Section 3, Exercise 1

It seems like there is no way one could use either insetting (putting a given set into another set) and pairing or pairing on two different inputs to obtain the same set. However, if one sees pairing the same set, then pairing $\emptyset$ with $\emptyset$ would result in $\{\emptyset\}$, which is also the result of insetting $\emptyset$.

Proof:

Insetting and pairing must have different results because insetting will always result in a set with 1 element, and pairing will always result in a set with 2 elements. Therefore, they can't be the same set.

Pairing the sets a, b and c, d can't result in the same set unless $a = c$ and $b = d$ or $a = d$ and $b = c$. Otherwise, $\{a, b\}$ would contain at least one element not in $\{c, d\}$.

$\square$

Section 4, Exercise 1

I am not exactly sure what I'm supposed to do here. I guess "observe" means "prove" here, so "prove that the condition has nothing to do with the set B".

Proof:

$$(A \cap B) \cup C = A \cap (B \cup C) \Leftrightarrow C \subset A$$

$$(A \cup C) \cap (B \cup C) = A \cap (B \cup C) \Leftrightarrow C \subset A$$

$$A \cup C = A \Leftrightarrow C \subset A$$

This is trivially true.

$\square$

Section 5, Some easy exercises

$$A - B = A \cap B'$$

Proof:

$$A - B = \{a | a \in A \wedge a \notin B\} = \{a | a \in A \wedge a \in B'\} = A \cap B'$$

$\square$

$$A \subset B \text{ if and only if } A - B = \emptyset$$

Proof:

$$A \subset B \Leftrightarrow \forall a \in A : a \in B \Leftrightarrow \exists C : B = A \cup C \Leftrightarrow A - (A \cup C) = \emptyset \Leftrightarrow A - B = \emptyset$$

$\square$

$$A - (A - B) = A \cap B$$

Proof:

$$A - (A - B) = A - (A \cap B') = A \cap (A \cap B')' = A \cap (A' \cup B) = A \cap A' \cup A \cap B = \emptyset \cup A \cap B = A \cap B$$

$\square$

$$A \cap (B - C) = (A \cap B) - (A \cap C)$$

Proof:

$$(A \cap B) - (A \cap C) = (A \cap B) \cap (A \cap C)' = (A \cap B) \cap (A' \cup C') = (A \cap B \cap A') \cup (A \cap B \cap C') = A \cap B \cap C' = A \cap (B - C)$$

$\square$

$$A \cap B \subset (A \cap C) \cup (B \cap C')$$

Proof:

$$\begin{aligned} A \cap B &\subset (A \cap C) \cup (B \cap C') \\ &= ((A \cap C) \cup B) \cap ((A \cap C) \cup C') \\ &= ((A \cap C) \cup B) \cap ((A \cup C') \cap (C \cup C')) \\ &= ((A \cap C) \cup B) \cap (A \cup C') \\ &= (A \cup B) \cap (C \cup B) \cap (A \cup C') \end{aligned}$$

$A \cap B \subset (A \cup B) \cap (C \cup B) \cap (A \cup C')$ is true because $A \subset (A \cup B)$ and $B \subset (C \cup B)$ and $A \subset (A \cup C')$.

$\square$

$$(A \cup C) \cap (B \cup C') \subset A \cup B$$

Proof:

$$(A \cup C) \cap (B \cup C')$$

$$= ((A \cup C) \cap B) \cup ((A \cup C) \cap C')$$

$$= ((A \cup C) \cap B) \cup A$$

$$= (A \cap B) \cup (C \cap B) \cup A \subset A \cup B$$

This is the case because $(A \cap B) \cup (C \cap B) \subset B$ (since intersections with B are subsets of B), and the union with A doesn't change the equation.

$\square$

Section 5, Exercise 1

To be shown: The power set of a set with n elements has 2^n elements. Proof by induction.

Proof:

Induction base: The power set of the empty set contains 1 element:

$$|P(\emptyset)| = |\emptyset| = 1 = 2^0 = 2^{|\emptyset|}$$

Induction assumption:

$$|P(A)| = 2^{|A|}$$

Induction step:

To be shown: $|P(A \cup \{a\}| = 2 * 2^{|A|} = 2^{|A|+1}$.

$P(A \cup \{a\})$ contains two disjunct subsets: $P(A)$ and $N = \{\{a\} \cup S | S \in P(A)\}$. Those are disjunct because every element in N contains a ($\forall n \in N : a \in n$), but there is no element of $P(A)$ that contains a. Also, it holds that $P(A) \cup N = P(A \cup \{a\})$, because elements in the power set can either contain a or not, there is no middle ground. It is clear that $|N| = |P(A)|$, therefore $|P(A \cup \{a\})| = |P(A)| + |N| = 2 * |P(A)| = 2 * 2^{|A|} = 2^{|A|+1}$.

$\square$

Section 5, Exercise 2

To be shown:

$$\mathcal{P}(E) \cap \mathcal{P}(F) = \mathcal{P}(E \cap F)$$

Proof:

If $S \in \mathcal{P}(E \cap F)$, then $\forall s \in S : s \in E \cap F$. Therefore, $S \subset E$ and $S \subset F$ and thereby $S \in \mathcal{P}(E)$ and $S \in \mathcal{P}(F)$. This means that $S \in \mathcal{P}(E) \cap \mathcal{P}(F)$.

If $S \in \mathcal{P}(E) \cap \mathcal{P}(F)$, then a very similar proof can be written: $S \subset E$ and $S \subset F$, so $\forall s \in S : s \in E$ and $\forall s \in S : s \in F$. Then $S \subset E \cap F$ and therefore $S \in \mathcal{P}(E \cap F)$.

$\square$

To be shown:

$$\mathcal{P}(E) \cup \mathcal{P}(F) \subset \mathcal{P}(E \cup F)$$

Proof:

If $S \in \mathcal{P}(E) \cup \mathcal{P}(F)$, then $S \in \mathcal{P}(E) \Leftrightarrow S \subset E$ or $S \in \mathcal{P}(F) \Leftrightarrow S \subset F$. Since it is true for any set X that $S \subset E \Rightarrow S \in \mathcal{P}(E \cup X)$, it is true that $S \in \mathcal{P}(E \cup F)$ (similar argumentation if $S \subset F$).

$\square$

A reasonable interpretation for the introduced notation: If $\mathcal{C} = X_1, X_2, \ldots, X_n$, then

$$\bigcap_{X \in \mathcal{C}} X = X_1 \cap X_2 \cap \ldots X_n$$

Similarly, if $\mathcal{C} = X_1, X_2, \ldots, X_n$, then

$$\bigcup_{X \in \mathcal{C}} X = X_1 \cup X_2 \cup \ldots X_n$$

The symbol $\mathcal{P}$ still stands for the power set.

To be shown:

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}(X) = \mathcal{P}(\bigcap_{X \in \mathcal{C}} X)$$

Proof by induction.

Induction base:

$$\mathcal{P}(E) \cap \mathcal{P}(F) = \mathcal{P}(E \cap F)$$

Induction assumption:

$$\bigcap_{X \in \mathcal{C}} \mathcal{P}(X) = \mathcal{P}(\bigcap_{X \in \mathcal{C}} X)$$

Induction step:

$$\mathcal{P}(Y) \cap \bigcap_{X \in \mathcal{C}} \mathcal{P}(X)$$

$$= \mathcal{P}(Y) \cap \mathcal{P}(\bigcap_{X \in \mathcal{C}} X)$$

$$= \mathcal{P}(Y \cap \bigcap_{X \in \mathcal{C}} X)$$

The last step uses $\mathcal{P}(E) \cap \mathcal{P}(F) = \mathcal{P}(E \cap F)$, since $\bigcap_{X \in \mathcal{C}} X$ is also just a set.

$\square$

To be shown:

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}(X) \subset \mathcal{P}(\bigcup_{X \in \mathcal{C}} X)$$

Proof by induction.

Induction base:

$$\mathcal{P}(E) \cup \mathcal{P}(F) \subset \mathcal{P}(E \cup F)$$

Induction assumption:

$$\bigcup_{X \in \mathcal{C}} \mathcal{P}(X) \subset \mathcal{P}(\bigcup_{X \in \mathcal{C}} X)$$

Induction step:

$$\mathcal{P}(Y) \cup \bigcup_{X \in \mathcal{C}} \mathcal{P}(X)$$

$$\subset \mathcal{P}(Y) \cup \mathcal{P}(\bigcup_{X \in \mathcal{C}} X)$$

$$\subset \mathcal{P}(Y \cup \bigcup_{X \in \mathcal{C}} X)$$

The last step uses $\mathcal{P}(E) \cup \mathcal{P}(F) \subset \mathcal{P}(E \cup F)$, since $\bigcup_{X \in \mathcal{C}} X$ is also just a set.

$\square$

To be shown:

$$\bigcup \mathcal{P}(E) = E$$

Proof:

$\forall X \in \mathcal{P}(E) : X \subset E$. Furthermore, $E \in \mathcal{P}(E)$. Since $A \subset E \Rightarrow A \cup E = E$, it holds that $E = \bigcup_{X \in \mathcal{P}(E)} = \bigcup \mathcal{P}(E)$.

$\square$

And "E is always equal to $\bigcup_{X \in \mathcal{P}(E)}$ (that is $\bigcup \mathcal{P}(E) = E$), but that the result of applying $\mathcal{P}$ and $\bigcup$ to E in the other order is a set that includes E as a subset, typically a proper subset" (p. 21).

I am not entirely sure what this is supposed to mean. If it means that we treat $\mathcal{E}$ as a collection, then $\forall X \in \mathcal{E} : \bigcup_{E \in \mathcal{E}} E \subset X$. But that doesn't mean that $\mathcal{E} \subset \mathcal{P}(\bigcup_{E \in \mathcal{E}} E)$: If $\mathcal{E} = \{\{a, b\}, \{b, c\}\}$, then $\bigcup_{E \in \mathcal{E}} E = \{b\}$, and $\mathcal{E} = \{\{a, b\}, \{b, c\}\} \not\subset \mathcal{P}(\{b\}) = \{\{b\}, \emptyset\}$.

If we treat E simply as a set, then $\bigcup E = E$, and it is of course clear that $E \subset \mathcal{P}(E)$, as for all other subsets of E.

$\square$

Section 6, A non-trivial exercise

"find an intrinsic characterization of those sets of subsets of A that correspond to some order in A"

Let $\mathcal{M} \subset \mathcal{P}(\mathcal{P}(A))$ be the set of all possible orderings of A. In the case of $A = \{a, b\}$, $\mathcal{M}$ would be $\{\{\{a\}, \{a, b\}\}, \{\{b\}, \{a, b\}\}\}$.

Some facts about every element $M \in \mathcal{M}$:

$\bigcap M = \{min\}$, where $\{min\}$ is the smallest element in the ordering M, the element in M for which there is no other element $a \in M$ so that $a \subset \{min\}$ (which means that $\emptyset \notin M$).

$\bigcup M = A$. A must therefore be in M and be the biggest element (no element $a \in M$ so that $a \supset A$).

For all elements $m \in M$ except $\{min\}$ there exists at least one element $n \in M$ so that $n \subset m$.

Similarly, for all elements $m \in M$ except A there exists at least one element $n \in M$ so that $n \supset m$.

For every $m \in M$ except A and $\{min\}$, there exist two unique elements $x, y \in M$ so that m is the only set in M for which it is true that $x \subset m \subset y$.

For every $a \in A$, there must exist two sets $m, n \in M$ so that $n = m \cup \{a\}$ (except for min). This means that the $|A| = |M|$, the size of A is the size of M.

These conditions characterise $\mathcal{M}$ intrinsically and are the solution to the question.

Section 6, Exercise 1

(i) To be shown: $(A \cup B) \times X = (A \times X) \cup (B \times X)$

Proof:

$(A \cup B) \times X = \{(e, x) : e \in A \vee e \in B, x \in X\} = \{(e, x) : e \in A, x \in X\} \cup \{(e, x) : e \in B, x \in X\} = (A \times X) \cup (B \times X)$

$\square$

(ii) To be shown: $(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$

Proof:

$$(A \times X) \cap (B \times Y) =$$

$$\{(x, y), x \in A, y \in X\} \cap \{(v, w), v \in B, w \in Y\} =$$

$$\{(x, y), x \in A \wedge x \in B, y \in X \wedge y \in Y\} =$$

$$\{(x, y), x \in A \cap B, y \in X \cap Y\} =$$

$$(A \cap B) \times (X \cap Y)$$

$\square$

(iii) To be shown: $(A - B) \times X = (A \times X) - (B \times X)$

Two-sided proof by contradiction:

1. $(A - B) \times X \subset (A \times X) - (B \times X)$

Let $(u, v) \in (A - B) \times X$. Then $u \in (A - B)$, and $v \in X$. Suppose $(u, v) \notin (A \times X) - (B \times X)$. Then $(u, v) \in (A \times X) \cap (B \times X)$. Then $(u, v) \in (A \cap B) \times (X \cap X)$. Then $u \in A \cap B$ and $v \in X$. But if $u \in A \cap B$, then $u \notin A - B$! Contradiction.

2. $(A \times X) - (B \times X) \subset (A - B) \times X$

Let $(u, v) \in (A \times X) - (B \times X)$. Then $(u, v) \in (A \times X)$ and $(u, v) \notin (B \times X)$. Because v must be in X, and there is no flexibility there, $u \notin B$. Suppose $(u, v) \notin (A - B) \times X$. Since necessariliy $v \in X$, $u \notin A - B$. But if $u \notin A - B$, u must be an element of $A \cap B$. Then $u \in B$, and there is a contradiction.

$\square$

Section 7, Exercise 1

Reflexive, but neither symmetric nor transitive (symmetry violation: $(b, a) \notin$, transitivity violation: $(a, c) \notin$): $\{(a, a), (a, b), (b, b), (b, c), (c, c)\}$

Symmetric, but neither reflexive nor transitive (reflexivity violation: $(a, a) \notin$, transitivity violation: $(a, c) \notin$): $\{(a, b), (b, a), (b, c), (c, b)\}$

Transitive, but neither reflexive nor symmetric (reflexivity violation: $(a, a) \notin$, symmetry violation: $(b, a) \notin$): $\{(a, b), (b, c), (a, c)\}$