Earlier, we introduced three special sets of numbers. These sets are $\mathbb{Z}$ (the integers), $\mathbb{N}$ (the natural numbers), and $\mathbb{Q}$ (the rational numbers).

An object that belongs to a set is called an *element* of the set.

Membership in a set is denoted with the symbol $\in$. The notation $x \in A$ means that the object x is a member of the set A. For example, $2 \in \{2, 3, \frac{1}{2}\}$ is true, but $5 \in \{2, 3, \frac{1}{2}\}$ is false. In the latter case, we can write $5 \notin \{2, 3, \frac{1}{2}\}$; the notation $x \notin A$ means x is not an element of A.

When read aloud, $\in$ is pronounced "is a member of" or "is an element of" or "is in." Often mathematicians write, "If $x \in \mathbb{Z}$, then. . . ." This means exactly the same thing as "If x is an integer, then. . . ."

However, the $\in$ symbol can also stand for "be a member of" or "be in." For example, if we write "Let $x \in \mathbb{Z}$," we mean "Let x be a member of $\mathbb{Z}$" or, more prosaically, "Let x be an integer."

Absolute value bars around a set stand for the *cardinality* or *size* of the set (i.e., the number of elements in that set). An alternative notation for the cardinality of a set is $\#A$.

The number of elements in a set A is denoted $|A|$. The *cardinality* of A is simply the number of objects in the set. The cardinality of the set $\{2, 3, \frac{1}{2}\}$ is 3. The cardinality of $\mathbb{Z}$ is infinite. We also call $|A|$ the *size* of the set A.

A set is called *finite* if its cardinality is an integer (i.e., is finite). Otherwise, it is called *infinite*.

The empty set is also known as the *null set*.

The *empty set* is the set with no members. The empty set may be denoted $\{\ \}$, but it is better to use the special symbol $\emptyset$. The statement "$x \in \emptyset$" is false regardless of what object x might represent. The cardinality of the empty set is zero (i.e., $|\emptyset| = 0$).

Please note that the symbol $\emptyset$ is not the same as the Greek letter phi: ϕ or Φ.

Set-builder notation.

There are two principal ways of specifying a set. The most direct way is to list the elements of the set between curly braces, as in $\{3, 4, 9\}$. This notation is appropriate for small sets. More often, *set-builder notation* is used. The form of this notation is

$$\{\text{dummy variable} : \text{conditions}\}\,.$$

For example, consider

$$\{x : x \in \mathbb{Z},\ x \geq 0\}\,.$$

This is the set of all objects x that satisfy two conditions: (1) $x \in \mathbb{Z}$ (i.e., x must be an integer) and (2) $x \geq 0$ (i.e., x is nonnegative). In other words, this set is $\mathbb{N}$, the natural numbers.

An alternative way of writing set-builder notation is

$$\{\text{dummy variable} \in \text{set} : \text{conditions}\}\,.$$

This is the set of all objects drawn from the set mentioned and subject to the conditions specified. For example,

$$\{x \in \mathbb{Z} : 2|x\}$$

is the set of all integers that are divisible by 2 (i.e., the set of even integers).

Proof Template 5 Proving two sets are equal.

Let A and B be the sets. To show $A = B$, we have the following template:

- Suppose $x \in A$. . . . Therefore $x \in B$.
- Suppose $x \in B$. . . . Therefore $x \in A$.

Therefore $A = B$. ■

It is often tempting to write a set by establishing a pattern to the elements and then using three dots (. . .) to indicate that the pattern continues. For example, we might write $\{1, 2, 3, \ldots, 100\}$ to denote the set of integers from 1 to 100 inclusive. In this case, the notation is clear, but it would be better to write $\{x \in \mathbb{Z} : 1 \leq x \leq 100\}$.

Here is another example, which is less clear: $\{3, 5, 7, \ldots\}$. What is intended? We have to guess whether we mean the set of odd integers greater than 1 or the set of odd primes. Use the ". . . " notation sparingly and only when there is absolutely no chance of confusion.

Equality of Sets

What does it mean for two sets to be *equal*? It means that the two sets have exactly the same elements. To prove that sets A and B are equal, one shows that every element of A is also an element of B, and vice versa.

Let us illustrate the use of Proof Template 5 on a simple statement.

Proposition 10.1 The following two sets are equal:

$$E = \{x \in \mathbb{Z} : x \text{ is even}\}, \quad \text{and}$$
$$F = \{z \in \mathbb{Z} : z = a + b \text{ where } a \text{ and } b \text{ are both odd}\}.$$

In other words, the set F is the set of all integers that can be written as the sum of two odd numbers. Using the template, the proof looks like this:

> Let $E = \{x \in \mathbb{Z} : x \text{ is even}\}$ and $F = \{z \in \mathbb{Z} : z = a+b$ where a and b are both odd$\}$. We seek to prove that $E = F$.
>
> Suppose $x \in E$. ... Therefore $x \in F$.
> Suppose $x \in F$. ... Therefore $x \in E$. ■

Start with the first half by unraveling definitions.

> Let $E = \{x \in \mathbb{Z} : x \text{ is even}\}$ and $F = \{z \in \mathbb{Z} : z = a+b$ where a and b are both odd$\}$. We seek to prove that $E = F$.
>
> Suppose $x \in E$. Therefore x is even, and hence divisible by 2, so $x = 2y$ for some integer y. ... Therefore x is the sum of two odd numbers and so $x \in F$.
>
> Suppose $x \in F$. ... Therefore $x \in E$. ■

We have that $x = 2y$, and we want x as the sum of two odd numbers. Here's a simple way to do this: $2y + 1$ is odd (see Definition 3.4) and so is -1 (because $-1 = 2 \times (-1) + 1$). So we can write

$$x = 2y = (2y + 1) + (-1).$$

Let's fold these ideas into the proof.

> Let $E = \{x \in \mathbb{Z} : x \text{ is even}\}$ and $F = \{z \in \mathbb{Z} : z = a+b$ where a and b are both odd$\}$. We seek to prove that $E = F$.
>
> Suppose $x \in E$. Therefore x is even, and hence divisible by 2, so $x = 2y$ for some integer y. Note that $2y + 1$ and -1 are both odd and since $x = 2y = (2y + 1) + (-1)$ we see that x is the sum of two odd numbers. Therefore $x \in F$.
>
> Suppose $x \in F$. ... Therefore $x \in E$. ■

The second part of the proof was already considered in Exercise 5.1 (and the solution to that exercise can be found in Appendix A). So we simply refer to that previously worked problem to complete the proof.

Let $E = \{x \in \mathbb{Z} : x \text{ is even}\}$ and $F = \{z \in \mathbb{Z} : z = a+b \text{ where } a \text{ and } b \text{ are both odd}\}$. We seek to prove that $E = F$.

Suppose $x \in E$. Therefore x is even, and hence divisible by 2, so $x = 2y$ for some integer y. Note that $2y + 1$ and -1 are both odd, and since $x = 2y = (2y + 1) + (-1)$, we see that x is the sum of two odd numbers. Therefore $x \in F$.

Suppose $x \in F$. Therefore x is the sum of two odd numbers. As we showed in Exercise 5.1, x must be even and so $x \in E$. ■

Note that Proposition 10.1 can be rewritten as follows: *An integer is even if and only if it can be expressed as the sum of two odd numbers.*

Subset

Next we define *subset*.

Definition 10.2 **(Subset)** Suppose A and B are sets. We say that A is a *subset* of B provided every element of A is also an element of B. The notation $A \subseteq B$ means A is a subset of B.

For example, $\{1, 2, 3\}$ is a subset of $\{1, 2, 3, 4\}$. For any set A, we have $A \subseteq A$ because every element of A is (of course) in A.

Furthermore, for any set A, we have $\emptyset \subseteq A$. This is because every element of $\emptyset$ is in A—since there are no elements in $\emptyset$, there are no elements of $\emptyset$ that fail to be in A. This is an example of a vacuous statement, but a useful one.

The symbol $\subset$ is often used for subset as well, but we do not use it in this book. We prefer $\subseteq$ because it looks more like $\leq$, and we want to emphasize that a set is always a subset of itself. (The symbol $\subseteq$ is a hybrid of the symbols $\subset$ and $=$.) If we want to rule out the equality of the two sets, we may say that A is a *strict* or *proper* subset of B; this means that $A \subseteq B$ and $A \neq B$. It would be tempting to let $\subset$ denote proper subset (because it looks like $<$), but the use of $\subset$ to mean ordinary subset has not completely fallen out of fashion in the mathematics community. We avoid controversy by not using the symbol $\subset$.

$\subseteq$ and $\in$ have related but different meanings. They cannot be interchanged!

It is important to distinguish between $\in$ and $\subseteq$. The notation $x \in A$ means that x is an element (or member) of A. The notation $A \subseteq B$ means that every element of A is also an element of B. Thus $\emptyset \subseteq \{1, 2, 3\}$ is true, but $\emptyset \in \{1, 2, 3\}$ is false.

The difference between $\in$ and $\subset$ is analogous to the difference between x and $\{x\}$. The symbol x refers to some object (a number or whatever), and the notation $\{x\}$ means the set whose one and only element is the object x. It is always correct to write $x \in \{x\}$, but it is incorrect to write $x = \{x\}$ or $x \subseteq \{x\}$. (Well, it *usually* is incorrect to write $x \subseteq \{x\}$; see Exercise 10.14.)

To prove that one set is a subset of another, we need to show that every element of the first set is also a member of the second set.

Proposition 10.3 Let x be anything and let A be a set; then $x \in A$ if and only if $\{x\} \subseteq A$.

Proof. Let x be any object and let A be a set.

($\Rightarrow$) Suppose that $x \in A$. We want to show $\{x\} \subseteq A$. To do this, we need to show that every element of $\{x\}$ is also an element of A. But the only element of $\{x\}$ is x, and we are given that $x \in A$. Therefore $\{x\} \subseteq A$.

($\Leftarrow$) Suppose that $\{x\} \subseteq A$. This means that every element of the first set ($\{x\}$) is also a member of the second set (A). But the only element of $\{x\}$ is certainly x and so $x \in A$. ■

The general method for showing that one set is a subset of another is outlined in Proof Template 6.