Probability and Random Variables

Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables

Utility and Probability

Utility

$U(A) > U(B)$ indicates A is preferable to B

$U(A) = U(B)$ " indifferent

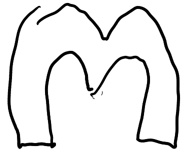
Probability

$P(A) > P(B)$ " A is more plausible than B

$P(A) = P(B)$ " A and B are equally plausible

Full story: https://projecteuclid.org/journals/statistical-science/volume-1/issue-3/The-Axioms-of-Subjective-Probability/10.1214/ss/1177013611.full

What is a Random Variable?

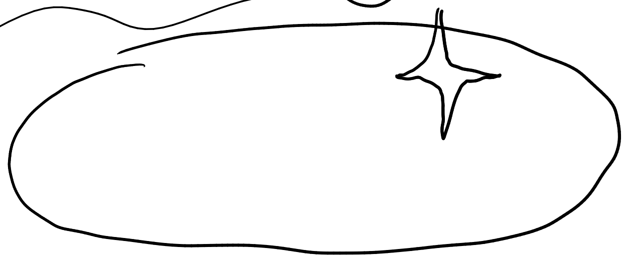


- Variable
  - finite set of values
  - probability

$P(X=1) = 0.5$

Today Only!

R.V. X



- Continuous/discrete
- related to other R.V.s

$P(X|Y)$

Blackbelly

sample space

measure

(Ω, Σ, μ)

σ algebra

$X : \Omega \rightarrow E$

Vocabulary/Notation

Vocabulary/Notation

Term	Definition	Coinflip Example	Uniform Example

Vocabulary/Notation

Term	Definition	Coinflip Example $\mathrm{Bernoulli}(0.5)$	Uniform Example

Vocabulary/Notation

Term	Definition	$\mathrm{Bernoulli}(0.5)$ Coinflip Example	$\mathcal{U}(0, 1)$ Uniform Example

Vocabulary/Notation

Term	Definition	$\text{Bernoulli}(0.5)$ Coinflip Example	$\mathcal{U}(0,1)$ Uniform Example
support(X)			

Vocabulary/Notation

Term	Definition	Coinflip Example	Uniform Example
		$\mathrm{Bernoulli}(0.5)$	$\mathcal{U}(0, 1)$
support(X)	All the values that X can take		

Vocabulary/Notation

Term	Definition	$\mathrm{Bernoulli}(0.5)$ **Coinflip Example**	$\mathcal{U}(0,1)$ **Uniform Example**
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Vocabulary/Notation

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Distribution			

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X	P(X)
0	0.5
1	0.5

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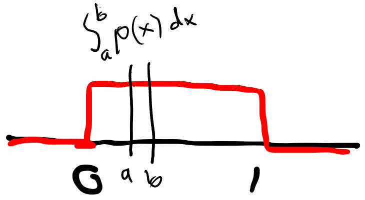

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$\text{Bernoulli}(0.5)$

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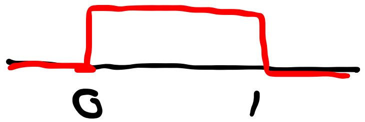

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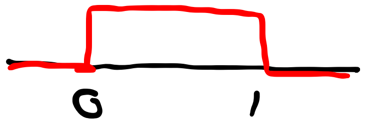

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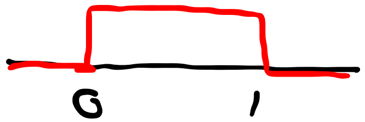

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Expectation			

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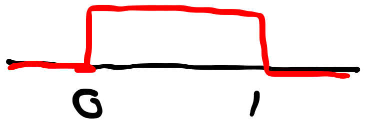

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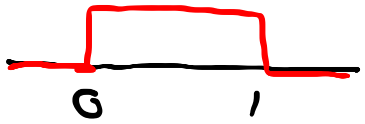

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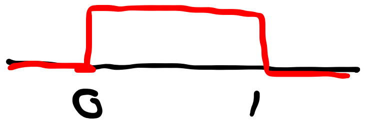

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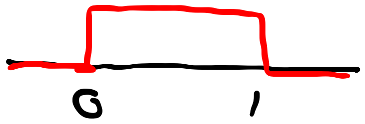

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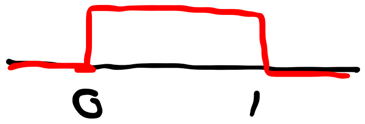
support(X) $x \in X$ $X \in [0,1]$	All the values that X can take	$\{h,t\}$ or $\{0,1\}$	$[0,1]$ 
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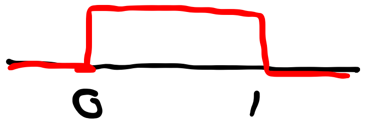
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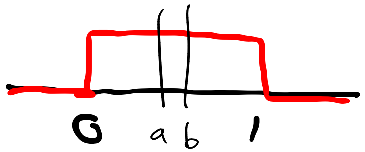

Distributions of related R.V.s

Distributions of related R.V.s

Joint Distribution

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Distributions of related R.V.s

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Conditional Distribution

Distributions of related R.V.s

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Conditional Distribution

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Distributions of related R.V.s

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Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
0	0.84
1	0.16

Distributions of related R.V.s

Joint Distribution

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Conditional Distribution

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(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
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Marginal Distribution

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

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0	1	0	0.09
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1	0	0	0.01
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Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

Distributions of related R.V.s

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(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
0	0.84
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Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

X	$P(X)$
0	0.85
1	0.15

Y	$P(Y)$
0	0.45
1	0.55

Z	$P(Z)$
0	0.20
1	0.80

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

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Distributions of related R.V.s

Joint Distribution

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Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

3 Rules

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

3 Rules

(Burrito-level)

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

3 Rules

(Burrito-level)

(Blackbelly Level: Axioms of Probability)

not on exam

AXIOM 1. STRUCTURE OF UNKNOWN REAL NUMBERS AND PLAUSIBLE VALUE. *We assume a set T of unknown numbers is a partially ordered commutative algebra over $\mathbb{R}$ with identity, 1.*

We assume in addition a given sub-Boolean algebra E of $E(T)$ with $0, 1 \in E$ and denote by E_0 the set of non-zero members of E. We assume that the partial ordering in $E(T)$ as a Boolean algebra coincides with the ordering that $E(T)$ inherits from the algebra T. Finally, we assume a function $PV : T \times E_0 \to \mathbb{R}$, called **PLAUSIBLE VALUE**, *whose value on the pair (x, e) is denoted $PV(x|e)$.*

AXIOM 2. STRONG RESCALING FOR PLAUSIBLE VALUE. *If a, b belong to $\mathbb{R}$, if x belongs to T, and if e belongs to E_0, then*

$$PV(ax + b|e) = aPV(x|e) + b. \tag{2}$$

AXIOM 3. ORDER CONSISTENCY FOR PLAUSIBLE VALUE. *If $x, y \in T$ and if $e \in E_0$, implies that $x \leq y$, then $PV(x|e) \leq PV(y|e)$.*

Notice that if $e \in E(T)$, then $0 \leq e \leq 1$, in T, as it is true in the lattice ordering of $E(T)$.

AXIOM 4. THE COX AXIOM FOR PLAUSIBLE VALUE: *If e, c are fixed in E, with $ec \in E_0$, if x_1, x_2 are in T, if $PV(x_1|ec) = PV(x_2|ec)$, then $PV(x_1e|c) = PV(x_2e|c)$. That is, we assume that as a function of x, the plausible value $PV(xe|c)$ depends only on $PV(x|ec)$.*

AXIOM 5. RESTRICTED ADDITIVITY OF PLAUSIBLE VALUE. *For each fixed $y \in T$ and $e \in E_0$, the plausible value $PV(x + y|e)$ as a function of $x \in T$ depends only on $PV(x|e)$, which is to say that if $x_1, x_2 \in T$ and $PV(x_1|e) = PV(x_2|e)$, then $PV(x_1 + y|e) = PV(x_2 + y|e)$.*

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

3 Rules

(Burrito-level)

1)

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2) "Law of total probability"

$$P(X) = \sum_{y \in Y} P(X, y)$$

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Marginal + Conditional $\rightarrow$ Joint

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Distributions of related R.V.s

Joint Distribution	Conditional Distribution	Marginal Distribution
$P(X, Y, Z)$	$P(X \mid Y, Z)$	$P(X)\ P(Y)\ P(Z)$

X	Y	Z	P(X, Y, Z)
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

$P(X)$

X	P(X)
0	$0.85 = 0.08 + 0.31 + 0.09 + 0.37$
1	0.15

$$P(X=0) = \sum_{y \in Y, z \in Z} P(X=0, Y=y, Z=z)$$

$$P(X=0 \mid Y=1, Z=1) = \frac{P(X=0, Y=1, Z=1)}{P(Y=1, Z=1)} = \frac{0.37}{0.37 + 0.07} =$$

Joint → Marginal

3 Rules

1) a) $0 \leq P(X \mid Y) \leq 1$

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Joint + Marginal → Conditional

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Break

$P(C=1) = P(C=1, P=1) + P(C=1, P=0)$

0.8   0.1   0.7

- $P \in \{0,1\}$: Powder Day
- $C \in \{0,1\}$: Pass Clear
- 1 in 5 days is a powder day
- The pass is clear 8 in 10 days
- If it is a powder day, there is a 50% chance the pass is blocked

$P(P=1) = 0.2$

$P(C=1) = 0.8$

$P(C=0 \mid P=1) = 0.5$

P	C	
0	0	0.1
0	1	0.7
1	0	0.1
1	1	0.1

$P(C=0 \mid P=1)\,P(P=1)$

$P(C=1 \mid P=1)\,P(P=1)$

0.5

$P(C=1 \mid P=1) + P(C=0 \mid P=1) = 1.0$

0.5

- Write out the joint probability distribution for P and C.
- Suppose it is a non-powder day, what is the probability that the pass is blocked?

$$P(C=0 \mid P=0) = \frac{P(C=0, P=0)}{P(P=0)} = \frac{0.1}{0.8} = 0.125$$

Bayes Rule

- Know: $P(B \mid A)$, $P(A)$, $P(B)$
- Want: $P(A \mid B)$

$$P(A \mid B) = \frac{P(A,B)}{P(B)} \qquad P(B \mid A) = \frac{P(A,B)}{P(A)}$$

$$P(A \mid B)\,P(B) = P(A,B) = P(B \mid A)\,P(A)$$

$$\boxed{P(A \mid B) = \frac{P(B \mid A)\,P(A)}{P(B)}}$$

$$P(A \mid B, C) = \frac{P(B \mid A, C)\,P(A \mid C)}{P(B \mid C)}$$

Independence

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Definition: X and Y are *conditionally independent* given Z iff
$P(X,Y \mid Z) = P(X \mid Z)\,P(Y \mid Z)$

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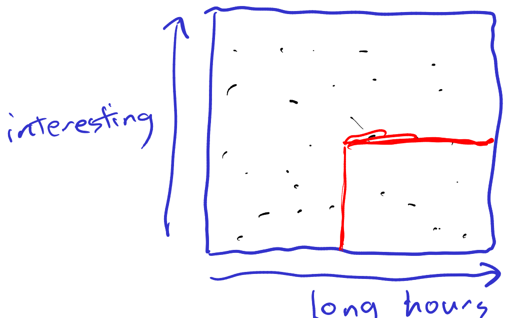
$X \perp Y \mid Z$



crash on I-70 · crash on US 40 · bad storm ?

$X \perp Y \mid Z \Rightarrow X \perp Y$ Not true

$X \perp Y \Rightarrow X \perp Y \mid Z$ Not true

considering

Rules for Continuous RVs

Discrete

1) a) $0 \leq P(X \mid Y) \leq 1$

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$$\int_X p(x|Y)\, dx = 1$$

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$$p(X) = \int_Y p(X, y) dy$$

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$$p(X, Y) = p(X \mid Y)\, p(Y)$$

Multivariate Gaussian Distribution

$x = [x_1, x_2]$

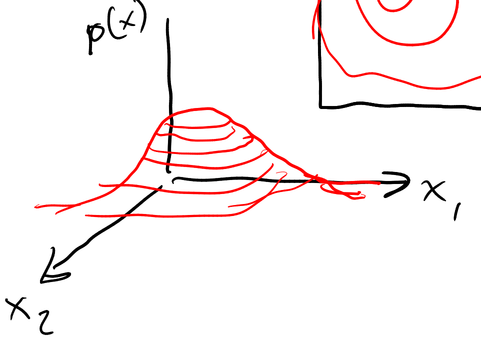
$\mathcal{N}(\mu, \Sigma)$

Joint Distribution

$$p(x) = \mathcal{N}(x \mid \mu, \Sigma)$$

$$= \frac{\exp\left(\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)}{(2\pi)^{n/2}|\Sigma|^{1/2}}$$

$\mu = [3,3]$ $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$



Conditional Distribution

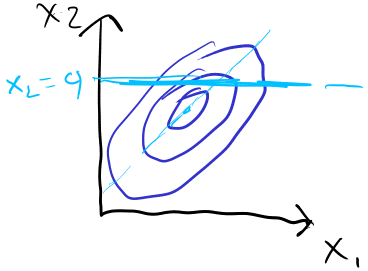
$\Sigma = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$$p(x_1 \mid x_2)$$

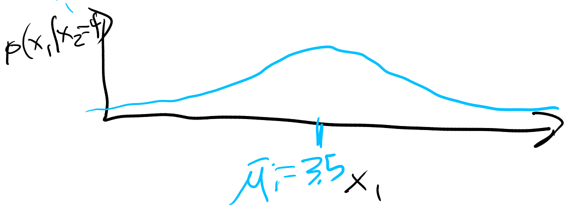
$$= \mathcal{N}(x_1 \mid \bar{\mu}, \bar{\Sigma}_1)$$

$$\bar{\mu}_1 = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$

$$\bar{\Sigma}_1 = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

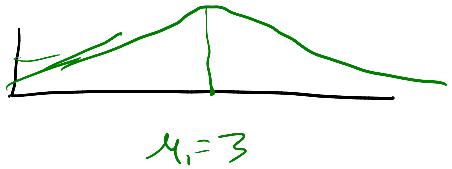


$$p(x_1 \mid x_2 = 4)$$



Marginal Distribution

$$p(x_1) = \mathcal{N}(x_1 \mid \mu_1, \Sigma_{11})$$



Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables