Probability and Random Variables

Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables

Utility and Probability

Full story: https://projecteuclid.org/journals/statistical-science/volume-1/issue-3/The-Axioms-of-Subjective-Probability/10.1214/ss/1177013611.full

What is a Random Variable?

R.V. X

Vocabulary/Notation

Vocabulary/Notation

Term	Definition	Coinflip Example	Uniform Example

Vocabulary/Notation

Term	Definition	$\text{Bernoulli}(0.5)$ Coinflip Example	Uniform Example

Vocabulary/Notation

Term	Definition	$\text{Bernoulli}(0.5)$ Coinflip Example	$\mathcal{U}(0,1)$ Uniform Example

Vocabulary/Notation

Term	Definition	Coinflip Example	Uniform Example
		$\mathrm{Bernoulli}(0.5)$	$\mathcal{U}(0,1)$
support(X)			

Vocabulary/Notation

Term	Definition	Coinflip Example $\text{Bernoulli}(0.5)$	Uniform Example $\mathcal{U}(0,1)$
support(X)	All the values that X can take		

Vocabulary/Notation

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Vocabulary/Notation

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Distribution			

Vocabulary/Notation

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Distribution • Discrete: PMF • Continuous: PDF			

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X	P(X)
0	0.5
1	0.5

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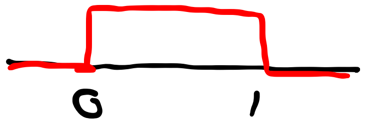

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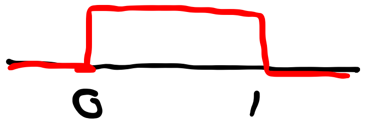

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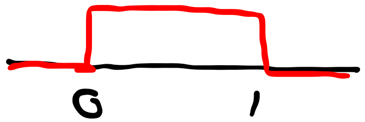

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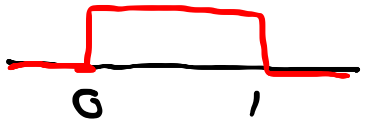

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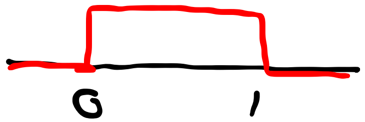
Expectation			



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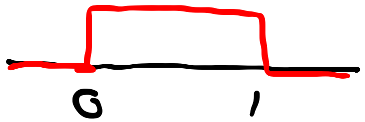

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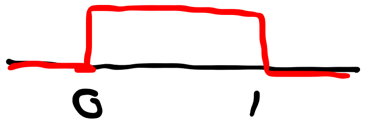
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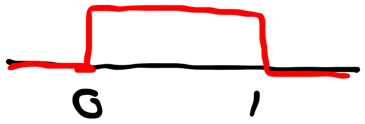

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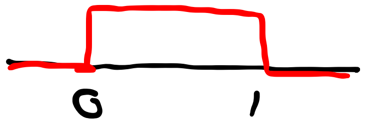
support(X) $x \in X$ $X \in [0,1]$	All the values that X can take	$\{h,t\}$ or $\{0,1\}$	$[0,1]$ 
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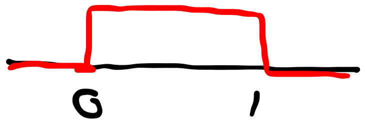</td></tr>
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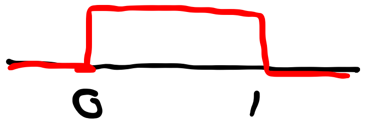
support(X) $x \in X$ $X \in [0,1]$	All the values that X can take	$\{h, t\}$ or $\{0, 1\}$	$[0, 1]$ 
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Distributions of related R.V.s

Distributions of related R.V.s

Joint Distribution

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

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Conditional Distribution

Distributions of related R.V.s

Joint Distribution

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Conditional Distribution

$$P(X \mid Y, Z)$$

Distributions of related R.V.s

Joint Distribution

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1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
0	0.84
1	0.16

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X, Y, Z)$
0	0	0	0.08
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Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
0	0.84
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Marginal Distribution

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
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Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
0	0.84
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Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
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Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
0	0.84
1	0.16

Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

X	$P(X)$
0	0.85
1	0.15

Y	$P(Y)$
0	0.45
1	0.55

Z	$P(Z)$
0	0.20
1	0.80

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X)\; P(Y)\; P(Z)$$

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

3 Rules

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

3 Rules

(Burrito-level)

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

3 Rules

(Burrito-level)

(Blackbelly Level: Axioms of Probability)

not on exam

AXIOM 1. STRUCTURE OF UNKNOWN REAL NUMBERS AND PLAUSIBLE VALUE. *We assume a set T of unknown numbers is a partially ordered commutative algebra over $\mathbb{R}$ with identity, 1.*

We assume in addition a given sub-Boolean algebra E of $E(T)$ with $0, 1 \in E$ and denote by E_0 the set of non-zero members of E. We assume that the partial ordering in $E(T)$ as a Boolean algebra coincides with the ordering that $E(T)$ inherits from the algebra T. Finally, we assume a function $PV : T \times E_0 \to \mathbb{R}$, called **PLAUSIBLE VALUE**, *whose value on the pair (x, e) is denoted $PV(x|e)$.*

AXIOM 2. STRONG RESCALING FOR PLAUSIBLE VALUE. *If a, b belong to $\mathbb{R}$, if x belongs to T, and if e belongs to E_0, then*

$$PV(ax + b|e) = aPV(x|e) + b. \tag{2}$$

AXIOM 3. ORDER CONSISTENCY FOR PLAUSIBLE VALUE. *If $x, y \in T$ and if $e \in E_0$, implies that $x \leq y$, then $PV(x|e) \leq PV(y|e)$.*

Notice that if $e \in E(T)$, then $0 \leq e \leq 1$, in T, as it is true in the lattice ordering of $E(T)$.

AXIOM 4. THE COX AXIOM FOR PLAUSIBLE VALUE: *If e, c are fixed in E, with $ec \in E_0$, if x_1, x_2 are in T, if $PV(x_1|ec) = PV(x_2|ec)$, then $PV(x_1e|c) = PV(x_2e|c)$. That is, we assume that as a function of x, the plausible value $PV(xe|c)$ depends only on $PV(x|ec)$.*

AXIOM 5. RESTRICTED ADDITIVITY OF PLAUSIBLE VALUE. *For each fixed $y \in T$ and $e \in E_0$, the plausible value $PV(x + y|e)$ as a function of $x \in T$ depends only on $PV(x|e)$, which is to say that if $x_1, x_2 \in T$ and $PV(x_1|e) = PV(x_2|e)$, then $PV(x_1 + y|e) = PV(x_2 + y|e)$.*

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\ P(Y)\ P(Z)$

3 Rules

(Burrito-level)

1)

Distributions of related R.V.s

Joint Distribution

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3 Rules

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1) a) $0 \leq P(X \mid Y) \leq 1$

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$$P(X) = \sum_{y \in Y} P(X, y)$$

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Marginal + Conditional $\rightarrow$ Joint

$$P(X, Y) = P(X|Y)\, P(Y)$$

Distributions of related R.V.s

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Marginal Distribution

$P(X)\, P(Y)\, P(Z)$

X	Y	Z	$P(X, Y, Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

3 Rules

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Break

- $P \in \{0, 1\}$: Powder Day
- $C \in \{0, 1\}$: Pass Clear
- 1 in 5 days is a powder day
- The pass is clear 8 in 10 days
- If it is a powder day, there is a 50% chance the pass is blocked

- Write out the joint probability distribution for P and C.
- Suppose it is a non-powder day, what is the probability that the pass is blocked?

Bayes Rule

- Know: $P(B \mid A)$, $P(A)$, $P(B)$
- Want: $P(A \mid B)$

Independence

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Definition: X and Y are *independent* iff $P(X,Y) = P(X)\,P(Y)$

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Definition: X and Y are *conditionally independent* given Z iff
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Rules for Continuous RVs

Discrete

1) a) $0 \leq P(X \mid Y) \leq 1$

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2) $P(X) = \sum_{y \in Y} P(X, y)$

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$$\int_X p(x|Y)\, dx = 1$$

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Multivariate Gaussian Distribution

Joint Distribution

Conditional Distribution

Marginal Distribution

Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables