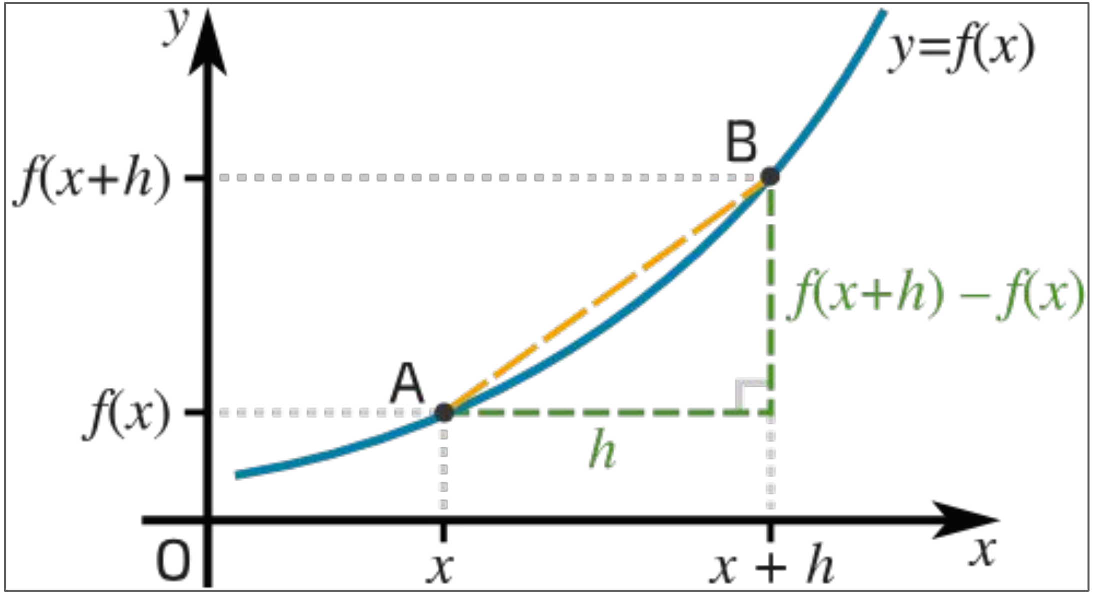
Differentiation



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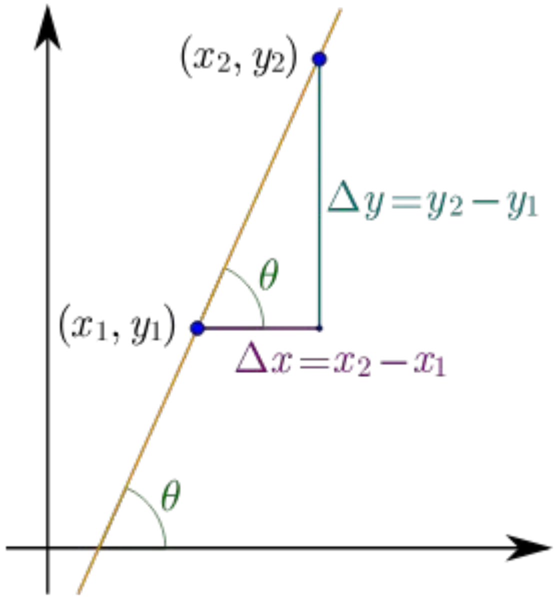
The Slope of a Curve

The slope of a straight line can be measured by dividing the change in the value of **y** between any two points on the line by the corresponding change in the value of **x** between the same two points:

$$\text{Slope of a straight line} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$$



Exercise 01

Find the slope of the line passing through the following pair of points.

1) (6, - 5) and (4, - 1)

2) (5, 3) and (- 1, - 4)

3) (3, 4) and (3, 7)

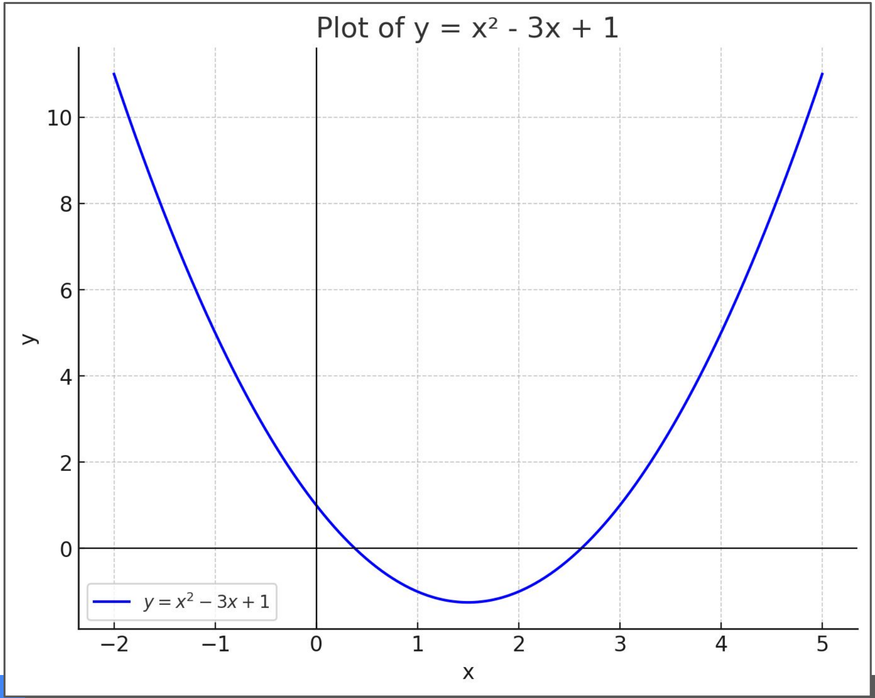
4) (- 2, 4) and (- 3, - 2)

For a curve, measuring the slope is more difficult because the slope changes as we move along the curve.

For example,

in the plot below we can see that the slope of the quadratic function $y = x^2 - 3x + 1$ becomes steeper as we move away from the point where

x=1.5 and y=−1.25



- A tangent to a curve is defined as a straight line which touches the curve at a single point.

- We can overcome the problem of measuring the slope of a curve by noticing that the slope of a curve is equal to the slope of the tangent to curve **$\Delta y/\Delta x$** at the point where they touch.

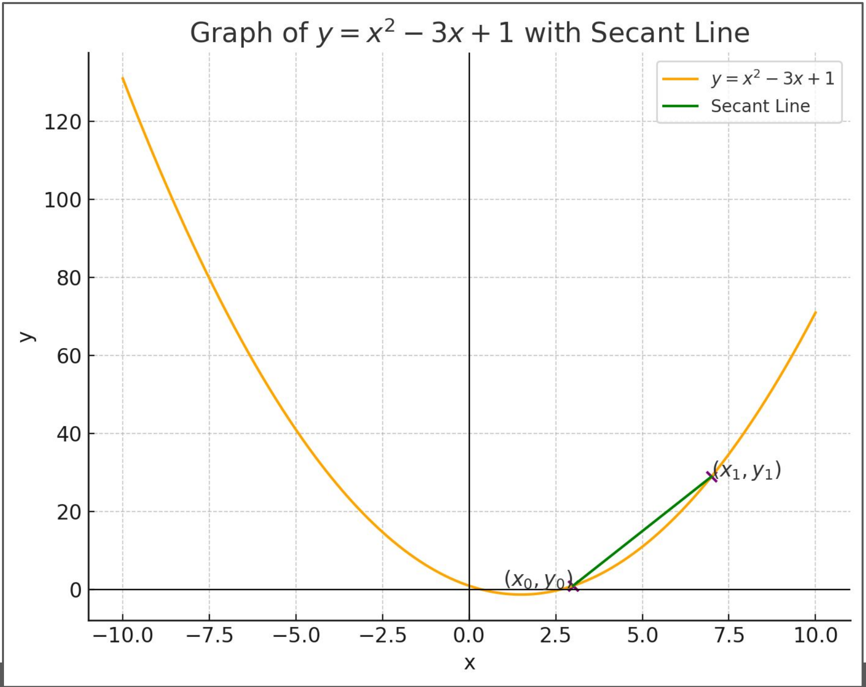
- The slope of the tangent to a curve at this point is known as the **derivative of the function with respect to x** and is denoted by: $\frac{\mathrm{d}y}{\mathrm{d}x}$

- This is the rate at which the output of a function y changes when we change the input by a very small amount.
- **Finding the derivative of a curve** is known as **differentiation**.

How can we measure the slope of the tangent at any point on the curve?

- A **chord** is a straight line connecting two points on a curve.
- The slope of a chord is equal to the average slope of the curve between the two points.

For example, in the plot below we have added a chord to the graph of $y = x^2 - 3x + 1$



between the points (X_0, Y_0) and (X_1, Y_1)
The slope of this chord is equal to:

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$$

- By bringing the points (X_0, Y_0) and (X_1, Y_1) closer and closer together ($\triangle x \to 0$), we can see that the chord will get nearer and nearer to being a tangent to the curve.
- In addition, the slope of the chord will become a better and better approximation to the slope of the tangent and hence, the slope of the curve.t

$$\lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = \frac{dy}{dx}$$

; where $\triangle x$ is a very small change in x and $\triangle y$ is a very small change in y

- The gradient of a curve at a given point is defined to be the gradient of the tangent at that point.
- We approximate the slope of this tangent by calculating the slope of a chord between 2 points on the curve and bringing these points closer and closer together.

Note: The gradient of a curve at a given point is defined to be the gradient of the tangent at that point. We approximate the slope of this tangent by calculating the slope of a chord between 2 points on the curve and bringing these points closer and closer together.

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Note:

- **dy/dx does not represent a fraction** such that you can cancel out terms in the numerator or denominator.
- **dy/dx** is a function of x and so should contain only x terms (no y terms).
- This means that once we have found the derivative of a function, we can find the slope at any point on its curve by substituting the value of x at that point into the derivative.

Differentiation

Differentiation means the rate of change of one quantity with respect to another.

The speed is calculated as the rate of change of distance with respect to time.
This speed at each instant is not the same as the average calculated. Speed is the same as the slope, which is nothing but the instantaneous rate of change of the distance over a period of time.

Differentiation Notation

- When given a variable (y) expressed as a function of another variable (x) such that $y = f(x)$, we can differentiate $\boldsymbol{y}$ with respect to $\boldsymbol{x}$.

- In terms of mathematical notation, the derivative (or the first derivative) of the general function $y = f(x)$ can be expressed as:

$$\frac{dy}{dx} \quad \textbf{or} \quad \frac{d}{dx}y \quad \textbf{or} \quad f'(x) \quad \textbf{or} \quad \frac{d}{dx}f(x) \quad \textbf{or} \quad y'$$

Derivative Functions

Let f be a function. The derivative function, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

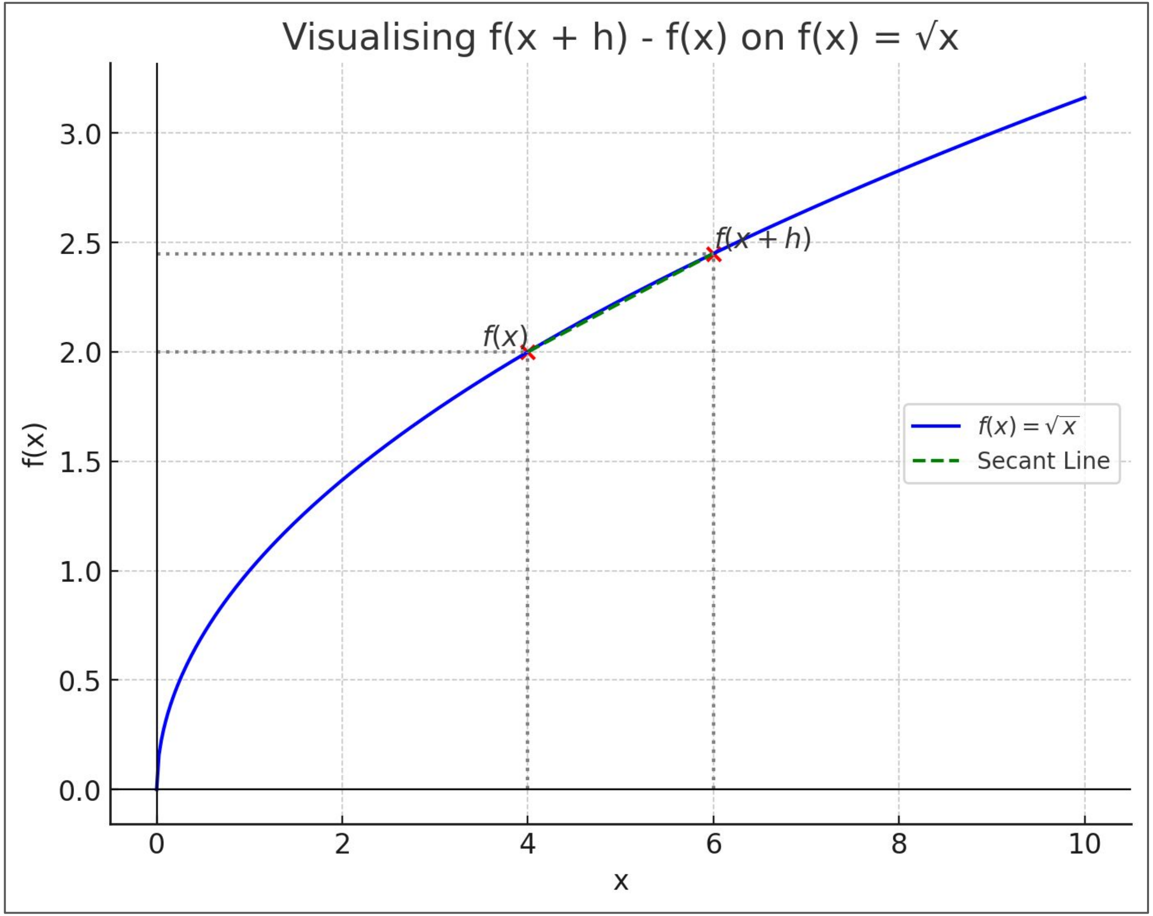
;

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

- The h represents a small change in x .
- x, often called Δx (delta x)
- A function $f(x)$ is said to be differentiable at a if $f'(a)$ exists.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\mathrm{d}y}{\mathrm{d}x}$$



Example :-

1)find the slope of the line tangent to the graph of $f(x) = x^2$ x=3

.

Answer

$$m_{\tan} = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{(3+h)^2 - 9}{h}$$

$$= \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h}$$

$$= \lim_{h\to 0} \frac{h(6+h)}{h} = \lim_{h\to 0}(6+h) = 6$$

Example :-

2) Find the derivative of $f(x) = \sqrt{x}.$

Answer

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{1}{2\sqrt{x}}$$

Exercise 02 :-

Use the limit definition to compute the derivative, f '(x)

$$f(x) = x^2 - 2x$$

$$f(x) = \frac{1}{2}x - \frac{3}{5}$$

$$f(x) = 5x^2 + 3x + 7$$

$$f(x) = 3x^2 - 4x + 1$$ find f'(2) .

The Basic Rules

1) Constant Rule

Let c be a constant. If f(x)=c , then f '(x)=0. Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

We first apply the limit definition of the derivative to find the derivative of the constant function, f(x)=c. For this function, both **f(x)=c** and **f(x+h)=c**, so we obtain the following result:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{c-c}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 = 0. \end{aligned}$$

- The rule for differentiating constant functions is called the constant rule.
- It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0.

Example :-

1) Find the derivative of $f(x)= 10$.

Answer

This is just a one-step application of the rule: $f'(10)=0$.

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2) Find the derivative of $f(x)= - 0.5$.

Answer

This is just a one-step application of the rule: $f'(- 0.5) =0$.

Exercise

3) Find the derivative of $g(x)= - 9$.

The Basic Rules
2) The Power Rule

Let n be a **positive integer**. If

$$f(x) = x^n$$

then the **derivative** of $f(x)$ is:

$$f'(x) = nx^{n-1}$$

Alternatively, we can write this rule using derivative notation:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

$$(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n,$$

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n.$$

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n}{h}.$$

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1}.$$

$$f'(x) = \lim_{h \to 0}\left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1}\right)$$

$$= nx^{n-1}.$$

Example :-

1) Find the derivative of the function $f(x) = x^9$

Answer

$$f'(x) = 9x^{9-1} = 9x^8$$

Exercise

2) Find the derivative of $f(x) = x^{17}$

3) Find the derivative of $f(x) = 10x^7$

4) Find the derivative of $f(x) = -6x^{-2}$

The Basic Rules

3) The Sum, Difference, and Constant Multiple Rules

Let f(x) and g(x) be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule:- The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g

$$\frac{d}{dx}(f(x)+g(x))=\frac{d}{dx}(f(x))+\frac{d}{dx}(g(x))$$

Which is, if $s(x)=f(x)+g(x)$ then $s'(x)=f'(x)+g'(x)$

Difference Rule:- The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g. (The derivative of a difference is the difference of the derivatives)

:

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$$

Which is, if $d(x) = f(x) - g(x)$ then $d'(x) = f'(x) - g'(x)$

Constant Multiple Rule:- The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}(kf(x)) = k \cdot \frac{d}{dx}(f(x)) = kf'(x)$$

Which is, if $m(x) = kf(x)$ then $m'(x) = kf'(x)$

Example :-

1) Find the derivative of $f(x) = 3x^6 + 8$

Answer

$$f'(x) = \frac{d}{dx}(3x^6 + 8)$$

$$= \frac{d}{dx}(3x^6) + \frac{d}{dx}(8)$$

$$= 3 \cdot \frac{d}{dx}(x^6) + 0$$

$$= 3(6x^5)$$

$$= 18x^5$$

Exercise

1) Find the derivative of $f(x) = 3x^4 - 6x^{-2} + 10$

2) Find the derivative of $f(x) = 5x^7 - 4x^{-3} + \dfrac{2}{x^2} - 9$

3) Find the derivative of $f(x) = 4x^5 - \dfrac{3}{x^3} + \sqrt{x} + 7$

The Basic Rules
4) The Product Rule

Let f(x) and g(x) be differentiable functions. Then:

$$\frac{d}{dx}(f(x)\cdot g(x)) = \frac{d}{dx}(f(x))\cdot g(x) + \frac{d}{dx}(g(x))\cdot f(x)$$

Which is, if $p(x) = f(x)\cdot g(x)$ then $p'(x) = f'(x)\cdot g(x) + g'(x)\cdot f(x)$

Example :-

find p′(x) by applying the **product rule**. Then check the result by first expanding the product and differentiating.

Answer

Let $f(x) = 2x^2 - 1, \quad g(x) = x^3 + 4x$

Differentiate both functions: $f'(x) = 4x, \quad g'(x) = 3x^2 + 4$

Now apply the product rule:

$$p'(x) = f'(x)g(x) + g'(x)f(x)$$

$$p'(x) = (4x)(x^3 + 4x) + (3x^2 + 4)(2x^2 - 1)$$

$$p'(x) = (4x^4 + 16x^2) + (6x^4 + 5x^2 - 4)$$

$$= 10x^4 + 21x^2 - 4$$

Expand the original expression:

Expand the original expression:

$$p(x) = (2x^2 - 1)(x^3 + 4x)$$

Multiply:

$$= 2x^2 \cdot x^3 + 2x^2 \cdot 4x - 1 \cdot x^3 - 1 \cdot 4x = 2x^5 + 8x^3 - x^3 - 4x = 2x^5 + 7x^3 - 4x$$

Differentiate:

$$p'(x) = \frac{d}{dx}(2x^5 + 7x^3 - 4x) = 10x^4 + 21x^2 - 4$$

Exercise

Find the derivative using product rule

1. $A(x) = \sqrt{x}(x^2 + 1)$
2. $R(x) = (x^3 - 2x)(5x^2 + 4)$

The Basic Rules
5) The Quotient Rule

Let f(x) and g(x) be **differentiable functions**. Then:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}$$

Which is, if $q(x) = \dfrac{f(x)}{g(x)}$ then, $q'(x) = \dfrac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}$

Example :-

Use the quotient rule to find the derivative of $$q(x) = \frac{2x^3 + 1}{x^2 - 5}$$

Answer

Let $$f(x) = 2x^3 + 1$$

$$g(x) = x^2 - 5$$

Now apply the Quotient rule:

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

$$= \frac{6x^2(x^2 - 5) - 2x(2x^3 + 1)}{(x^2 - 5)^2}$$

$$q'(x) = \frac{2x^4 - 30x^2 - 2x}{(x^2 - 5)^2}$$

Exercise

Find the derivative using Quotient rule

1. $q(x) = \dfrac{x^2 + 1}{3x - 2}$
2. $r(x) = \dfrac{4x^3 - x}{x^2 + 5}$

The Basic Rules
6) The Chain Rule

Let f and g be functions. For all xin the domain of g for which g is differentiable at x and f is differentiable at g(x) , the derivative of the composite function

IF $y = f(g(x))$ then , $\dfrac{dy}{dx} = f'(g(x)) \cdot g'(x)$

Example :-

Let:

$$y = (3x+2)^5$$

- **Outer function:** u^5
- **Inner function:** $u = 3x + 2$

Apply chain rule:

$$\frac{dy}{dx} = 5(3x+2)^4 \cdot 3 = 15(3x+2)^4$$

Exercise :-

1. $y = (3x + 1)^4$
2. $y = \sqrt{5x^2 - 2x}$
3. $y = (x^2 + 4)^6$
4. $y = \left(2x^3 - x + 1\right)^5$

Answers:-

1. $\frac{dy}{dx} = 12(3x+1)^3$
2. $\frac{dy}{dx} = \frac{10x-2}{2\sqrt{5x^2-2x}}$
3. $\frac{dy}{dx} = 12x(x^2+4)^5$
4. $\frac{dy}{dx} = 5(2x^3-x+1)^4 \cdot (6x^2-1)$

The Basic Rules

7) Differentiation of trigonometric functions

(i) $\dfrac{d}{dx}\sin x = \cos x$

(ii) $\dfrac{d}{dx}\cos x = -\sin x$

(iii) $\dfrac{d}{dx}\tan x = \sec^2 x$

(iv) $\dfrac{d}{dx}\sec x = \sec x \tan x$

(v) $\dfrac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$

(vi) $\dfrac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

Exercise :-

1. $y = 2\sin(x) + 3\cos(x)$
2. $y = 5\tan(x) - 4\cot(x)$
3. $y = 4\sec(x) + 6\csc(x)$

Answers:-

1. $y = 2\sin(x) + 3\cos(x)$

$$\frac{dy}{dx} = 2\cos(x) - 3\sin(x)$$

2. $y = 5\tan(x) - 4\cot(x)$

$$\frac{dy}{dx} = 5\sec^2(x) + 4\csc^2(x)$$

3. $y = 4\sec(x) + 6\csc(x)$

$$\frac{dy}{dx} = 4\sec(x)\tan(x) - 6\csc(x)\cot(x)$$

The Basic Rules

8) Differentiation of logarithmic and exponential functions

1) If $f(x) = e^x$ then $f'(x) = e^x$ (Exponential functions with base e are their own derivative!)

With chain rule,

If $f(x) = e^{g(x)}$ then $f'(x) = e^{g(x)} \cdot g'(x)$

The Basic Rules

Natural logarithm : $\ln(x) = \log_e(x)$

8) Differentiation of logarithmic and exponential functions

2) If $f(x) = \ln(x)$ then $f'(x) = \dfrac{1}{x}$

With chain rule,

If $f(x) = \ln(g(x))$ then $\dfrac{d}{dx}(\ln(g(x))) = \dfrac{g'(x)}{g(x)}$

Exercise :-

1. $y = e^x + \ln x$

2. $y = 3e^x - 5\ln x$

3. $y = e^x \cdot x$

Answers:-

1. $y = e^x + \ln x$

$$\frac{dy}{dx} = e^x + \frac{1}{x}$$

2. $y = 3e^x - 5\ln x$

$$\frac{dy}{dx} = 3e^x - \frac{5}{x}$$

3. $y = e^x \cdot x$

   (Product Rule)

$$\frac{dy}{dx} = e^x \cdot x + e^x = e^x(x+1)$$

Exercise :-

1. $y = (3x^2 + 1)^4$
2. $y = x^2 \cdot \sin(x)$
3. $y = \frac{e^x}{x^2+1}$
4. $y = \ln(5x^2 + 3)$
5. $y = \tan(2x + 1)$
6. $y = x \cdot e^{x^2}$
7. $y = \frac{\ln x}{\sin x}$
8. $y = \sqrt{\cos(x)}$
9. $y = \sec(x^2 + 1)$
10. $y = \frac{(x^2+1)^3}{e^x}$

Summary

GENERAL DERIVATIVE RULES

Constant Rule	$\frac{d}{dx}[c] = 0$
Constant Multiple Rule	$\frac{d}{dx}[cf(x)] = cf'(x)$
Sum Rule	$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
Difference Rule	$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$
Product Rule	$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$
Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$
Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

Rule Type	Formula
Constant Rule	$\frac{d}{dx}[c] = 0$
Power Rule	$\frac{d}{dx}[x^n] = nx^{n-1}$
Exponential Rule	$\frac{d}{dx}[e^x] = e^x$
Logarithmic Rule	$\frac{d}{dx}[\ln x] = \frac{1}{x}$
Trig Functions	$\frac{d}{dx}[\sin x] = \cos x$
	$\frac{d}{dx}[\cos x] = -\sin x$
	$\frac{d}{dx}[\tan x] = \sec^2 x$
	$\frac{d}{dx}[\csc x] = -\csc x \cot x$
	$\frac{d}{dx}[\sec x] = \sec x \tan x$
	$\frac{d}{dx}[\cot x] = -\csc^2 x$

Partial Derivative Definition

- Consider a function **f(x,y)** that depends on two independent variables, **x** and **y**.

- The function is said to be partially dependent on both x and y. When we compute the derivative of this function with respect to one variable, while treating the other variable as constant, it is called the partial derivative of f.

- Specifically, when differentiating f with respect to x, we treat y as a constant. Similarly, when differentiating f with respect to y, we treat x as a constant.

Example

Let's take the function $f(x, y) = x^2 + y^2$

Partial Derivative with Respect to x:

When we take the partial derivative of f with respect to x, we treat y as a constant.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2)$$

Since y^2 is treated as a constant with respect to x, it disappears during differentiation.

$$\frac{\partial f}{\partial x} = 2x$$

Partial Derivative with Respect to y:

Now, when we take the partial derivative with respect to y, we treat x as a constant.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2)$$

Since x^2 is treated as a constant, it disappears during differentiation.

$$\frac{\partial f}{\partial y} = 2y$$

Exercise

01) Given the function $f(x, y) = e^{x^2+y^2}$, find:

1. The partial derivative of $f(x, y)$ with respect to x.
2. The partial derivative of $f(x, y)$ with respect to y.

02) Given the function:

$$f(x, y) = \sin(xy) + \cos(x^2 + y^2)$$

1. Find the partial derivative of $f(x, y)$ with respect to x.
2. Find the partial derivative of $f(x, y)$ with respect to y.

Answers

01)

1. The partial derivative of $f(x,y)$ with respect to x:

$$\frac{\partial f}{\partial x} = 2xe^{x^2+y^2}$$

2. The partial derivative of $f(x,y)$ with respect to y:

$$\frac{\partial f}{\partial y} = 2ye^{x^2+y^2}$$

02)

1. The partial derivative of $f(x,y)$ with respect to x:

$$\frac{\partial f}{\partial x} = -2x\sin(x^2+y^2) + y\cos(xy)$$

2. The partial derivative of $f(x,y)$ with respect to y:

$$\frac{\partial f}{\partial y} = x\cos(xy) - 2y\sin(x^2+y^2)$$