Probability and Random Variables

Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables

Utility and Probability

Utility

$U(A) > U(B)$ indicates A is preferable to B

$U(A) = U(B)$ " indifferent

Probability

$P(A) > P(B)$ " A is more plausible than B

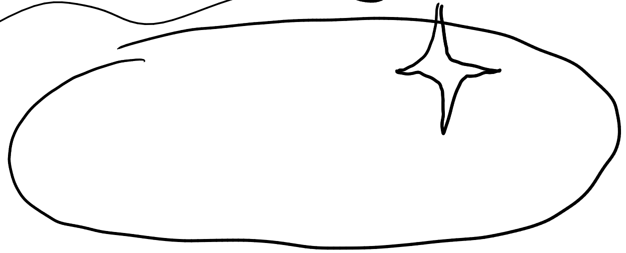
$P(A) = P(B)$ " A and B are equally plausible

Full story: https://projecteuclid.org/journals/statistical-science/volume-1/issue-3/The-Axioms-of-Subjective-Probability/10.1214/ss/1177013611.full

What is a Random Variable?

Today Only!

R.V. X



- Variable
  - finite set of values
  - probability

$P(X=1) = 0.5$

- Continuous/discrete
- related to other R.V.s

$P(X|Y)$

Blackbelly

sample space

measure

(Ω, Σ, μ)

σ algebra

$X : \Omega \to E$

Vocabulary/Notation

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Term	Definition	Coinflip Example	Uniform Example

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Term	Definition	Coinflip Example	Uniform Example
		$\mathrm{Bernoulli}(0.5)$	

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support(X)			

Vocabulary/Notation

Term	Definition	$\mathrm{Bernoulli}(0.5)$ **Coinflip Example**	$\mathcal{U}(0,1)$ **Uniform Example**
support(X)	All the values that X can take		

Vocabulary/Notation

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Vocabulary/Notation

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support(X) $x \in X$ $X \in [0,1]$	All the values that X can take		

Vocabulary/Notation

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support(X) $x \in X$ $X \in [0,1]$	All the values that X can take	$\{h, t\}$ or $\{0, 1\}$	

Vocabulary/Notation

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Distribution			

Vocabulary/Notation

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X	P(X)
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1	0.5

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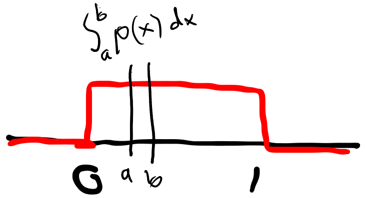

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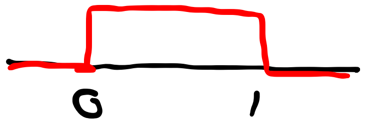

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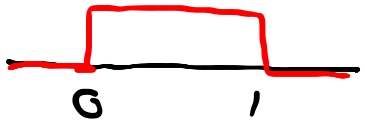

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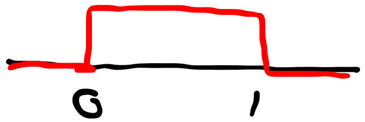

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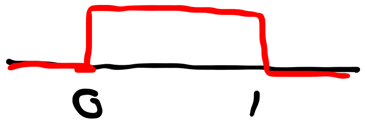
Expectation			



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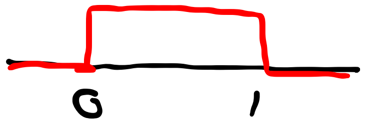
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X=1)=0.5$ $P(X=0)=0.5$ $P(X)$ is a table	 $p(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$ $P(X=1)=0$ $P(X \in [a,b]) = \int_a^b p(x)dx$
Expectation	First moment of the random variable, "mean"		

X	P(X)
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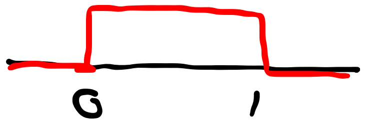

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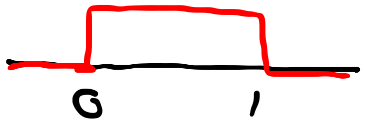
Expectation $\boxed{E[X]}$	First moment of the random variable, "mean"	$E[X] = \sum_{\underbrace{x \in X}} \underbrace{xP(x)}$	



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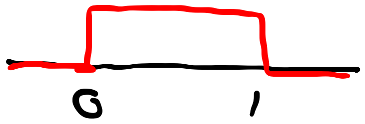
support(X) $x \in X$ $X \in [0,1]$	All the values that X can take	$\{h, t\}$ or $\{0, 1\}$	$[0, 1]$ 
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Expectation $E[X]$	First moment of the random variable, "mean"	$E[X] = \sum_{x \in X} x P(x)$ $= 0.5$	

Vocabulary/Notation

Term | **Definition** | $\text{Bernoulli}(0.5)$ **Coinflip Example** | $\mathcal{U}(0,1)$ **Uniform Example**

support(X)

$x \in X$

$X \in [0,1]$

All the values that X can take

$\{h, t\}$ or $\{0, 1\}$

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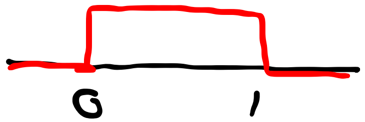

Distribution

- Discrete: PMF
- Continuous: PDF

Maps each value in the support to a real number indicating its probability

$$P(X=1) = 0.5$$
$$P(X=0) = 0.5$$

$P(X)$ is a table

X	P(X)
0	0.5
1	0.5

$$p(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$$

$$P(X=1) = 0$$

$$P(X \in [a,b]) = \int_a^b p(x)dx$$

Expectation

$E[X]$

First moment of the random variable, "mean"

$$E[X] = \sum_{x \in X} xP(x)$$
$$= 0.5$$

$$E[X] = \int_{x \in X} xp(x)dx$$

Vocabulary/Notation

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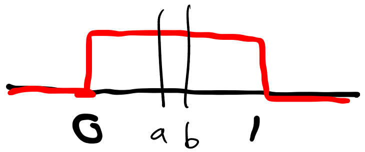
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Distributions of related R.V.s

Distributions of related R.V.s

Joint Distribution

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Distributions of related R.V.s

Joint Distribution

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Conditional Distribution

Distributions of related R.V.s

Joint Distribution

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Conditional Distribution

$$P(X \mid Y, Z)$$

Distributions of related R.V.s

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1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
0	0.84
1	0.16

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
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Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
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1	0.16

Marginal Distribution

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
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Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

Distributions of related R.V.s

Joint Distribution

$$P(X,Y,Z)$$

X	Y	Z	$P(X,Y,Z)$
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Conditional Distribution

$$P(X \mid Y,Z)$$

(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
0	0.84
1	0.16

Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

X	$P(X)$
0	0.85
1	0.15

Y	$P(Y)$
0	0.45
1	0.55

Z	$P(Z)$
0	0.20
1	0.80

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

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Distributions of related R.V.s

Joint Distribution

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Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

3 Rules

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

3 Rules (Burrito-level)

Distributions of related R.V.s

Joint Distribution	Conditional Distribution	Marginal Distribution
$P(X, Y, Z)$	$P(X \mid Y, Z)$	$P(X)\ P(Y)\ P(Z)$

3 Rules

(Burrito-level)

(Blackbelly Level: Axioms of Probability)

not on exam

AXIOM 1. STRUCTURE OF UNKNOWN REAL NUMBERS AND PLAUSIBLE VALUE. *We assume a set T of unknown numbers is a partially ordered commutative algebra over $\mathbb{R}$ with identity, 1.*

We assume in addition a given sub-Boolean algebra E of $E(T)$ with $0, 1 \in E$ and denote by E_0 the set of non-zero members of E. We assume that the partial ordering in $E(T)$ as a Boolean algebra coincides with the ordering that $E(T)$ inherits from the algebra T. Finally, we assume a function $PV : T \times E_0 \to \mathbb{R}$, called **PLAUSIBLE VALUE**, *whose value on the pair (x, e) is denoted $PV(x|e)$.*

AXIOM 2. STRONG RESCALING FOR PLAUSIBLE VALUE. *If a, b belong to $\mathbb{R}$, if x belongs to T, and if e belongs to E_0, then*

$$PV(ax + b|e) = aPV(x|e) + b. \tag{2}$$

AXIOM 3. ORDER CONSISTENCY FOR PLAUSIBLE VALUE. *If $x, y \in T$ and if $e \in E_0$, implies that $x \leq y$, then $PV(x|e) \leq PV(y|e)$.*

Notice that if $e \in E(T)$, then $0 \leq e \leq 1$, in T, as it is true in the lattice ordering of $E(T)$.

AXIOM 4. THE COX AXIOM FOR PLAUSIBLE VALUE: *If e, c are fixed in E, with $ec \in E_0$, if x_1, x_2 are in T, if $PV(x_1|ec) = PV(x_2|ec)$, then $PV(x_1 e|c) = PV(x_2 e|c)$. That is, we assume that as a function of x, the plausible value $PV(xe|c)$ depends only on $PV(x|ec)$.*

AXIOM 5. RESTRICTED ADDITIVITY OF PLAUSIBLE VALUE. *For each fixed $y \in T$ and $e \in E_0$, the plausible value $PV(x + y|e)$ as a function of $x \in T$ depends only on $PV(x|e)$, which is to say that if $x_1, x_2 \in T$ and $PV(x_1|e) = PV(x_2|e)$, then $PV(x_1 + y|e) = PV(x_2 + y|e)$.*

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

3 Rules

(Burrito-level)

1)

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Marginal + Conditional $\rightarrow$ Joint

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X	Y	Z	P(X,Y,Z)
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

$P(X)$

X	P(X)
0	0.85 = 0.08 + 0.31 + 0.09 + 0.37
1	0.15

$$P(X=0) = \sum_{y\in Y, z\in Z} P(X=0, Y=y, Z=z)$$

$$P(X=0 \mid Y=1, Z=1) = \frac{P(X=0, Y=1, Z=1)}{P(Y=1, Z=1)} = \frac{0.37}{0.37+0.07} =$$

Joint $\rightarrow$ Marginal

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Break

$P(C=1) = P(C=1, P=1) + P(C=1, P=0)$

$0.8 \quad 0.1$

P	C	
0	0	0.1
0	1	0.7
1	0	0.1
1	1	0.1

$P(C=0|P=1)P(P=1)$

$P(C=1|P=1)P(P=1)$

0.5

- $P \in \{0,1\}$: Powder Day
- $C \in \{0,1\}$: Pass Clear
- 1 in 5 days is a powder day $P(P=1) = 0.2$
- The pass is clear 8 in 10 days $P(C=1) = 0.8$
- If it is a powder day, there is a 50% chance the pass is blocked $P(C=0 \mid P=1) = 0.5$

$P(C=1|P=1) + P(C=0|P=1) = 1.0$

0.5

- Write out the joint probability distribution for P and C.
- Suppose it is a non-powder day, what is the probability that the pass is blocked?

$P(C=0 \mid P=0) = \frac{P(C=0, P=0)}{P(P=0)} = \frac{0.1}{0.8} = 0.125$

Bayes Rule

- Know: $P(B \mid A)$, $P(A)$, $P(B)$
- Want: $P(A \mid B)$

$$P(A|B) = \frac{P(A,B)}{P(B)} \qquad P(B|A) = \frac{P(A,B)}{P(A)}$$

$$P(A|B)\,P(B) = P(A,B) = P(B|A)\,P(A)$$

$$\boxed{P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}}$$

$$P(A|B,C) = \frac{P(B|A,C)\,P(A|C)}{P(B|C)}$$

Independence

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Definition: X and Y are *independent* iff $P(X, Y) = P(X)\,P(Y)$

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$P(X \mid Y) = P(X)$

Definition: X and Y are *conditionally independent* given Z iff $P(X, Y \mid Z) = P(X \mid Z)\,P(Y \mid Z)$

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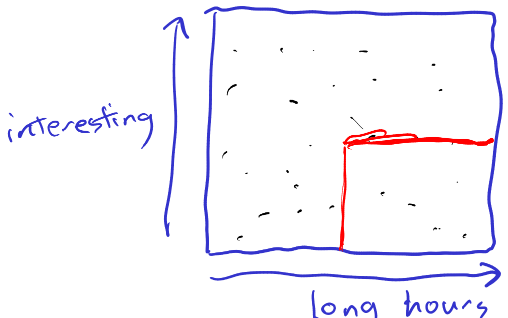
Definition: X and Y are *conditionally independent* given Z iff
$P(X,Y \mid Z) = P(X \mid Z)\,P(Y \mid Z)$

$X \perp Y \mid Z$



crash on I-70   crash on US 40   bad storm ?

$X \perp Y \mid Z \Rightarrow X \perp Y$ Not true

$X \perp Y \Rightarrow X \perp Y \mid Z$ Not true

considering

Rules for Continuous RVs

Discrete

1) a) $0 \leq P(X \mid Y) \leq 1$

   b) $\sum_{x \in X} P(x \mid Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

3) $P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$

$$P(X, Y) = P(X \mid Y)\, P(Y)$$

Continuous

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$$\int_X p(x|Y)\, dx = 1$$

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Continuous

1) $0 \leq p(X \mid Y)$

$$\int_X p(x|Y)\, dx = 1$$

2)

$$p(X) = \int_Y p(X, y) dy$$

Rules for Continuous RVs

Discrete	Continuous
1) a) $0 \leq P(X \mid Y) \leq 1$ b) $\sum_{x \in X} P(x \mid Y) = 1$	1) $0 \leq p(X \mid Y)$ $\int_X p(x \vert Y)\, dx = 1$
2) $P(X) = \sum_{y \in Y} P(X, y)$	2) $p(X) = \int_Y p(X, y) dy$
3) $P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$ $P(X, Y) = P(X \mid Y)\, P(Y)$	3) $p(X \mid Y) = \frac{p(X,Y)}{p(Y)}$ $p(X, Y) = p(X \mid Y)\, p(Y)$

Multivariate Gaussian Distribution

$x = [x_1, x_2]$

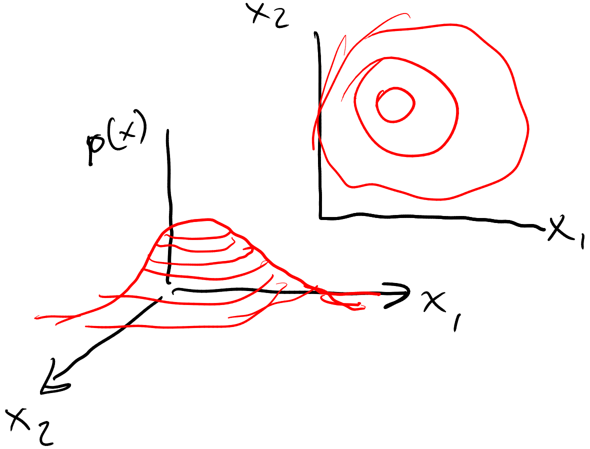
$\mathcal{N}(\mu, \Sigma)$

Joint Distribution

$$p(x) = \mathcal{N}(x \mid \mu, \Sigma)$$

$$= \frac{\exp\left(\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)}{(2\pi)^{n/2}|\Sigma|^{1/2}}$$

$\mu = [3, 3]$ $\quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$



Conditional Distribution

$\Sigma = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

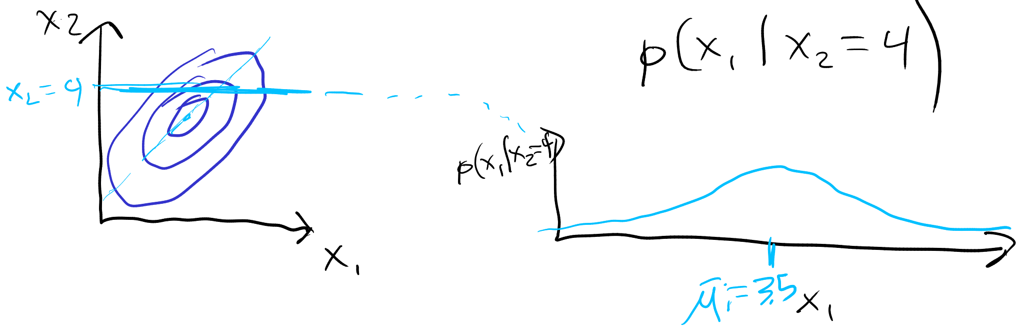
$$p(x_1 \mid x_2)$$

$$= \mathcal{N}(x_1 \mid \bar{\mu}, \bar{\Sigma}_1)$$

$$\bar{\mu}_1 = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$

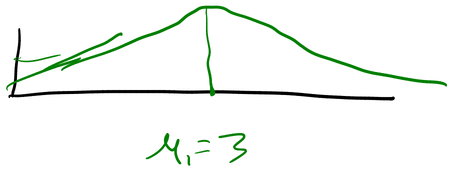
$$\bar{\Sigma}_1 = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$p(x_1 \mid x_2 = 4)$$



Marginal Distribution

$$p(x_1) = \mathcal{N}(x_1 \mid \mu_1, \Sigma_{11})$$



Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables