

# Bayesian Networks: Inference and Independence

# Bayesian Networks

Today:

- Bayesian Networks
- How do we perform inference on Bayesian Networks?
- How do we reason about independence in Bayesian Networks?

# Review

Joint

$$P(A, B, C)$$

$$P(A=1, B=3, C=5)$$

Conditional

$$P(X|Y) \leftarrow \text{distribution-valued function of } Y$$

$$P(X=1|Y=1)$$

Marginal

$$P(X)$$

Independence

$$P(X, Y) = P(X)P(Y)$$

$$X \perp Y$$

$$P(X|Y) = P(X)$$

Conditional Indep.

$$P(X, Y|Z) = P(X|Z)P(Y|Z)$$

$$X \perp Y|Z$$

$$P(X|Z) = P(X|Y, Z)$$

# Joint Distribution Complexity

Binary Random Variables  $X_1, X_2, X_3$

$X_0$	$P(X_0)$
0	0.7
1	0.3

How many independent parameters ( $\theta$ ) to specify joint distribution?

$2^3 = 8$

$X_1$	$X_2$	$X_3$	$P(X_1, X_2, X_3)$
0	0	0	
0	0	1	
0	1	0	

# Joint Distribution Complexity

Binary Random Variables  $X_1, X_2, X_3$

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For  $n$  binary R.V.s,  $2^n - 1$  independent parameters specify the joint distribution.

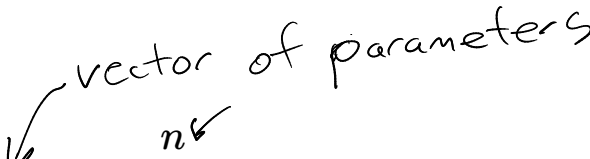
# Joint Distribution Complexity

Binary Random Variables  $X_1, X_2, X_3$

How many independent parameters ( $\theta$ ) to specify joint distribution? 7

For  $n$  binary R.V.s,  $2^n - 1$  independent parameters specify the joint distribution.

In general


$$\dim(\theta) = \prod_{i=1}^n |\text{support}(X_i)| - 1$$

# Bayesian Network



# Bayesian Network

Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**

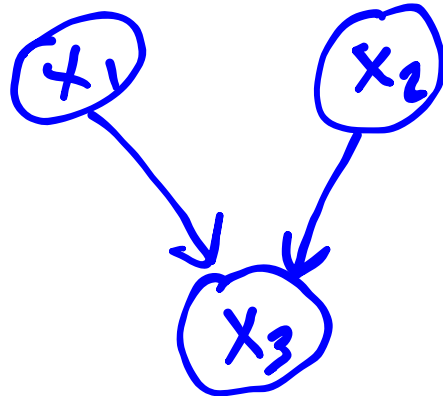
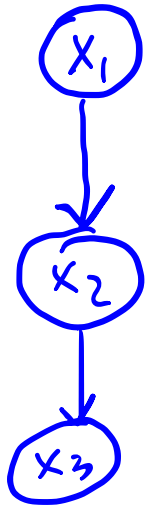
# Bayesian Network

Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**



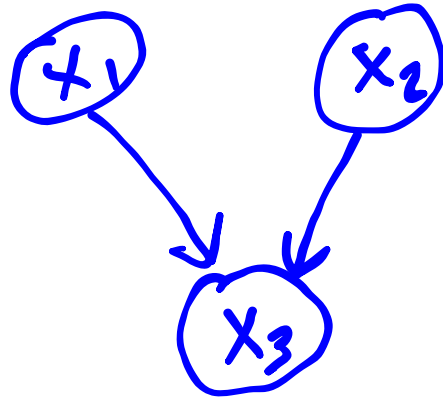
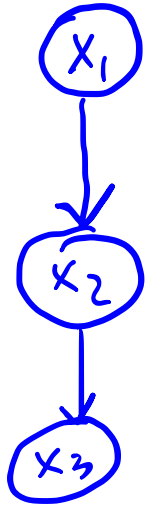
# Bayesian Network

Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**



# Bayesian Network

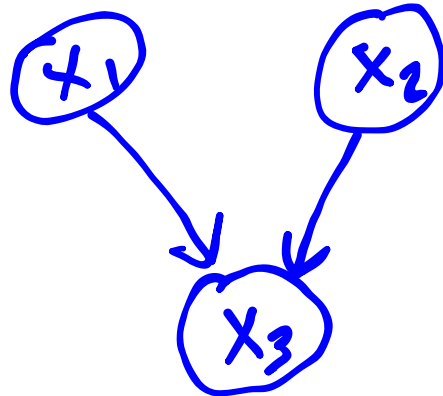
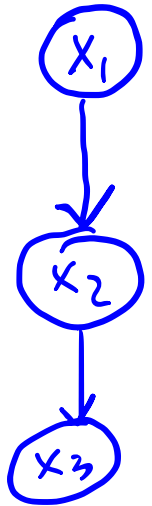
Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**



- Node:
- Edges encode:

# Bayesian Network

Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**



- Node: Random Variable
- Edges encode:

# Bayesian Network

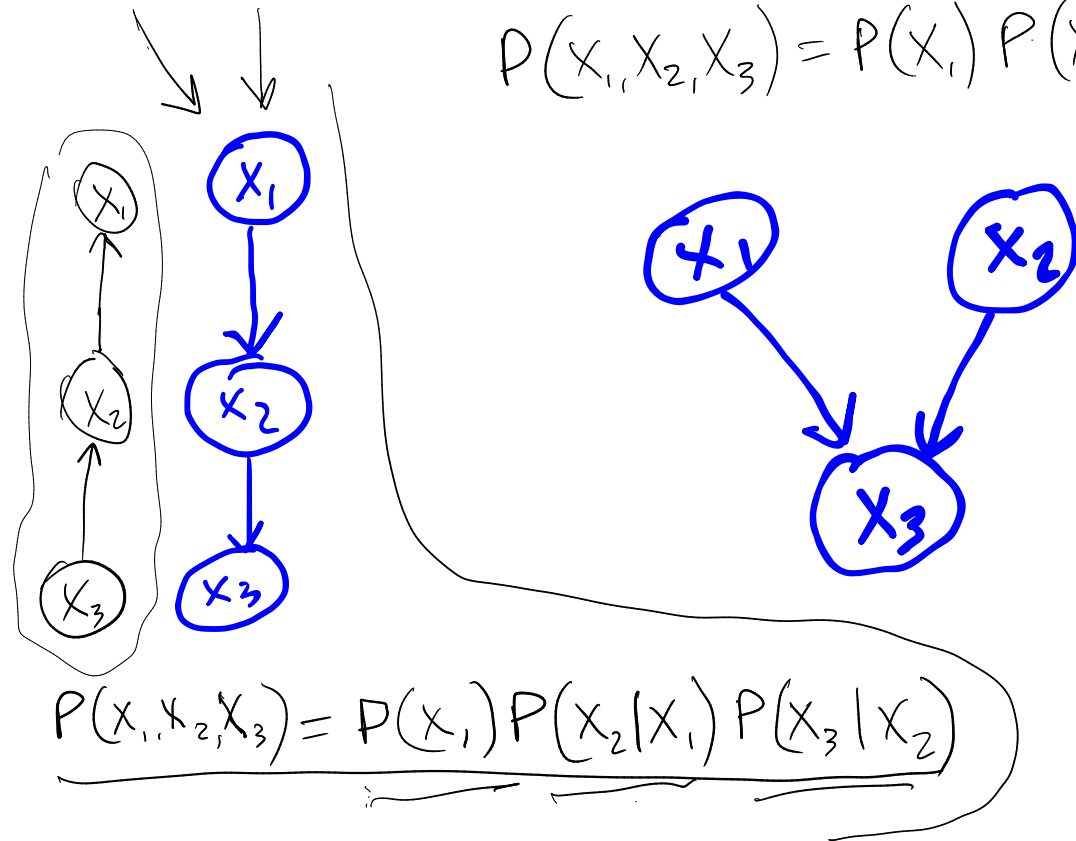
Bayesian Network: Directed Acyclic Graph (DAG) that represents a **joint probability distribution**

$$P(X_1, X_2, X_3) = P(X_1) P(X_2) P(X_3 | X_1, X_2)$$

- Node: Random Variable
- Edges encode:

$$P(X_{1:n}) = \prod_{i=1}^n P(\underline{X_i} | \underline{\text{pa}(X_i)})$$

Joint



$$P(X_1, X_2, X_3) = P(X_1) P(X_2 | X_1) P(X_3 | X_2)$$

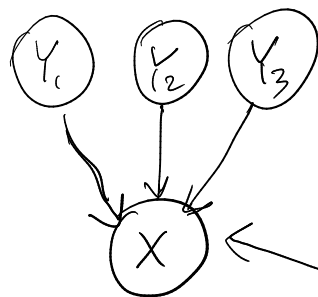
$X_1, X_2, X_3$	$P(X_1, X_2, X_3)$

# Counting Parameters

For discrete R.V.s:

$$\dim(\theta_X) = \underbrace{(|\text{support}(X)| - 1)} \prod_{\underbrace{Y \in Pa(X)}} |\text{support}(Y)|$$

# Counting Parameters



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$$\prod_{Y \in \text{Pa}(X)} |\text{support}(Y)|$$

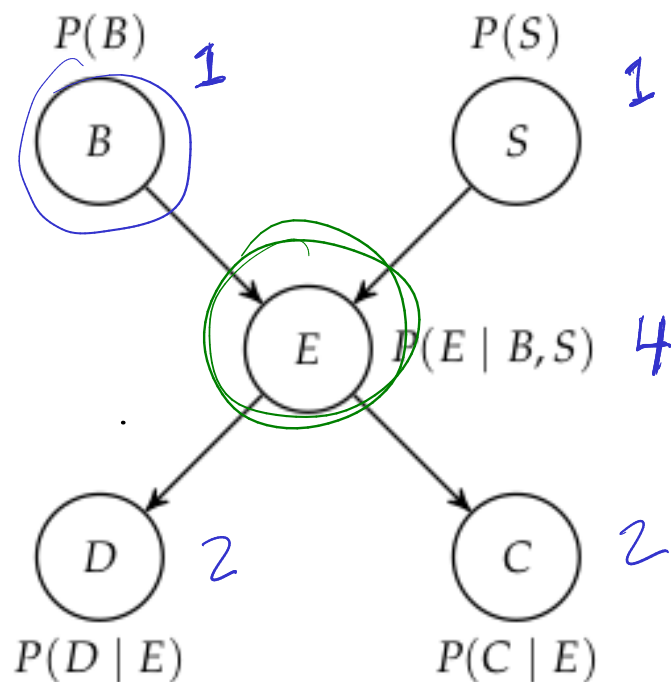
Handwritten annotations: A blue bracket above the product symbol, a green bracket below the product symbol, and a green '2' below the product symbol.

support(B) = {0, 1}  
|support(B)| = 2

$P(X | Y_1, Y_2, Y_3)$

0	0	0
0	0	1

$\rightarrow P(X | Y_1=0, Y_2=0, Y_3=0) \Rightarrow 1_{\text{param}}$   
 $\Rightarrow 1_{\text{param}}$



support(E) = {0, 1}

l = 2

Pa(E) = {B, S}

|support(B)| = 2

|support(S)| = 2

B.N.

Number of parameters: 10

Number of parameters for naive representation  $2^5 - 1 = 31$



# Inference

**Inputs**

**Outputs**

# Inference

## Inputs

- Bayesian network structure

## Outputs

# Inference

## Inputs

- Bayesian network structure
- Bayesian network parameters

## Outputs

# Inference

## Inputs

- Bayesian network structure
- Bayesian network parameters
- Values of *evidence variables*

## Outputs

# Inference

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## Outputs

- Posterior distribution of *query variables*

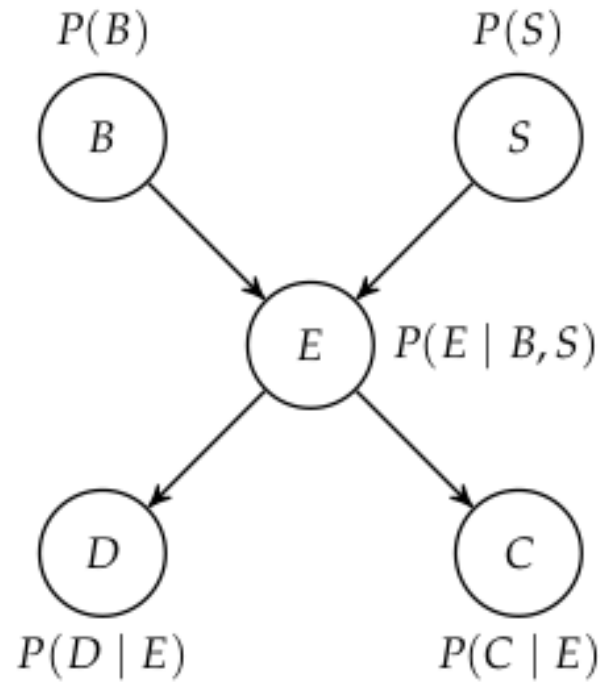
# Inference

## Inputs

- Bayesian network structure
- Bayesian network parameters
- Values of *evidence variables*

## Outputs

- Posterior distribution of *query variables*



$B$  battery failure  
 $S$  solar panel failure  
 $E$  electrical system failure  
 $D$  trajectory deviation  
 $C$  communication loss

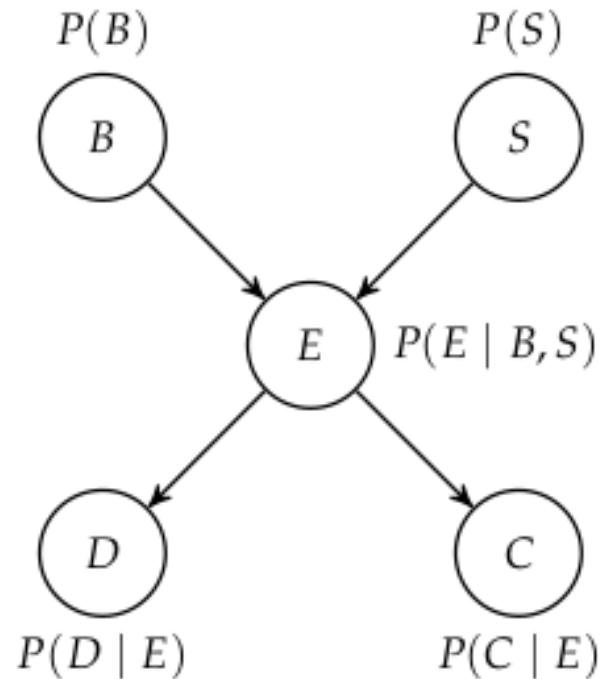
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Given that you have detected a trajectory deviation, and the battery has not failed what is the probability of a solar panel failure?

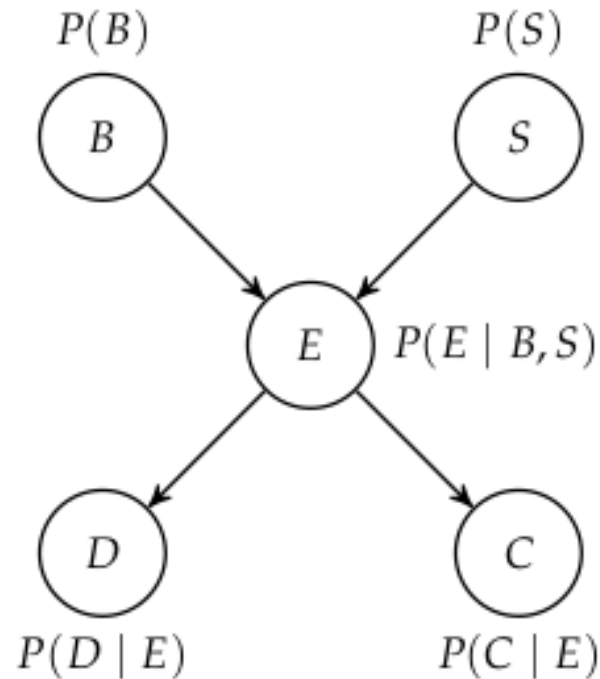
# Inference

## Inputs

- Bayesian network structure
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## Outputs

- Posterior distribution of *query variables*



B battery failure  
S solar panel failure  
E electrical system failure  
D trajectory deviation  
C communication loss

Given that you have detected a trajectory deviation, and the battery has not failed what is the probability of a solar panel failure?

$$P(\underbrace{S = 1}_{\text{query}} \mid \underbrace{D = 1, B = 0}_{\text{Evidence}})$$



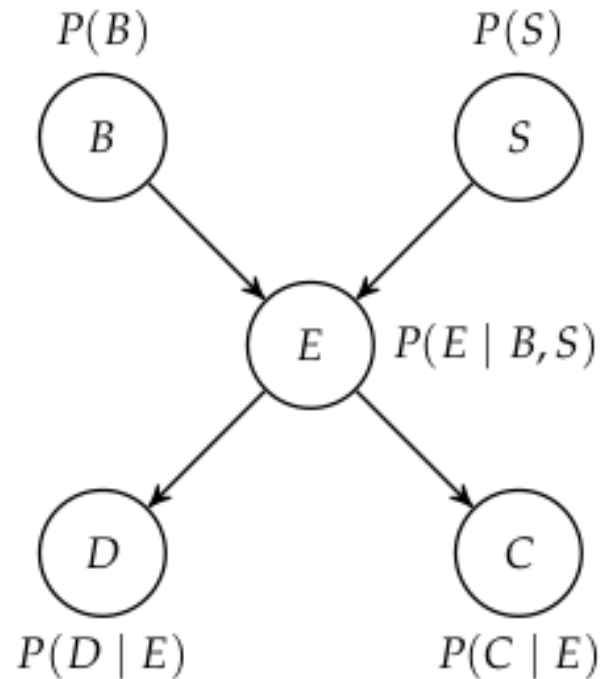
# Inference

## Inputs

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$$P(S = 1 \mid D = 1, B = 0)$$

Exact

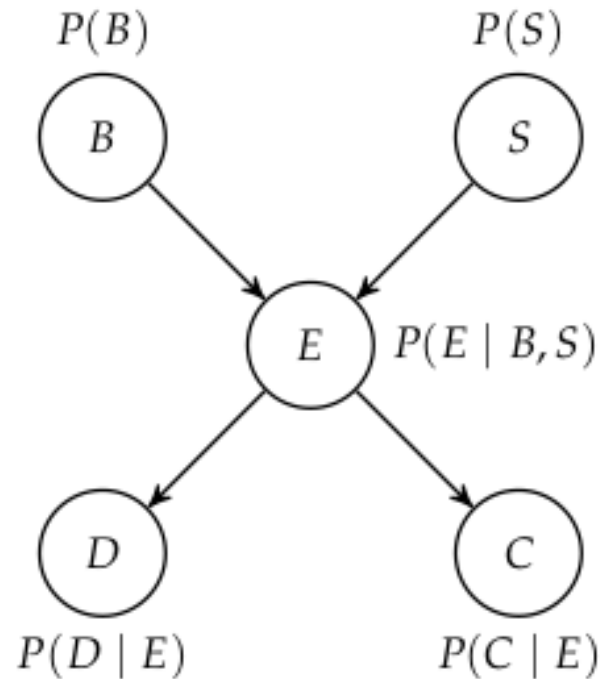
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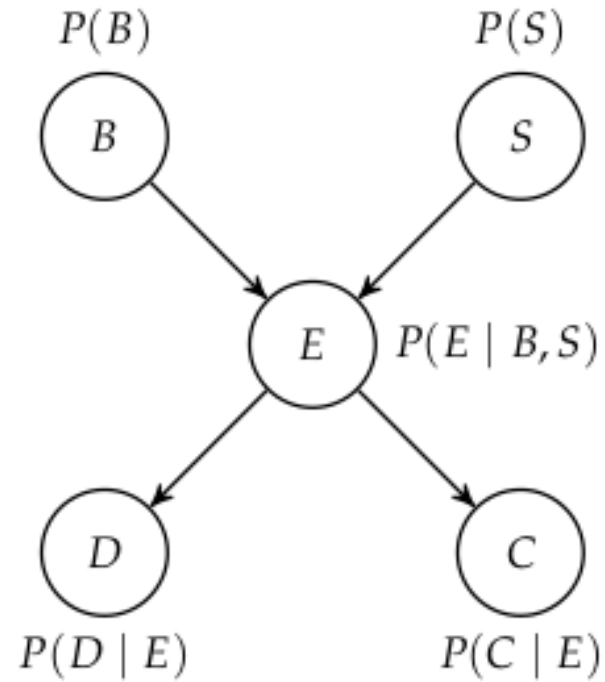
$$P(S = 1 \mid D = 1, B = 0)$$

Exact

Approximate

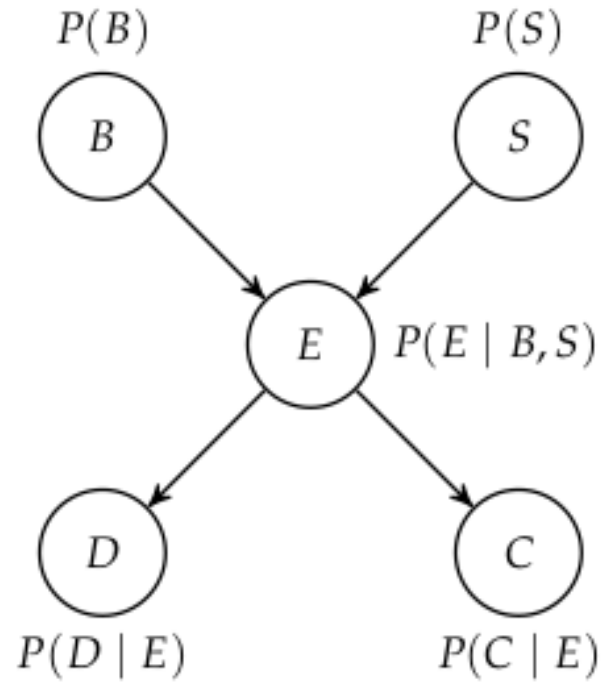
# Exact Inference

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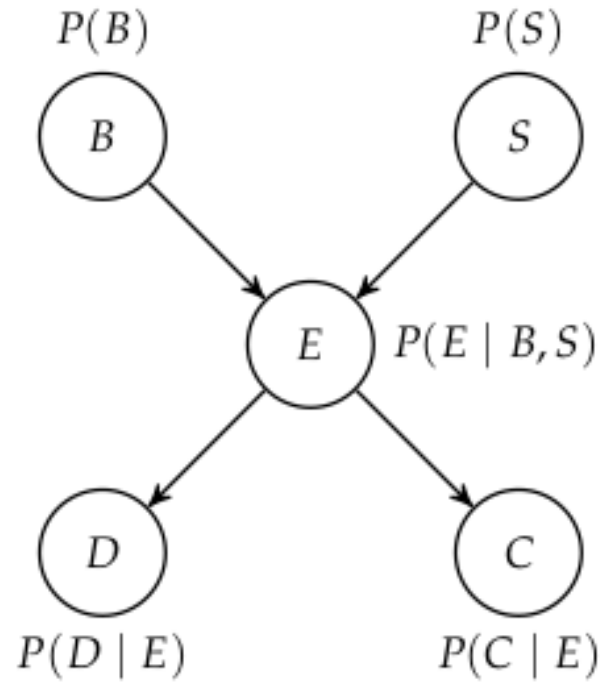
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$$P(S=1 \mid D=1, B=0)$$

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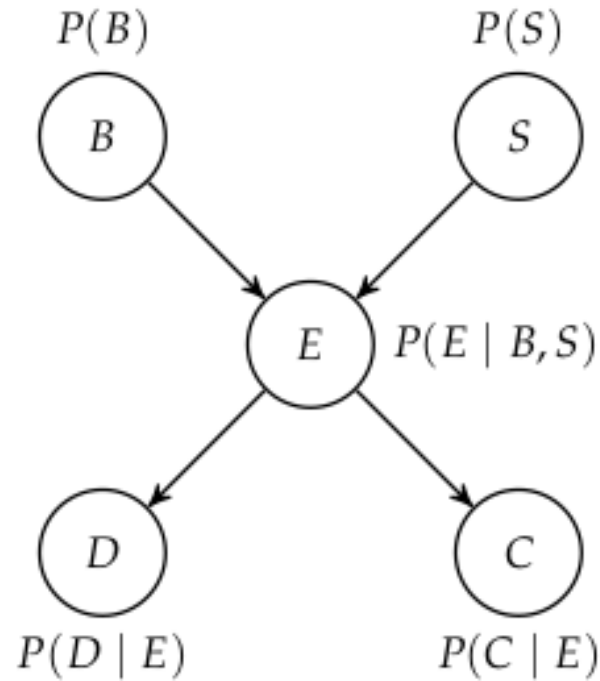
# Exact Inference



$$P(S=1 \mid D=1, B=0) = \frac{P(S=1, D=1, B=0)}{P(D=1, B=0)}$$

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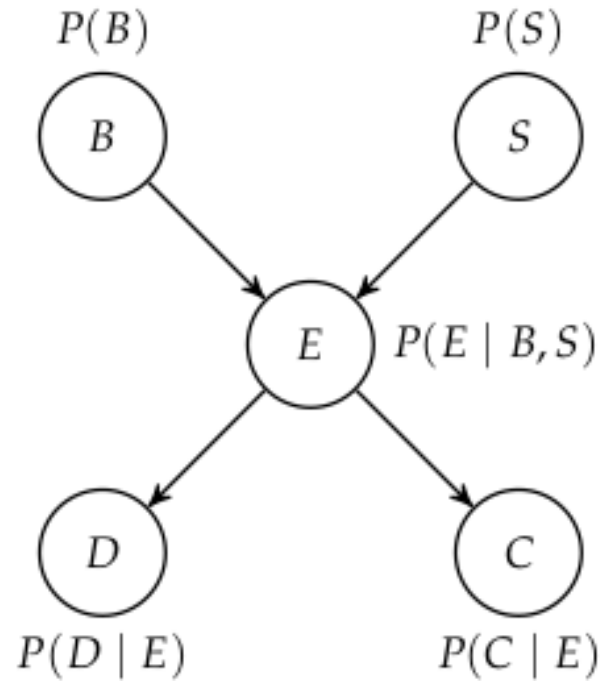


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$$P(S=1, D=1, B=0) = \sum_{\underline{e, c}} P(B=0, S=1, E=e, D=1, C=c)$$

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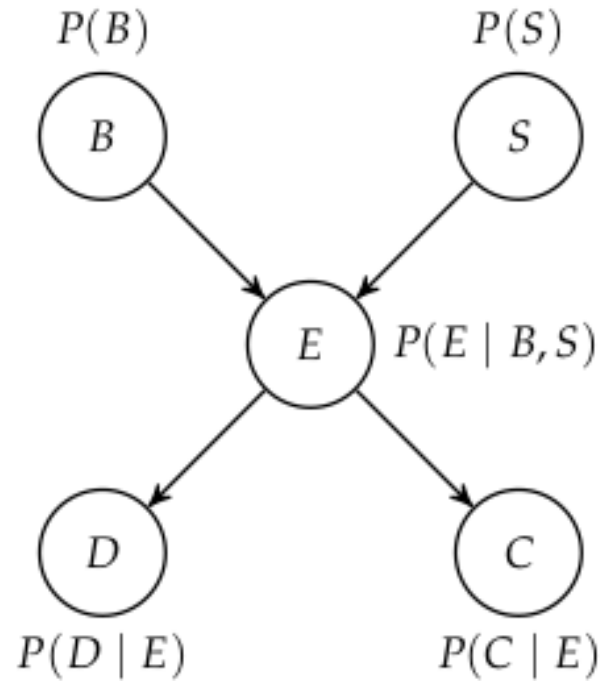
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$$P(B=0, S=1, E, D=1, C)$$



# Exact Inference



B battery failure  
 S solar panel failure  
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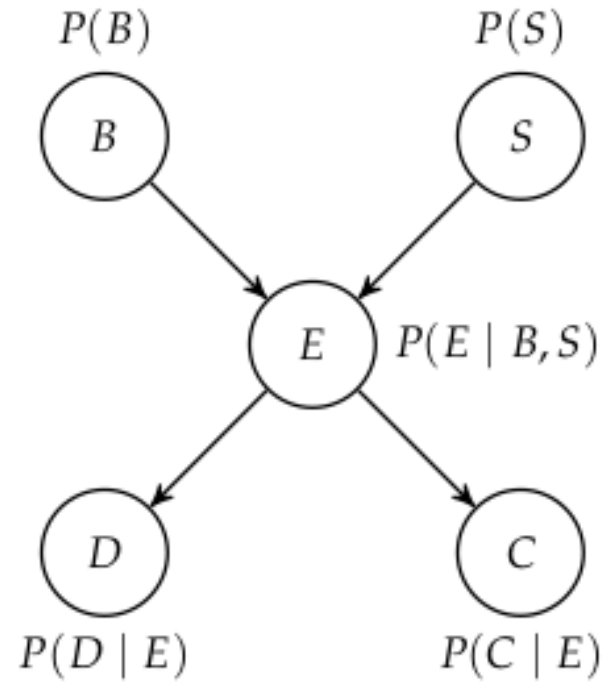
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$$P(B=0, S=1, E, D=1, C) = P(B=0) P(S=1) P(E \mid B=0, S=1) P(D=1 \mid E) P(C=1 \mid E)$$

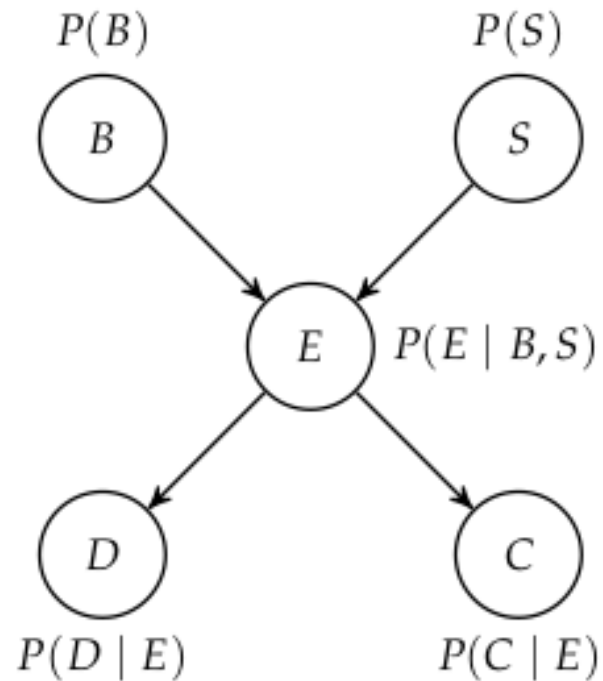
Bayesian Network

# Exact Inference



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# Exact Inference



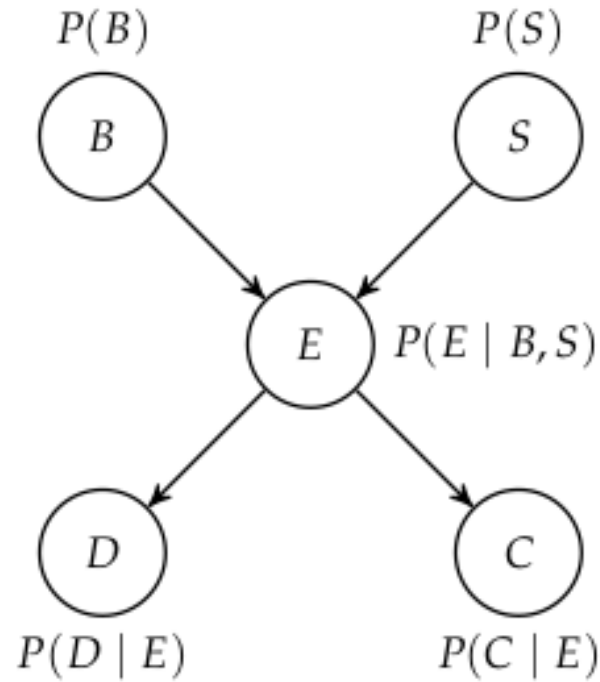
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Product

$X$	$Y$	$\phi_1(X, Y)$
0	0	0.3
0	1	0.4
1	0	0.2
1	1	0.1

$X$	$Y$	$Z$	$\phi_3(X, Y, Z)$
0	0	0	0.06
0	0	1	0.00
0	1	0	0.12
0	1	1	0.20
1	0	0	0.04
1	0	1	0.00
1	1	0	0.03
1	1	1	0.05

# Exact Inference



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0	1	0.0
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$X$	$Y$	$Z$	$\phi_3(X, Y, Z)$
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## Condition

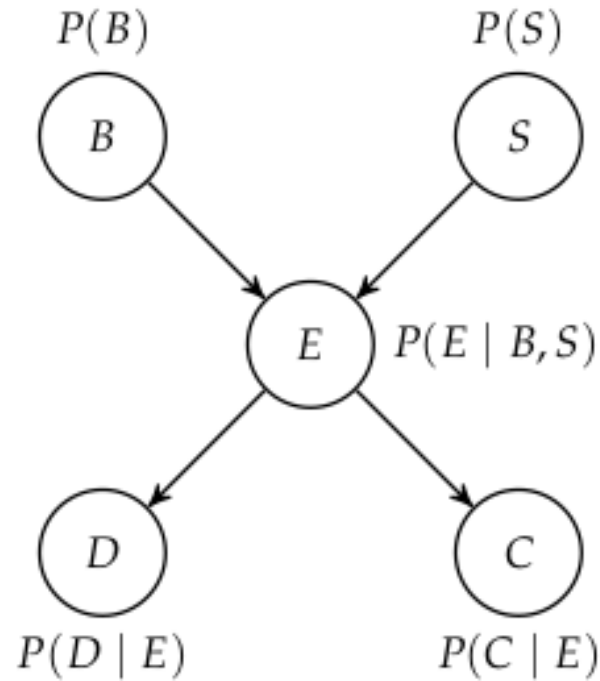
$X$	$Y$	$Z$	$\phi(X, Y, Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

$Y = 1$

$X$	$Z$	$\phi(X, Z)$
0	0	0.09
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# Exact Inference



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0	1	1	0.20
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Condition

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0	0	0	0.08
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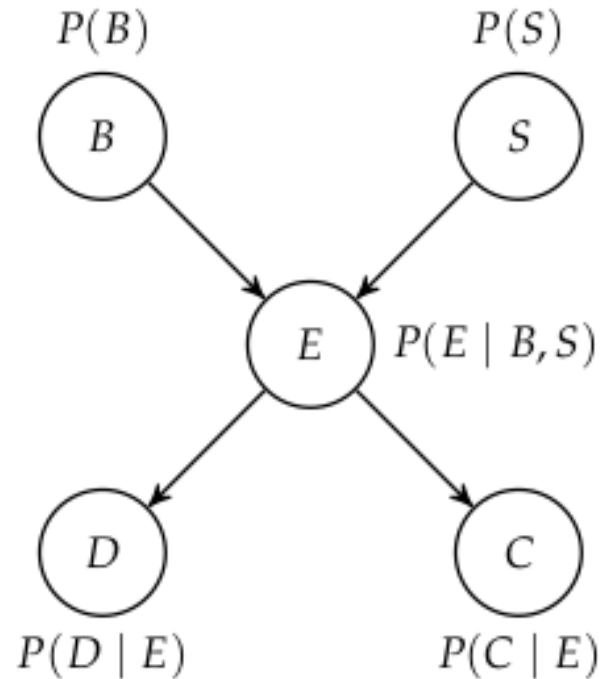
Marginalize

$X$	$Y$	$Z$	$\phi(X, Y, Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

$X$	$Z$	$\phi(X, Z)$
0	0	0.17
0	1	0.68
1	0	0.03
1	1	0.12

# Exact Inference



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$Y = 1$	$X$	$Z$	$\phi(X, Z)$
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	$X$	$Z$	$\phi(X, Z)$
→	0	0	0.17
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→	1	1	0.12

```
struct ExactInference end
```

```
function infer(M::ExactInference, bn, query, evidence)
```

```
     $\phi$  = prod(bn.factors)
```

```
     $\phi$  = condition( $\phi$ , evidence)
```

```
    for name in setdiff(variablenames( $\phi$ ), query)
```

```
         $\phi$  = marginalize( $\phi$ , name)
```

```
    end
```

```
    return normalize!( $\phi$ )
```

```
end
```

# Exact Inference

```
struct ExactInference end

function infer(M::ExactInference, bn, query, evidence)
     $\phi$  = prod(bn.factors)
     $\phi$  = condition( $\phi$ , evidence)
    for name in setdiff(variablenames( $\phi$ ), query)
         $\phi$  = marginalize( $\phi$ , name)
    end
    return normalize!( $\phi$ )
end
```

```
struct VariableElimination
    ordering # array of variable indices
end

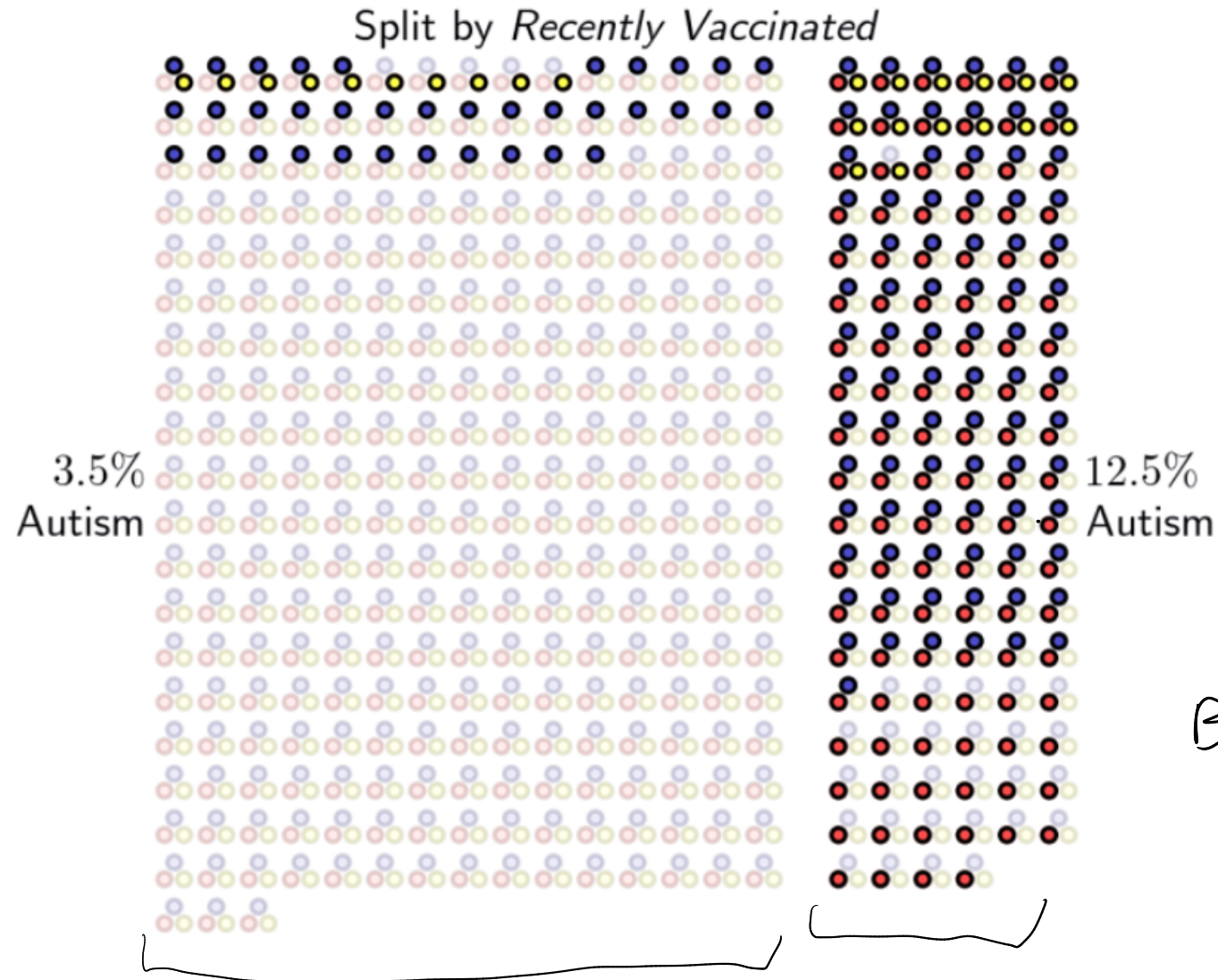
function infer(M::VariableElimination, bn, query, evidence)
     $\Phi$  = [condition( $\phi$ , evidence) for  $\phi$  in bn.factors]
    for i in M.ordering
        name = bn.vars[i].name
        if name  $\notin$  query
            inds = findall( $\phi \rightarrow$  in_scope(name,  $\phi$ ),  $\Phi$ )
            if !isempty(inds)
                 $\phi$  = prod( $\Phi$ [inds])  $\leftarrow$  product over
                deleteat!( $\Phi$ , inds) smaller number
                 $\phi$  = marginalize( $\phi$ , name) of variables
                push!( $\Phi$ ,  $\phi$ )
            end
        end
    end
    return normalize!(prod( $\Phi$ ))
end
```

$\nwarrow$  choosing order to eliminate variables  
is difficult  
(NP-hard)

# Break

Yellow: Autism

Red: recently vaccinated



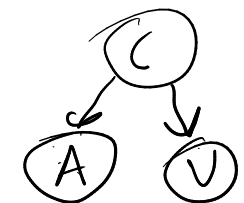
Does this imply  
a link <sup>between</sup> autism and  
vaccination

$$P(A) \quad P(V)$$

$$P(A, V) \neq P(A)P(V)$$

$$A \not\perp V$$

Blue: Child



$$A \perp V | C$$



# What does conditional independence mean?

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$$X \perp Y \mid Z$$

# What does conditional independence mean?

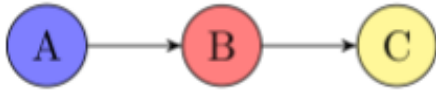
$$X \perp Y \mid Z \implies$$

# What does conditional independence mean?

$X \perp Y \mid Z \implies$  All of  $X$ 's influence on  $Y$  comes through  $Z$   $P(X \mid Z) = P(X \mid Y, Z)$

# What does conditional independence mean?

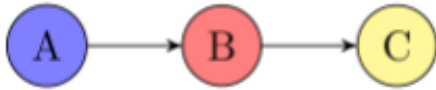
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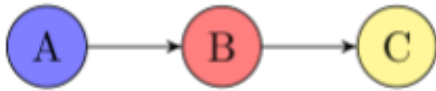


$A \perp C \mid B ?$

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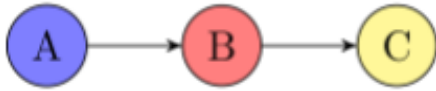


$A \perp C \mid B$  ? Yes

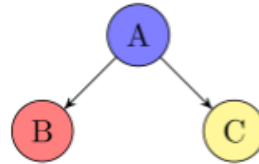
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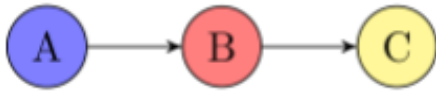




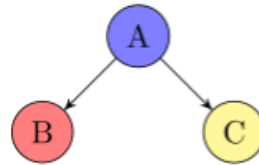
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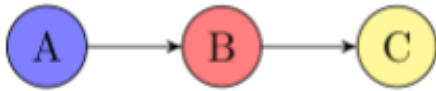


$B \perp C \mid A$  ?

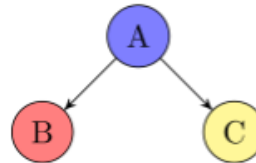
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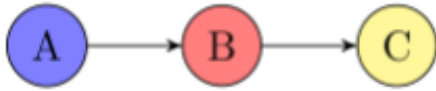


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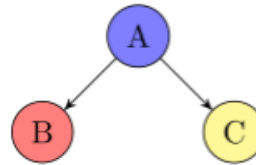
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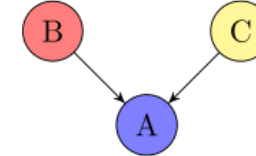
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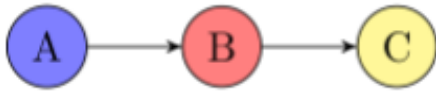
$B \perp C \mid A$  ? Yes



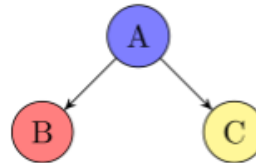
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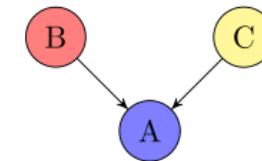
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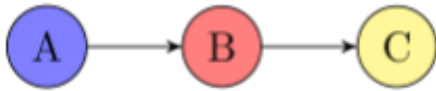


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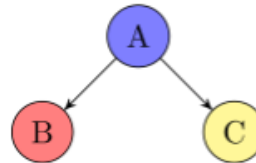
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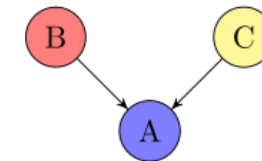
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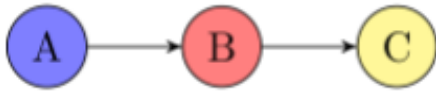


$B \perp C \mid A$  ? Inconclusive

# What does conditional independence mean?

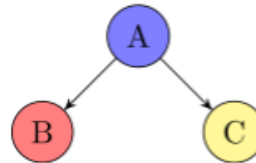
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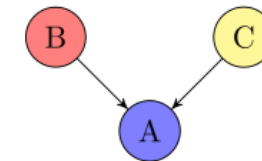


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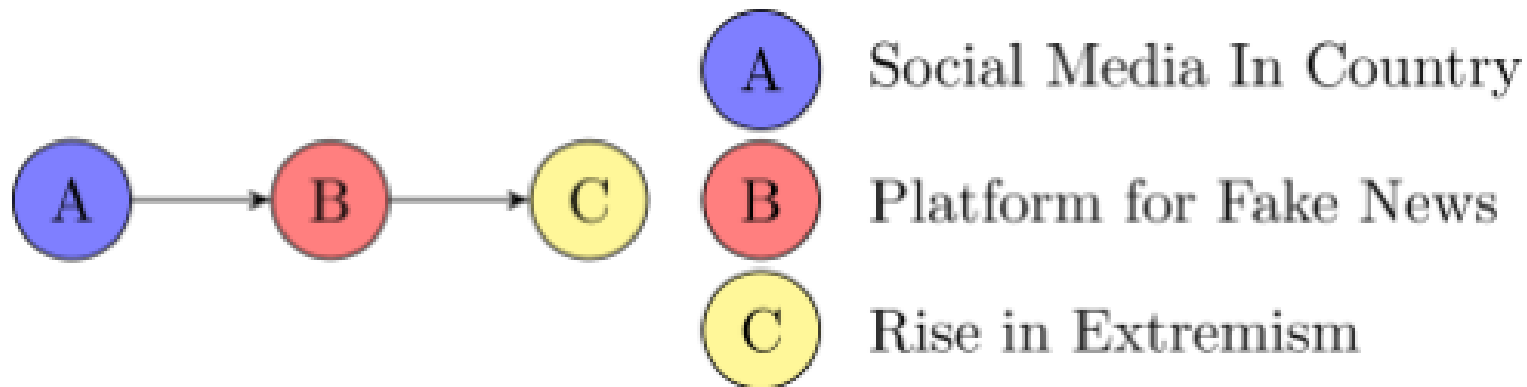
Mediator



$B \perp C \mid A$  ? Yes



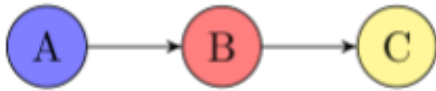
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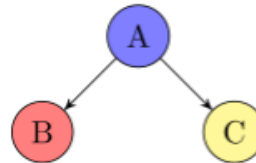
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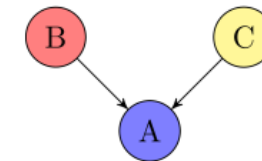
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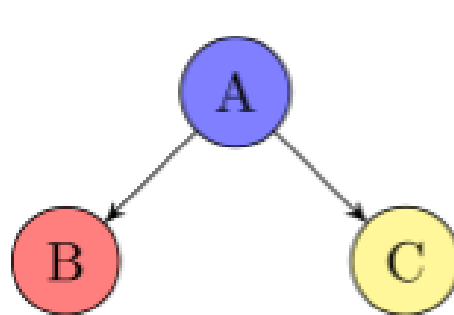


$B \perp C \mid A$  ? Yes

Confounder



$B \perp C \mid A$  ? Inconclusive



Is a Child



Recently Vaccinated

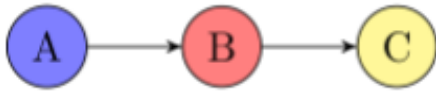


Diagnosed with Autism

# What does conditional independence mean?

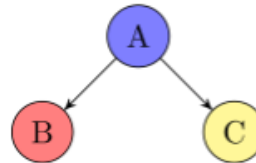
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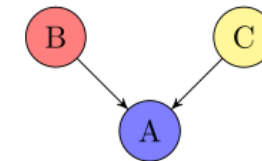
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Mediator



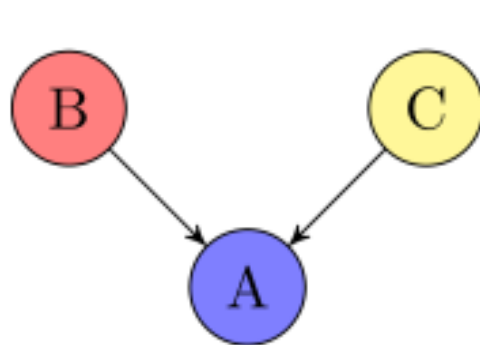
$B \perp C \mid A$  ? Yes

Confounder



$B \perp C \mid A$  ? Inconclusive

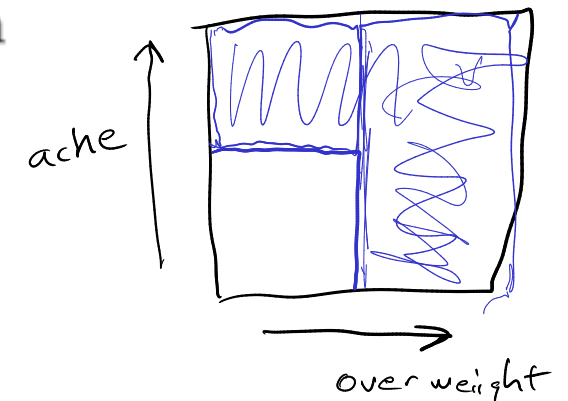
Collider



Saw the Dietician

Is Overweight

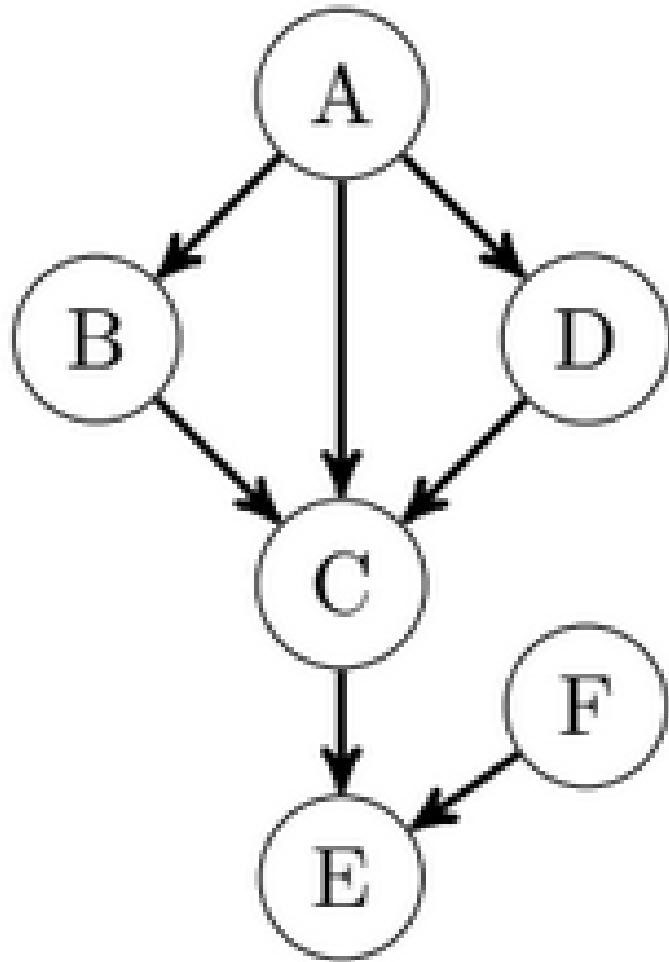
Has Acne





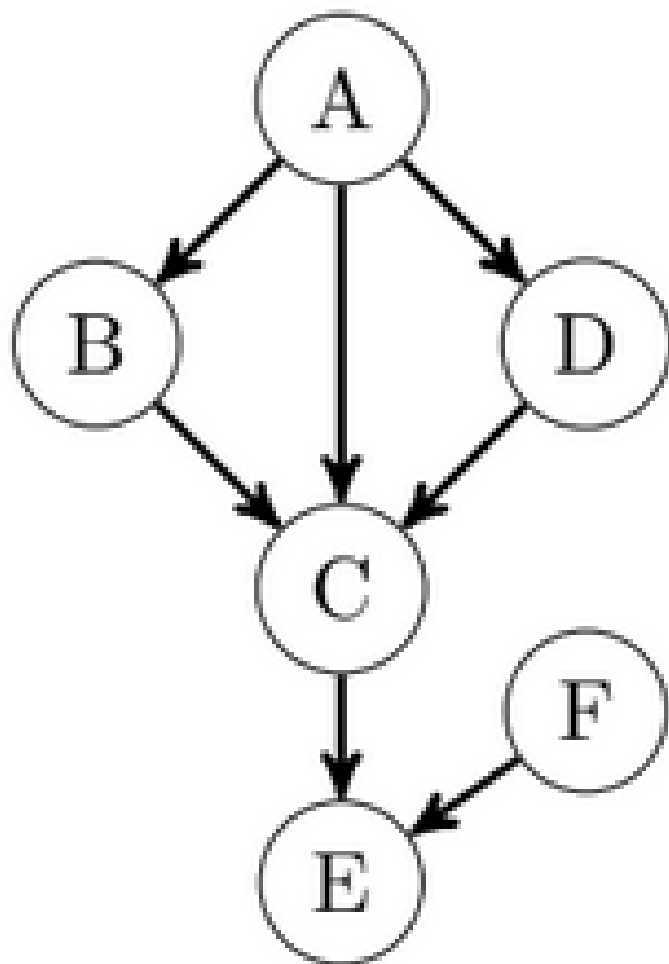
# More Complex Example

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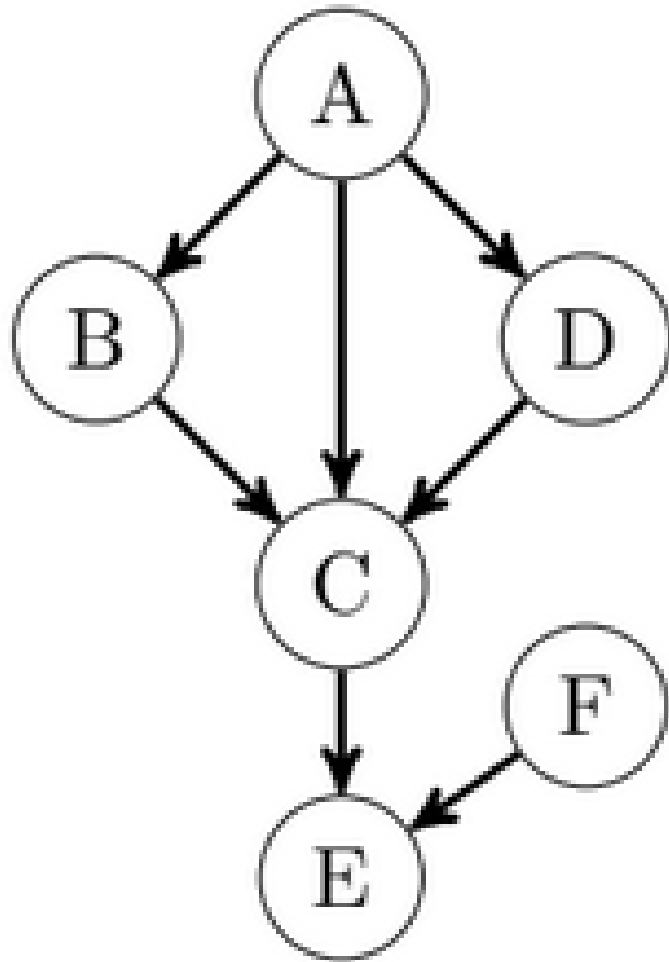
$(B \perp D \mid A) ?$



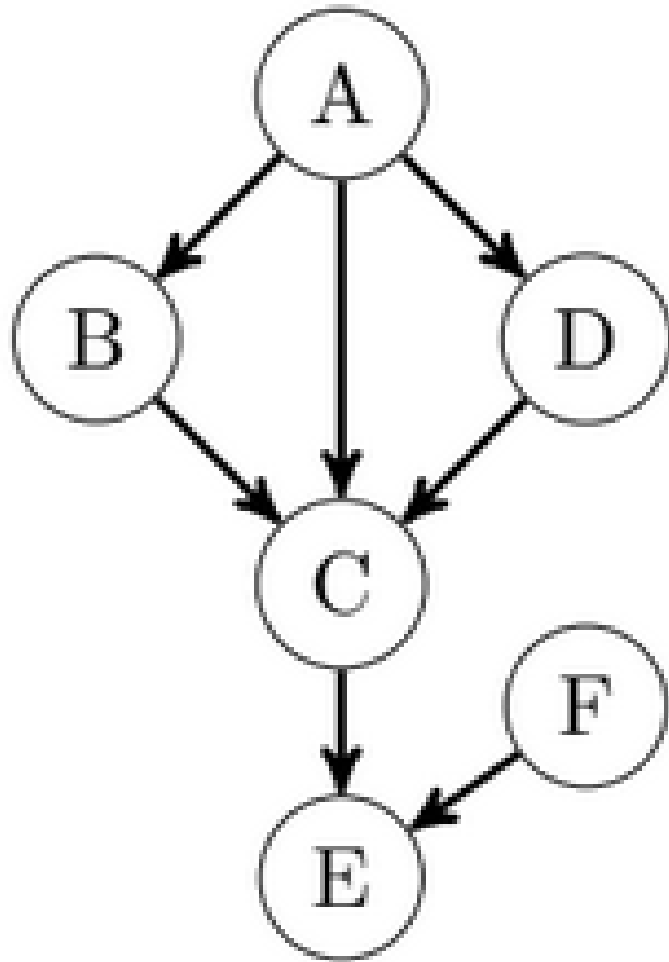
# More Complex Example

$(B \perp D \mid A) ?$

Yes!



# More Complex Example

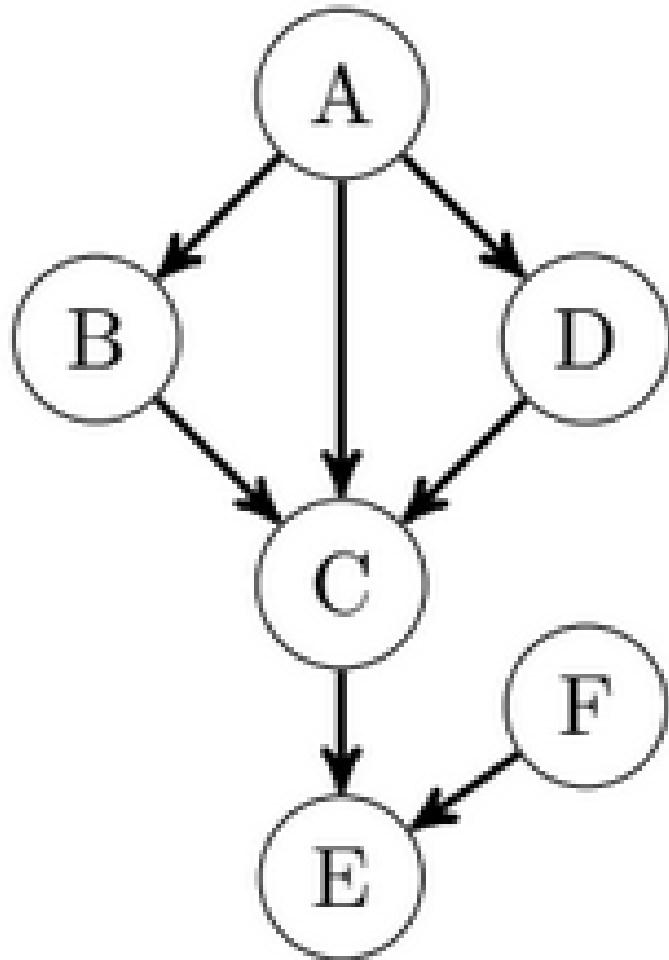


$(B \perp D \mid A) ?$

Yes!

$(B \perp D \mid E) ?$

# More Complex Example



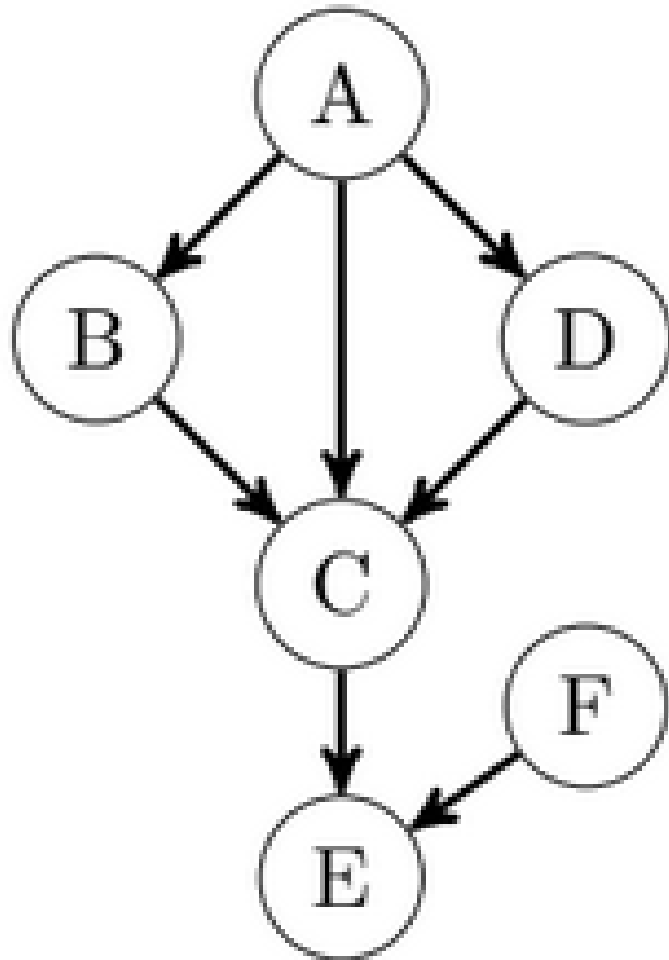
$(B \perp D \mid A) ?$

Yes!

$(B \perp D \mid E) ?$

Inconclusive

# More Complex Example



$(B \perp D \mid A) ?$

Yes!

$(B \perp D \mid E) ?$

Inconclusive

Why is this relevant to decision making?

$A \perp B \mid C$   
↑                    ↑

# d-Separation

\*short for "directionally separated"



# d-Separation

A, B, C, D, E

$$\mathcal{C} = \{C, D\}$$

Let  $\mathcal{C}$  be a set of random variables.

\*short for "directionally separated"

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Let  $\mathcal{C}$  be a set of random variables.

A *path* between  $A$  and  $B$  is *d-separated*\* by  $\mathcal{C}$  if any of the following are true

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We say that  $A$  and  $B$  are *d-separated* by  $\mathcal{C}$  if all paths between  $A$  and  $B$  are d-separated by  $\mathcal{C}$ .

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We say that  $A$  and  $B$  are *d-separated* by  $\mathcal{C}$  if all paths between  $A$  and  $B$  are d-separated by  $\mathcal{C}$ .

If  $A$  and  $B$  are d-separated by  $\mathcal{C}$  then  $A \perp B \mid \mathcal{C}$

\*short for "directionally separated"

# Proving Conditional Independence

1. The path contains a *chain*  $X \rightarrow Y \rightarrow Z$  such that  $Y \in \mathcal{C}$
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# Proving Conditional Independence

1. Enumerate all (non-cyclic) paths between nodes in question

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# Proving Conditional Independence

1. Enumerate all (non-cyclic) paths between nodes in question
2. Check all paths for d-separation

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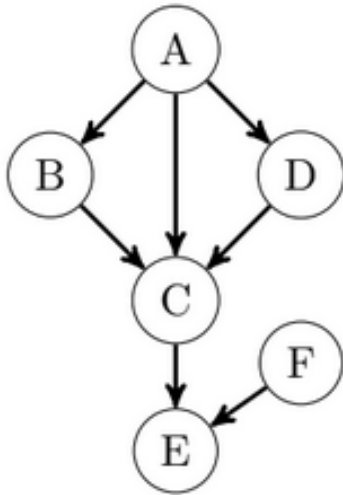
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# Proving Conditional Independence

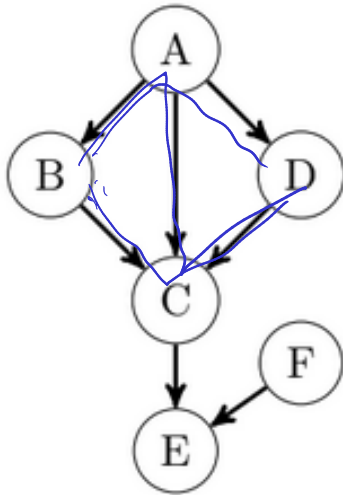
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# Proving Conditional Independence

- ✓ 1. Enumerate all (non-cyclic) paths between nodes in question
2. Check all paths for d-separation
3. If all paths d-separated, then CE ← conditionally independent



Example:  $(B \perp D \mid C, E)$  ?

$\rightarrow B \leftarrow A \rightarrow D$   
 $\rightarrow B \rightarrow C \leftarrow D$   
 $B \leftarrow A \rightarrow C \leftarrow D$   
 $B \rightarrow C \leftarrow A \rightarrow D$

rule 3  
 rule 2  
 rule 3  
 rule 3  
 rule 2

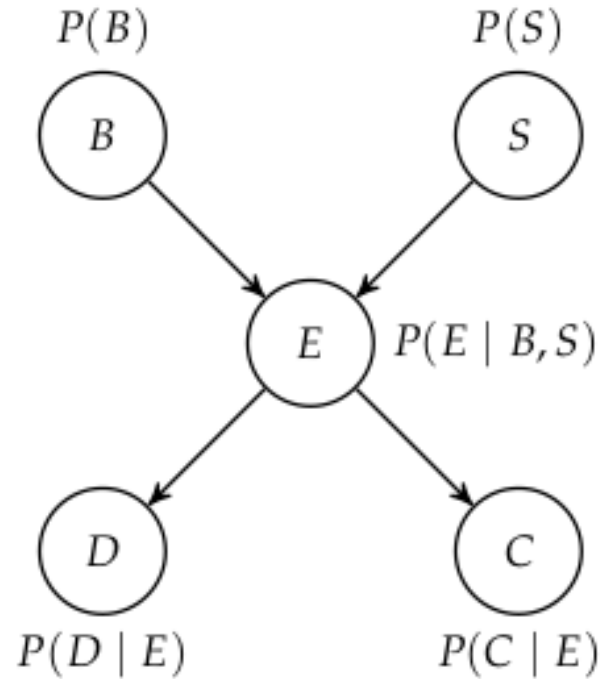
None true  $\rightarrow$  not d separated  $\rightarrow$  inconclusive based only on structure  
 not d-separated  
 not d-sep  
 not d-sep  
 not d-sep  
 not d-sep

- No
1. The path contains a *chain*  $X \rightarrow Y \rightarrow Z$  such that  $Y \in \mathcal{C}$
  - No 2. The path contains a *fork*  $X \leftarrow Y \rightarrow Z$  such that  $Y \in \mathcal{C}$
  - No 3. The path contains an *inverted fork* (v-structure)  $X \rightarrow Y \leftarrow Z$  such that  $Y \notin \mathcal{C}$  and no descendant of  $Y$  is in  $\mathcal{C}$ .

# Exercise

1. The path contains a *chain*  $X \rightarrow Y \rightarrow Z$  such that  $Y \in \mathcal{C}$
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# Exercise

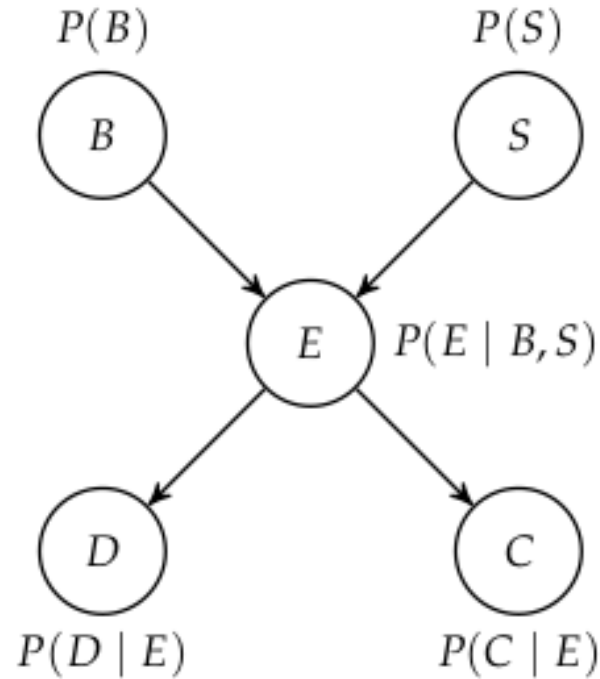


$B$  battery failure  
 $S$  solar panel failure  
 $E$  electrical system failure  
 $D$  trajectory deviation  
 $C$  communication loss

1. The path contains a *chain*  $X \rightarrow Y \rightarrow Z$  such that  $Y \in \mathcal{C}$
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# Exercise

$$D \perp C \mid B ?$$

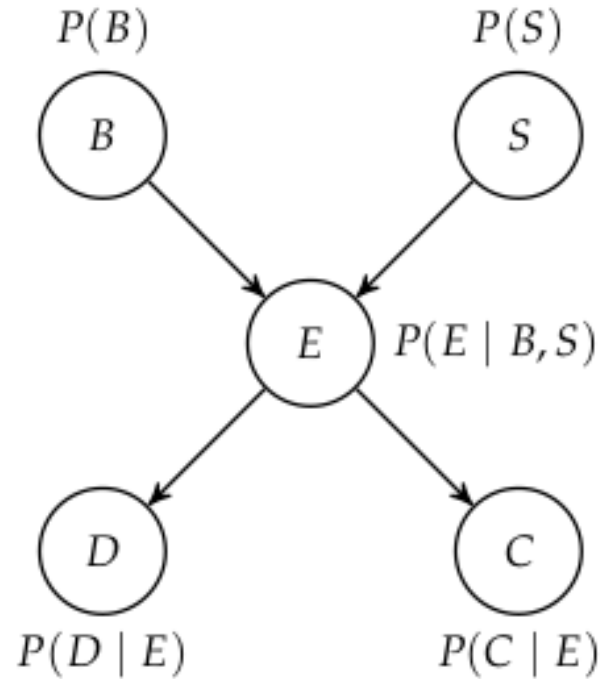


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# Recap