

Section 3, Exercise 1

It seems like there is no way one could use either insetting (putting a given set into another set) and pairing or pairing on two different inputs to obtain the same set. However, if one sees pairing the same set, then pairing \emptyset with \emptyset would result in $\{\emptyset\}$, which is also the result of insetting \emptyset .

Proof:

Insetting and pairing must have different results because insetting will always result in a set with 1 element, and pairing will always result in a set with 2 elements. Therefore, they can't be the same set.

Pairing the sets a, b and c, d can't result in the same set unless $a = c$ and $b = d$ or $a = d$ and $b = c$. Otherwise, $\{a, b\}$ would contain at least one element not in $\{c, d\}$.

□

Section 4, Exercise 1

I am not exactly sure what I'm supposed to do here. I guess "observe" means "prove" here, so "prove that the condition has nothing to do with the set B".

Proof:

$$(A \cap B) \cup C = A \cap (B \cup C) \Leftrightarrow C \subset A$$

$$(A \cup C) \cap (B \cup C) = A \cap (B \cup C) \Leftrightarrow C \subset A$$

$$A \cup C = A \Leftrightarrow C \subset A$$

This is trivially true.

□

Section 5, Some easy exercises

$$A - B = A \cap B'$$

Proof:

$$A - B = \{a | a \in A \wedge a \notin B\} = \{a | a \in A \wedge a \in B'\} = A \cap B'$$

□

$$A \subset B \text{ if and only if } A - B = \emptyset$$

Proof:

$$A \subset B \Leftrightarrow \forall a \in A : a \in B \Leftrightarrow \exists C : B = A \cup C \Leftrightarrow A - (A \cup C) = \emptyset \Leftrightarrow A - B = \emptyset$$

□

$$A - (A - B) = A \cap B$$

Proof:

$$A - (A - B) = A - (A \cap B') = A \cap (A \cap B')' = A \cap (A' \cup B) = A \cap A' \cup A \cap B = \emptyset \cup A \cap B = A \cap B$$

□

$$A \cap (B - C) = (A \cap B) - (A \cap C)$$

Proof:

$$(A \cap B) - (A \cap C) = (A \cap B) \cap (A \cap C)' = (A \cap B) \cap (A' \cup C') = (A \cap B \cap A') \cup (A \cap B \cap C') = A \cap B \cap C' = A \cap (B - C)$$

□

$$A \cap B \subset (A \cap C) \cup (B \cap C')$$

Proof:

$$\begin{aligned} A \cap B &\subset (A \cap C) \cup (B \cap C') \\ &= ((A \cap C) \cup B) \cap ((A \cap C) \cup C') \\ &= ((A \cap C) \cup B) \cap ((A \cup C') \cap (C \cup C')) \\ &= ((A \cap C) \cup B) \cap (A \cup C') \\ &= (A \cup B) \cap (C \cup B) \cap (A \cup C') \end{aligned}$$

$$A \cap B \subset (A \cup B) \cap (C \cup B) \cap (A \cup C') \text{ is true because } A \subset (A \cup B) \text{ and } B \subset (C \cup B) \text{ and } A \subset (A \cup C').$$

□

$$(A \cup C) \cap (B \cup C') \subset A \cup B$$

Proof:

$$(A \cup C) \cap (B \cup C')$$

$$\begin{aligned}
&= ((A \cup C) \cap B) \cup ((A \cup C) \cap C') \\
&= ((A \cup C) \cap B) \cup A \\
&= (A \cap B) \cup (C \cap B) \cup A \subset A \cup B
\end{aligned}$$

This is the case because $(A \cap B) \cup (C \cap B) \subset B$ (since intersections with B are subsets of B), and the union with A doesn't change the equation.

□

Section 5, Exercise 1

To be shown: The power set of a set with n elements has 2^n elements. Proof by induction.

Proof:

Induction base: The power set of the empty set contains 1 element:

$$|P(\emptyset)| = |\emptyset| = 1 = 2^0 = 2^{|\emptyset|}$$

Induction assumption:

$$|P(A)| = 2^{|A|}$$

Induction step:

To be shown: $|P(A \cup \{a\})| = 2 * 2^{|A|} = 2^{|A|+1}$.

$P(A \cup \{a\})$ contains two disjoint subsets: $P(A)$ and $N = \{\{a\} \cup S \mid S \in P(A)\}$. Those are disjoint because every element in N contains a ($\forall n \in N : a \in n$), but there is no element of $P(A)$ that contains a . Also, it holds that $P(A) \cup N = P(A \cup \{a\})$, because elements in the power set can either contain a or not, there is no middle ground. It is clear that $|N| = |P(A)|$, therefore $|P(A \cup \{a\})| = |P(A)| + |N| = 2 * |P(A)| = 2 * 2^{|A|} = 2^{|A|+1}$.

□

Section 5, Exercise 2

To be shown:

$$\mathcal{P}(E) \cap \mathcal{P}(F) = \mathcal{P}(E \cap F)$$

Proof:

If $S \in \mathcal{P}(E \cap F)$, then $\forall s \in S : s \in E \cap F$. Therefore, $S \subset E$ and $S \subset F$ and thereby $S \in \mathcal{P}(E)$ and $S \in \mathcal{P}(F)$. This means that $S \in \mathcal{P}(E) \cap \mathcal{P}(F)$.

If $S \in \mathcal{P}(E) \cap \mathcal{P}(F)$, then a very similar proof can be written: $S \subset E$ and $S \subset F$, so $\forall s \in S : s \in E$ and $\forall s \in S : s \in F$. Then $S \subset E \cap F$ and therefore $S \in \mathcal{P}(E \cap F)$.

□

To be shown:

$$\mathcal{P}(E) \cup \mathcal{P}(F) \subset \mathcal{P}(E \cup F)$$

Proof:

If $S \in \mathcal{P}(E) \cup \mathcal{P}(F)$, then $S \in \mathcal{P}(E) \Leftrightarrow S \subset E$ or $S \in \mathcal{P}(F) \Leftrightarrow S \subset F$. Since it is true for any set X that $S \subset E \Rightarrow S \in \mathcal{P}(E \cup X)$, it is true that $S \in \mathcal{P}(E \cup F)$ (similar argumentation if $S \subset F$).

□

A reasonable interpretation for the introduced notation: If $\mathcal{C} = X_1, X_2, \dots, X_n$, then

$$\bigcap_{X \in \mathcal{C}} X = X_1 \cap X_2 \cap \dots \cap X_n$$

Similarly, if $\mathcal{C} = X_1, X_2, \dots, X_n$, then

$$\bigcup_{X \in \mathcal{C}} X = X_1 \cup X_2 \cup \dots \cup X_n$$

The symbol \mathcal{P} still stands for the power set.

To be shown:

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}(X) = \mathcal{P}\left(\bigcap_{X \in \mathcal{C}} X\right)$$

Proof by induction.

Induction base:

$$\mathcal{P}(E) \cap \mathcal{P}(F) = \mathcal{P}(E \cap F)$$

Induction assumption:

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}(X) = \mathcal{P}\left(\bigcap_{X \in \mathcal{C}} X\right)$$

Induction step:

$$\mathcal{P}(Y) \cap \bigcap_{X \in \mathcal{C}} \mathcal{P}(X)$$

$$\begin{aligned}
&= \mathcal{P}(Y) \cap \mathcal{P}\left(\bigcap_{X \in \mathcal{C}} X\right) \\
&= \mathcal{P}\left(Y \cap \bigcap_{X \in \mathcal{C}} X\right)
\end{aligned}$$

The last step uses $\mathcal{P}(E) \cap \mathcal{P}(F) = \mathcal{P}(E \cap F)$, since $\bigcap_{X \in \mathcal{C}} X$ is also just a set.

□

To be shown:

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}(X) \subset \mathcal{P}\left(\bigcup_{X \in \mathcal{C}} X\right)$$

Proof by induction.

Induction base:

$$\mathcal{P}(E) \cup \mathcal{P}(F) \subset \mathcal{P}(E \cup F)$$

Induction assumption:

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}(X) \subset \mathcal{P}\left(\bigcup_{X \in \mathcal{C}} X\right)$$

Induction step:

$$\begin{aligned}
&\mathcal{P}(Y) \cup \bigcup_{X \in \mathcal{C}} \mathcal{P}(X) \\
&\subset \mathcal{P}(Y) \cup \mathcal{P}\left(\bigcup_{X \in \mathcal{C}} X\right) \\
&\subset \mathcal{P}\left(Y \cup \bigcup_{X \in \mathcal{C}} X\right)
\end{aligned}$$

The last step uses $\mathcal{P}(E) \cup \mathcal{P}(F) \subset \mathcal{P}(E \cup F)$, since $\bigcup_{X \in \mathcal{C}} X$ is also just a set.

□

To be shown:

$$\bigcup \mathcal{P}(E) = E$$

Proof:

$\forall X \in \mathcal{P}(E) : X \subset E$. Furthermore, $E \in \mathcal{P}(E)$. Since $A \subset E \Rightarrow A \cup E = E$, it holds that $E = \bigcup_{X \in \mathcal{P}(E)} X = \bigcup \mathcal{P}(E)$.

□

And "E is always equal to $\bigcup_{X \in \mathcal{P}(E)} X$ (that is $\bigcup \mathcal{P}(E) = E$), but that the result of applying \mathcal{P} and \bigcup to E in the other order is a set that includes E as a subset, typically a proper subset" (p. 21).

I am not entirely sure what this is supposed to mean. If it means that we treat \mathcal{E} as a collection, then $\forall X \in \mathcal{E} : \bigcup_{E \in \mathcal{E}} E \subset X$. But that doesn't mean that $\mathcal{E} \subset \mathcal{P}(\bigcup_{E \in \mathcal{E}} E)$: If $\mathcal{E} = \{\{a, b\}, \{b, c\}\}$, then $\bigcup_{E \in \mathcal{E}} E = \{a, b, c\}$, and $\mathcal{E} = \{\{a, b\}, \{b, c\}\} \not\subset \mathcal{P}(\{a, b, c\}) = \{\{a, b, c\}, \emptyset\}$.

If we treat E simply as a set, then $\bigcup E = E$, and it is of course clear that $E \subset \mathcal{P}(E)$, as for all other subsets of E .

□

Section 6, A non-trivial exercise

"find an intrinsic characterization of those sets of subsets of A that correspond to some order in A "

Let $\mathcal{M} \subset \mathcal{P}(\mathcal{P}(A))$ be the set of all possible orderings of A . In the case of $A = \{a, b\}$, \mathcal{M} would be $\{\{\{a\}, \{a, b\}\}, \{\{b\}, \{a, b\}\}\}$.

Some facts about every element $M \in \mathcal{M}$:

$\bigcap M = \{\min\}$, where $\{\min\}$ is the smallest element in the ordering M , the element in M for which there is no other element $a \in M$ so that $a \subset \{\min\}$ (which means that $\emptyset \notin M$).

$\bigcup M = A$. A must therefore be in M and be the biggest element (no element $a \in M$ so that $a \supset A$).

For all elements $m \in M$ except $\{\min\}$ there exists at least one element $n \in M$ so that $n \subset m$.

Similarly, for all elements $m \in M$ except A there exists at least one element $n \in M$ so that $n \supset m$.

For every $m \in M$ except A and $\{\min\}$, there exist two unique elements $x, y \in M$ so that m is the only set in M for which it is true that $x \subset m \subset y$.

For every $a \in A$, there must exist two sets $m, n \in M$ so that $n = m \cup \{a\}$ (except for \min). This means that the $|A| = |M|$, the size of A is the size of M .

These conditions characterise \mathcal{M} intrinsically and are the solution to the question.

Section 6, Exercise 1

(i) To be shown: $(A \cup B) \times X = (A \times X) \cup (B \times X)$

Proof:

$$(A \cup B) \times X = \{(e, x) : e \in A \vee e \in B, x \in X\} = \{(e, x) : e \in A, x \in X\} \cup \{(e, x) : e \in B, x \in X\} = (A \times X) \cup (B \times X)$$

□

(ii) To be shown: $(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$

Proof:

$$\begin{aligned} (A \times X) \cap (B \times Y) &= \\ \{(x, y), x \in A, y \in X\} \cap \{(v, w), v \in B, w \in Y\} &= \\ \{(x, y), x \in A \wedge x \in B, y \in X \wedge y \in Y\} &= \\ \{(x, y), x \in A \cap B, y \in X \cap Y\} &= \\ (A \cap B) \times (X \cap Y) \end{aligned}$$

□

(iii) To be shown: $(A - B) \times X = (A \times X) - (B \times X)$

Two-sided proof by contradiction:

1. $(A - B) \times X \subset (A \times X) - (B \times X)$

Let $(u, v) \in (A - B) \times X$. Then $u \in (A - B)$, and $v \in X$. Suppose $(u, v) \notin (A \times X) - (B \times X)$. Then $(u, v) \in (A \times X) \cap (B \times X)$. Then $(u, v) \in (A \cap B) \times (X \cap X)$. Then $u \in A \cap B$ and $v \in X$. But if $u \in A \cap B$, then $u \notin A - B$! Contradiction.

2. $(A \times X) - (B \times X) \subset (A - B) \times X$

Let $(u, v) \in (A \times X) - (B \times X)$. Then $(u, v) \in (A \times X)$ and $(u, v) \notin (B \times X)$. Because v must be in X , and there is no flexibility there, $u \notin B$. Suppose $(u, v) \notin (A - B) \times X$. Since necessarily $v \in X$, $u \notin A - B$. But if $u \notin A - B$, u must be an element of $A \cap B$. Then $u \in B$, and there is a contradiction.

□

Section 7, Exercise 1

Reflexive, but neither symmetric nor transitive (symmetry violation: $(b, a) \notin$, transitivity violation: $(a, c) \notin$): $\{(a, a), (a, b), (b, b), (b, c), (c, c)\}$

Symmetric, but neither reflexive nor transitive (reflexivity violation: $(a, a) \notin$, transitivity violation: $(a, c) \notin$): $\{(a, b), (b, a), (b, c), (c, b)\}$

Transitive, but neither reflexive nor symmetric (reflexivity violation: $(a, a) \notin$, symmetry violation: $(b, a) \notin$): $\{(a, b), (b, c), (a, c)\}$