Stochastic Processes and Simple Decisions

Review

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?

Stochastic Process

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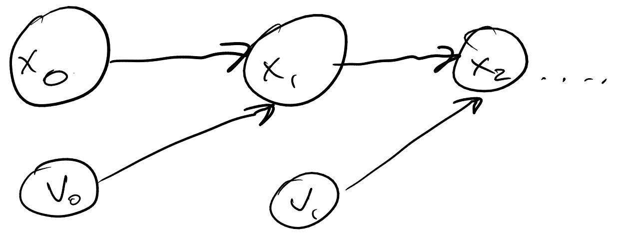

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$$P(x' \mid x)$$
$$= \mathrm{SparseCat}([x, x+1], [0.5, 0.5])$$

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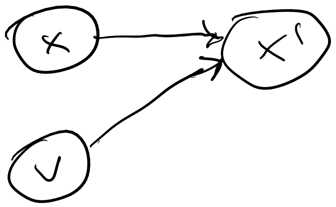
Bayes Net

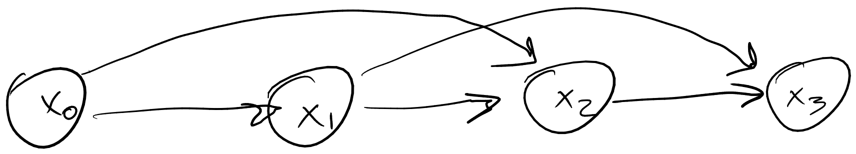


Dynamic Bayes Net (DBN)

Causal Stochastic Processes

In general, stochastic processes may have connections between any times in their Bayesian Network.

In a *causal* stochastic process, x_t may depend on any x_τ with $\tau < t$.



Simulating a Causal Stochastic Process

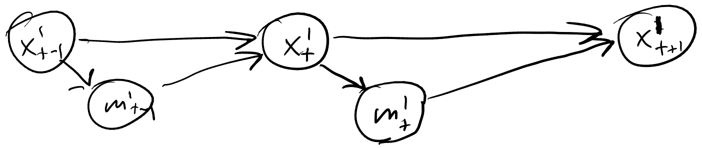
030-Stochastic-Processes.ipynb

Markov Process

Markov Process

- A stochastic process $\{s_t\}$ is *Markov* if
$$P(s_{t+1} \mid s_t, s_{t-1}, \ldots, s_0) = P(s_{t+1} \mid s_t)$$
$$s_{t+1} \perp s_{t-\tau} \mid s_t \quad \forall \tau \in 1 : t$$

Markov Process



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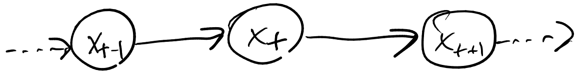
- s_t is called the "state" of the process

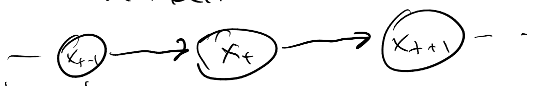
Positive Uniform Random Walk



Is $\{x_t\}$ Markov?

$x_{t+1} \perp x_{t-\tau} \mid \underline{x_t}$ ✓

all paths would contain



all paths d-sep

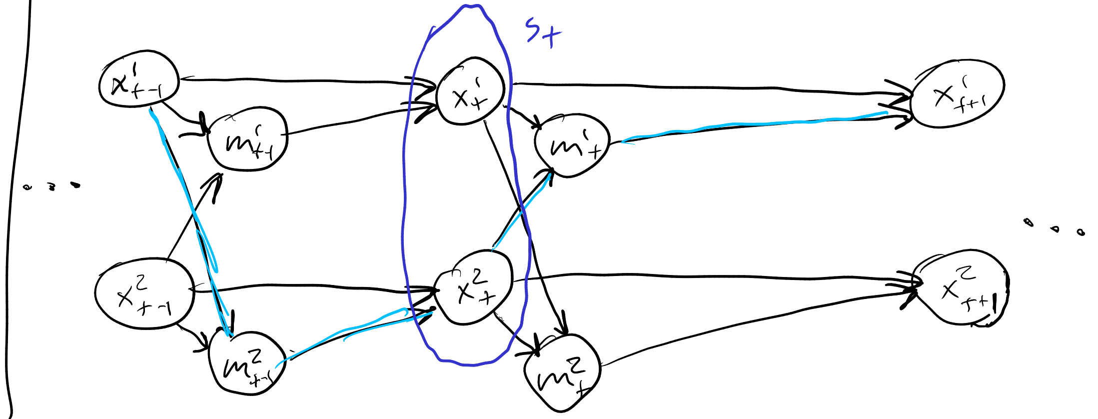
Airborne Collision Avoidance

x^1: physical state of A/C 1

x^2: physical state of A/C 2

m^1: maneuver of A/C 1

m^2: " " " 2



Is $\{x_t^1\}$ Markov?

Inconclusive given only Bayes Net structure

(With reasonable pilots, No)

$s_t = (x_t^1, x_t^2)$

Is $\{s_t\}$ Markov?

Yes

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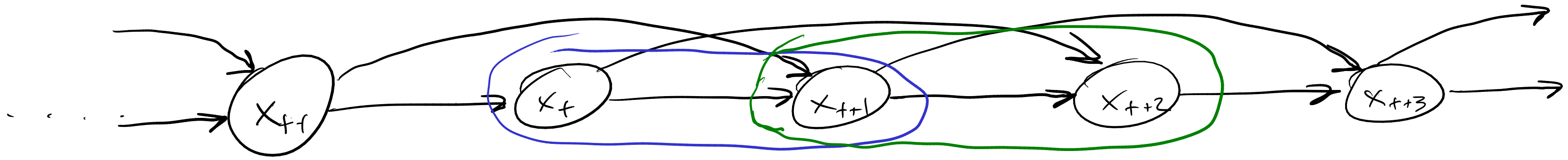
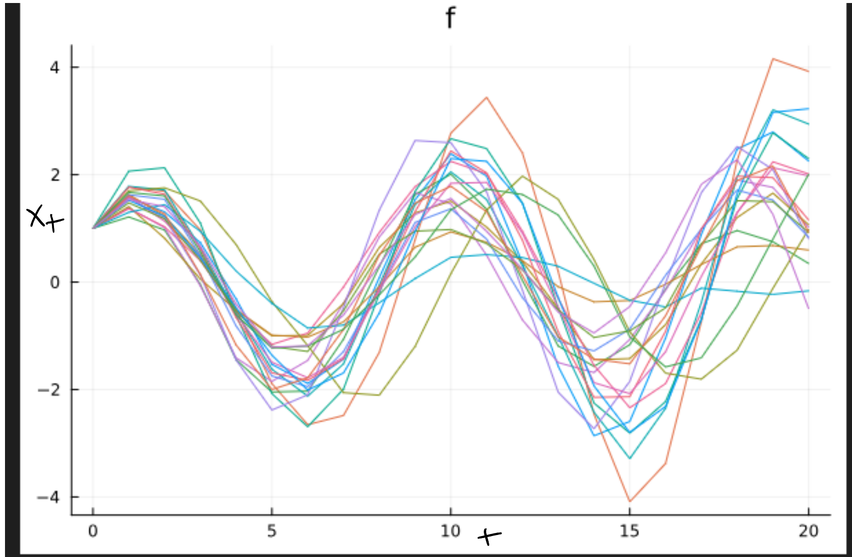
Another example



Cannot conclude $\{X_t\}$ is Markov based on structure

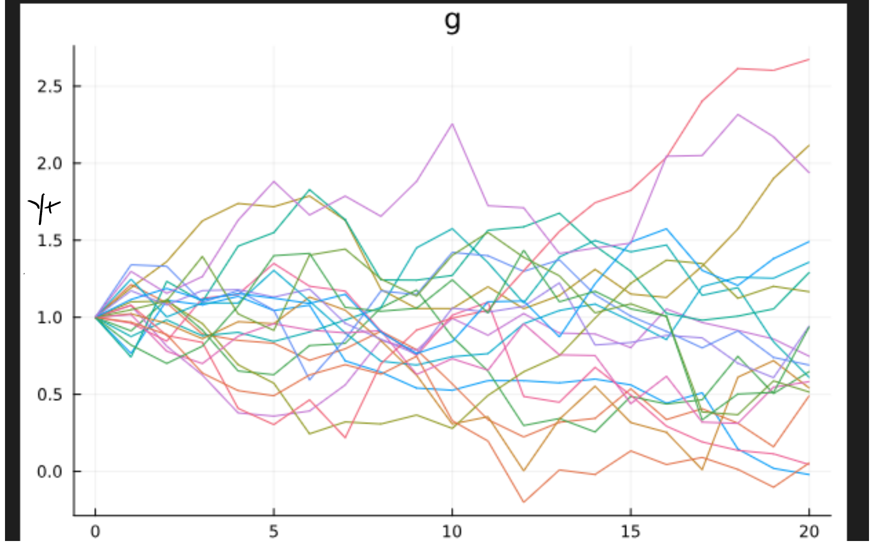
$s_t = (x_t, x_{t+1})$ $\quad$ $s_{t+1} = (x_{t+1}, x_{t+2})$ $\quad$ $P(s_{t+1} \mid s_t)$

then $\{s_t\}$ is Markov $\quad$ $P(s' \mid s)$




```
plot(0:20, sim(g, 20), title = "g", legend=nothing)
```

✓ 0.0s



$\{x_t\}$ is not Markov

$z_t = (x_t, x_{t-1})$

$\{z_t\}$ is Markov

cisions

$\{y_t\}$ is Markov

Simple Decisions

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$S_1 \ldots S_n$ or A, B, C

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$S_1 \ldots S_n$ or A, B, C	$p_1 \ldots p_n$

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$[S_1 : p_1; \ldots ; S_n : p_n]$

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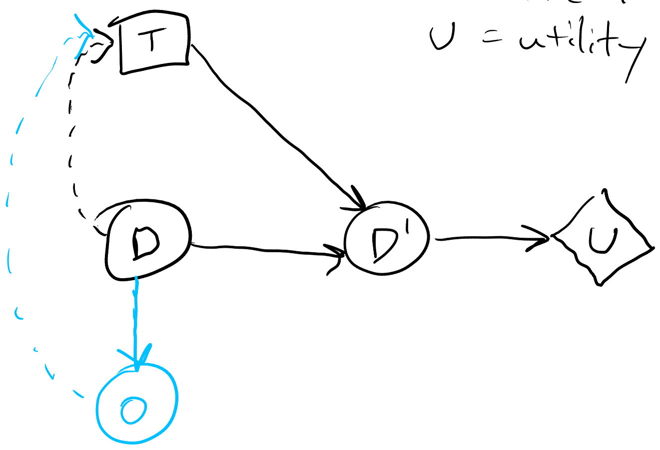
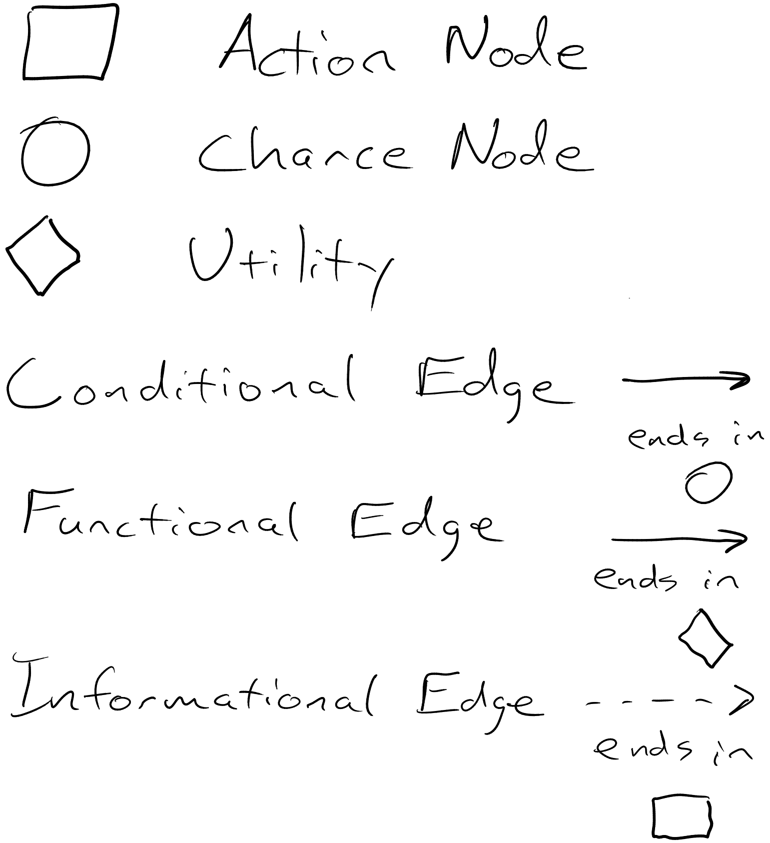
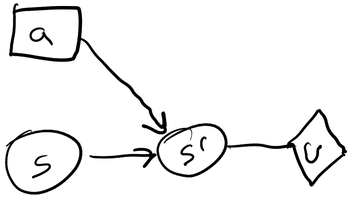
- $U([S_1 : p_1; \dots; S_n : p_n]) = \sum_{i=1}^{n} p_i U(S_i)$

Decision Networks

D = disease state
T = treatment
U = utility







$$a^* = \operatorname{argmax} E\left[U(s') \mid a, s\right]$$

Markov Decision Process

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4. Terminal States

Infinite time, but a terminal state (no reward, no leaving) is always reached with probability 1.

Break

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Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 14 days after they contract the disease.
- The number of people infected by each person on day d of their illness is roughly $\mathcal{N}(\mu_d, \sigma^2)$

n_t = number of people newly infected on day t

$$P(n_{15} \mid n_0 \dots n_{14}) = \mathcal{N}(\mu_{14} n_0 + \mu_{13} n_1 \dots \mid 14\sigma^2)$$

Guiding Question

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