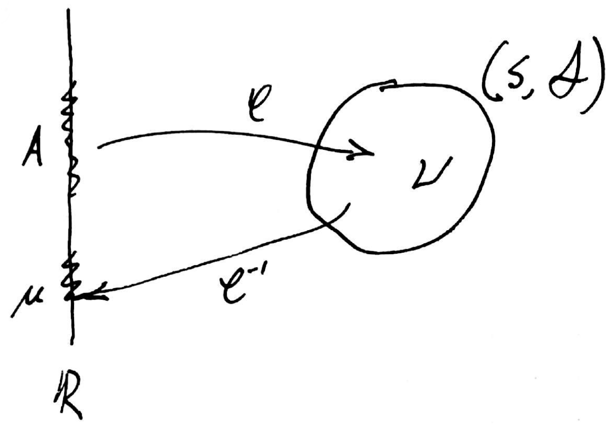
Remark 3.2. ϕ identity map on $(S_0, \mathcal{S}_1)$ to $(S_0, \mathcal{S}_2)$ is measurable iff $\mathcal{S}_2 \subset cS_1$. Same for ϕ^{-1} iff $\mathcal{S}_1 \subset \mathcal{S}_2$.

Both ϕ and ϕ^{-1} measurable $\iff$ $\mathcal{S}_1 = \mathcal{S}_2$.

Outsource to analysis:

Theorem 3.3. *Every complete separable metric space is a Borel space.*

Consider a PM ν on a Borel space $(S,\mathcal{S})$. Let μ be the PM on A, the push-forward of ν under ϕ^{-1}.



$X = F^{-1}(U)$ is a RV with dist $= \mu$. ν is the push-forward of ν under ϕ

$$\implies \phi(F_\mu^{-1}(U)) \text{ has distribution } \nu \tag{3.1}$$

Have proved:

Lemma 3.4. *Given a PM ν on a Borel space $(S, \mathcal{S})$, there exists measurable $h : [0,1] \to S$ such that $H(U)$ has distribution ν.*

Remark 3.5. $\pi_k = k$th prime number.

$$I^{(k)} = \{\pi_k, \pi_k^2, \cdots\} \text{ infinite set} \qquad I^{(1)}, I^{(2)}, \cdots \text{ disjoint} \tag{3.2}$$

Given sequence μ_k of PMs on $\mathbb{R}$, define $U_k = \sum_{i=1}^{\infty} 2^{-i} b_{\pi_k^i}(U)$. Then $U_k \sim \text{Uniform}[0,1]$, independent as k varies.