AS/4602/WKC/5

THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH4602 Scientific Computing
Mimi Project

Due Date: 24 April 2023 (5:00 pm)

In this MATLAB coding assignment, we focus on solving the following **structured** linear system of equations:

$$A\mathbf{x} = \mathbf{b}$$

where A is an $n^2 \times n^2$ matrix taking the following form:

$$A = \begin{bmatrix} A_n & -I_n & & & & 0 \\ -I_n & A_n & -I_n & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -I_n & A_n & -I_n \\ 0 & & & & -I_n & A_n \end{bmatrix}. \tag{1}$$

Here I_n is the $n \times n$ identity matrix and

$$A_n = \begin{bmatrix} 2^d & -1 & & & & 0 \\ -1 & 2^d & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -1 & 2^d & -1 \\ 0 & & & & -1 & 2^d \end{bmatrix}. \tag{2}$$

It is a block-Toeplitz matrix with Toeplitz blocks. We first consider solving the linear system by using an iterative method, the **Gauss-Seidel method**.

(a) (i) Show that A is a symmetric positive definite matrix for $d \geq 2$. Therefore Gauss-Seidel method converges when applied to solving the linear system $A\mathbf{x} = \mathbf{b}$.
(ii) Let the right-hand side vector $\mathbf{b} = A\mathbf{e}$ where

$$\mathbf{e} = [1\ 1\ \ldots\ 1]^T$$

and the true solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{e}$. Write a MATLAB program to implement the **Gauss-Seidel method** for solving $A\mathbf{x} = \mathbf{b}$ with the initial guess $\mathbf{x}_0 = \mathbf{0}$ and the stopping criterion is as follows:

$$||\mathbf{x}_n - \mathbf{e}||_2 < 10^{-6}$$

where $\mathbf{x}_n$ is the approximate solution obtained in the nth iteration.
(iii) What is the computational cost in each iteration of the Gauss-Seidel method?
(iv) Report the number of iterations for convergence for the following pairs of (n, d), $n = 10, 20, 30, 40$ and $d = 2, 3, 4, 5$. Discuss your observations.

(b) We then consider solving the same linear system of equations by using the **Block Jacobi method**. The idea is to consider the splitting of the matrix A as the sum of the diagonal block bimatrix

$$D = \begin{bmatrix} A_n & & & & & 0 \\ & A_n & & & & \\ & \ddots & \ddots & \ddots & & \\ & & & & A_n & \\ 0 & & & & & A_n \end{bmatrix}$$

and the bi-diagonal block matrix $(A - D)$.

(i) Write a MATLAB program to implement the **Block Jacobi method** for solving $A\mathbf{x} = \mathbf{b}$ with the initial guess $\mathbf{x}_0 = \mathbf{0}$ and the same stopping criterion

$$||\mathbf{x}_n - \mathbf{e}||_2 < 10^{-6}$$

where $\mathbf{x}_n$ is the approximate solution obtained in the nth iteration.
(ii) What is the computational cost in each iteration of the block Jacobi method?
(iii) Report the number of iterations for convergence for the following pairs of (n, d), $n = 10, 20, 30, 40$ and $d = 2, 3, 4, 5$. Discuss your observations.

(c) We then consider solving the same linear system of equations by using the **Block Gauss-Seidel method**. The idea is to consider the splitting of the matrix A as the sum of the lower triangular blocks

$$L = \begin{bmatrix} A_n & & & & & 0 \\ -I_n & A_n & & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -I_n & A_n & \\ 0 & & & & -I_n & A_n \end{bmatrix}$$

and the upper triangular blocks $(A - L)$.

(i) Write a MATLAB program to implement the **block Gauss-Seidel method** for solving $A\mathbf{x} = \mathbf{b}$ with the initial guess $\mathbf{x}_0 = \mathbf{0}$ and the same stopping criterion

$$||\mathbf{x}_n - \mathbf{e}||_2 < 10^{-6}$$

where $\mathbf{x}_n$ is the approximate solution obtained in the nth iteration.
(ii) What is the computational cost in each iteration of the block Gauss-Seidel method?
(iii) Report the number of iterations for convergence for the following pairs of (n, d), $n = 10, 20, 30, 40$ and $d = 2, 3, 4, 5$. Discuss your observations.