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Maps Between Some Different Kinds of Contraction Function: The Finite Case

Abstract. In some recent papers, the authors and Peter Gärdenfors have defined and studied two different kinds of formal operation, conceived as possible representations of the intuitive process of contracting a theory to eliminate a proposition. These are partial meet contraction (including as limiting cases full meet contraction and maxichoice contraction) and safe contraction. It is known, via the representation theorem for the former, that every safe contraction operation over a theory is a partial meet contraction over that theory. The purpose of the present paper is to study the relationship more finely, by seeking an explicit map between the component orderings involved in each of the two kinds of contraction. It is shown that at least in the finite case a suitable map exists, with the consequence that the relational, transitively relational, and antisymmetrically relational partial meet contraction functions form identifiable subclasses of the safe contraction functions, over any theory finite modulo logical equivalence. In the process of constructing the map, as the composition of four simple transformations, mediating notions of bottom and top contraction are introduced. The study of the infinite case remains open.

1. Background

The notion of partial meet contraction was introduced by the authors and Peter Gärdenfors in [2], and the notion of safe contraction by the authors alone in [3]. The special case of partial meet contraction known as maxichoice contraction had been studied earlier by the authors in [1]. There is also a general survey of these notions, without proofs, in [4]. To avoid lengthy repetition we shall assume that the reader is familiar with [2] and [3] or at least with the survey [4] for concepts and motivation, and we recall here only the most salient definitions.

Let A be a set of propositions, and Cn a consequence operation. We assume that Cn includes classical tautological implication, is compact, and satisfies the rule of "introduction of disjunction into the premisses", that is, that $y \in Cn(A \cup \{x_1 \vee x_2\})$ whenever $y \in Cn(A \cup \{x_1\})$ and $y \in Cn(A \cup \cup \{x_2\})$. We write $A \vdash x$ as an alternative notation for $x \in Cn(A)$. $A \perp x$ is defined to be the set of all maximal subsets $B \subseteq A$ such that B *non* $\vdash x$. A *selection function* over A is a function γ such that for every proposition x, if $A \perp x$ is non-empty then $\gamma(A \perp x)$ is a non-empty subset of $A \perp x$, and if $A \perp x = \emptyset$ then $\gamma(A \perp x) = \{A\}$. The *partial meet contraction* function $\dot{-}$ determined by γ is defined by putting $A \dot{-} x = \bigcap \gamma(A \perp x)$ for every proposition x. As a special case, if $\gamma(A \perp x)$ is a singleton $\{B\}$ where $B \in A \perp x$ for all values of $x \notin Cn(\emptyset)$, then the selection function γ is called a *maxichoice selection function*, and the partial meet contraction function $\dot{-}$ that it determines is called a *maxichoice contraction function*.

A selection function γ is called *relational* iff there is a relation $\leqslant$ over $\bigcup_x \{A \perp x\}$ such that for all $x \notin Cn(\emptyset)$, $\gamma(A \perp x)$ consists of just those elements of $A \perp x$ that "dominate" $A \perp x$ under $\leqslant$, i.e. $\gamma(A \perp x) = \{B \in A \perp x\colon B' \leqslant B$ for all $B' \in A \perp x\}$. Equivalently and more conveniently when A is a theory, i.e. when $A = Cn(A)$: iff there is a relation $\leqslant$ over $U_A = \bigcup_{x \in A} \{A \perp x\}$ such that for all $x \in A$, $x \notin Cn(\emptyset)$, the same identity, which we call the "marking off" identity, holds.

On the other hand, the concept of safe contraction is defined as follows. A *hierarchy* over A is a relation $<$ over A that is *non-circular* in the sense that for no $a_1, \ldots, a_n \in A$ $(n \geqslant 1)$, $a_1 < a_2 < \ldots < a_n < a_1$; and the *safe contraction function* $\dot{-}$ over A determined by a hierarchy $<$ over A is defined by putting $A \dot{-} x = A \cap Cn(A/x)$. Here A/x is the set of all elements $a \in A$ that are *safe* with respect to x, in the sense that a is not a minimal element (under $<$) of any minimal subset (under set inclusion) of A that implies x. When A is a theory this reduces to $A \dot{-} x = Cn(A/x)$.

Now we know from [3] Observation 3.2 that every safe contraction function over a set A of propositions satisfies the Gärdenfors postulates ($\dot{-}$1) to ($\dot{-}$6) for contraction; and we also know from the first Representation Theorem 2.5 of [2] that when A is a theory, then any operation satisfying those postulates is a partial meet contraction function over A. Thus, as noted in Corollary 3.3 of [3], every safe contraction function over a theory is a partial meet contraction function over that theory.

Our present purpose is to study more directly and deeply the interconnection between the two kinds of contraction, by constructing an explicit map between the relations involved, in the finite case. We give constructions in detail, but abbreviate and sometimes omit the more routine verifications.

2.1 Dominance and Maximality

The first step in constructing the desired map is to reformulate the definition of the relationality of a selection function in terms of maximality rather than dominance, as follows: a selection function γ over a theory A is relational iff there is some relation $<$ over U_A such that for all $x \in A$, $x \notin Cn(\emptyset)$, we have $\gamma(A \perp x) = \{B \in A \perp x\colon$ for no $B' \in A \perp x$, $B < B'\}$.

The equivalence of this to the original definition of relationality is easily verified using the map f_1 that takes a relation $\leqslant$ over U_A to the complement of its converse, that is, putting $B' < B$ iff $B \nleqslant B'$. Clearly f_1 is a one-one transformation of the set of all relations over U_A onto itself; and clearly we have that $\{B \in A \perp x\colon$ for all $B' \in A \perp x$, $B' \leqslant B\}$ $= \{B \in A \perp x\colon$ for no $B' \in A \perp x$, $B < B'\}$, where $<$ is $f_1(\leqslant)$.

Now suppose from this point onwards to the end of the paper that the theory A is finite modulo Cn, that is, that there are only finitely many elements $a \in A$ mutually non-equivalent under the relation of logical equivalence $a \simeq a'$ defined to hold iff $Cn(a) = Cn(a')$. Suppose that γ is a relational selection function over A, with $<$ a relation over U_A such that for all $x \in A$, $x \notin Cn(\emptyset)$, we have $\gamma(A \perp x) = \{B \in A \perp x\colon$ for no $B' \in A \perp x$, $B < B'\}$. Then it is not difficult to show that $<$ must be non-circular over U_A, that is, for no $B_1, \ldots, B_n \in U_A$ $(n \geqslant 1)$ do we have $B_1 < B_2 < \ldots < B_n < B_1$. For suppose that $B_1, \ldots, B_n \in U_A$. Now by finiteness it is easy to show that for each $i \leqslant n$ there is a $b_i \in A$ with $A \perp b_i = \{B_i\}$ — take for example b_i to be the disjunction of all (up to logical equivalence) the $a \in A$ such that B_i *non* $\vdash a$, and use the fact that for each such a, $A \subseteq Cn(B_i \cup \{a\})$. And $A \perp b_1 \wedge \ldots \wedge b_n = A \perp b_1 \cup \ldots \cup A \perp b_n$ by Observation 4.1 of [2], so $A \perp b_1 \wedge \ldots \wedge b_n = \{B_1, \ldots, B_n\}$, so since γ is a selection function, $\gamma(A \perp b_1 \wedge \ldots \wedge b_n)$ is a non-empty subset of $\{B_1, \ldots, B_n\}$ and so contains some B_i. Hence B_i is maximal in $\{B_1, \ldots, B_n\}$ under $<$, i.e. $B_i \not< B_j$ for all $j \leqslant n$ and so in particular $B_i \not< B_{i+1}$ (reading $i+1$ to be 1 if $i = n$) as desired.

Thus when A is a theory finite modulo Cn, we know that if a function γ is a relational selection function over A, then there is a non-circular relation $<$ over U_A such that for all $x \notin A$, $x \notin Cn(\emptyset)$, $\gamma(A \perp x)$ consists of precisely those elements of $A \perp x$ that are maximal under $<$, i.e. $\gamma(A \perp x) = \{B \in A \perp x\colon$ for no $B' \in A \perp x$, $B < B'\}$.

Moreover, again under the assumption that A is theory finite modulo Cn, and supposing further that $\gamma(A \perp x) = \{A\}$ in the limiting cases that $x \notin A$ or $x \in Cn(\emptyset)$, we also have the converse implication. For if $\gamma(A \perp x) = \{A\}$ in the limiting cases, and there is such a relation $<$ over U_A, then using the non-circularity of $<$ and the finiteness of $A \perp x$ we can easily verify that $\gamma(A \perp x)$ is non-empty whenever $A \perp x$ is non-empty, so that γ is indeed a selection function, and so, as shown in the first paragraph of this section, is a relational selection function.

Putting these two implications together we can therefore say: the relational partial meet contraction functions over a theory A finite modulo Cn are just the functions $\dot{-}$ such that $A \dot{-} x = A$ in the limiting cases that $x \notin A$ or $x \in Cn(\emptyset)$, and such that there is a non-circular relation $<$ over U_A with $A \dot{-} x = \bigcap \{B \in A \perp x\colon$ for no $B' \in A \perp x$, $B < B'\}$, or more briefly expressed, $A \dot{-} x = \bigcap max(A \perp x)$, for all $x \in A$, $x \notin Cn(\emptyset)$.

We note for later use that there is a systematic correspondence between properties of $\leqslant$ and properties of $f_1(\leqslant) = <$. In particular: $\leqslant$ is reflexive (over U_A) iff $<$ is irreflexive; $\leqslant$ is connected iff $<$ is asymmetric; $\leqslant$ is transitive iff $<$ is *virtually connected* in the sense of [3], i.e. for all B, B', $B'' \in U_A$, if $B < B'$ then either $B < B''$ or $B'' < B'$; and $\leqslant$ is antisymmetric (which suffices to render its partial meet contraction function a maxichoice contraction function) iff $<$ is connected up to identity (i.e. either $B < B'$ or $B' < B$ or $B = B'$ for all $B, B' \in U_A$).

2.2 Bottom Contraction

When A is finite modulo Cn, the elements of U_A can of course be identified, up to equivalence modulo Cn, with the atoms of A. Clearly then, there is a natural one-one correspondence f_2 between the class of all non-circular relations $<$ over U_A and the class of all normal non-circular relations over the set $\perp_A$ of all atoms of A. Here a relation $<$ is said to be *normal* over a subset A' of A iff it respects logical equivalence over that set, i.e. iff for all $a, a', b \in A'$, if $Cn(a) = Cn(a')$ then $a < b$ iff $a' < b$ and likewise $b < a$ iff $b < a'$. It is thus easy to verify that an operation $\dot{-}$ is a relational partial meet contraction function over a theory A finite modulo Cn iff there is a normal non-circular relation over the atoms of A such that $A \dot{-} x = A$ in the limiting cases that $x \notin A$ or $x \in Cn(\emptyset)$, and such that in the remaining (principal) cases $A \dot{-} x = Cn(\bigvee max \perp(x))$. Here $\perp(x)$ is the set of all atoms k of A such that k *non* $\vdash x$, and $max \perp(x)$ is thus the set of all atoms of A that are maximal among the atoms that do not imply x. This set will be infinite, but as it is finite modulo Cn, disjunction over it may be understood as a disjunction of arbitrarily chosen representatives of its finitely many cells.

We call such an operation $\dot{-}$, for a normal and non-circular relation $<$ over $\perp_A$, the *bottom contraction* over A determined by $<$. Thus, cumulating our results so far, we know that the relational partial meet contraction functions over a theory A finite modulo Cn are just the bottom contraction functions over A.

2.3 Top Contraction

It appears to be difficult to construct a map directly from bottom contraction to safe contraction. So we proceed via a dual notion of top contraction.

Clearly, the function $\sim$ on $A = Cn(a_0)$ defined by putting $\sim a = a_0 \vee \neg a$ sets up a one-one correspondence (up to logical equivalence) between the atoms and co-atoms of A. Given a relation $<$ over the atoms, we define the relation $f_3(<) = <'$ over the set T_A of co-atoms by putting $s <' t$ iff $\sim t < \sim s$. Clearly f_3 establishes a one-one correspondence between the class of all normal relations over $\perp_A$ and the class of all normal relations over T_A, and moreover, a normal relation $<$ over $\perp_A$ is non-circular iff $<'$ over T_A is likewise so.

Nor is the map f_3 a merely formal transformation; it has an epistemological rationale. For suppose that the relation $<$ is read as indicating degree of security, as suggested in the discussion in Section 4 of [3]. More specifically, suppose that we understand $b < c$ as indicating that in the circumstance that A as a whole is regarded as not entirely true, b is more open to doubt, or more in danger of falsehood, than c, or in other words that c is more secure than b. Now of all the situations or "possible worlds" that fail to render A as a whole true, that is which render a_0 false, those which make an element $a \in A$ false are precisely those which make its relative negation $\sim a = a_0 \vee \neg a$ true, as is evident from the classical truth conditions for disjunction and negation. The danger of a being false, given the falsehood of a_0, is thus the same as the susceptibility of $\sim a$ being true. Consequently, when $b < c$ so that b is more in danger of being false than c, then $\sim b$ is more susceptible of being true than is $\sim c$, which is to say that $\sim c$ is more in danger of falsehood than $\sim b$. Similarly for the converse. Thus if $<$ over $\perp_A$ is understood as indicating degree of security given the falsehood of a_0, then the relation $<' = f_3(<)$ over T_A also indicates degree of security in the same circumstance.

It is not difficult to verify, using standard boolean considerations and the normality of the relations $<$, $f_3(<)$ involved, that $Cn(\bigvee max \perp(x)) = Cn(T_A - min\, T(x))$ whenever $x \notin Cn(\emptyset)$ and $x \in A$, where $T(x)$ is the set of all co-atoms $t \in T_A$ such that $x \vdash t$, and where maximality on the left means maximality under $<$ whilst minimality on the right is under $<' = f_3(<)$.

Hence we can say that an operation $\dot{-}$ is a bottom contraction over A iff there is a normal and non-circular relation over T_A such that $A \dot{-} x = A$ in the limiting cases that $x \notin A$ or $x \in Cn(\emptyset)$, and such that $A \dot{-} x = Cn(T_A - min\, T(x))$ in the remaining (principal) cases. We call such an operation $\dot{-}$, for a normal and non-circular relation $<$ over T_A, the *top contraction function* over A determined by $<$. The bottom contraction functions are thus just the top contraction functions. Cumulating our results so far, we can say that the relational partial meet contraction functions over a theory A finite modulo Cn are just the top contraction functions over A.

2.4 Regular Safe Contraction

The passage from top contraction to safe contraction is rather different from the transformations of the previous sections. This is essentially because a finite theory A with n mutually non-equivalent co-atoms will have 2^n non-equivalent elements, and so instead of a one-one correspondence between relations over the former set and relations over the latter set, we shall construct a slightly more complex functional connection.

Using the terminology of [3], we say that a relation $<$ over A *continues up* $\vdash$ over A iff whenever $a, b, c \in A$ and $a < b$ whilst $b \vdash c$ then $a < c$. Analogously we say that $<$ *continues down* $\vdash$ over A iff whenever $a, b, c \in A$ and $a \vdash b$ whilst $b < c$ then $a < c$. A relation over A is said to be *regular* over A iff it continues both up and down $\vdash$ over A. It is easy to check that when $<$ is regular over A then it is normal over A

In this section we wish to show that for a theory A finite modulo Cn, an operation $\dot{-}$ is a top contraction function over A iff it is a safe contraction function determined by some regular hierarchy over A. To do this we make use of two maps, f_4 and g. When $<$ is a relation over T_A, we define $f_4(<)$ to be the relation $<'$ over A such that $a <' b$ iff there is some $s \in T(a)$ such that for all $t \in T(b)$, $s < t$. When $<$ is a relation over A, we define $g(<)$ to be simply the restriction of $<$ to T_A.

It is easy to check that when $<$ is a normal and non-circular relation over T_A, then $f_4(<)$ is also normal and non-circular over A, and also continues up and down $\vdash$ over A, and so is a regular hierarchy over A. Moreover it is clear that when $<$ is normal over T_A then $f_4(<)$ is a conservative extension of $<$, that is, $g(f_4(<)) = <$. In view of this, to prove the desired equivalence it will suffice to verify more specifically the following:

LEMMA. *Let A be a theory finite modulo Cn and $<$ a regular hierarchy over A. Then an operation $\dot{-}$ is the safe contraction function determined by $<$ iff $\dot{-}$ is the top contraction function determined by $g(<)$.*

To see that this lemma will suffice, note first that if $\dot{-}$ is a safe contraction function determined by a regular hierarchy $<$ over A, then by the lemma $\dot{-}$ is the top contraction function determined by the (normal and non-circular) relation $g(<)$ over A. Conversely, suppose that $\dot{-}$ is a top contraction function over A, and so determined by a normal and non-circular relation $< = g(f_4(<))$ over T_A. Then $f_4(<)$ is a regular hierarchy over A, and so by the lemma again, $\dot{-}$ is the safe contraction function determined by $f_4(<)$.

PROOF OF THE LEMMA. Let $<$ be a regular hierarchy over A. Then its restriction $g(<)$ to T_A is normal and non-circular, so we need only show that $Cn(A/x) = Cn(T_A - min\,T(x))$ for all $x \in A$, $x \notin Cn(\emptyset)$, where A/x on the left is determined by $<$ and $min\,T(x)$ on the right is determined by $g(<)$, which for simplicity we henceforth also write as $<$. By boolean considerations it will suffice to show that $LHS \cap T_A = RHS \cap T_A$, and clearly since $<$ is normal over T_A, $RHS \cap T_A = T_A - min\,T(x)$. Thus we need only show that $Cn(A/x) \cap T_A = T_A - min\,T(x)$ whenever $x \in A$, $x \notin Cn(\emptyset)$.

Suppose $t \in Cn(A/x) \cap T_A$. If $t \notin T(x)$ then $t \in T_A - min\,T(x)$ as desired, so we may suppose $t \in T(x)$. Since t is a co-atom of A and $t \in Cn(A/x)$, then by a boolean property of co-atoms, there is an $a \in A/x$ with $a \vdash t$. But since $<$ is regular over A and so continues up $\vdash$ over A, we can infer by Lemma 4.1 of [3] that $t \in A/x$, so that t is not a minimal element of any minimal subset $B \subseteq A$ with $B \vdash x$. Now the relation $\simeq$ of logical equivalence under Cn partitions $T(x)$ into cells; let B be a choice set containing just one s from each of these cells, with $t \in T(x)$ chosen as the representative of its cell. It is easy to see that B is a minimal subset of A that implies x, so since $t \in B$, we can say that t is not minimal in B. Hence since $B \subseteq T(x)$, t is not minimal in $T(x)$, so $t \in T_A - min T(x)$ as desired.

For the converse, suppose that $t \in T_A - min T(x)$. Then t is a co-atom of A and either x *non* $\vdash t$ or there is an $s \in T(x)$ with $s < t$. In the case that x *non* $\vdash t$, it is not difficult to show that since t is a co-atom of A, t is not an element of any minimal $B \subseteq A$ with $B \vdash x$, so that $t \in A/x$ and so $t \in Cn(A/x) \cap T_A$ as desired. Suppose then $x \vdash t$, so that there is an $s \in T(x)$ with $s < t$. Again, we show that $t \in A/x$ and thus $t \in Cn(A/x) \cap T_A$. Let B be any minimal subset of A with $B \vdash x$ and $t \in B$; it will suffice to show that t is not minimal in B. Since $B \vdash x \vdash s$ and s is a co-atom of A, there is a $b \in B$ with $b \vdash s$. But since $<$ is regular over A, it continues down $\vdash$ over A, so since $b \vdash s < t$ we have $b < t$, and so t is not minimal in B, as desired. $\square$

3. Conclusions

Putting together the chain of equivalences established in Section 2 we thus have:

THEOREM 1. *For any theory A finite modulo Cn, the following conditions are equivalent*:

(1) $\dot{-}$ *is a relational partial meet contraction function over* A;

(2) $\dot{-}$ *is a bottom contraction function over* A;

(3) $\dot{-}$ *is a top contraction function over* A;

(4) $\dot{-}$ *is a regular safe contraction function over* A;

Whilst the principal interest of conditions (2) and (3) lies in the finite case, conditions (1) and (4) are of interest whether or not A is finite modulo Cn. However it remains an open question whether (1) implies (4), or conversely, in the infinite case.

We recall in this connection that several conditions on a selection function γ that are equivalent to the relationality of γ, and so imply condition (1) above for the partial meet contraction $\dot{-}$ determined by γ, are given in [2] Observation 4.10 and its final note added in proof.

THEOREM 2. *For any theory A finite modulo Cn, the following conditions are equivalent:*
(1) $\dot{-}$ *is a transitively relational partial meet contraction function over A;*
(2) $\dot{-}$ *is a virtually connected bottom contraction function over A;*
(3) $\dot{-}$ *is a virtually connected top contraction function over A;*
(4) $\dot{-}$ *is a safe contraction function determined by a regular hierarchy that is virtually connected over A.*

PROOF. The meaning of (2) should be clear: $\dot{-}$ is a bottom contraction function determined a (normal and non-circular) relation over $\perp_A$ that is also virtually connected over $\perp_A$. Similarly for (3).

We need only make some minor additions to the verification of Theorem 1. Specifically, as already noted in Section 2.1, a relation $\leqslant$ over U_A is transitive iff $f_1(\leqslant)$ over U_A and thus also $f_2(f_1(\leqslant))$ over $\perp_A$ is virtually connected; this establishes the equivalence of (1) and (2). It is easy to check that any normal relation $<$ over $\perp_A$ is virtually connected iff $f_3(<)$ over T_A is virtually connected, which establishes (2)$\Leftrightarrow$(3). Clearly, if $<$ over A is virtually connected over all of A then it is virtually connected over T_A, so that $g(<)$ is virtually connected over T_A, which establishes (4)$\Rightarrow$(3). Finally, to show that (3)$\Rightarrow$(4) we need only check that when a normal relation $< \, = g(f_4(<))$ over T_A is virtually connected over T_A then $<' = f_4(<)$ is virtually connected over all of A. For this, suppose $a <' b$ but $a \not<' c$. From the former, there is an $s \in T(a)$ such that $s < t$ for all $t \in T(b)$. From the latter, there is a $u \in T(c)$ with $s \not< u$. Since $<$ is virtually connected over T_A, $s < t$ and $s \not< u$ gives us $u < t$ for all $t \in T(b)$, so that $c <' b$ as desired. □

Note that part of Theorem 2, namely the implication (4)$\Rightarrow$(1), is also known to hold quite generally for theories A finite or infinite. For if (4) holds, then as shown in Corollary 6.3 of [3], $\dot{-}$ satisfies the Gärdenfors "supplementary postulates" ($\dot{-}$7) and ($\dot{-}$8), and so by the second representation Theorem 4.4 of [2], $\dot{-}$ is a transitively relational partial meet contraction function. However it remains an open question whether the converse implication (1)$\Rightarrow$(4) also holds in the infinite case.

Several further conditions on a partial meet contraction function that are equivalent to (1), in both the finite and the infinite cases, are given in [2], notably in Corollary 4.5 and Observation 6.5.

Following the terminology of [3], we say that a relation $<$ over A is *quotient connected* over a set $B \subseteq A$ of propositions iff for all $b, c \in B$ either $b < c$ or $c < b$ or $Cn(b) = Cn(c)$. As noted in Remark 6.1 (a) of [3], if $<$ is a normal hierarchy over B and is quotient connected over B, then it is also transitive over B. This concept is useful for our third theorem.

THEOREM 3. *For any theory A finite modulo Cn, the following conditions are equivalent:*

(*1a*) $\dot{-}$ *is a relational maxichoice contraction function over* A;
(*1b*) $\dot{-}$ *is an antisymmetrically relational partial meet contraction function over* A;
(*2*) $\dot{-}$ *is a quotient connected bottom contraction function over* A;
(*3*) $\dot{-}$ *is a quotient connected top contraction function over* A;
(*4*) $\dot{-}$ *is a safe contraction function determined by a regular hierarchy that is quotient connected over* A.

PROOF. We need to make minor additions to the proof of Theorem 1, and also replace the map f_4 by a more refined map f_4^* in order to verify the implication (3) $\Rightarrow$ (4).

Clearly (1b) implies (1a) even in the infinite case. For suppose $B, B' \in \gamma(A \perp x)$. Then by the marking off identity we have both $B' \leqslant B$ and $B \leqslant B'$, so that antisymmetry implies $B = B'$, so that $\gamma(A \perp x)$ is a singleton as desired. The converse implication holds in the finite case. For let γ be a selection function over a theory A finite modulo Cn, such that γ is relational under $\leqslant$, and let $\dot{-}$ be the partial meet contraction function that it determines. By Observation 5.2 of [2], $\leqslant$ is connected and so also reflexive over U_A. Let $B, B' \in U_A$ and suppose both $B' \leqslant B$ and $B \leqslant B'$; we need to show $B = B'$. Since A is finite modulo Cn, there is an $x \in A$, $x \notin Cn(\emptyset)$, with $A \perp x = \{B, B'\}$. Since $B' \leqslant B$ and $B \leqslant B'$ and $\leqslant$ is reflexive over U_A, we have by the marking off identity that $\gamma(A \perp x) = \{B, B'\}$. Hence if γ is a maxichoice selection function, so that $\gamma(A \perp x)$ is a singleton, we have $B = B'$ as desired.

As already noted in Section 2.1, it is easy to check that $\leqslant$ over U_A is antisymmetric over U_A iff $f_1(\leqslant)$ is connected up to identity over U_A, and clearly this in turn holds iff $f_2(f_1(\leqslant))$ is quotient connected over $\perp_A$. It is also easy to check that a normal relation $<$ over $\perp_A$ is quotient connected over $\perp_A$ iff $f_3(<)$ is quotient connected over T_A. This establishes (1b) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Clearly using the lemma of Section 2.4, (4) $\Rightarrow$ (3). It remains therefore to verify (3) $\Rightarrow$ (4).

Suppose (3) holds, i.e. $\dot{-}$ is a top contraction function determined by a normal and non-circular relation $<$ over T_A that is quotient connected over T_A. Clearly this condition implies the transitivity of $<$ over T_A, as already noted. Define $f_4^*(<) = <'$ by putting $a <' b$, for any $a, b \in A$, iff there is an $s \in T(a) - T(b)$ such that for all $t \in T(b) - T(a)$, $s < t$.

An equivalent formulation of the definition, useful in some of the verifications, is as follows. Note that from the conditions imposed on $<$ it follows that every non-empty subset S of T_A has a least element up to logical equivalence, i.e. an $s \in S$ such that for all $s' \in S$ either $s < s'$ or $Cn(s) = Cn(s')$, and moreover this least element of S is unique up to logical equivalence. It is then easy to verify that the above definition of $f_4^*(<)$ is equivalent to the following: $a <' b$ iff $T(a) + T(b)$ is non-empty and all of its least (up to logical equivalence) elements are in $T(a)$. Equivalently again, given the normality of $<$ over T_A, iff $T(a)+T(b)$ is non-empty and at least one of its least (up to logical equivalence) elements is in $T(a)$. Here $+$ is symmetric difference, i.e. $T(a)+T(b) = (T(a)-T(b))\cup(T(b)-T(a))$; note that $T(a)+T(b) = \emptyset$ iff $Cn(a) = Cn(b)$.

Clearly $f_4^*(<)$ is both normal and asymmetric over A. Using the transitivity of $<$ over T_A, we can verify that $f_4^*(<) = <'$ is transitive over A. For suppose that $a <' b$, $b <' c$. Then there is an $s \in T(a)-T(b)$ such that for all $u \in T(b)-T(a)$, $s < u$, and likewise there is a $t \in T(b)-T(c)$ such that for all $u \in T(c)-T(b)$, $t < u$. To show that $a <' c$, it will suffice to find an $r \in T(a)-T(c)$ such that for all $u \in T(c)-T(a)$, $r < u$. Define r by putting $r = s$ in the case that both $s \notin T(c)$ and either $t \notin T(a)$ or there is a $w \in (T(b)\cap T(c))-T(a)$ with $w < t$, and putting $r = t$ otherwise. Then it is easy to verify that r has the desired properties.

Transitivity and asymmetry of $f_4^*(<)$ give us non-circularity, and so we can say that $f_4^*(<)$ is indeed a hierarchy over A. Clearly from the last of the equivalent definitions, $f_4^*(<)$ is quotient connected over A. Moreover it is straightforward to verify that it continues both up and down $\vdash$ over A, and so is regular over A. Finally, it is clear that $f_4^*(<)$ is a conservative extension of the normal $<$ over T_A, i.e. $g(f_4^*(<)) = <$. Consequently by the lemma of Section 2.4, the top contraction function $\dot{-}$ over A determined by $<$ is identical with the safe contraction function determined by the regular quotient connected hierarchy $f_4^*(<)$; so that condition (4) holds as desired. $\square$

The function f_4^* defined in the proof of Theorem 3 has a number of interesting features. In terms of the epistemological reading of $<$, the definition of $f_4^*(<)$ is a very natural one. For it amounts to saying that if a, b are elements of A, and s is the worst co-atom (under the given ordering $<$ of the co-atoms) under which they differ, then $a < b$ or $b < a$ according as this worst co-atom is implied by a or by b. Elements of A are thus compared by the worst co-atoms on which they differ.

Formally, it is clear that the relation $f_4^*(<)$ includes the relation $f_4(<)$. For if a stands in the latter relation to b, then there is an $s \in T(a)$ with $s < t$ for all $t \in T(b)$, so by the irreflexivity of $<$, $s \notin T(b)$ so that $s \in T(a)-T(b)$ and $s < t$ for all $t \in T(b)-T(a) \subseteq T(b)$. The relation $f_4^*(<)$ also has the property of being *strictly order preserving*, i.e. when $a \vdash b$ but $Cn(a) \neq Cn(b)$ then $a <' b$. For if $a \vdash b$ but $Cn(a) \neq Cn(b)$ then $T(b)$ is a proper subset of $T(a)$, so $T(a)-T(b)$ is non-empty whilst $T(b)-T(a)$ is empty, and so vacuously $a <' b$. Finally, $f_4^*(<) = <'$ has the *invertibility* property that for all $a, b \in A$, $a <' b$ iff ${\sim}b <' {\sim}a$, where $\sim$ is the relative negation operation defined in Section 2.3 by the equation ${\sim}c = a_0 \vee \neg c$. For suppose that $a <' b$; then there is an $s \in T(a)-T(b)$ such that for all $t \in T(b)-T(a)$, $s < t$. Since $a \vdash s$ and b *non* $\vdash s$, we have

$\sim a$ *non* $\vdash s$ and $\sim b \vdash s$, so $s \in T(\sim b) - T(\sim a)$. Likewise $t \in T(\sim a) - T(\sim b)$ implies $t \in T(b) - T(a)$, so $\sim b < \sim a$. The converse is similar.

Given these features of the map f_4^*, it would be pleasant to be able to use it in the proof of Theorem 1, taking it to be defined in the first of the three ways mentioned, since in the more general context where quotient connectivity is no longer assumed, this is no longer equivalent to the other two. However, it is not clear whether this is possible, for in this more general context there is no longer any apparent way of verifying the non-circularity of $f_4^*(<)$. It remains an open question whether $f_4^*(<)$ is non-circular, for an arbitrary normal non-circular relation $<$ over T_A.

We remark that the implication $(4) \Rightarrow (1a)$ of Theorem 3 can also be established under the same hypotheses by a variant argument that uses no maps. For suppose condition (4) holds. Then $\dot{-}$ satisfies the Gärdenfors supplementary postulates ($\dot{-}7$) and ($\dot{-}8$), as shown by [3] Observation 6.2 and Remark 6.1 (a), and so $\dot{-}$ is a transitively relational partial meet contraction function, by the representation Theorem 4.4 of [2]. But since A is finite modulo Cn and condition (4) holds, we also know from Observation 6.4 of [3] that $\dot{-}$ is a maxichoice contraction function. So putting these together (as we may, for if $\dot{-}$ is determined by a selection function that is a maxichoice selection function, then it is determined by a unique selection function) we know that $\dot{-}$ is a relational maxichoice contraction function and (1a) holds as desired.

However it remains an open question whether either $(4) \Rightarrow (1a)$ or its converse holds in the infinite case.

Several further conditions which, for arbitrary partial meet contraction functions, are equivalent to (1a) in both the finite and infinite cases are given in Observation 6.3 of [2]. These include the "decomposition condition" that $A \dot{-} (x \wedge y) = A \dot{-} x$ or $A \dot{-} (x \wedge y) = A \dot{-} y$ for all x, y. When the domain under consideration is restricted to arbitrary *maxichoice* contraction functions, then even more conditions become equivalent to (1a), i.e. in this context to relationality, as is shown in [1] Section 8.

Added in Proof

The authors have shown that the map f_4^* does preserve non-circularity, so that it can indeed be used in place of f_4 in the proof of Theorem 1. This answers the question raised at the top of page 197.

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