Prove Prop. 2.18(ii) by induction. (F 2/14)

> **Prop. 2.18(ii).**
> $\forall k \in \mathbb{N},\ k^4 - 6k^3 + 11k^2 - 6k$ is divisible by 4.

Proof.
Definition of divisibility:
m is divisible by n if there exists a $j \in \mathbb{Z}$ such that $m = jn$.

Let $P(k)$ be the statement: There exists a $z \in \mathbb{Z}$ such that $k^4 - 6k^3 + 11k^2 - 6k = 4z$.

Prove $P(1)$.
$1^4 - 6(1)^3 + 11(1)^2 - 6(1) = 4z$.
The left-hand side can be rewritten as:
$1 - 6 + 11 - 6 = 0$
To obtain:
$0 = 4z$. We can set $z = 0$, therefore $P(1)$ is true.

Induction step:
Assume $P(k)$ is true:
$k^4 - 6k^3 + 11k^2 - 6k = 4z$, where $z \in \mathbb{Z}$.

Prove $P(k+1)$.
Prove there exists a $m \in \mathbb{Z}$ such that $(k+1)^4 - 6(k+1)^3 + 11(k+1)^2 - 6(k+1) = 4m$.
$(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$
$(k+1)^3 = k^3 + 3k^2 + 3k + 1$
$(k+1)^2 = k^2 + 2k + 1$

The left-hand side can be rewritten as:

$$(k^4 + 4k^3 + 6k^2 + 4k + 1) - 6(k^3 + 3k^2 + 3k + 1) + 11(k^2 + 2k + 1) - 6(k+1)$$
$$= k^4 + 4k^3 + 6k^2 + 4k + 1 - 6k^3 - 18k^2 - 18k - 6 + 11k^2 + 22k + 11 - 6k - 6$$
$$= k^4 - 2k^3 - k^2 + 2k$$
$$= (k^4 - 2k^3 - k^2 + 2k) + (4z - (k^4 - 6k^3 + 11k^2 - 6k))$$
$$= (k^4 - 2k^3 - k^2 + 2k) - (k^4 - 6k^3 + 11k^2 - 6k) + 4z$$

$$
\begin{aligned}
&= 4k^3 - 12k^2 + 8k + 4z \\
&= 4(k^3 - 3k^2 + 2k + z)
\end{aligned}
$$

We can set $l = (k^3 - 3k^2 + 2k + z) \in \mathbb{Z}$, therefore $P(k)$ is true by mathematical induction.