CHAPTER

2 Collections

This chapter deals with collections. We consider two types of collections: ordered collections (lists) and unordered collections (sets).

8 Lists

A *list* is an ordered sequence of objects. We write lists by starting with an open parenthesis, following with the elements of the list separated by commas, and finishing with a close parenthesis. For example, $(1, 2, \mathbb{Z})$ is a list whose first element is the number 1, whose second element is the number 2, and whose third element is the set of integers.

The order in which elements appear in a list is significant. The list $(1, 2, 3)$ is not the same as the list $(3, 2, 1)$.

Elements in a list might be repeated, as in $(3, 3, 2)$.

The number of elements in a list is called its *length*. For example, the list $(1, 1, 2, 1)$ is a list of length four.

A list of length two has a special name; it is called an *ordered pair*.

A list of length zero is called the *empty list* and is denoted $()$.

What it means for two lists to be equal.

Two lists are *equal* provided they have the same length, and elements in the corresponding positions on the two lists are equal. Lists (a, b, c) and (x, y, z) are equal iff $a = x$, $b = y$, and $c = z$.

Mathspeak!

Another word mathematicians use for lists is *tuple*. A list of n elements is known as an n-tuple.

Lists are all-pervasive in mathematics and beyond. A point in the plane is often specified by an ordered pair of real numbers (x, y). A natural number, when written in standard notation, is a list of digits; you can think of the number 172 as the list $(1, 7, 2)$. An English word is a list of letters. An identifier in a computer program is a list of letters and digits (where the first element of the list is a letter).

Counting Two-Element Lists

In this section, we address questions of the form “How many lists can we make?”

Example 8.1 Suppose we wish to make a two-element list where the entries in the list may be any of the digits 1, 2, 3, and 4. How many such lists are possible?

The most direct approach to answering this question is to write out all the possibilities.

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{array}$$

There are 16 such lists.

We organized the lists in a manner that ensures we have neither repeated nor omitted a list. The first row of the chart contains all the possible lists that begin with 1, the second row those that begin with 2, and so on. Thus there are $4 \times 4 = 16$ length-two lists whose elements are any one of the digits 1 through 4.

Mathspeak!

The mathematical use of the word *choice* is strange. If a restaurant has a menu with only one entrée, the mathematician would say that this menu offers one choice. The rest of the world probably would say that the menu offers no choices! The mathematical use of the word *choice* is similar to *option*.

Let's generalize this example a little bit. Suppose we wish to know the number of two-element lists where there are n possible choices for each entry in the list. We may assume the possible elements are the integers 1 through n. As before, we organize all the possible lists into a chart.

$$\begin{array}{cccc} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \vdots & \vdots & \ddots & \vdots \\ (n,1) & (n,2) & \cdots & (n,n) \end{array}$$

The first row contains all the lists that begin with 1, the second those that begin with 2, and so forth. There are n rows in all. Each row has exactly n lists. Therefore there are $n \times n = n^2$ possible lists.

When a list is formed, the options for the second position may be different from the options for the first position. Imagine that a meal is a two-element list consisting of an entrée followed by a dessert. The number of possible entrées might be different from the number of possible desserts.

Therefore let us ask: How many two-element lists are possible in which there are n choices for the first element and m choices for the second element? Suppose that the possible entries in the first position of the list are the integers 1 through n, and the possible entries in the second position are 1 through m.

We construct a chart of all the possibilities as before.

$$\begin{array}{cccc} (1,1) & (1,2) & \cdots & (1,m) \\ (2,1) & (2,2) & \cdots & (2,m) \\ \vdots & \vdots & \ddots & \vdots \\ (n,1) & (n,2) & \cdots & (n,m) \end{array}$$

There are n rows (for each possible first choice), and each row contains m entries. Thus the number of possible such lists is

$$\underbrace{m + m + \cdots + m}_{n \text{ times}} = m \times n.$$

Sometimes the elements of a list satisfy special properties. In particular, the choice of the second element might depend on what the first element is. For example, suppose we wish to count the number of two-element lists we can form from the integers 1 through 5, in which the two numbers on the list must be different. For example, we want to count $(3, 2)$ and $(2, 5)$ but not $(4, 4)$. We make a chart of the possible lists.

$$\begin{array}{ccccc} - & (1,2) & (1,3) & (1,4) & (1,5) \\ (2,1) & - & (2,3) & (2,4) & (2,5) \\ (3,1) & (3,2) & - & (3,4) & (3,5) \\ (4,1) & (4,2) & (4,3) & - & (4,5) \\ (5,1) & (5,2) & (5,3) & (5,4) & - \end{array}$$

As before, the first row contains all the possible lists that begin with 1, the second row those lists that start with 2, and so on, so there are 5 rows. Notice that each row contains exactly $5 - 1 = 4$ lists, so the number of lists is $5 \times 4 = 20$.

Let us summarize and generalize what we have learned in a general principle.

Theorem 8.2 **(Multiplication Principle)** Consider two-element lists for which there are n choices for the first element, and for each choice of the first element there are m choices for the second element. Then the number of such lists is nm.

Proof. Construct a chart of all the possible lists. Each row of this chart contains all the two-element lists that begin with a particular element. Since there are n choices for the first element, there are n rows in the chart. Since, for each choice of the first element, there are m choices for the second element, we know that every row of the chart has m entries. Therefore the number of lists is

$$\underbrace{m + m + \cdots + m}_{n \text{ times}} = n \times m.$$

■

Let us consider some examples.

Example 8.3 A person's initials are the two-element list consisting of the initial letters of their first and last names. For example, the author's initials are ES. In how many ways can we form a person's initials? In how many ways can we form initials where the two letters are different?

The first question asks for the number of two-element lists where there are 26 choices for each element. There are 26^2 such lists.

The second question asks for the number of two-element lists where there are 26 choices for the first element and, for each choice of first element, 25 choices for the second element. Thus there are 26×25 such lists.

Another way to answer the second question in Example 8.3 is as follows: There are 26^2 ways to form initials (repetitions allowed). Of these, there are 26 "bad" sets of initials in which there is a repetition, namely, AA, BB, CC, ..., ZZ. The remaining lists are the ones we want to count, so there are $26^2 - 26$ possibilities. Since $26 \times 25 = 26 \times (26 - 1) = 26^2 - 26$, the two answers agree.

Please note that we reported the answers to these questions as 26^2 and 26×25, and not as 676 and 650. Although the latter pair of answers are correct, the answers 26^2 and 26×25 are preferred because they retain the essence of the reasoning used to derive them. Furthermore, the conversion of 26^2 and 26×25 to 676 and 650, respectively, is not interesting and can be done easily by anyone with a calculator.

Example 8.4 A club has ten members. The members wish to elect a president and to elect someone else as a vice president. In how many ways can these posts be filled?

We recast this question as a list-counting problem. How many two-element lists of people can be formed in which the two people in the list are selected from a collection of ten candidates and the same person may not be selected twice?

There are ten choices for the first element of the list. For each choice of the first element (for each president), there are nine possible choices for the second element of the list (vice president). By the Multiplication Principle, there are 10×9 possibilities.

Longer Lists

Let us explore how to use the Multiplication Principle to count longer lists.

Consider the following problem. How many lists of three elements can we make using the numbers 1, 2, 3, 4, and 5? Let us write out all the possibilities. Here is a way we might organize our work:

(1,1,1)	(1,1,2)	(1,1,3)	(1,1,4)	(1,1,5)
(1,2,1)	(1,2,2)	(1,2,3)	(1,2,4)	(1,2,5)
(1,3,1)	(1,3,2)	(1,3,3)	(1,3,4)	(1,3,5)
(1,4,1)	(1,4,2)	(1,4,3)	(1,4,4)	(1,4,5)
(1,5,1)	(1,5,2)	(1,5,3)	(1,5,4)	(1,5,5)
(2,1,1)	(2,1,2)	(2,1,3)	(2,1,4)	(2,1,5)
(2,2,1)	(2,2,2)	(2,2,3)	(2,2,4)	(2,2,5)
		and so forth until		
(5,5,1)	(5,5,2)	(5,5,3)	(5,5,4)	(5,5,5)

The first line of this chart contains all lists that begin $(1, 1, \ldots)$. The second line is all lists that begin $(1, 2, \ldots)$ and so forth. Clearly, each line has five lists. The question becomes:

How many lines are there in this chart?

This is a problem we have already solved! Notice that each line of the chart begins, effectively, with a different two-element list; the number of two-element lists where each element is one of five possible values is 5×5, so this chart has 5×5 lines. Therefore, since each line of the chart has five entries, the number of three-element lists is $(5 \times 5) \times 5 = 5^3$.

We can think of a three-element list as the concatenation of a two-element list and a one-element list. In this problem, there are 25 possible two-element lists to occupy the front of the three-element list, and for each choice of the front part, there are five choices for the back.

Suppose A and B are lists. Their *concatenation* is the new list formed by listing first the elements of A and then the elements of B. The concatenation of the lists $(1, 2, 1)$ and $(1, 3, 5)$ is the list $(1, 2, 1, 1, 3, 5)$.

Next, let us count three-element lists whose elements are the integers 1 through 5 in which no number is repeated. As before, we make a chart.

(1,2,3)	(1,2,4)	(1,2,5)
(1,3,2)	(1,3,4)	(1,3,5)
(1,4,2)	(1,4,3)	(1,4,5)
(1,5,2)	(1,5,3)	(1,5,4)
(2,1,3)	(2,1,4)	(2,1,5)
	and so forth until	
(5,4,1)	(5,4,2)	(5,4,3)

The first line of the chart contains all the lists that begin $(1, 2, \ldots)$. (There can be no lines that begin $(1, 1, \ldots)$ because repetitions are disallowed.) The second line contains all lists that begin $(1, 3, \ldots)$, and so on. Each line of the chart contains just three lists; once we have chosen the first and second elements of the list (from a world of only five choices), there are exactly three ways to finish the list. So, as before, the question becomes: How many lines are on this chart? And as before, this is a problem we have already solved!

The first two elements of the list form, unto themselves, a two-element list with each element chosen from a list of five possible objects and without repetition. So, by the Multiplication Principle, there are 5×4 lines on the chart. Since each line has three elements, there is a total of $5 \times 4 \times 3$ possible lists in all.

These three-element lists are a concatenation of a two-element list (20 choices), and, for each two-element list, a one-element list (3 choices), giving a total of 20×3 lists.

We extend the Multiplication Principle to count longer lists. Consider length-three lists. Suppose we have a choices for the first element of the list, and for each choice of first element, there are b choices for the second element, and for each choice of first and second elements, there are c choices for the third element. Thus, in all, there are abc possible lists. To see why, imagine that the three-element list consists of two parts: the initial two elements and the final element. There are ab ways to fill in the first two elements (by the Multiplication Principle!) and there are c ways to complete the last element once the first two are specified. So, by the Multiplication Principle again, there are $(ab)c$ ways to make the lists. The extension of these ideas to lists of length-four or more is analogous.

A useful way to think about list-counting problems is to make a diagram with boxes. Each box stands for a position in the list, so if the length of the list is four, there are four boxes. We write the number of possible entries in each box. The number of possible lists is computed by multiplying these numbers together.

Example 8.5 Let us revisit Example 8.4. We have a club with ten members. We want to elect an executive board consisting of a president, a vice president, a secretary, and a treasurer. In how many ways can we do this (assuming no member of the club can fill two offices)? We draw the following diagram.

Pres.	V.P.	Sec.	Treas.
10	9	8	7

This shows there are ten choices for president. Once the president is selected, there are are nine choices for vice president, so there are 10×9 ways to fill in the first two elements of the list. Once these are filled, there are eight ways to fill in the next element of the list (secretary), so there are $(10 \times 9) \times 8$ ways to complete the first three slots. Finally, once the first three offices are filled, there are seven ways to select a treasurer, so there are $(10 \times 9 \times 8) \times 7$ ways to select the entire slate of officers.

Two particular list-making problems recur often enough to warrant special attention. These problems both involve making a list of length k in which each element of the list is selected from among n possibilities. In the first problem, we count all such lists; in the second problem, we count those without repeated elements.

This formula gives the number of lists of length k where there are n possible entries in each position of the list and repetitions are allowed.

When repetitions are allowed, we have n choices for the first element of the list, n choices for the second element of the list, and so on, and n choices for the last element of the list. All told, there are

$$\underbrace{n \times n \times \cdots \times n}_{k \text{ times}} = n^k \tag{1}$$

possible lists.

Lists without repetitions are sometimes called *permutations*. However, in this book, the word *permutation* has another meaning described later (Section 27).

This formula counts the number of lists of length k where the elements are chosen from a pool of n possibilities and no two elements on the list are the same.

Now suppose we fill in the length-k list with n possible values, but in this case, repetition is not allowed. There are n ways to select the first element of the list. Once this is done, there are $n-1$ choices for the second element of the list. There are $n-2$ ways to fill in position three, $n-3$ ways to fill in position four, and so on, and finally, there are $n-(k-1) = n-k+1$ ways to fill in position k. Therefore, the number of ways to make a list of length k where the elements are chosen from a pool of n possibilities and no two elements on the list may be the same is

$$n \times [n-1] \times [n-2] \times \cdots \times [n-(k-1)]. \tag{2}$$

This formula is correct, but there is a minor mistake in our reasoning! How many length-six lists can we make where each element of the list is one of the digits 1, 2, 3, or 4 and repetition is not allowed? The answer, obviously, is zero; you cannot make a length-six list using only four possible elements and not repeat an element! What does the formula give? Equation (2) says the number of such lists is

$$4 \times 3 \times 2 \times 1 \times 0 \times -1$$

which equals 0. However, the reasoning behind the formula breaks down. Although it is true that there are 4, 3, 2, 1, and 0 choices for positions one through five, it does not make any sense to say there are -1 choices for the last position! Formula (2) gives the correct answer, but the reasoning used to arrive there needs to be rechecked.

In this paragraph, we use Exercise 5.16: If $a, b \in \mathbb{Z}$, then $a < b \iff a \le b - 1$.

If the number of elements from which we select entries in the list, n, is less than the length of the list, k, no repetition-free list is possible. But since $n < k$, we know that $n - k < 0$ and so $n - k + 1 < 1$. Since $n - k + 1$ is an integer, we know that $n - k + 1 \le 0$. Therefore, in the product $n \times (n-1) \times \cdots \times (n-k+1)$, we know that at least one of the factors is zero. Therefore the whole expression evaluates to zero, which is what we wanted!

On the other hand, if $n \ge k$, our reasoning makes sense (all the numbers are positive), and the formula in (2) gives the correct answer.

Length-zero lists.

One case is worth special mention: $k = 0$. We ask: How many lists of length zero can we form from a pool of n elements? The answer is one since the empty list (a list with no elements) is a legitimate list.

The special notation for $n(n-1)\cdots(n-k+1)$ is $(n)_k$. An alternative notation, still in use on some calculators, is ${}_nP_k$.

Because the expression $n(n-1)(n-2)\cdots(n-k+1)$ occurs fairly often, there is a special notation for it. The notation is

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1).$$

This notation is called *falling factorial*. We summarize our results on lists with or without repetition concisely using this notation.

Theorem 8.6 The number of lists of length k whose elements are chosen from a pool of n possible elements

$$= \begin{cases} n^k & \text{if repetitions are permitted} \\ (n)_k & \text{if repetitions are forbidden.} \end{cases}$$

We do not recommend that you memorize this result because it is too easy to get confused between the meanings of n and k. Rather, rederive it in your mind when you need it. Imagine the k boxes written out in front of you, put the appropriate numbers in the boxes, and multiply.

Recap

This section deals with counting lists of objects. The central tool is the Multiplication Principle. A general formula is developed for counting length-k lists of elements selected from a pool of n elements either with or without repetitions.

8 Exercises

8.1. Write out all the possible two-letter "words" one can make using only the vowels A, E, I, O, and U. These will be mostly nonsense words from "AA" to "UU".

How many of these have no repeated letter?

8.2. Airports have names, but they also have three-letter codes. For example, the airport serving Baltimore is BWI, and the code YYY is for the airport in Mont Joli, Québec, Canada. How many different airport codes are possible?

8.3. A *bit string* is a list of 0s and 1s. How many length-k bit strings can be made?

8.4. A car's ventilation system has various controls. The fan control has four settings: off, low, medium, and high. The air stream can be set to come out at the floor, through the vents, or through the defroster. The air conditioning button can be either on or off. The temperature control can be set to cold, cool, warm, or hot. And finally, the recirculate button can be either on or off.

In how many different ways can these various controls be set?

Note: Several of these settings result in the same effect since nothing happens if the fan control is off. However, the problem asks for the number of different settings of the controls, not the number of different ventilation effects possible.

8.5. I want to create two play lists on my MP3 player from my collection of 500 songs. One play list is titled "Exercise" for listening in the gym and the other is titled "Relaxing" for leisure time at home. I want 20 different songs on each of these lists.

In how many ways can I load songs onto my MP3 player I if allow a song to be on both play lists?

And how many ways can I load the songs if I want the two lists to have no overlap?

8.6. How many 3-element lists can be formed whose entries are drawn from a pool of n possible elements if we require that the first and last entries of the list must be the same?

How many such lists can be formed if we require that the first and last entries must be different?

(In both cases, there is no restriction on the middle entry in the list.)

8.7. I have 30 photos to post on my website. I'm planning to post these on two web pages, one marked "Friends" and the other marked "Family". No photo may go on both pages, but every photo will end up on one or the other. Conceivably, one of the pages may be empty.

Please answer these two questions:

a. In how many ways can I post these photos to the web pages if the order in which the photos appear on those pages matters?

b. In how may ways can I post these photos to the web pages if the order in which the photos appear on those pages does not matter?

8.8. You own three different rings. You wear all three rings, but no two of the rings are on the same finger, nor are any of them on your thumbs. In how many ways can you wear your rings? (Assume any ring will fit on any finger.)

8.9. In how many ways can a black rook and a white rook be placed on different squares of a chess board such that neither is attacking the other? (In other words, they cannot be in the same row or the same column of the chess board. A standard chess board is 8×8.)

8.10. License plates in a certain state consist of six characters: The first three characters are uppercase letters (A–Z), and the last three characters are digits (0–9).

a. How many license plates are possible?

b. How many license plates are possible if no character may be repeated on the same plate?

8.11. A telephone number (in the United States and Canada) is a ten-digit number whose first digit cannot be a 0 or a 1. How many telephone numbers are possible?

8.12. A U.S. Social Security number is a nine-digit number. The first digit(s) may be 0.

a. How many Social Security numbers are available?

b. How many of these are even?

c. How many have all of their digits even?

d. How many read the same backward and forward (e.g., 122979221)?

e. How many have none of their digits equal to 8?

f. How many have at least one digit equal to 8?

g. How many have exactly one 8?

8.13. Let n be a positive integer. Prove that $n^2 = (n)_2 + n$ in two different ways.

First (and more simply) show this equation is true algebraically.

Second (and more interestingly) interpret the terms n^2, $(n)_2$, and n in the context of list counting and use that to argue why the equation must be true.

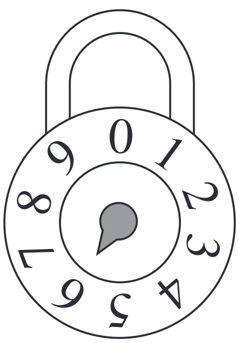
The word *character* means a letter or a digit.

8.14. A computer operating system allows files to be named using any combination of uppercase letters (A–Z) and digits (0–9), but the number of characters in the file name is at most eight (and there has to be at least one character in the file name). For example, `X23`, `W`, `4AA`, and `ABCD1234` are valid file names, but `W-23` and `WONDERFUL` are not valid (the first has an improper character, and the second is too long).

How many different file names are possible in this system?

8.15. How many five-digit numbers are there that do not have two consecutive digits the same? For example, you would count 12104 and 12397 but not 6321 (it is not five digits) or 43356 (it has two consecutive 3s).

Note: The first digit may not be a zero.



8.16. A padlock has the digits 0 through 9 arranged in a circle on its face. A combination for this padlock is four digits long. Because of the internal mechanics of the lock, no pair of consecutive numbers in the combination can be the same or one place apart on the face. For example 0-2-7-1 is a valid combination, but neither 0-4-4-7 (repeated digit 4) nor 3-0-9-5 (adjacent digits 0-9) are permitted.

How many combinations are possible?

8.17. A bookshelf contains 20 books. In how many different orders can these books be arranged on the shelf?

8.18. A class contains ten boys and ten girls. In how many different ways can they stand in a line if they must alternate in gender (no two boys and no two girls are standing next to one another)?

8.19. Four cards are drawn from a standard deck of 52 cards. In how many ways can this be done if the cards are all of different values (e.g., no two 5s or two jacks) and all of different suits? (For this problem, the order in which the cards are drawn matters, so drawing A♠-K♡-3♢-6♣ is not the same as drawing 6♣-K♡-3♢-A♠ even though the same cards are selected.)

9 Factorial

In Section 8, we counted lists of elements of various lengths in which we were either allowed or forbidden to repeat elements. A special case of this problem is to count the number of length-n lists chosen from a pool of n objects in which repetition is forbidden. In other words, we want to arrange n objects into a list, using each object exactly once. By Theorem 8.6, the number of such lists is

$$(n)_n = n(n-1)(n-2)\cdots(n-n+1) = n(n-1)(n-2)\cdots(1).$$

The quantity $(n)_n$ occurs frequently in mathematics and has a special name and notation; it is called *n factorial* and is written $n!$. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

Two special cases of the factorial function require special attention.

First, let us consider $1!$. This is the result of multiplying all the numbers starting from 1 all the way down to, well, 1. The answer is 1. Just in case this isn't clear, let's return to the list-counting application. In how many ways can we make a length-1 list where there is only one possible element to fill the first (and only!) position? Obviously, there is only one possible list. So $1! = 1$.

The other special case is $0!$.

Much Ado About 0!

0! is 1. Students' reactions to this statement typically range from "That doesn't make sense" to "That's wrong!" There seems to be an overwhelming urge to evaluate 0! as 0.

Because of this confusion, I feel I owe you a clear and unambiguous explanation of why $0! = 1$. Here it is: Because I said so!

No, that wasn't a terribly satisfying answer, and I will endeavor to do a better job in a moment, but the simple fact is that mathematicians have defined 0! to be 1, and we are all in agreement on this point. Just as we declared (via our definition) that the number 1 is not prime, we can also declare $0! = 1$. Mathematics is a human invention, and as long as we are consistent, we can set things up pretty much however we please.

So now the burden falls on me to explain why it is a good idea to have $0! = 1$ and a bad idea for it to be 0, $\sqrt{17}$, or anything else.

To begin, let us rethink the list-counting problem. The number 0! ought to be the answer to the following problem:

> In how many ways can we make a length-0 list whose elements come from a pool of 0 elements in which there is no repetition?

It is tempting to say that no such list is possible, but this is not correct. There is a list whose length is zero: the empty list ( ). The empty list has zero length, and (vacuously!) its elements satisfy the conditions of the problem. So the answer to the problem is $0! = 1$.

Here is another explanation why $0! = 1$. Consider the equation

$$n! = n \times (n-1)! \tag{3}$$

For example, $5! = 5 \times (4 \times 3 \times 2 \times 1) = 5 \times 4!$. Equation (3) makes sense for $n = 2$ since $2! = 2 \times 1! = 2 \times 1$. The question becomes: Does Equation (3) make sense for $n = 1$? If we want Equation (3) to work when $n = 1$, we need $1! = 1 \times 0!$. This forces us to choose $0! = 1$.

Here is another explanation why $0! = 1$. We can think of $n!$ as the result of multiplying n numbers together. For example, 5! is the result of multiplying the numbers on the list $(5, 4, 3, 2, 1)$. What should it mean to multiply together the numbers on the empty list ( )? Let me try to convince you that the sensible answer is 1. We begin by considering what it means to add the numbers on the empty list.

Alice and Bob are to add the numbers on the list $(2, 3, 3, 5, 4)$. The answer should be 17.

Alice and Bob work in a number factory and are given a list of numbers to add. They are both quite adept at addition, so they decide to break the list in two. Alice will add her numbers, Bob will add his numbers, and then they will add their results to get the final answer. This is a sensible procedure, and they ask Carlos to break the list in two for them.

Carlos gives Alice the list $(2, 3, 3, 5, 4)$ and Bob the list $()$. Alice adds her numbers and gets 17. What should Bob say?

Carlos, perhaps because he is feeling mischievous, decides to give Alice all of the numbers and Bob none of the numbers. Alice receives the full list and Bob receives the empty list. Alice adds her numbers as usual, but what is Bob to report as the sum of the numbers on his list? If Bob gives any answer other than 0, the final answer to the problem will be incorrect. The only sensible thing for Bob to say is that his list—the empty list—sums to 0.

The sum of the numbers in the empty list is 0.

Alice and Bob are to multiply the numbers on the list $(2, 3, 3, 5, 4)$. The answer should be 360.

Now, all three of them have received a promotion and are working on multiplication. Their multiplication procedure is the same as their addition procedure. They are asked to multiply lists of numbers. When they receive a list, they ask Carlos to break the list into two parts. Alice multiplies the numbers on her list, and Bob multiplies the numbers on his. They then multiply together their individual results to get the final answer.

Carlos gives Alice the list $()$ and Bob the list $(2, 3, 3, 5, 4)$. Bob multiplies his numbers and gets 360. What should Alice say?

But of course Carlos decides to have some fun and gives all the numbers to Bob; to Alice, he gives the empty list. Bob reports the product of his numbers as usual. What should Alice say? What is the product of the numbers on $()$? If she says 0, then when her answer is multiplied by Bob's answer, the final result will be 0, and this is likely to be the wrong answer. Indeed, the only sensible reply that Alice can give is 1.

The product of the numbers in the empty list is 1. Since 0! "asks" you to multiply together a list containing no numbers, the sensible answer is 1.

This reasoning is akin to taking $2^0 = 1$.

The final reason why we declare $0! = 1$ is that as we move on, other formulas work out better if we take $0! = 1$. If we did not set $0! = 1$, these other results would have to treat 0 as a special case, different from other natural numbers.

Product Notation

Here is another way to write $n!$:

$$n! = \prod_{k=1}^{n} k.$$

What does this mean? The symbol $\prod$ is the uppercase form of the Greek letter pi (π), and it stands for *product* (i.e., multiply). This notation is similar to using $\sum$ for summation.

The letter k is called a *dummy variable* and is a place holder that ranges from the lower value (written below the $\prod$ symbol) to the upper value (written at the top). The variable k takes on the values $1, 2, \ldots, n$.

To the right of the $\prod$ symbol are the quantities we multiply. In this case, it is simple: We just multiply the values of k as k goes from 1 to n; that is, we multiply

$$1 \times 2 \times \cdots \times n.$$

The expression on the right of the $\prod$ symbol can be more complex. For example, consider the product

$$\prod_{k=1}^{5} (2k + 3).$$

This specifies that we multiply together the various values of $(2k + 3)$ for $k = 1, 2, 3, 4, 5$. In other words,

$$\prod_{k=1}^{5} (2k + 3) = 5 \times 7 \times 9 \times 11 \times 13.$$

The expression on the right of the $\prod$ can be simpler. For example,

$$\prod_{k=1}^{n} 2$$

is a fancy way to write 2^n.

Consider the following way of writing 0!:

$$\prod_{k=1}^{0} k.$$

This means that k starts at 1 and goes up to 0. Since there is no possible value of k with $1 \le k \le 0$, there are no terms to multiply. Therefore the product is empty and evaluates to 1.

Recap

In this section, we introduced factorial, discussed why $0! = 1$, and presented product notation.

9 Exercises

9.1. Solve the equation $n! = 720$ for n.

9.2. There are six different French books, eight different Russian books, and five different Spanish books.
- **a.** In how many different ways can these books be arranged on a bookshelf?
- **b.** In how many different ways can these books be arranged on a bookshelf if all books in the same language are grouped together?

9.3. Give an Alice-and-Bob discussion about what it means to add (and to multiply) a list of numbers that only contains one number.

9.4. Consider the formula

$$(n)_k = \frac{n!}{(n-k)!}.$$

This formula is mostly correct. For what values of n and k is it correct? Prove the formula is correct under a suitable hypothesis; that is, this problem asks you to find and prove a theorem of the form "If (conditions on n and k), then $(n)_k = n!/(n-k)!$."

9.5. Evaluate $\frac{100!}{98!}$ without calculating 100! or 98!.

9.6. Order the following integers from least to greatest: 2^{100}, 100^2, 100^{100}, 100!, 10^{10}.

9.7. The Scottish mathematician James Stirling found an approximation formula for $n!$. Stirling's formula is

$$n! \approx \sqrt{2\pi n}\, n^n e^{-n}$$

where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$. (Scientific calculators have a key that computes e^x; this key might be labeled $\boxed{\exp x}$.)

Compute $n!$ and Stirling's approximation to $n!$ for $n = 10, 20, 30, 40, 50$. What is the relative error in the approximations?

9.8. Calculate the following products:
- **a.** $\prod_{k=1}^{4}(2k+1)$.
- **b.** $\prod_{k=-3}^{4} k$.
- **c.** $\prod_{k=1}^{n} \frac{k+1}{k}$, where n is a positive integer.
- **d.** $\prod_{k=1}^{n} \frac{1}{k}$, where n is a positive integer.

9.9. Please calculate the following:
- **a.** $1 \times 1!$.
- **b.** $1 \times 1! + 2 \times 2!$.
- **c.** $1 \times 1! + 2 \times 2! + 3 \times 3!$.
- **d.** $1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4!$.
- **e.** $1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + 5 \times 5!$.

Now, make a conjecture. That is, guess the value of

$$\sum_{k=1}^{n} k \cdot k!.$$

You do not have to prove your answer.

9.10. When 100! is written out in full, it equals

$$100! = 9332621\ldots000000.$$

Without using a computer, determine the number of 0 digits at the end of this number.

9.11. Prove that all of the following numbers are composite: $1000!+2$, $1000!+3$, $1000!+4$, $\ldots$, $1000!+1002$.

The point of this problem is to present a long list of consecutive integers, all of which are composite.

9.12. A *factorion* is a positive integer with the following cute property. When written in ordinary base-10, it equals the sum of the factorials of its digits.

For example, 145 is a factorion because

$$1! + 4! + 5! = 1 + 24 + 120 = 145.$$

The numbers 1 and 2 are also factorions (because $1! = 1$ and $2! = 2$). There is only one other factorion; find it!

We know of no nice "pencil and paper" solution to this problem. You're best off solving this with the help of a computer program.

9.13. Can factorial be extended to negative integers? On the basis of equation (3), what value should $(-1)!$ be given?

9.14. Evaluate: 0^0.

9.15. The *double factorial* $n!!$ is defined for odd positive integers n; it is the product of all the odd numbers from 1 to n inclusive. For example, $7!! = 1 \times 3 \times 5 \times 7 = 105$. Please answer the following.

a. Evaluate 9!!.

b. For an odd integer n, are $n!!$ and $(n!)!$ equal?

c. Write an expression for $n!!$ using product notation.

d. Explain why this formula works:

$$(2k-1)!! = \frac{(2k)!}{k!2^k}.$$

The remaining exercises in this section require calculus.

16. Let n be a positive integer. What is the n^{th} derivative of x^n?

17. The following formula appears in W.A. Granville's *Elements of the Differential and Integral Calculus* (revised) published in 1911:

$$f(x) = f(a) + \frac{(x-a)}{\underline{|1}} f'(a) + \frac{(x-a)^2}{\underline{|2}} f''(a) + \frac{(x-a)^3}{\underline{|3}} f'''(a) + \cdots$$

Explain the notation used in the denominators.

18. Evaluate the following integral for $n = 0, 1, 2, 3, 4$:

$$\int_0^\infty x^n e^{-x}\, dx.$$

Note: The case $n = 0$ is easiest. Do the remaining values of n in order (first 1, then 2, etc.) and use integration by parts.

What is the value of this integral for an arbitrary natural number n?

Extra for experts: Evaluate the integral with $n = \frac{1}{2}$.

10 Sets I: Introduction, Subsets

A set is a repetition-free, unordered collection of objects. A given object either is a member of a set or it is not—an object cannot be in a set "more than once." There is no order to the members of a set. The simplest way to specify a set is to list its elements between curly braces. For example, $\{2, 3, \frac{1}{2}\}$ is a set with exactly three members: the integers 2 and 3, and the rational number $\frac{1}{2}$. No other objects are in this set. All of the following sets are the same:

$$\{2, 3, \tfrac{1}{2}\} \qquad \{3, \tfrac{1}{2}, 2\} \qquad \{2, 2, 3, \tfrac{1}{2}\}.$$

It does not matter in what order we list the objects, nor does it matter if we repeat an object. All that matters is what objects are members of the set and what objects are not. In this example, exactly three objects are members of the set; no other objects are members.

Earlier, we introduced three special sets of numbers. These sets are $\mathbb{Z}$ (the integers), $\mathbb{N}$ (the natural numbers), and $\mathbb{Q}$ (the rational numbers).

An object that belongs to a set is called an *element* of the set.

Membership in a set is denoted with the symbol $\in$. The notation $x \in A$ means that the object x is a member of the set A. For example, $2 \in \{2, 3, \frac{1}{2}\}$ is true, but $5 \in \{2, 3, \frac{1}{2}\}$ is false. In the latter case, we can write $5 \notin \{2, 3, \frac{1}{2}\}$; the notation $x \notin A$ means x is not an element of A.

When read aloud, $\in$ is pronounced "is a member of" or "is an element of" or "is in." Often mathematicians write, "If $x \in \mathbb{Z}$, then...." This means exactly the same thing as "If x is an integer, then...."

However, the $\in$ symbol can also stand for "be a member of" or "be in." For example, if we write "Let $x \in \mathbb{Z}$," we mean "Let x be a member of $\mathbb{Z}$" or, more prosaically, "Let x be an integer."

Absolute value bars around a set stand for the *cardinality* or *size* of the set (i.e., the number of elements in that set). An alternative notation for the cardinality of a set is $\#A$.

The number of elements in a set A is denoted $|A|$. The *cardinality* of A is simply the number of objects in the set. The cardinality of the set $\{2, 3, \frac{1}{2}\}$ is 3. The cardinality of $\mathbb{Z}$ is infinite. We also call $|A|$ the *size* of the set A.

A set is called *finite* if its cardinality is an integer (i.e., is finite). Otherwise, it is called *infinite*.

The empty set is also known as the *null set*.

The *empty set* is the set with no members. The empty set may be denoted $\{\ \}$, but it is better to use the special symbol $\emptyset$. The statement "$x \in \emptyset$" is false regardless of what object x might represent. The cardinality of the empty set is zero (i.e., $|\emptyset| = 0$).

Please note that the symbol $\emptyset$ is not the same as the Greek letter phi: ϕ or Φ.

There are two principal ways of specifying a set. The most direct way is to list the elements of the set between curly braces, as in $\{3, 4, 9\}$. This notation is appropriate for small sets. More often, *set-builder notation* is used. The form of this notation is

Set-builder notation.

$$\{\text{dummy variable} : \text{conditions}\}.$$

For example, consider

$$\{x : x \in \mathbb{Z},\ x \geq 0\}.$$

This is the set of all objects x that satisfy two conditions: (1) $x \in \mathbb{Z}$ (i.e., x must be an integer) and (2) $x \geq 0$ (i.e., x is nonnegative). In other words, this set is $\mathbb{N}$, the natural numbers.

An alternative way of writing set-builder notation is

$$\{\text{dummy variable} \in \text{set} : \text{conditions}\}.$$

This is the set of all objects drawn from the set mentioned and subject to the conditions specified. For example,

$$\{x \in \mathbb{Z} : 2|x\}$$

is the set of all integers that are divisible by 2 (i.e., the set of even integers).

Proof Template 5 Proving two sets are equal.

Let A and B be the sets. To show $A = B$, we have the following template:

- Suppose $x \in A$. ... Therefore $x \in B$.
- Suppose $x \in B$. ... Therefore $x \in A$.

Therefore $A = B$. ■

It is often tempting to write a set by establishing a pattern to the elements and then using three dots (...) to indicate that the pattern continues. For example, we might write $\{1, 2, 3, \ldots, 100\}$ to denote the set of integers from 1 to 100 inclusive. In this case, the notation is clear, but it would be better to write $\{x \in \mathbb{Z} : 1 \leq x \leq 100\}$.

Here is another example, which is less clear: $\{3, 5, 7, \ldots\}$. What is intended? We have to guess whether we mean the set of odd integers greater than 1 or the set of odd primes. Use the "..." notation sparingly and only when there is absolutely no chance of confusion.

Equality of Sets

What does it mean for two sets to be *equal*? It means that the two sets have exactly the same elements. To prove that sets A and B are equal, one shows that every element of A is also an element of B, and vice versa.

Let us illustrate the use of Proof Template 5 on a simple statement.

Proposition 10.1 The following two sets are equal:

$$E = \{x \in \mathbb{Z} : x \text{ is even}\}, \quad \text{and}$$
$$F = \{z \in \mathbb{Z} : z = a + b \text{ where } a \text{ and } b \text{ are both odd}\}.$$

In other words, the set F is the set of all integers that can be written as the sum of two odd numbers. Using the template, the proof looks like this:

> Let $E = \{x \in \mathbb{Z} : x \text{ is even}\}$ and $F = \{z \in \mathbb{Z} : z = a+b \text{ where } a \text{ and } b \text{ are both odd}\}$. We seek to prove that $E = F$.
>
> Suppose $x \in E$. ... Therefore $x \in F$.
> Suppose $x \in F$. ... Therefore $x \in E$. ■

Start with the first half by unraveling definitions.

> Let $E = \{x \in \mathbb{Z} : x \text{ is even}\}$ and $F = \{z \in \mathbb{Z} : z = a+b \text{ where } a \text{ and } b \text{ are both odd}\}$. We seek to prove that $E = F$.
>
> Suppose $x \in E$. Therefore x is even, and hence divisible by 2, so $x = 2y$ for some integer y. ... Therefore x is the sum of two odd numbers and so $x \in F$.
>
> Suppose $x \in F$. ... Therefore $x \in E$. ■

We have that $x = 2y$, and we want x as the sum of two odd numbers. Here's a simple way to do this: $2y + 1$ is odd (see Definition 3.4) and so is -1 (because $-1 = 2 \times (-1) + 1$). So we can write

$$x = 2y = (2y + 1) + (-1).$$

Let's fold these ideas into the proof.

> Let $E = \{x \in \mathbb{Z} : x \text{ is even}\}$ and $F = \{z \in \mathbb{Z} : z = a+b \text{ where } a \text{ and } b \text{ are both odd}\}$. We seek to prove that $E = F$.
>
> Suppose $x \in E$. Therefore x is even, and hence divisible by 2, so $x = 2y$ for some integer y. Note that $2y + 1$ and -1 are both odd and since $x = 2y = (2y + 1) + (-1)$ we see that x is the sum of two odd numbers. Therefore $x \in F$.
>
> Suppose $x \in F$. ... Therefore $x \in E$. ■

The second part of the proof was already considered in Exercise 5.1 (and the solution to that exercise can be found in Appendix A). So we simply refer to that previously worked problem to complete the proof.

Let $E = \{x \in \mathbb{Z} : x \text{ is even}\}$ and $F = \{z \in \mathbb{Z} : z = a+b \text{ where } a \text{ and } b \text{ are both odd}\}$. We seek to prove that $E = F$.

Suppose $x \in E$. Therefore x is even, and hence divisible by 2, so $x = 2y$ for some integer y. Note that $2y + 1$ and -1 are both odd, and since $x = 2y = (2y + 1) + (-1)$, we see that x is the sum of two odd numbers. Therefore $x \in F$.

Suppose $x \in F$. Therefore x is the sum of two odd numbers. As we showed in Exercise 5.1, x must be even and so $x \in E$. ■

Note that Proposition 10.1 can be rewritten as follows: *An integer is even if and only if it can be expressed as the sum of two odd numbers.*

Subset

Next we define *subset*.

Definition 10.2 **(Subset)** Suppose A and B are sets. We say that A is a *subset* of B provided every element of A is also an element of B. The notation $A \subseteq B$ means A is a subset of B.

For example, $\{1, 2, 3\}$ is a subset of $\{1, 2, 3, 4\}$. For any set A, we have $A \subseteq A$ because every element of A is (of course) in A.

Furthermore, for any set A, we have $\emptyset \subseteq A$. This is because every element of $\emptyset$ is in A—since there are no elements in $\emptyset$, there are no elements of $\emptyset$ that fail to be in A. This is an example of a vacuous statement, but a useful one.

The symbol $\subset$ is often used for subset as well, but we do not use it in this book. We prefer $\subseteq$ because it looks more like $\leq$, and we want to emphasize that a set is always a subset of itself. (The symbol $\subseteq$ is a hybrid of the symbols $\subset$ and $=$.) If we want to rule out the equality of the two sets, we may say that A is a *strict* or *proper* subset of B; this means that $A \subseteq B$ and $A \neq B$. It would be tempting to let $\subset$ denote proper subset (because it looks like $<$), but the use of $\subset$ to mean ordinary subset has not completely fallen out of fashion in the mathematics community. We avoid controversy by not using the symbol $\subset$.

$\subseteq$ and $\in$ have related but different meanings. They cannot be interchanged!

It is important to distinguish between $\in$ and $\subseteq$. The notation $x \in A$ means that x is an element (or member) of A. The notation $A \subseteq B$ means that every element of A is also an element of B. Thus $\emptyset \subseteq \{1, 2, 3\}$ is true, but $\emptyset \in \{1, 2, 3\}$ is false.

The difference between $\in$ and $\subset$ is analogous to the difference between x and $\{x\}$. The symbol x refers to some object (a number or whatever), and the notation $\{x\}$ means the set whose one and only element is the object x. It is always correct to write $x \in \{x\}$, but it is incorrect to write $x = \{x\}$ or $x \subseteq \{x\}$. (Well, it *usually* is incorrect to write $x \subseteq \{x\}$; see Exercise 10.14.)

To prove that one set is a subset of another, we need to show that every element of the first set is also a member of the second set.

Proposition 10.3 Let x be anything and let A be a set; then $x \in A$ if and only if $\{x\} \subseteq A$.

Proof. Let x be any object and let A be a set.

($\Rightarrow$) Suppose that $x \in A$. We want to show $\{x\} \subseteq A$. To do this, we need to show that every element of $\{x\}$ is also an element of A. But the only element of $\{x\}$ is x, and we are given that $x \in A$. Therefore $\{x\} \subseteq A$.

($\Leftarrow$) Suppose that $\{x\} \subseteq A$. This means that every element of the first set ($\{x\}$) is also a member of the second set (A). But the only element of $\{x\}$ is certainly x and so $x \in A$. ■

The general method for showing that one set is a subset of another is outlined in Proof Template 6.

Proof Template 6 Proving one set is a subset of another.

To show $A \subseteq B$:
Let $x \in A$. ... Therefore $x \in B$. Therefore $A \subseteq B$. ■

We illustrate the use of Proof Template 6 using the following concept.

Definition 10.4 **(Pythagorean Triple)** A list of three integers (a, b, c) is called a *Pythagorean triple* provided $a^2 + b^2 = c^2$.

Please note that $(\sqrt{2}, \sqrt{3}, \sqrt{5})$ is not a Pythagorean triple because the numbers in the list are not integers; the term *Pythagorean triple* only applies to lists of integers.

For example, $(3, 4, 5)$ is a Pythagorean triple because $3^2 + 4^2 = 5^2$. Pythagorean triples are so named because they are the lengths of the sides of a right triangle.

Proposition 10.5 Let P be the set of Pythagorean triples; that is,

$$P = \{(a, b, c) : a, b, c \in \mathbb{Z} \text{ and } a^2 + b^2 = c^2\}$$

and let T be the set

$$T = \{(p, q, r) : p = x^2 - y^2,\ q = 2xy,\ \text{and } r = x^2 + y^2 \text{ where } x, y \in \mathbb{Z}\}.$$

Then $T \subseteq P$.

For example, if we let $x = 3$ and $y = 2$ and we calculate

$$p = x^2 - y^2 = 9 - 4 = 5, \qquad q = 2xy = 12, \qquad r = x^2 + y^2 = 9 + 4 = 13$$

we find that $(5, 12, 13) \in T$. Proposition 10.5 asserts that $T \subseteq P$, which implies $(5, 12, 13) \in P$. Indeed, this is correct since

$$5^2 + 12^2 = 25 + 144 = 169 = 13^2.$$

We now develop the proof of Proposition 10.5 by utilizing Proof Template 6.

> Let P and T be as in the statement of Proposition 10.5.
> Let $(p, q, r) \in T$. ... Therefore $(p, q, r) \in P$. ■

Unravel the meaning of $(p, q, r) \in T$.

> Let P and T be as in the statement of Proposition 10.5.
> Let $(p, q, r) \in T$. Therefore there are integers x and y such that $p = x^2 - y^2$, $q = 2xy$, and $r = x^2 + y^2$. ... Therefore $(p, q, r) \in P$. ■

To verify that $(p, q, r) \in P$, we simply have to check that all three are integers (which is clear) and that $p^2 + q^2 = r^2$. We can write p, q, and r in terms of x and y, so the problem reduces to an algebraic computation. We finish the proof.

Let P and T be as in the statement of Proposition 10.5.

Let $(p, q, r) \in T$. Therefore there are integers x and y such that $p = x^2 - y^2$, $q = 2xy$, and $r = x^2 + y^2$. Note that p, q, and r are integers because x and y are integers. We calculate

$$\begin{aligned} p^2 + q^2 &= (x^2 - y^2)^2 + (2xy)^2 \\ &= (x^4 - 2x^2y^2 + y^4) + 4x^2y^2 \\ &= x^4 + 2x^2y^2 + y^4 \\ &= (x^2 + y^2)^2 = r^2. \end{aligned}$$

Therefore (p, q, r) is a Pythagorean triple and so $(p, q, r) \in P$. ■

The symbols $\in$ and $\subseteq$ may be written backward: $\ni$ and $\supseteq$. The notation $A \ni x$ means exactly the same thing as $x \in A$. The symbol $\ni$ can be read, "contains the element." The notation $B \supseteq A$ means exactly the same thing as $A \subseteq B$. We say that B is a *superset* of A.

(We also say that B contains A and A is contained in B, but the word *contains* can be a bit ambiguous. If we say "B contains A," we generally mean that $B \supseteq A$, but it might mean $B \ni A$. We avoid this term unless the meaning is utterly clear from context.)

Counting Subsets

How many subsets does a set have? Let us consider an example.

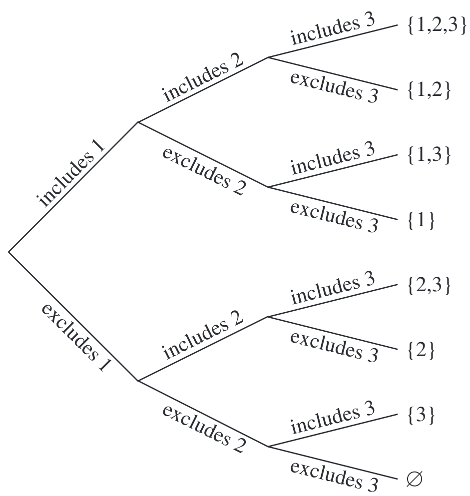
Example 10.6 How many subsets does $A = \{1, 2, 3\}$ have?

The easiest way to do this is to list all the possibilities. Since $|A| = 3$, a subset of A can have anywhere from zero to three elements. Let's write down all the possibilities organized this way.

Number of elements	Subsets	Number
0	$\emptyset$	1
1	$\{1\}, \{2\}, \{3\}$	3
2	$\{1, 2\}, \{1, 3\}, \{2, 3\}$	3
3	$\{1, 2, 3\}$	1
	Total:	8

Therefore, there are eight subsets of $\{1, 2, 3\}$.

There is another way to analyze this problem. Each element of the set $\{1, 2, 3\}$ either is a member of or is not a member of a subset. Look at the following diagram.



For each element, we have two choices: to include or not to include that element in the subset. We can "ask" each element if it "wants" to be in the subset. The list of answers uniquely determines the subset. So if we ask elements 1, 2, and 3 in turn if they are in the subset and the answers we receive are (yes, yes, no), then the subset is $\{1, 2\}$.

The problem of counting subsets of $\{1, 2, 3\}$ reduces to the problem of counting lists, and we know how to count lists! The number of lists of length three where each entry on the list is either "yes" or "no" is $2 \times 2 \times 2 = 8$.

This list-counting method gives us the solution to the general problem.

Theorem 10.7 Let A be a finite set. The number of subsets of A is $2^{|A|}$.

Proof. Let A be a finite set and let $n = |A|$. Let the n elements of A be named $a_1, a_2, \ldots, a_n$. To each subset B of A we can associate a list of length n; each element of the list is one of the words "yes" or "no." The k^{th} element of the list is "yes" precisely when $a_k \in B$. This establishes a correspondence between length-n yes-no lists and subsets of A. Observe that each subset of A gives a yes-no list, and every yes-no list determines a different subset of A. Therefore the number of subsets of A is exactly the same as the number of length-n yes-no lists. The number of such lists is 2^n, so the number of subsets of A is 2^n where $n = |A|$. ■

This style of proof is called a *bijective* proof. To show that two counting problems have the same answer, we establish a one-to-one correspondence between the two sets we want to count. If we know the answer to one of the counting problems, then we know the answer to the other.

Power Set

A set can be an element of another set. For example, $\{1, 2, \{3, 4\}\}$ is a set with three elements: the number 1, the number 2, and the set $\{3, 4\}$. A special example of this is called the *power set* of a set.

Definition 10.8 **(Power set)** Let A be a set. The *power set* of A is the set of all subsets of A.

For example, the power set of $\{1, 2, 3\}$ is the set

$$\Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}.$$

The power set of A is denoted 2^A. However, some authors also use the notation $\mathcal{P}(A)$.

Theorem 10.7 tells us that if a set A has n elements, its power set contains 2^n elements (the subsets of A). As a mnemonic, the notation for the power set of A is 2^A. This is a special notation; there is no general meaning for raising a number to an exponent that is a set. The only case in which this makes sense is writing the set as a superscript on the number 2; the meaning of the notation is the power set of A. This notation was created so that we would have

$$\left|2^A\right| = 2^{|A|}$$

for any finite set A. The left side of this equation is the cardinality of the power set of A; the right side is 2 raised to the cardinality of A. On the left, the superscript on 2 is a set, so the notation means power set; on the right, the superscript on 2 is a number, so the notation means ordinary exponentiation.

Recap

In this section, we introduced the concept of a set and the notation $x \in A$. We presented the set-builder notation $\{x \in A : \ldots\}$. We discussed the concepts of empty set ($\emptyset$), subset ($\subseteq$), and superset ($\supseteq$). We distinguished between finite and infinite sets and presented the notation $|A|$ for the cardinality of A. We considered the problem of counting the number of subsets of a finite set and defined the power set of a set, 2^A.

10 Exercises

10.1. Write out the following sets by listing their elements between curly braces.

a. $\{x \in \mathbb{N} : x \leq 10 \text{ and } 3|x\}$.
b. $\{x \in \mathbb{Z} : x \text{ is prime and } 2|x\}$.
c. $\{x \in \mathbb{Z} : x^2 = 4\}$.
d. $\{x \in \mathbb{Z} : x^2 = 5\}$.
e. $2^{\emptyset}$.
f. $\{x \in \mathbb{Z} : 10|x \text{ and } x|100\}$.
g. $\{x : x \subseteq \{1, 2, 3, 4, 5\} \text{ and } |x| \leq 1\}$.

10.2. For each of the following sets, find a way to rewrite the set using set-builder notation (rather than listing the elements).

a. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
b. $\{-8, -6, -4, -2, 0, 2, 4, 6, 8\}$.
c. $\{1, 3, 5, 7, 9, 11, 13, \ldots\}$.
d. $\{1, 4, 9, 16, 25, 36, 64, 81, 100\}$.

10.3. Find the cardinality of the following sets.

a. $\{x \in \mathbb{Z} : |x| \leq 10\}$.
b. $\{x \in \mathbb{Z} : 1 \leq x^2 \leq 2\}$.
c. $\{x \in \mathbb{Z} : x \in \emptyset\}$.
d. $\{x \in \mathbb{Z} : \emptyset \in x\}$.
e. $\{x \in \mathbb{Z} : \emptyset \subseteq \{x\}\}$.
f. $2^{2^{\{1,2,3\}}}$.
g. $\{x \in 2^{\{1,2,3,4\}} : |x| = 1\}$.
h. $\big\{\{1, 2\}, \{3, 4, 5\}\big\}$.

10.4. Complete each of the following by writing either $\in$ or $\subseteq$ in place of the $\bigcirc$.

a. $2 \bigcirc \{1, 2, 3\}$.
b. $\{2\} \bigcirc \{1, 2, 3\}$.
c. $\{2\} \bigcirc \big\{\{1\}, \{2\}, \{3\}\big\}$.
d. $\emptyset \bigcirc \{1, 2, 3\}$.
e. $\mathbb{N} \bigcirc \mathbb{Z}$.
f. $\{2\} \bigcirc \mathbb{Z}$.
g. $\{2\} \bigcirc 2^{\mathbb{Z}}$.

10.5. In each part of this exercise, please find three different sets and/or numbers A, B, and C to make the statement true:

a. $A \subseteq B \subseteq C$.
b. $A \in B \subseteq C$.
c. $A \in B \in C$.
d. $A \subseteq B \in C$

10.6. In each part of this exercise, find a set A that makes the sentence true or explain why no solution can be found.

a. $\emptyset \subseteq A$.
b. $\emptyset \in A$.
c. $A \subseteq \emptyset$.
d. $A \in \emptyset$.

10.7. For each of the following statements about sets A, B, and C, either prove the statement is true or give a counterexample to show that it is false.

a. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
b. If $A \in B$ and $B \subseteq C$, then $A \subseteq C$.
c. If $A \in B$ and $B \subseteq C$, then $A \in C$.
d. If $A \in B$ and $B \in C$, then $A \in C$.

10.8. Let A and B be sets. Prove that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(This gives a slightly different proof strategy for showing two sets are equal; compare to Proof Template 5.)

10.9. Let A, B, and C be sets and suppose $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$. Prove that $A = C$.

10.10. Let $A = \{x \in \mathbb{Z} : 4|x\}$ and let $B = \{x \in \mathbb{Z} : 2|x\}$. Prove that $A \subseteq B$.

10.11. Generalize the previous problem. Let a and b be integers and let $A = \{x \in \mathbb{Z} : a|x\}$ and $B = \{x \in \mathbb{Z} : b|x\}$.

Find and prove a necessary and sufficient condition for $A \subseteq B$. In other words, given the notation developed, find and prove a theorem of the form "$A \subseteq B$ if and only if *some condition involving a and b*."

10.12. Let $C = \{x \in \mathbb{Z} : x|12\}$ and let $D = \{x \in \mathbb{Z} : x|36\}$. Prove that $C \subseteq D$.

10.13. Generalize the previous problem. Let c and d be integers and let $C = \{x \in \mathbb{Z} : x|c\}$ and $D = \{x \in \mathbb{Z} : x|d\}$.

Find and prove a necessary and sufficient condition for $C \subseteq D$.

10.14. Give an example of an object x that makes the sentence $x \subseteq \{x\}$ true.

10.15. Please refer to Proposition 10.5, in which we proved that $T \subseteq P$. Show that $T \neq P$.

11 Quantifiers

There are certain phrases that appear frequently in theorems, and the purpose of this section is to clarify and formalize those phrases. At first glance, these phrases are simple, but we'll do our best to try to make them complicated. The expressions are *there is* and *every*.

There Is

Consider a sentence such as the following:

There is a natural number that is prime and even.

The general form of this sentence is "There is an object x, a member of set A, that has the following properties." The example sentence can be rewritten to adhere more strictly to this form as follows:

There is an x, a member of $\mathbb{N}$, such that x is prime and even.

The meaning of the sentence is, we hope, clear. It says that at least one element in $\mathbb{N}$ has the required properties. In this case, there is only one possible x (the number 2), but the phrase *there is* does not rule out the possibility that there can be more than one object with the requisite properties.

The phrase *there exists* is synonymous with *there is*.

Because the phrase *there is* occurs so often, mathematicians have developed a formal notation for statements of the form "There is an x in set A such that . . . ." We write a backward, uppercase E (i.e., $\exists$) that we pronounce *there is* or *there exists*. The general form for using this notation is

$$\exists x \in A, \text{ assertions about } x.$$

This is read, "There is an x, a member of the set A, for which the assertions hold." The sentence "There is a natural number that is prime and even" would be written

$$\exists x \in \mathbb{N}, \ x \text{ is prime and even.}$$

The letter x is a dummy variable—simply a placeholder. It is similar to the index of summation in $\sum$ notation.

At times, we abbreviate the statement "$\exists x \in A$, assertions about x" to "$\exists x$, assertions about x" when context makes it clear what sort of object x ought to be.

The backward E is called the *existential quantifier*.

To prove a statement of the form "$\exists x \in A$, assertions about x," we have to show that some element in A satisfies the assertions. The general form for such a proof is given in Proof Template 7.

Proof Template 7 Proving existential statements.

To prove $\exists x \in A$, assertions about x:

Let x be (give an explicit example) ... (Show that x satisfies the assertions.) ... Therefore x satisfies the required assertions. ■

Proving an existential statement is akin to finding a counterexample. We simply have to find one object with the required properties.

Example 11.1 Here is a proof (very short!) that there is an integer that is even and prime.

Statement: $\exists x \in \mathbb{Z},\ x$ is even and x is prime.

Proof. Consider the integer 2. Clearly 2 is even and 2 is prime. ■

For All

The other phrase we consider in this section is *every*, as in "Every integer is even or odd." There are alternative phrases we use in place of *every*, including *all*, *each*, and *any*. All of the following sentences mean the same thing:

- *Every* integer is either even or odd.
- *All* integers are either even or odd.
- *Each* integer is either even or odd.
- Let x be *any* integer. Then x is even or odd.

In all cases, we mean that the condition applies to all integers without exception.

There is a fancy notation for these types of sentences. Just as we used the backward E for *there is*, we use an upside-down A ($\forall$) as a notation for *all*. The general form for this notation is

$$\forall x \in A,\ \text{assertions about } x.$$

This means that all elements of the set A satisfy the assertions, as in

$$\forall x \in \mathbb{Z},\ x \text{ is odd or } x \text{ is even.}$$

When the context makes clear what sort of object x is, the notation may be shortened to "$\forall x$, assertions about x."

The upside-down A is called the *universal quantifier*.

To prove an "all" theorem, we need to show that every element of the set satisfies the required assertions. The general form for this sort of proof is given in Proof Template 8.

Proof Template 8 Proving universal statements.

To prove $\forall x \in A$, assertions about x:

Let x be any element of A.... (Show that x satisfies the assertions using only the fact that $x \in A$ and no further assumptions on x.) ... Therefore x satisfies the required assertions. ■

Example 11.2 To prove: Every integer that is divisible by 6 is even.

More formally, let $A = \{x \in \mathbb{Z} : 6|x\}$. Then the statement we want to prove is

$$\forall x \in A,\ x \text{ is even.}$$

Proof. Let $x \in A$; that is, x is an integer that is divisible by 6. This means there is an integer y such that $x = 6y$, which can be rewritten $x = (2 \cdot 3)y = 2(3y)$. Therefore x is divisible by 2 and therefore even. ■

Mathspeak!

Mathematicians use the word *arbitrary* in a slightly nonstandard way. When we say that x is an arbitrary element of a set A, we mean that x might be any element of A, and one should not assume anything about x other than it is an element of A. To say x is an arbitrary even number means that x is even, but we make no further assumptions about x.

Note that this proof is not significantly different from proving an ordinary if-then: "If x is divisible by 6, then x is even." The point we are trying to stress is that in the proof, we assume that x is an arbitrary element of A and then move on to show that x satisfies the condition.

Negating Quantified Statements

Consider the statements

- There is no integer that is both even and odd.
- Not all integers are prime.

Symbolically, these can be written

- $\neg\,(\exists x \in \mathbb{Z},\ x \text{ is even and } x \text{ is odd})$.
- $\neg\,(\forall x \in \mathbb{Z},\ x \text{ is prime})$.

In both cases, we have negated a quantified statement. What do these negations mean?

Let us first consider a statement of the form

$$\neg\,(\exists x \in A,\ \text{assertions about } x)\,.$$

This means that none of the elements of A satisfy the assertions, and this is equivalent to saying that *all* of the elements of A fail to satisfy the assertions. In other words, the following two sentences are equivalent:

$$\neg\,(\exists x \in A,\ \text{assertions about } x)$$
$$\forall x \in A,\ \neg\,(\text{assertions about } x)\,.$$

For example, the statement "There is no integer that is both even and odd" says the same thing as "Every integer is not both even and odd."

Next we consider the negation of universal statements. Consider a statement of the form

$$\neg\,(\forall x \in A,\ \text{assertions about } x)\,.$$

This means that not all of the elements of x have the requisite assertions (i.e., some don't). Thus the following two statements are equivalent:

$$\neg\,(\forall x \in A,\ \text{assertions about } x)$$
$$\exists x \in A,\ \neg\,(\text{assertions about } x)\,.$$

For example, the statement "Not all integers are prime" is equivalent to the statement "There is an integer that is not prime."

The mnemonic I use to remember these equivalences is

$$\neg\forall \ldots = \exists\neg \ldots \qquad \text{and} \qquad \neg\exists \ldots = \forall\neg \ldots.$$

When the $\neg$ "moves" inside the quantifier, it toggles the quantifier between $\forall$ and $\exists$.

Combining Quantifiers

Quantified statements can become difficult and confusing when there are two (or more!) quantifiers in the same statement. For example, consider the following statements about integers:

- For every x, there is a y, such that $x + y = 0$.
- There is a y, such that for every x, we have $x + y = 0$.

In symbols, these statements are written

- $\forall x,\ \exists y,\ x + y = 0$.
- $\exists y,\ \forall x,\ x + y = 0$.

What do these mean?

The first sentence makes a claim about an arbitrary integer x. It says that no matter what x is, something is true—namely, we can find an integer y such that $x + y = 0$. Let's say $x = 12$. Can we find a y such that $x + y = 0$? Of course! We just want $y = -12$. Say

$x = -53$. Can we find a y such that $x + y = 0$? Yes! Take $y = 53$. Notice that the y that satisfies $x = 12$ is different from the y that satisfies $x = -53$. The statement just requires that no matter how we pick x ($\forall x$), we can find a y ($\exists y$) such that $x + y = 0$. And this is a true statement. Here is the proof:

Let x be any integer. Let y be the integer $-x$. Then $x + y = x + (-x) = 0$. ■

Since the overall statement begins $\forall x$, we begin the proof by considering an arbitrary integer x. We now have to prove something about this number x—namely, we can find a number y such that $x + y = 0$. The choice for y is obvious, just take $y = -x$. The statement $\forall x, \exists y, x + y = 0$ is true.

Now let us examine the similar statement

$$\exists y, \forall x, x + y = 0.$$

This sentence is similar to the previous sentence; the only difference is the order of the quantifiers. This sentence alleges that there is an integer y with a certain property—namely, no matter what number we add to y ($\forall x$), we get 0 ($x + y = 0$). This sentence is blatantly false! There is no such integer y. No matter what integer y you might think of, we can always find an integer x such that $x + y$ is not zero.

The statements $\forall x, \exists y, x+y = 0$ and $\exists y, \forall x, x+y = 0$ are made a bit clearer through the use of parentheses. They may be rewritten as follows:

$$\forall x, (\exists y, x + y = 0)$$
$$\exists y, (\forall x, x + y = 0).$$

These additional parentheses are not strictly necessary, but if they make these statements clearer to you, please feel free to use them.

In general, the two sentences

$$\forall x, \exists y, \text{ assertions about } x \text{ and } y$$
$$\exists y, \forall x, \text{ assertions about } x \text{ and } y$$

are not equivalent to one another.

Recap

We analyzed statements of the form "For all ..." and "There exists ..." and introduced the formal quantifier notation for them. We presented basic proof templates for such sentences. We examined the negation of quantified sentences, and we studied statements with more than one quantifier.

11 Exercises

11.1. Write the following sentences using the quantifier notation (i.e., use the symbols $\exists$ and/or $\forall$). *Note*: We do not claim these statements are true, so please do not try to prove them!

a. Every integer is prime.
b. There is an integer that is neither prime nor composite.
c. There is an integer whose square is 2.
d. All integers are divisible by 5.
e. Some integer is divisible by 7.
f. The square of any integer is nonnegative.
g. For every integer x, there is an integer y such that $xy = 1$.
h. There are an integer x and an integer y such that $x/y = 10$.
i. There is an integer that, when multiplied by any integer, always gives the result 0.
j. No matter what integer you choose, there is always another integer that is larger.
k. Everybody loves somebody sometime.

11.2. Write the negation of each of the sentences in the previous problem. You should "move" the negation all the way inside the quantifiers. Give your answer in English and symbolically. For example, the negation of part (a) would be "There is an integer that is not prime" (English) and "$\exists x \in \mathbb{Z},\ x$ is not prime" (symbolic).

11.3. What does the sentence "Everyone is not invited to my party" mean?

Presumably the meaning of this sentence is not what the speaker intended. Rewrite this sentence to give the intended meaning.

11.4. *True or False*: Please label each of the following sentences about integers as either true or false. (You do not need to prove your assertions.)

a. $\forall x,\ \forall y,\ x + y = 0$.
b. $\forall x,\ \exists y,\ x + y = 0$.
c. $\exists x,\ \forall y,\ x + y = 0$.
d. $\exists x,\ \exists y,\ x + y = 0$.
e. $\forall x,\ \forall y,\ xy = 0$.
f. $\forall x,\ \exists y,\ xy = 0$.
g. $\exists x,\ \forall y,\ xy = 0$.
h. $\exists x,\ \exists y,\ xy = 0$.

11.5. For each of the following sentences, write the negation of the sentence, but place the $\neg$ symbol as far to the right as possible. Then rewrite the negation in English.

For example, for the sentence

$$\forall x \in \mathbb{Z}, x \text{ is odd}$$

the negation would be

$$\exists x \in \mathbb{Z}, \neg(x \text{ is odd}),$$

which in English is "There is an integer that is not odd."

a. $\forall x \in \mathbb{Z}, x < 0$.
b. $\exists x \in \mathbb{Z}, x = x + 1$.
c. $\exists x \in \mathbb{N}, x > 10$.
d. $\forall x \in \mathbb{N}, x + x = 2x$.
e. $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x > y$.
f. $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x = y$.
g. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 0$.

11.6. Do the following two statements mean the same thing?

$$\forall x,\ \forall y,\ \text{assertions about } x \text{ and } y$$
$$\forall y,\ \forall x,\ \text{assertions about } x \text{ and } y$$

Explain.

Likewise, do the following two statements mean the same thing?

$$\exists x,\ \exists y,\ \text{assertions about } x \text{ and } y$$
$$\exists y,\ \exists x,\ \text{assertions about } x \text{ and } y$$

Explain.

11.7. The notation $\exists!$ is sometimes used to indicate that there is exactly one object that satisfies the condition. For example, $\exists! x \in \mathbb{N}, x^2 = 1$ means there is a natural number x whose square is equal to 1 *and there is only one such* x. Of course, the realm of the integers, there are two numbers whose squares equal 1, so the statement $\exists! x \in \mathbb{Z}, x^2 = 1$ is false.

The notation $\exists!$ can be pronounced "there is a unique".

Which of the following statements are true? Support your answer with a brief explanation.

a. $\exists! x \in \mathbb{N}, x^2 = 4$.
b. $\exists! x \in \mathbb{Z}, x^2 = 4$.
c. $\exists! x \in \mathbb{N}, x^2 = 3$.
d. $\exists! x \in \mathbb{Z}, \forall y \in \mathbb{Z}, xy = x$.
e. $\exists! x \in \mathbb{Z}, \forall y \in \mathbb{Z}, xy = y$.