11.8. A subset of the plane is called a *convex region* provided that, given any two points in the region, every point on the line segment is also in that region.

a. Rewrite the definition of *convex region* using quantifiers. We suggest you use the letter R to stand for the region and the notation $L(a,b)$ to stand for the line segment whose endpoints are a and b. Your answer should use three $\forall$ quantifiers.

b. Using quantifier notation, write what it means for a region not to be convex. Your answer should use three $\exists$ quantifiers.

c. Rewrite your answer to (b) in English and without using notation. (At the beginning of this exercise, we defined what it means for a region to be convex purely with words. Here, you are asked to explain what it means for a region *not* to be convex purely with words.)

d. Illustrate your answer to (b) [and (c)] with a suitably labeled diagram.

12 Sets II: Operations

Just as numbers can be added or multiplied, and truth values can be combined with $\wedge$ and $\vee$, there are various operations we perform on sets. In this section, we discuss several set operations.

Union and Intersection

The most basic set operations are *union* and *intersection*.

Definition 12.1 **(Union and intersection)** Let A and B be sets.

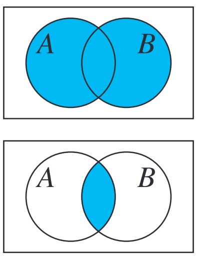
The *union* of A and B is the set of all elements that are in A or B (or both). The union of A and B is denoted $A \cup B$.

The *intersection* of A and B is the set of all elements that are in both A and B. The intersection of A and B is denoted $A \cap B$.

In symbols, we can write this as follows:

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \quad \text{and} \quad A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Example 12.2 Suppose $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$. Then $A \cup B = \{1,2,3,4,5,6\}$ and $A \cap B = \{3,4\}$.



It is useful to have a mental picture of union and intersection. A *Venn diagram* depicts sets as circles or other shapes. In the figure, the shaded region in the upper diagram is $A \cup B$, and the shaded region in the lower diagram is $A \cap B$.

The operations of $\cup$ and $\cap$ obey various algebraic properties. We list some of them here.

Theorem 12.3 Let A, B, and C denote sets. The following are true:

- $A \cup B = B \cup A$ and $A \cap B = B \cap A$. (Commutative properties)
- $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$. (Associative properties)
- $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. (Distributive properties)

Proof. Most of the proof is left as Exercise 12.5. Theorem 7.2 is extremely useful in proving this result.

Here we prove the associative property for union. You may use this as a template for proving the other parts of this theorem.

Let A, B, and C be sets. We have the following:

$$\begin{aligned} A \cup (B \cup C) &= \{x : (x \in A) \vee (x \in B \cup C)\} && \text{definition of union} \\ &= \{x : (x \in A) \vee ((x \in B) \vee (x \in C))\} && \text{definition of union} \\ &= \{x : ((x \in A) \vee (x \in B)) \vee (x \in C)\} && \text{associative property of } \vee \\ &= \{x : (x \in A \cup B) \vee (x \in C)\} && \text{definition of union} \\ &= (A \cup B) \cup C && \text{definition of union.} \end{aligned}$$

■

How did we think up this proof? We used the technique of writing the beginning and end of the proof and working toward the middle. Imagine a long sheet of paper. On the left, we write $A \cup (B \cup C) = \ldots$; on the right, we write $\ldots = (A \cup B) \cup C$. On the left, we unravel the definition of $\cup$ for the first $\cup$, obtaining $A \cup (B \cup C) = \{x : (x \in A) \vee (x \in B \cup C)\}$. We unravel the definition of $\cup$ again (this time on the $B \cup C$) to transform the set into

$$\{x : (x \in A) \vee ((x \in B) \vee (x \in C))\}.$$

Meanwhile, we do the same thing on the right. We unravel the second $\cup$ in $(A \cup B) \cup C$ to yield $\{x : (x \in A \cup B) \vee (x \in C)\}$ and then unravel $A \cup B$ to get $\{x : ((x \in A) \vee (x \in B)) \vee (x \in C)\}$.

Now we ask: What do we have and what do we want? On the left, we have

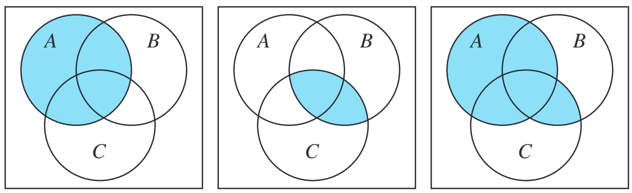
$$\{x : (x \in A) \vee ((x \in B) \vee (x \in C))\}$$

and on the right, we need

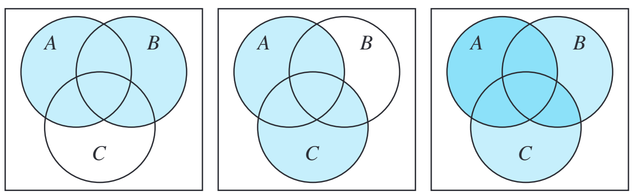
$$\{x : ((x \in A) \vee (x \in B)) \vee (x \in C)\}.$$

Finally, we stare at these two sets and realize that the conditions after the colon are logically equivalent (by Theorem 7.2) and we have our proof.

Venn diagrams are also useful for visualizing why these properties hold. For example, the following diagrams illustrate the distributive property $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.



A $B \cap C$ $A \cup (B \cap C)$



$A \cup B$ $A \cup C$ $(A \cup B) \cap (A \cup C)$

First examine the top row of pictures. On the left, we see the set A highlighted; in the center, the region for $B \cap C$ is shaded; and finally, on the right, we show $A \cup (B \cap C)$.

Next examine the bottom row. The left and center pictures show $A \cup B$ and $A \cup C$ highlighted, respectively. The rightmost picture superimposes the first two, and the darkened region shows $(A \cup B) \cap (A \cup C)$.

Notice that exactly the same two shapes on the right panels (top and bottom) are dark, illustrating that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

The Size of a Union

Suppose A and B are finite sets. There is a simple relationship between the quantities $|A|$, $|B|$, $|A \cup B|$, and $|A \cap B|$.

Proposition 12.4 Let A and B be finite sets. Then

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

Proof. Imagine we assign labels to every object. We attach a label A to objects in the set A, and we attach a label B to objects in B.

Question: How many labels have we assigned?

On the one hand, the answer to this question is $|A| + |B|$ because we assign $|A|$ labels to the objects in A and $|B|$ labels to the objects in B.

On the other hand, we have assigned at least one label to the elements in $|A \cup B|$. So $|A \cup B|$ counts the number of objects that get at least one label. Elements in $A \cap B$ receive two labels. Thus $|A \cup B| + |A \cap B|$ counts all elements that receive a label and double counts those elements that receive two labels. This gives the number of labels.

Since these two quantities, $|A| + |B|$ and $|A \cup B| + |A \cap B|$, answer the same question, they must be equal. ■

This proof is an example of a *combinatorial proof*. Typically a combinatorial proof is used to demonstrate that an equation (such as the one in Proposition 12.4) is true. We do this by creating a question and then arguing that both sides of the equation give a correct answer to the question. It then follows, since both sides are correct answers, that the two sides of the alleged equation must be equal. This technique is summarized in Proof Template 9.

Proof Template 9 Combinatorial proof.

To prove an equation of the form LHS = RHS:

Pose a question of the form, "In how many ways . . . ?"

On the one hand, argue why LHS is a correct answer to the question.

On the other hand, argue why RHS is a correct answer.

Therefore LHS = RHS. ■

Finding the correct question to ask can be difficult. Writing combinatorial proofs is akin to playing the television game *Jeopardy!*. You are given the answer (indeed, two answers) to a counting question; your job is to find a question whose answers are the two sides of the equation you are trying to prove.

We shall do more combinatorial proofs, but for now, let us return to Proposition 12.4. One useful way to rewrite this result is as follows:

Basic inclusion-exclusion.

$$|A \cup B| = |A| + |B| - |A \cap B|. \tag{4}$$

This is a special case of a counting method called *inclusion-exclusion*. It can be interpreted as follows: Suppose we want to count the number of things that have one property or another. Imagine that set A contains those things that have the one property and set B contains those that have the other. Then the set $A \cup B$ contains those things that have one property or the other, and we can count those things by calculating $|A| + |B| - |A \cap B|$. This is useful when calculating $|A|$, $|B|$, and $|A \cap B|$ is easier than calculating $|A \cup B|$. We develop the concept of inclusion-exclusion more extensively in Section 19.

Example 12.5 How many integers in the range 1 to 1000 (inclusive) are divisible by 2 or by 5?

Let

$$A = \{x \in \mathbb{Z} : 1 \le x \le 1000 \text{ and } 2|x\} \qquad \text{and}$$
$$B = \{x \in \mathbb{Z} : 1 \le x \le 1000 \text{ and } 5|x\}.$$

The problem asks for $|A \cup B|$.

It is not hard to see that $|A| = 500$ and $|B| = 200$. Now $A \cap B$ are those numbers (in the range from 1 to 1000) that are divisible by both 2 and 5. Now an integer is divisible by both 2 and 5 if and only if it is divisible by 10 (this can be shown rigorously using ideas developed in Section 39; see Exercise 39.3), so

$$A \cap B = \{x \in \mathbb{Z} : 1 \le x \le 1000 \text{ and } 10|x\}$$

and it follows that $|A \cap B| = 100$. Finally, we have

$$|A \cup B| = |A| + |B| - |A \cap B| = 500 + 200 - 100 = 600.$$

There are 600 integers in the range 1 to 1000 that are divisible by 2 or by 5.

In case $A \cap B = \emptyset$, Equation (4) simplifies to $|A \cup B| = |A| + |B|$. In words, if two sets have no elements in common, then the size of their union equals the sum of their sizes. There is a special term for sets with no elements in common.

Definition 12.6 **(Disjoint, pairwise disjoint)** Let A and B be sets. We call A and B *disjoint* provided $A \cap B = \emptyset$.

Let $A_1, A_2, \ldots, A_n$ be a collection of sets. These sets are called *pairwise disjoint* provided $A_i \cap A_j = \emptyset$ whenever $i \ne j$. In other words, they are pairwise disjoint provided no two of them have an element in common.

Example 12.7 Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, and $C = \{7, 8, 9\}$. These sets are pairwise disjoint because $A \cap B = A \cap C = B \cap C = \emptyset$.

However, let $X = \{1, 2, 3\}$, $Y = \{4, 5, 6, 7\}$, and $Z = \{7, 8, 9, 10\}$. This collection of sets is not pairwise disjoint because $Y \cap Z \ne \emptyset$ (all other pairwise intersections are empty).

Corollary 12.8 **(Addition Principle)** Let A and B be finite sets. If A and B are disjoint, then $|A \cup B| = |A| + |B|$.

Corollary 12.8 follows immediately from Proposition 12.4. There is an extension of the Addition Principle to more than two sets.

If $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

This can be shown formally using the methods from Section 21; see Exercise 21.11.

A fancy way to write this is

$$\left| \bigcup_{k=1}^{n} A_k \right| = \sum_{k=1}^{n} |A_k|.$$

The big $\bigcup$ is analogous to the $\sum$ and $\prod$ symbols. It means, as k goes from 1 to n (the lower and upper values), take the union of the expression to the right (in this case A_k). So the big $\bigcup$ notation is just a shorthand for $A_1 \cup A_2 \cup \cdots \cup A_n$. This is surrounded by vertical bars, so we want the size of that set. On the right, we see an ordinary summation symbol telling us to add up the cardinalities of $A_1, A_2, \ldots, A_n$.

Difference and Symmetric Difference

Definition 12.9 **(Set difference)** Let A and B be sets. The *set difference*, $A - B$, is the set of all elements of A that are not in B:

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

The *symmetric difference* of A and B, denoted $A \mathbin{\Delta} B$, is the set of all elements in A but not B or in B but not A. That is,

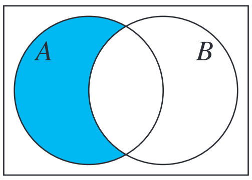
$$A \mathbin{\Delta} B = (A - B) \cup (B - A).$$

Example 12.10 Suppose $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Then $A - B = \{1, 2\}$, $B - A = \{5, 6\}$, and $A \mathbin{\Delta} B = \{1, 2, 5, 6\}$.

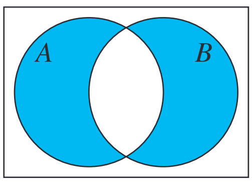
The figures show Venn diagram for these operations.
In general, the sets $A - B$ and $B - A$ are different (but see Exercise 12.18).
Here is another way to express symmetric difference:

Proposition 12.11 Let A and B be sets. Then

$$A \mathbin{\Delta} B = (A \cup B) - (A \cap B).$$



$A - B$



$A \mathbin{\Delta} B$

Let us illustrate the various proof techniques by developing the proof of Proposition 12.11 step by step. The proposition asks us to prove that two sets are equal, namely, $A \mathbin{\Delta} B$ and $(A \cup B) - (A \cap B)$. We use Proof Template 5 to form the skeleton of the proof.

Let A and B be sets.
(1) Suppose $x \in A \mathbin{\Delta} B$. ... Therefore $x \in (A \cup B) - (A \cap B)$.
(2) Suppose $x \in (A \cup B) - (A \cap B)$. ... Therefore $x \in A \mathbin{\Delta} B$.
Therefore $A \mathbin{\Delta} B = (A \cup B) - (A \cap B)$. ■

We begin with part (1) of the proof. We unravel definitions from both ends. We know that $x \in A \mathbin{\Delta} B$. By definition of Δ, this means $x \in (A - B) \cup (B - A)$. The proof now reads as follows:

Let A and B be sets.

(1) Suppose $x \in A \mathbin{\Delta} B$. Thus $x \in (A - B) \cup (B - A)$. ... Therefore $x \in (A \cup B) - (A \cap B)$.

(2) Suppose $x \in (A \cup B) - (A \cap B)$. ... Therefore $x \in A \mathbin{\Delta} B$.
Therefore $A \mathbin{\Delta} B = (A \cup B) - (A \cap B)$. ■

Now we know that $x \in (A - B) \cup (B - A)$. What does this mean? By definition of union, it means that $x \in (A - B)$ or $x \in (B - A)$. We have to consider both possibilities since we don't know in which of these sets x lies. This means that part (1) of the proof breaks into cases depending on whether $x \in A - B$ or $x \in B - A$. In both cases, we need to show that $x \in (A \cup B) - (A \cap B)$.

> Let A and B be sets.
>
> (1) Suppose $x \in A \,\Delta\, B$. Thus $x \in (A - B) \cup (B - A)$. This means either $x \in A - B$ or $x \in B - A$. We consider both cases.
>
> - Suppose $x \in A - B$. ... Therefore $x \in (A \cup B) - (A \cap B)$.
> - Suppose $x \in B - A$. ... Therefore $x \in (A \cup B) - (A \cap B)$.
>
> ... Therefore $x \in (A \cup B) - (A \cap B)$.
>
> (2) Suppose $x \in (A \cup B) - (A \cap B)$. ... Therefore $x \in A \,\Delta\, B$.
> Therefore $A \,\Delta\, B = (A \cup B) - (A \cap B)$. ■

Let's focus on the first case, $x \in A - B$. This means that $x \in A$ and $x \notin B$. We put that in.

> Let A and B be sets.
>
> (1) Suppose $x \in A \,\Delta\, B$. Thus $x \in (A - B) \cup (B - A)$. This means either $x \in A - B$ or $x \in B - A$. We consider both cases.
>
> - Suppose $x \in A - B$. So $x \in A$ and $x \notin B$. ... Therefore $x \in (A \cup B) - (A \cap B)$.
> - Suppose $x \in B - A$. ... Therefore $x \in (A \cup B) - (A \cap B)$.
>
> ... Therefore $x \in (A \cup B) - (A \cap B)$.
>
> (2) Suppose $x \in (A \cup B) - (A \cap B)$. ... Therefore $x \in A \,\Delta\, B$.
> Therefore $A \,\Delta\, B = (A \cup B) - (A \cap B)$. ■

We appear to be stuck. We have unraveled definitions down to $x \in A$ and $x \notin B$. To proceed, we work backward from our goal; we want to show that $x \in (A \cup B) - (A \cap B)$. To do that, we need to show that $x \in A \cup B$ and $x \notin A \cap B$. We add this language to the proof.

> Let A and B be sets.
>
> (1) Suppose $x \in A \,\Delta\, B$. Thus $x \in (A - B) \cup (B - A)$. This means either $x \in A - B$ or $x \in B - A$. We consider both cases.
>
> - Suppose $x \in A - B$. So $x \in A$ and $x \notin B$. ...
>   Thus $x \in A \cup B$, but $x \notin A \cap B$. Therefore $x \in (A \cup B) - (A \cap B)$.
> - Suppose $x \in B - A$. ... Therefore $x \in (A \cup B) - (A \cap B)$.
>
> ... Therefore $x \in (A \cup B) - (A \cap B)$.
>
> (2) Suppose $x \in (A \cup B) - (A \cap B)$. ... Therefore $x \in A \,\Delta\, B$.
> Therefore $A \,\Delta\, B = (A \cup B) - (A \cap B)$. ■

Now the two parts of this proof are moving closer together. Let's record what we know and what we want.

We already know:	$x \in A$ and $x \notin B$.
We want to show:	$x \in A \cup B$ and $x \notin A \cap B$.

The gap is now easy to close! Since we know $x \in A$, certainly x is in A or B (we just said it's in A!), so $x \in A \cup B$. Since $x \notin B$, x is not in both A and B (we just said it's not in B!), so $x \notin A \cap B$. We add this to the proof.

> Let A and B be sets.
>
> (1) Suppose $x \in A \,\Delta\, B$. Thus $x \in (A - B) \cup (B - A)$. This means either $x \in A - B$ or $x \in B - A$. We consider both cases.
>
> - Suppose $x \in A - B$. So $x \in A$ and $x \notin B$.
>   Since $x \in A$, we have $x \in A \cup B$. Since $x \notin B$, we have $x \notin A \cap B$.
>   Thus $x \in A \cup B$, but $x \notin A \cap B$. Therefore $x \in (A \cup B) - (A \cap B)$.
> - Suppose $x \in B - A$. ... Therefore $x \in (A \cup B) - (A \cap B)$.
>
> ... Therefore $x \in (A \cup B) - (A \cap B)$.
>
> (2) Suppose $x \in (A \cup B) - (A \cap B)$. ... Therefore $x \in A \,\Delta\, B$.
> Therefore $A \,\Delta\, B = (A \cup B) - (A \cap B)$. ■

We can now return to the second case of part (1) of the proof: "Suppose $x \in B - A$. ... Therefore $x \in (A \cup B) - (A \cap B)$." We have good news! This case looks just like the previous case, except we have A and B switched around. The argument in this case is going to proceed exactly as before. Since the steps are (essentially) the same, we don't really have to write them out. (If you are not 100% certain that the steps in this second case are exactly the same as before, I urge you to write out this portion of the proof for yourself using the previous case as a guide.) We can now complete part (1) of the proof.

> Let A and B be sets.
> (1) Suppose $x \in A \,\Delta\, B$. Thus $x \in (A - B) \cup (B - A)$. This means either $x \in A - B$ or $x \in B - A$. We consider both cases.
>
> - Suppose $x \in A - B$. So $x \in A$ and $x \notin B$. Since $x \in A$, we have $x \in A \cup B$. Since $x \notin B$, we have $x \notin A \cap B$. Thus $x \in A \cup B$, but $x \notin A \cap B$. Therefore $x \in (A \cup B) - (A \cap B)$.
> - Suppose $x \in B - A$. By the same argument as above, we have $x \in (A \cup B) - (A \cap B)$.
>
> Therefore $x \in (A \cup B) - (A \cap B)$.
> (2) Suppose $x \in (A \cup B) - (A \cap B)$. ... Therefore $x \in A \,\Delta\, B$.
> Therefore $A \,\Delta\, B = (A \cup B) - (A \cap B)$. ■

Now we are ready to work on part (2). We begin by unraveling $x \in (A \cup B) - (A \cap B)$. This means that $x \in A \cup B$, but $x \notin A \cap B$ (by the definition of set difference).

> Let A and B be sets.
> (1) Suppose $x \in A \,\Delta\, B$. Thus $x \in (A - B) \cup (B - A)$. This means either $x \in A - B$ or $x \in B - A$. We consider both cases.
>
> - Suppose $x \in A - B$. So $x \in A$ and $x \notin B$. Since $x \in A$, we have $x \in A \cup B$. Since $x \notin B$, we have $x \notin A \cap B$. Thus $x \in A \cup B$, but $x \notin A \cap B$. Therefore $x \in (A \cup B) - (A \cap B)$.
> - Suppose $x \in B - A$. By the same argument as above, we have $x \in (A \cup B) - (A \cap B)$.
>
> Therefore $x \in (A \cup B) - (A \cap B)$.
> (2) Suppose $x \in (A \cup B) - (A \cap B)$. Thus $x \in A \cup B$ and $x \notin A \cap B$. ... Therefore $x \in A \,\Delta\, B$.
> Therefore $A \,\Delta\, B = (A \cup B) - (A \cap B)$. ■

Now let's work backward from the end of part (2). We want to show $x \in A \,\Delta\, B$, so we need to show $x \in (A - B) \cup (B - A)$.

Let A and B be sets.

(1) Suppose $x \in A \Delta B$. Thus $x \in (A-B) \cup (B-A)$. This means either $x \in A-B$ or $x \in B-A$. We consider both cases.

- Suppose $x \in A-B$. So $x \in A$ and $x \notin B$. Since $x \in A$, we have $x \in A \cup B$. Since $x \notin B$, we have $x \notin A \cap B$. Thus $x \in A \cup B$, but $x \notin A \cap B$. Therefore $x \in (A \cup B) - (A \cap B)$.
- Suppose $x \in B-A$. By the same argument as above, we have $x \in (A \cup B) - (A \cap B)$.

Therefore $x \in (A \cup B) - (A \cap B)$.

(2) Suppose $x \in (A \cup B) - (A \cap B)$. Thus $x \in A \cup B$ and $x \notin A \cap B$. ... So $x \in (A-B) \cup (B-A)$. Therefore $x \in A \Delta B$.

Therefore $A \Delta B = (A \cup B) - (A \cap B)$. ■

To show $x \in (A-B) \cup (B-A)$, we need to show that either $x \in A-B$ or $x \in B-A$. Let's pause and write down what we know and what we want.

We already know:	$x \in A \cup B$ and $x \notin A \cap B$.
We want to show:	$x \in A-B$ or $x \in B-A$.

What we know says: x is in A or B but not both. In other words, either x is in A, in which case it's not in B, or x is in B, in which case it's not in A. In other words, $x \in A-B$ or $x \in B-A$, and that's what we want to show! Let's work this into the proof.

Let A and B be sets.

(1) Suppose $x \in A \Delta B$. Thus $x \in (A-B) \cup (B-A)$. This means either $x \in A-B$ or $x \in B-A$. We consider both cases.

- Suppose $x \in A-B$. So $x \in A$ and $x \notin B$. Since $x \in A$, we have $x \in A \cup B$. Since $x \notin B$, we have $x \notin A \cap B$. Thus $x \in A \cup B$, but $x \notin A \cap B$. Therefore $x \in (A \cup B) - (A \cap B)$.
- Suppose $x \in B-A$. By the same argument as above, we have $x \in (A \cup B) - (A \cap B)$.

Therefore $x \in (A \cup B) - (A \cap B)$.

(2) Suppose $x \in (A \cup B) - (A \cap B)$. Thus $x \in A \cup B$ and $x \notin A \cap B$.

This means that x is in A or B but not both. Thus either x is in A but not B or x is in B but not A. That is, $x \in (A-B)$ or $x \in (B-A)$.

So $x \in (A-B) \cup (B-A)$. Therefore $x \in A \Delta B$.

Therefore $A \Delta B = (A \cup B) - (A \cap B)$. ■

And this completes the proof.

More properties of difference and symmetric difference are developed in the exercises. One particularly worthwhile result, however, is the following:

Proposition 12.12 **(DeMorgan's Laws)** Let A, B, and C be sets. Then

$$A-(B \cup C) = (A-B) \cap (A-C) \quad \text{and} \quad A-(B \cap C) = (A-B) \cup (A-C).$$

The proof is left to you (Exercise 12.19).

Cartesian Product

We close this section with one more set operation.

Definition 12.13 **(Cartesian product)** Let A and B be sets. The *Cartesian product* of A and B, denoted $A \times B$, is the set of all ordered pairs (two-element lists) formed by taking an element from A together with an element from B in all possible ways. That is,

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Example 12.14 Suppose $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Then

$$A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5)\} \quad \text{and}$$
$$B \times A = \{(3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3)\}.$$

Notice that for the sets in Example 12.14, $A \times B \neq B \times A$, so Cartesian product of sets is not a commutative operation.

In what sense does Cartesian product "multiply" the sets? Why do we use a times sign $\times$ to denote this operation? Notice, in the example, that the two sets both had three elements, and their product had $3 \times 3 = 9$ elements. In general, we have the following:

Proposition 12.15 Let A and B be finite sets. Then $|A \times B| = |A| \times |B|$.

The proof is left for Exercise 12.29.

Recap

In this section we discussed the following set operations:

- Union: $A \cup B$ is the set of all elements in A or B (or both).
- Intersection: $A \cap B$ is the set of all elements in both A and B.
- Set difference: $A - B$ is the set of all elements in A but not B.
- Symmetric difference: $A \,\Delta\, B$ is the set of all elements in A or B, but not both.
- Cartesian product: $A \times B$ is the set of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

12 Exercises

12.1. Let $A = \{1, 2, 3, 4, 5\}$ and let $B = \{4, 5, 6, 7\}$. Please compute:
 a. $A \cup B$.
 b. $A \cap B$.
 c. $A - B$.
 d. $B - A$.
 e. $A \,\Delta\, B$.
 f. $A \times B$.
 g. $B \times A$.

12.2. Let A and B be sets with $|A| = 10$ and $|B| = 7$. Calculate $|A \cap B| + |A \cup B|$ and justify your answer.

12.3. Let A and B be sets with $|A| = 10$ and $|B| = 7$. What can we say about $|A \cup B|$?

In particular, find two numbers x and y for which we can be sure that $x \leq |A \cup B| \leq y$ and then find specific sets A and B so that $|A \cup B| = x$ and another pair of sets so that $|A \cup B| = y$.

Finally, answer the same question about $|A \cap B|$ (find upper and lower bounds as well as examples to show that your bounds are tight).

12.4. **a.** A line in the plane is a set of points. If ℓ_1 and ℓ_2 are two different lines, what can we say about $|\ell_1 \cap \ell_2|$? In particular, find all possible values of $|\ell_1 \cap \ell_2|$ and interpret them geometrically.

b. A circle in the plane is also a set of points. If C_1 and C_2 are two different circles, what can we say about $|C_1 \cap C_2|$? Again, interpret your answer geometrically.

12.5. Prove Theorem 12.3.

12.6. Let A and B be sets. Explain why $A \cap B$ and $A \mathbin{\Delta} B$ are disjoint.

12.7. Earlier we presented a Venn diagram of the distributive property $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Please give a Venn diagram of the other distributive property, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

12.8. Is a Venn diagram illustration a proof? (This is a philosophical question.)

12.9. Suppose A, B, and C are sets with $A \cap B \cap C = \emptyset$. Prove or disprove: $|A \cup B \cup C| = |A| + |B| + |C|$.

12.10. Suppose A, B, and C are pairwise disjoint sets. Prove or disprove: $|A \cup B \cup C| = |A| + |B| + |C|$.

12.11. Let A and B be sets. Prove or disprove: $A \cup B = A \cap B$ if and only if $A = B$.

12.12. Let A and B be sets. Prove or disprove: $|A \mathbin{\Delta} B| = |A| + |B| - |A \cap B|$.

12.13. Let A and B be sets. Prove or disprove: $|A \mathbin{\Delta} B| = |A - B| + |B - A|$.

12.14. Let A be a set. Prove: $A - \emptyset = A$ and $\emptyset - A = \emptyset$.

12.15. Let A be a set. Prove: $A \mathbin{\Delta} A = \emptyset$ and $A \mathbin{\Delta} \emptyset = A$.

12.16. Prove that $A \subseteq B$ if and only if $A - B = \emptyset$.

12.17. Let A and B be nonempty sets. Prove: $A \times B = B \times A$ if and only if $A = B$.

Why do we need the condition that A and B are nonempty?

12.18. State and prove necessary and sufficient conditions for $A - B = B - A$. In other words, create a theorem of the form "Let A and B be sets. We have $A - B = B - A$ if and only if (a condition on A and B)." Then prove your result.

12.19. Give a standard proof of Proposition 12.12 and illustrate it with a Venn diagram.

12.20. In Exercise 11.8 we defined what it means for a region in the plane to be convex, namely, a region R is convex provided given any two points in R, the line segment joining those points is entirely contained in R.

Prove or disprove the following assertions:

a. The union of two convex regions is convex.

b. The intersection of two convex regions is convex.

12.21. *True or False*: For each of the following statements, determine whether the statement is true or false and then prove your assertion. That is, for each true statement, please supply a proof, and for each false statement, present a counterexample (with explanation).

In the following, A, B, and C denote sets.

a. $A - (B - C) = (A - B) - C$.

b. $(A - B) - C = (A - C) - B$.

c. $(A \cup B) - C = (A - C) \cap (B - C)$.

d. If $A = B - C$, then $B = A \cup C$.

e. If $B = A \cup C$, then $A = B - C$.

f. $|A - B| = |A| - |B|$.

g. $(A - B) \cup B = A$.

h. $(A \cup B) - B = A$.

Set complement.

12.22. Let A be a set. The *complement* of A, denoted $\overline{A}$, is the set of all objects that are not in A. STOP! This definition needs some amending. Taken literally, the complement of the set $\{1, 2, 3\}$ includes the number -5, the ordered pair $(3, 4)$, and the sun, moon, and stars! After all, it says "*. . . all objects* that are not in A." This is not what is intended.

When mathematicians speak of set complements, they usually have some overarching set in mind. For example, during a given proof or discussion about the integers, if A is a set containing just integers, $\overline{A}$ stands for the set containing all integers not in A.

If U (for "universe") is the set of all objects under consideration and $A \subseteq U$, then the complement of A is the set of all objects in U that are not in A. In other words, $\overline{A} = U - A$. Thus $\overline{\emptyset} = U$.

Prove the following about set complements. Here the letters A, B, and C denote subsets of a universe set U.

a. $A = B$ if and only if $\overline{A} = \overline{B}$.

b. $\overline{\overline{A}} = A$.

c. $\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$.

The notation $U - A$ is much clearer than $\overline{A}$.

The notation $\overline{A}$ is handy, but it can be ambiguous. Unless it is perfectly clear what the "universe" set U should be, it is better to use the set difference notation rather than complement notation.

12.23. Design a four-set Venn diagram. Notice that the three-set Venn diagram we have been using has eight regions (including the region surrounding the three circles) corresponding to the eight possible memberships an object might have. An object might be in or not in A, in or not in B, and in or not in C.

Explain why this gives eight possibilities.

Your Venn diagram should show four sets, A, B, C, and D. How many regions should your diagram have?

On your Venn diagram, shade in the set $A \Delta B \Delta C \Delta D$.

Note: Your diagram does not have to use circles to demark sets. Indeed, it is impossible to create a Venn diagram for four sets using circles! You need to use other shapes.

An expanded version of inclusion-exclusion.

12.24. Let A, B, and C be sets. Prove that

$$\begin{aligned}|A \cup B \cup C| = |A| + |B| + |C| \\ - |A \cap B| - |A \cap C| - |B \cap C| \\ + |A \cap B \cap C|.\end{aligned}$$

The connection between set operations and Boolean algebra.

12.25. There is an intimate connection between set concepts and Boolean algebra concepts. The symbols $\wedge$ and $\vee$ are pointy versions of $\cap$ and $\cup$, respectively. This is more than a coincidence. Consider:

$$\begin{aligned} x \in A \cap B &\iff (x \in A) \wedge (x \in B) \\ x \in A \cup B &\iff (x \in A) \vee (x \in B) \end{aligned}$$

Find similar connections between the set-theoretic notions of $\subseteq$ and Δ with notions from Boolean algebra.

12.26. Prove that symmetric difference is a commutative operation; that is, for sets A and B, we have $A \Delta B = B \Delta A$.

12.27. Prove that symmetric difference is an associative operation; that is, for any sets A, B, and C, we have $A \Delta (B \Delta C) = (A \Delta B) \Delta C$.

12.28. Give a Venn diagram illustration of $A \Delta (B \Delta C) = (A \Delta B) \Delta C$.

12.29. Prove Proposition 12.15.

12.30. Let A, B, and C denote sets. Prove the following:

a. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
b. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
c. $A \times (B - C) = (A \times B) - (A \times C)$.
d. $A \times (B \Delta C) = (A \times B) \Delta (A \times C)$.

13 Combinatorial Proof: Two Examples

In Section 12 we introduced the concept of combinatorial proof of equations. This technique works by showing that both sides of an equation are answers to a common question. This method was used to prove Proposition 12.4 (for finite sets A and B we have $|A| + |B| = |A \cup B| + |A \cap B|$). See Proof Template 9.

In this section we give two examples that further illustrate this technique. One is based on a set-counting problem and the other on a list-counting problem.

Proposition 13.1 Let n be a positive integer. Then

$$2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1.$$

For example, $2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$.

We seek a question to which both sides of the equation give a correct answer. The right hand side is simpler, so let us begin there. The 2^n term answers the question "How many subsets does an n-element set have?" However, the term is $2^n - 1$, not 2^n. We can modify the question to rule out all but one of the subsets. Which subset should we ignore? A natural choice is to skip the empty set. The rephrased question is "How many nonempty subsets does an n-element set have?" Now it is clear that the right hand side of the equation, $2^n - 1$, is a correct answer. But what of the left?

The left hand side is a long sum, with each term of the form 2^j. This is a hint that we are considering several subset-counting problems. Somehow, the question of how many nonempty subsets an n-element set has must be broken down into disjoint cases (each a subset-counting problem unto itself) and then combined to give the full answer.

We know we are counting nonempty subsets of an n-element set. For the sake of specificity, suppose the set is $\{1, 2, \ldots, n\}$. Let's start writing down the nonempty subsets of this set. It's natural to start with $\{1\}$. Next we write down $\{1, 2\}$ and $\{2\}$—these are the sets whose largest element is 2. Next we write down the sets whose largest element is 3. Let's organize this into a chart.

Largest element	Subsets of $\{1, 2, \ldots, n\}$
1	$\{1\}$
2	$\{2\}, \{1, 2\}$
3	$\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$
4	$\{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \ldots, \{1, 2, 3, 4\}$
$\vdots$	$\vdots$
n	$\{n\}, \{1, n\}, \{2, n\}, \{1, 2, n\}, \ldots, \{1, 2, 3, \ldots, n\}$

We neglected to write out all the subsets on line 4 of the chart. How many are there? The sets on this line must contain 4 (since that's the largest element). The other elements of these sets are chosen from among 1, 2, and 3. Because there are $2^3 = 8$ possible ways to form a subset of $\{1, 2, 3\}$, there must be 8 sets on this line. Please take a moment to verify this for yourself by completing line 4 of the chart.

Now skip to the last line of the chart. How many subsets of $\{1, 2, \ldots, n\}$ have largest element n? We must include the element n together with each subset of $\{1, 2, \ldots, n-1\}$, for a total of 2^{n-1} choices.

Notice that every nonempty subset of $\{1, 2, \ldots, n\}$ must appear exactly once in the chart. Totaling the row sizes gives

$$1 + 2 + 4 + 8 + \cdots + 2^{n-1}.$$

Aha! This is precisely the left hand side of the equation we seek to prove.

Armed with these insights, we are ready to write the proof.

Proof (of Proposition 13.1)

Let n be a positive integer, and let $N = \{1, 2, \ldots, n\}$. How many nonempty subsets does N have?

Answer 1: Since N has 2^n subsets, when we disregard the empty set, we see that N has $2^n - 1$ nonempty subsets.

Answer 2: We consider the number of subsets of N whose largest element is j where $1 \le j \le n$. Such subsets must be of the form $\{\ldots, j\}$ where the other elements are chosen from $\{1, \ldots, j-1\}$. Since this latter set has 2^{j-1} subsets, N has 2^{j-1} subsets whose largest element is j. Summing these answers over all j gives

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1}$$

nonempty subsets of N.

Since answers 1 and 2 are both correct solutions to the same counting problem, we have

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

■

We now turn to a second example (an equation you were led to discover in Exercise 9.9).

Proposition 13.2 Let n be a positive integer. Then

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1.$$

For example, with $n = 4$, observe that

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! &= 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 6 + 4 \cdot 24 \\ &= 1 + 4 + 18 + 96 \\ &= 119 = 120 - 1 = 5! - 1. \end{aligned}$$

The key to proving Proposition 13.2 is to find a question to which both sides of the equation give a correct answer. As with the first example, the right hand side is simpler, so we begin there.

The $(n+1)!$ term reminds us of counting lists without replacement. Specifically, it answers the question "How many lists can we form using the elements of $\{1, 2, \ldots, n+1\}$ in which every element is used exactly once?" Because the right hand side also includes a -1 term, we need to discard one of these lists. Which? A natural choice is to skip the list $(1, 2, 3, \ldots, n+1)$; this is the only list in which every element j appears in position j for every $j = 1, 2, \ldots, n$. In every other list, some element j is not in the j^{th} position on this list. Alternatively, the discarded list is the only one in which the elements appear in increasing order.

We therefore consider the question "How many lists can we form using the elements of $\{1, 2, \ldots, n+1\}$ in which every element appears exactly once and in which the elements do not appear in increasing order?"

Clearly $(n+1)! - 1$ is one solution to this problem; we need to show that the left hand side is also a correct answer. If the elements in the list are not in increasing order, then some element, say k, will not be in position k. We can organize this counting problem by considering where this first happens.

Let us consider the case $n = 4$. We form a chart containing all length-5 repetition-free lists we can form from the elements of $\{1, 2, 3, 4, 5\}$ that are not in increasing order. We organize the chart by considering the first time slot k is not element k. For example, when $k = 3$ the lists are 12$\underline{4}$35, 12$\underline{4}$53, 12$\underline{5}$34, and 12$\underline{5}$43 since the entries in positions 1 and 2 are elements 1 and 2, respectively, but entry 3 is not 3. (We have omitted the commas and parentheses for the sake of clarity.)

The chart for $n = 4$ follows.

k	first "misplaced" element at position k	
1	21345 21354 21435 21453 21534 21543 24135 24153 24315 24351 24513 24531 31245 31254 31425 31452 31524 31542 34125 34152 34215 34251 34512 34521 41235 41253 41325 41352 41523 41532 43125 43152 43215 43251 43512 43521 51234 51243 51324 51342 51423 51432 53124 53142 53214 53241 53412 53421	23145 23154 23415 23451 23514 23541 25134 25143 25314 25341 25413 25431 32145 32154 32415 32451 32514 32541 35124 35142 35214 35241 35412 35421 42135 42153 42315 42351 42513 42531 45123 45132 45213 45231 45312 45321 52134 52143 52314 52341 52413 52431 54123 54132 54213 54231 54312 54321
2	13245 13254 13425 13452 13524 13542 14235 14253 14325 14352 14523 14532 15234 15243 15324 15342 15423 15432	
3	12435 12453 12534 12543	
4	12354	
5	—	

Notice that row 5 of the chart is empty; why? This row should contain all repetition-free lists in which the first slot k that does not contain element k is $k = 5$. Such a list must be of the form $(1, 2, 3, 4, ?)$, but then there is no valid way to fill in the last position.

Next, count the number of lists in each portion of the chart. Working from the bottom, there are $1 + 4 + 18 + 96 = 119$ lists (all $5! = 120$ except the list $(1, 2, 3, 4, 5)$). The sum $1 + 4 + 18 + 96$ should be familiar; it is precisely $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4!$. Of course, this is not a coincidence. Consider the first row of the chart. The lists in this row must not begin with a 1 but may begin with any element of $\{2, 3, 4, 5\}$; there are 4 choices for the first element. Once the first element is chosen, the remaining four elements in the lists may be chosen in any way we like. Since there are 4 elements remaining (after selecting the first), these 4 elements can be arranged in 4! ways. Thus, by the Multiplication Principle, there are $4 \cdot 4!$ lists in which the first element is not 1.

The same analysis works for the second row. Lists on this row must begin with a 1, and then the second element must not be a 2. There are 3 choices for the second element because we must choose it from $\{3, 4, 5\}$. Once the second element has been selected, the remaining three elements may be arranged in any way we wish, and there are 3! ways to do so. Thus the second row of the chart contains $3 \cdot 3! = 18$ lists.

We are ready to complete the proof.

Proof (of Proposition 13.2)

Let n be a positive integer. We ask, "How many repetition-free lists can we form using all the elements in $\{1, 2, \ldots, n+1\}$ in which the elements do not appear in increasing order?"

Answer 1: There are $(n+1)!$ repetition-free lists, and in only one such list do the elements appear in order, namely $(1, 2, \ldots, n, n+1)$. Thus the answer to the question is $(n+1)! - 1$.

Answer 2: Let j be an integer between 1 and n, inclusive. Let us consider those lists in which the first $j-1$ elements are $1, 2, \ldots, j-1$, respectively, but for which the j^{th} element is not j. How many such lists are there? For element j there are $n+1-j$ choices because elements 1 through $j-1$ have already been chosen and we may not use element j. The remaining $n+1-j$ elements may fill in the remaining slots on the list in any order, giving $(n+1-j)!$ possibilities. By the Multiplication Principle, there are $(n+1-j) \cdot (n+1-j)!$ such lists. Summing over $j = 1, 2, \ldots, n$ gives

$$n \cdot n! + (n-1) \cdot (n-1)! + \cdots 3 \cdot 3! + 2 \cdot 2! + 1 \cdot 1!.$$

Since answers 1 and 2 are both correct solutions to the same counting problem, we have

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1.$$

■

Recap

In this section we illustrated the concept of combinatorial proof by applying the technique to demonstrate two identities.

13 Exercises

13.1. Give an alternative proof of Proposition 13.1 in which you use list counting instead of subset counting.

13.2. Let n be a positive integer. Use algebra to simplify the following expression:

$$(x-1)(1 + x + x^2 + \cdots + x^{n-1}).$$

Use this to give another proof of Proposition 13.1.

13.3. Substituting $x = 3$ into your expression in the previous problem yields

$$2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1.$$

Prove this equation combinatorially.

Next, substitute $x = 10$ and illustrate the result using ordinary base-10 numbers.

13.4. Let a and b be positive integers with $a > b$. Give a combinatorial proof of the identity $(a+b)(a-b) = a^2 - b^2$.

13.5. Let n be a positive integer. Give a combinatorial proof that $n^2 = n(n-1) + n$.

13.6. In this problem we want to calculate the number of two-element lists (a, b) we can form using the numbers $0, 1, \ldots, n$ with $a < b$.

a. Show that the answer is $(n+1)n/2$ by considering the number of two-element lists (a, b) in which $a < b$ or $a > b$.

b. Show that the answer is also $1 + 2 + \cdots + n$.

Taken together, (a) and (b) prove the formula

$$\sum_{k=1}^{n} k = \frac{(n+1)n}{2}.$$

13.7. How many two-element lists can we form using the integers from 1 to n in which the largest element in the list is a (where a is some integer between 1 and n)?

Use your answer to show:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

Chapter 2 Self Test

1. The call sign for a radio station in the United States is a list of three or four letters, such as WJHU or WJZ. The first letter must be a W or a K, and there is no restriction on the other letters. In how many ways can the call sign of a radio station be formed?
2. In how many ways can we make a list of three integers (a, b, c) where $0 \leq a, b, c \leq 9$ and $a + b + c$ is even?
3. In how many ways can we make a list of three integers (a, b, c) where $0 \leq a, b, c \leq 9$ and abc is even?
4. Without the use of any computational aid, simplify the following expression:
$$\frac{20!}{17! \cdot 3!}$$
5. In how many ways can we arrange a standard deck of 52 cards so that all cards in a given suit appear contiguously (e.g., first all the spades appear, then all the diamonds, then all the hearts, and then all the clubs)?
6. Ten married couples are waiting in line to enter a restaurant. Husbands and wives stand next to each other, but either one might be ahead of the other. How many such arrangements are possible?
7. Evaluate the following:
$$\prod_{k=0}^{100} \frac{k^2}{k+1}.$$
8. Let $A = \{x \in \mathbb{Z} : |x| < 10\}$. Evaluate $|A|$.
9. Let $A = \big\{1, 2, \{3, 4\}\big\}$. Which of the following are true and which false? No proof is required.
   a. $1 \in A$.
   b. $\{1\} \in A$.
   c. $3 \in A$.
   d. $\{3\} \in A$.
   e. $\{3\} \subseteq A$.
10. Let A and B be finite sets. Determine whether the following statements are true or false. Justify your answer with a proof or counterexample, as appropriate.
    a. $2^{A \cap B} = 2^A \cap 2^B$.
    b. $2^{A \cup B} = 2^A \cup 2^B$.
    c. $2^{A \Delta B} = 2^A \Delta 2^B$.
11. Let A be a set. Which of the following are true and which false?
    a. $x \in A$ iff $x \in 2^A$.
    b. $T \subseteq A$ iff $T \in 2^A$.
    c. $x \in A$ iff $\{x\} \in 2^A$.
    d. $\{x\} \in A$ iff $\big\{\{x\}\big\} \in 2^A$.
12. Which of the following statements are true and which are false? No proof is required.
    a. $\forall x \in \mathbb{Z}, x^2 \geq x$.
    b. $\exists x \in \mathbb{Z}, x^3 = x$.
    c. $\forall x \in \mathbb{Z}, 2x \geq x$.
    d. $\exists x \in \mathbb{Z}, x^2 + x + 1 = 0$.

13. Which of the following statements are true and which are false? No proof is required.

a. $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \le y$.
b. $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \le y$.
c. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \le y$.
d. $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \le y$.
e. $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, x \le y$.
f. $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, x \le y$.
g. $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, x \le y$.
h. $\exists x \in \mathbb{N}, \exists y \in \mathbb{N}, x \le y$.

14. Let $p(x, y)$ stand for a sentence about two integers, x and y. For example, $p(x, y)$ could mean "$x - y$ is a perfect square."

Assume the statement $\forall x, \exists y, p(x, y)$ is true. Which of the following statements about integers must also be true?

a. $\forall x, \exists y, \neg p(x, y)$.
b. $\neg\big(\exists x, \forall y, \neg p(x, y)\big)$.
c. $\exists x, \exists y, p(x, y)$.

15. Let A and B be sets and suppose $A \times B = \{(1,2), (1,3), (2,2), (2,3)\}$. Find $A \cup B$, $A \cap B$, and $A - B$.

16. Let A, B, and C denote sets. Prove that $(A \cup B) - C = (A - C) \cup (B - C)$ and give a Venn diagram illustration.

17. Consider the following argument: *All cats are mammals. I am a mammal. Therefore, I am a cat.* Show that this is fallacious using the language of set theory. Illustrate the fallacy with a Venn diagram.

18. Suppose A and B are finite sets. Given that $|A| = 10$, $|A \cup B| = 15$, and $|A \cap B| = 3$, determine $|B|$.

19. Let A and B be sets. Create an expression that evaluates to $A \cap B$ that uses only the operations union and set difference. That is, find a formula that uses only the symbols A, B, $\cup$, $-$, and parentheses; this formula should equal $A \cap B$ for all sets A and B.

20. Let n be a positive integer. Give a combinatorial proof of the identity

$$n^3 = n(n-1)(n-2) + 3n(n-1) + n.$$