Stochastic Processes and Simple Decisions

Review

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$ or $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

Example: Positive, Uniform Random Walk

$x_0 = 0$

$x_{t+1} = x_t + v_t$

$v_t \sim \mathcal{U}(\{0,1\})$ (i.i.d.)

Shorthand:

$x' = x + v$

$$P(x' \mid x)$$
$$= \mathrm{SparseCat}([x, x+1], [0.5, 0.5])$$

In a *stationary* stochastic process (all in this class), this relationship does not change with time

Bayes Net

Dynamic Bayes Net (DBN)

Causal Stochastic Processes

In general, stochastic processes may have connections between any times in their Bayesian Network.

In a *causal* stochastic process, x_t may depend on any x_τ with $\tau < t$.

Simulating a Causal Stochastic Process

030-Stochastic-Processes.ipynb

Markov Process

- A stochastic process $\{s_t\}$ is *Markov* if
  $$P(s_{t+1} \mid s_t, s_{t-1}, \ldots, s_0) = P(s_{t+1} \mid s_t)$$
  $$s_{t+1} \perp s_{t-\tau} \mid s_t \quad \forall \tau \in 1:t$$
- s_t is called the "state" of the process

Simple Decisions

Simple Decisions

Outcomes	Probabilities	Lottery
$S_1 \dots S_n$ or A, B, C	$p_1 \dots p_n$	$[S_1 : p_1; \dots ; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1-p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1-p] \succeq [B : p; C : 1-p]$

von Neumann - Morgenstern Axioms

These constraints imply a utility function U with the properties:

- $U(A) > U(B)$ iff $A \succ B$
- $U(A) = U(B)$ iff $A \sim B$
- $U([S_1 : p_1; \dots ; S_n : p_n]) = \sum_{i=1}^{n} p_i U(S_i)$

Decision Networks

Markov Decision Process

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

$$\lim_{n\to\infty} \mathbf{E}\left[\sum_{t=0}^{n} r_t\right]$$

3. Discounting

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

discount $\gamma \in [0, 1)$

typically 0.9, 0.95, 0.99

if $\underline{r} \le r_t \le \bar{r}$

then

$$\frac{\underline{r}}{1-\gamma} \le \sum_{t=0}^{\infty} \gamma^t r_t \le \frac{\bar{r}}{1-\gamma}$$

4. Terminal States

Infinite time, but a terminal state (no reward, no leaving) is always reached with probability 1.

Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start on a particular day. **What information should be in the state of the model?**

Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 14 days after they contract the disease.
- The number of people infected by each person on day d of their illness is roughly $\mathcal{N}(\mu_d, \sigma^2)$

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