Prove Props. 2.10, 2.21, 2.33. Do Project 2.28. Prove Props. 2.18(iii) by induction, and 2.18.1(i, ii).

> **Prop. 2.10.**
> The equation $x^2 = -1$ has no solution in $\mathbb{Z}$.

Proof.
Let $m \in \mathbb{Z}$.
The proposition can be rewritten as:
There does not exist any m where $m^2 = -1$.

Assume there *does* exist a $m \in \mathbb{Z}$ where $m^2 = -1$.

Proposition 2.9 states that if $m \neq 0$, then $m^2 \in \mathbb{N}$.
We know $m \neq 0$ because for any $n \in \mathbb{Z}$, $n \cdot 0 = 0$, and $0 \neq -1$, therefore $m \neq 0$.
Therefore, proposition 2.9 implies that $-1 \in \mathbb{N}$.

However, we know $-1 \notin \mathbb{N}$.
Proof.
By **proposition 2.3**, we know $1 \in \mathbb{N}$.
$1 \in \mathbb{N}$ and $-1 \in \mathbb{N}$ cannot *both* be true, since **proposition 2.2** states for $m \in \mathbb{Z}$, only one of the following is true: $m \in \mathbb{N}, -m \in \mathbb{N}, m = 0$.
Therefore, $-1 \notin \mathbb{N}$.

By contradiction, there cannot exist a $m \in \mathbb{Z}$ where $m^2 = -1$.

> **Prop. 2.21.**
> There exists no integer x such that $0 < x < 1$.

Proof.
Let $k \in \mathbb{Z}$.
Assume there *does* exist an k where $0 < k < 1$.
We rewrite the assumption as follows:
There exists an k where $0 < k$ and $k < 1$
$\Rightarrow$ There exists an k where $(k - 0) \in \mathbb{N}$ and $(1 - k) \in \mathbb{N}$

$\Rightarrow$ There exists an k where $k \in \mathbb{N}$ and $(1-k) \in \mathbb{N}$.

Let A be the assumption (for which we will be proving by contradiction):
There exists an k where $k \in \mathbb{N}$ and $(1-k) \in \mathbb{N}$.

In order to prove assumption A is false, we can prove that there is *no* $k \in \mathbb{N}$ where $(1-k) \in \mathbb{N}$.
For this, we use induction - let $P(n)$ be the statement:
$(1-n) \notin \mathbb{N}$.

Prove $P(1)$.

- $(1-1) \notin \mathbb{N}$
- $0 \notin \mathbb{N}$

$0 \notin \mathbb{N}$ is true, by Axiom 2.1(iii).

Induction step.
We assume $P(n)$ is true:
$(1-k) \notin \mathbb{N}$.

Prove $P(n+1)$.

- $(1-(n+1)) \notin \mathbb{N}$
- $(1-n-1) \notin \mathbb{N}$
- $(-n) \notin \mathbb{N}$

$(-n) \notin \mathbb{N}$ is true, because $n \in \mathbb{N}$ by the induction axiom, and by **proposition 2.2**, if $n \in \mathbb{N}$, then $(-n) \notin \mathbb{N}$.

Therefore, since for all $k \in \mathbb{N}$, $(1-k) \notin \mathbb{N}$, and since the contrapositive of a true statement holds, there exists no $k \in \mathbb{N}$ where $(1-k) \in \mathbb{N}$.

Therefore, assumption A is false by contradiction. That is, there *does not* exist a k where $k \in \mathbb{N}$ and $(1-k) \in \mathbb{N}$, so there cannot be an $x \in \mathbb{Z}$ where $0 < x < 1$.

> **Prop. 2.33.**
> Let $A \subseteq \mathbb{Z}$, $A \neq \emptyset$. Let $b \in \mathbb{Z}$ such that $(\forall a \in A)\ b \leq a$. Then A has a smallest element.

The well-ordering principle states that every non-empty subset of $\mathbb{N}$ has a smallest element.

uh...

`UNFINISHED`

> **Project 2.28.**
> Determine for which natural numbers $k^2 - 3k \geq 4$. Prove it.

Reduce the original expression:

$k(k-3) \geq 4$

`UNFINISHED`

> **Prop. 2.18(iii).**
> $\forall k \in \mathbb{N},\ k^3 + 5k$ is divisible by 6.

Proof.

m is divisible by n if there exists a $j \in \mathbb{Z}$ such that $m = jn$.

$P(k) = k^3 + 5k$ is divisible by 6.

Prove $P(1)$.

$(1)^3 + 5(1) = 1 + 5 = 6$.

6 is divisible by 6, therefore $P(1)$ is true.

Assume $P(k)$ is true for some $k \in \mathbb{N}$:

There exists a $n \in \mathbb{Z}$ such that $k^3 + 5k = 6n$

Prove $P(k+1)$.

Show there exists a $m \in \mathbb{Z}$ where $(k+1)^3 + 5(k+1) = 6n$.

We rewrite the left-hand side:

$(k+1)^3 + 5(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 5k + 5$

$= (k^3 + 5k) + 3k^2 + 3k + 6$

$= 6n + 3k^2 + 3k + 6$

$= 6n + 3(k^2 + k) + 6$

We now prove that $3(k^2 + k)$ is divisible by 6.

Intermediate proposition: If $n \in \mathbb{Z}$ is divisible by 2, then $3n$ is divisible by 6.

Proof.

If $n \in \mathbb{Z}$ is divisible by 2, then there exists a $l \in \mathbb{Z}$ where $n = 2l$.
We multiply both sides by 3 to obtain $3n = 6l$, and since this follows the definition of divisibility, $3n$ is divisible by 6.
Therefore, $3(k^2 + k)$ is divisible by 6, that is, there exists a $d \in \mathbb{Z}$ where $6d = 3(k^2 + k)$.

We now rewrite the left-hand side as:
$6n + 6d + 6 = 6(n + d + 1)$

We let $q = (n + d + 1) \in \mathbb{Z}$.
$(k + 1)^3 + 5(k + 1) = 6q$ where $q \in \mathbb{Z}$, therefore $P(k + 1)$ is true.

Prop 2.18(iii) proved by mathematical induction.

Let $n \in \mathbb{N}$ for the following propositions.

> **Prop 2.18.1(i).**
> $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof.
Let $P(n)$ be the statement:
$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Prove $P(1)$.

- $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$
- $1 = \frac{1(2)}{2}$
- $1 = 1$

Therefore $P(1)$ is true.

Induction step.
Assume $P(n)$ is true.

Prove $P(n + 1)$.

- $\sum_{i=1}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$
- $\sum_{i=1}^{n} i + (n + 1) = \frac{(n+1)(n+2)}{2}$
- $\frac{n(n+1)}{2} + (n + 1) = \frac{(n+1)(n+2)}{2}$
- $n(n + 1) + 2(n + 1) = (n + 1)(n + 2)$

- $n^2 + n + 2n + 2 = n^2 + 2n + n + 2$
- $n^2 + 3n + 2 = n^2 + 3n + 2$

Therefore $P(n+1)$ is true.

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by mathematical induction.

> **Prop 2.18.1(ii).**
> $2 \mid n(n+1)$

Proof.
We rewrite the proposition:
There exists $z \in \mathbb{Z}$ where $n(n+1) = 2z$

By **proposition 2.18.1(i)**, $\frac{n(n+1)}{2} \in \mathbb{N}$ because the sum of natural numbers is in $\mathbb{N}$.
If we let $z = \frac{n(n+1)}{2}$, then multiplying both sides by 2 results in $2z = n(n+1)$.
Since there exists a $z \in \mathbb{Z}$ where $n(n+1) = 2z$, $n(n+1)$ is divisible by 2.