

Markov Decision Processes

Last Time

- What does "Markov" mean in "Markov Process"?

Guiding Questions

Guiding Questions

- What is a **Markov decision process**?

Guiding Questions

- What is a **Markov decision process**?
- What is a **policy**?

Guiding Questions

- What is a **Markov decision process**?
- What is a **policy**?
- How do we **evaluate** policies?

Decision Networks and MDPs

Decision Networks and MDPs

Decision Network




 Chance node

 Decision node

 Utility node

Decision Networks and MDPs




Decision Network

-  Chance node
-  Decision node
-  Utility node

MDP Dynamic Decision Network

Decision Networks and MDPs

Decision Network

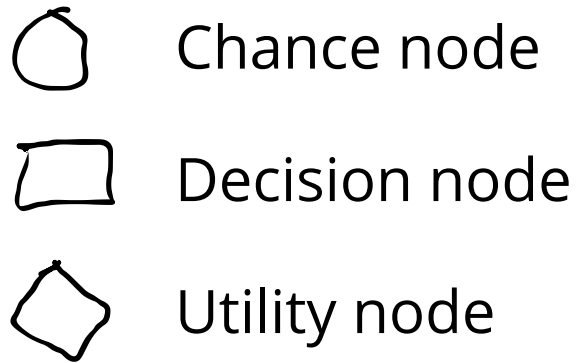
-  Chance node
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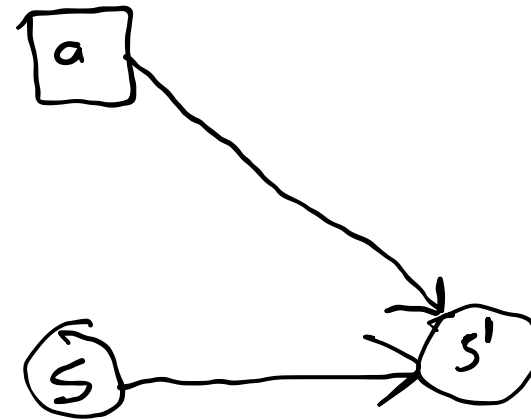


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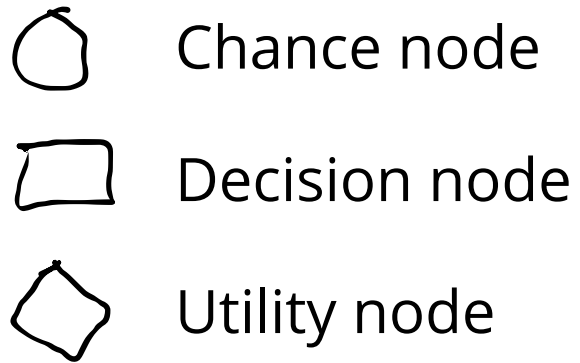


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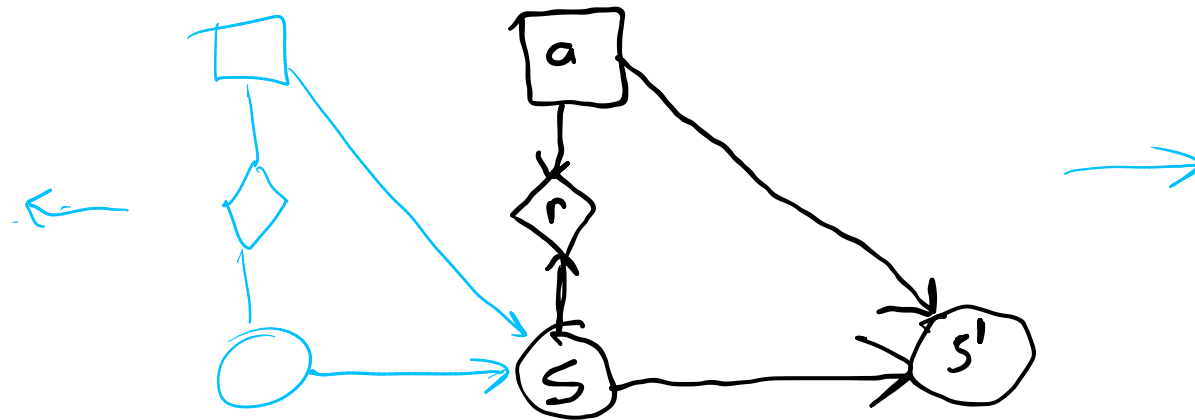


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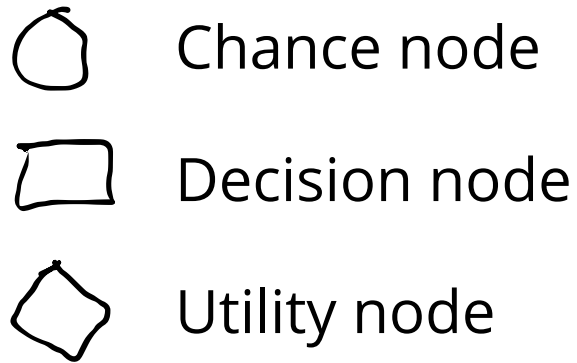


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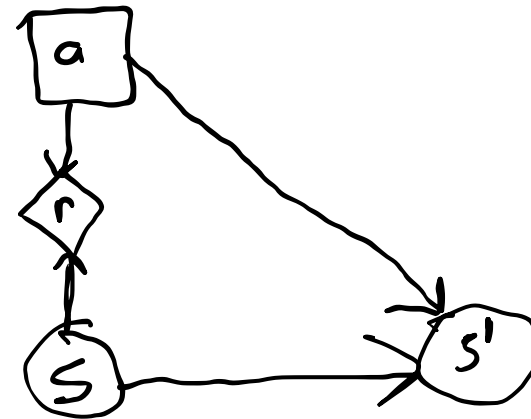


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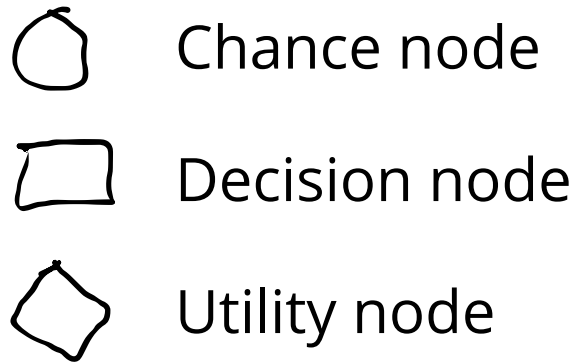
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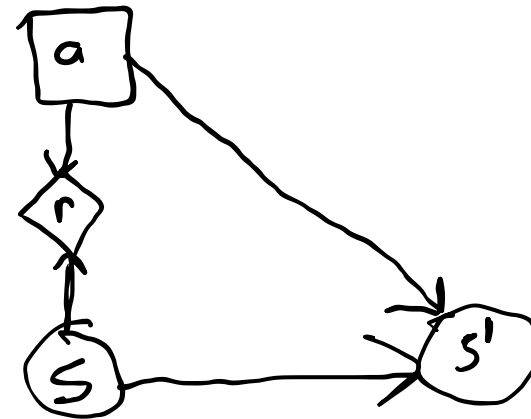
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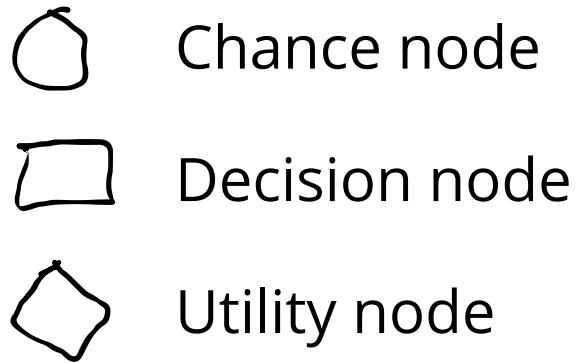


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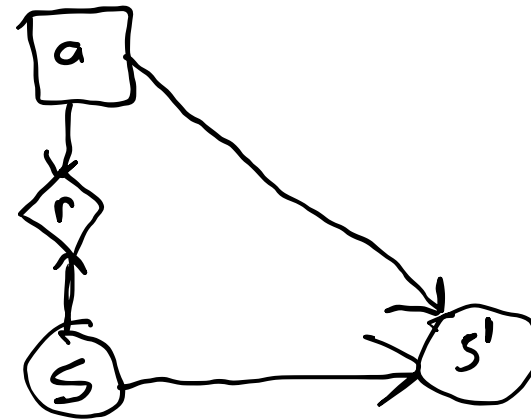
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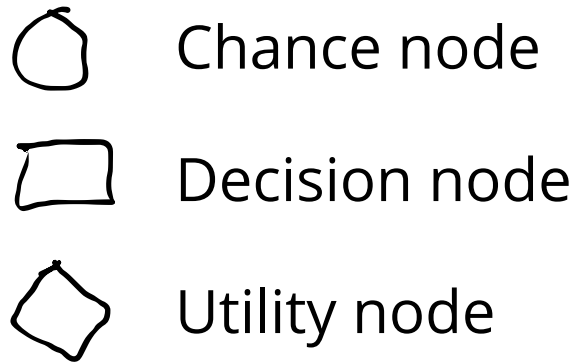


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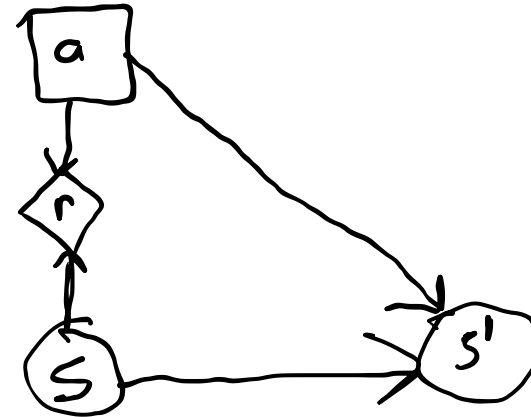
maximize $E \left[\sum_{t=0}^{\infty} r_t \right]$ Not well formulated!

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Infinite

Finite MDP Objectives

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 $R(s, a)$ or $R(s, a, s')$
- γ : discount factor
- b : initial state distribution
- S_T : set of terminal states

$$s', r = G(s, a)$$

MDP Example

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- If you drive, you will have to pay \$15 for parking; biking is free.
- On 1% of cold days, the ground is covered in ice and you will crash if you bike, but you can't discover this until you start riding. After your crash, you limp home with pain equivalent to losing \$100.

Policies and Simulation

Policies and Simulation

- A *policy*, denoted with $\pi(a_t \mid s_t)$, is a conditional distribution of actions given states.
- $a_t = \pi(s_t)$ is used as shorthand when a policy is deterministic.
- When a policy is combined with an MDP, it becomes a Markov stochastic process with

$$P(s' \mid s) = \sum_{a_t} T(s' \mid s, a_t) \pi(a_t \mid s_t)$$

Break

- Suggest a policy that you think is optimal for the icy day problem

Policy Evaluation

Naive Policy Evaluation not on Exam

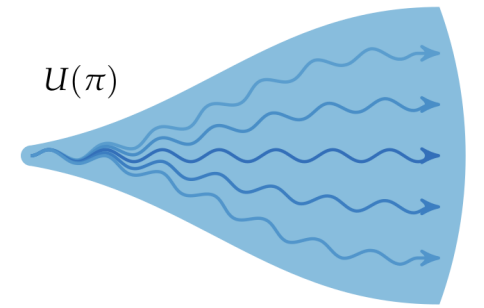
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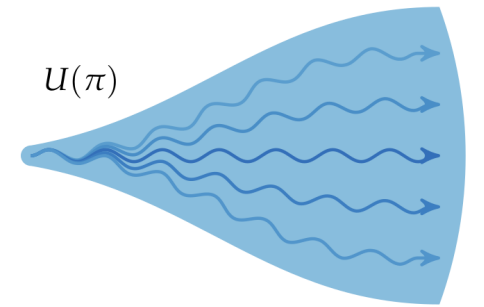
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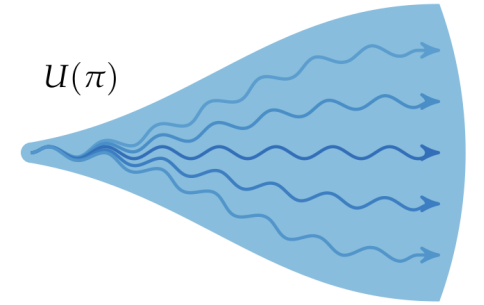


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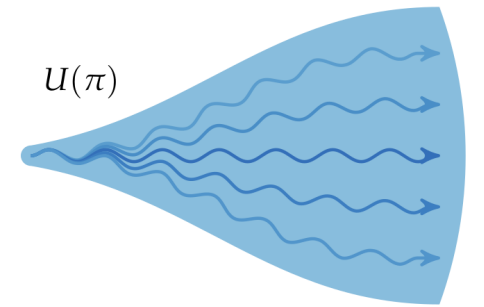
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where $\hat{u}^{(i)}$ is generated by a rollout simulation



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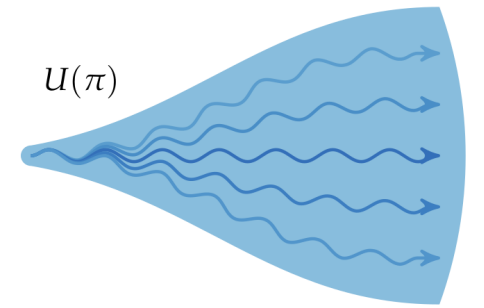
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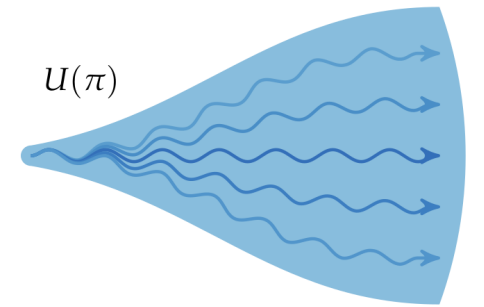
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also an R.V.

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Value Function-Based Policy Evaluation

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