# Bayesian Logistic Regression

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IIT Gandhinagar

# Outline

MLE

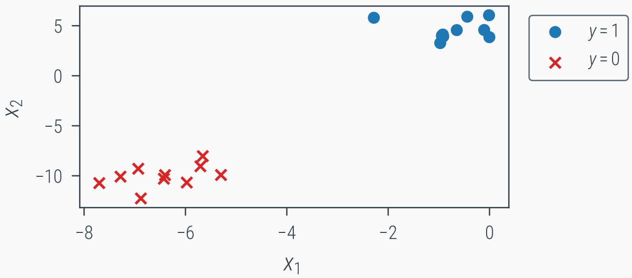
MAP

Fully Bayesian

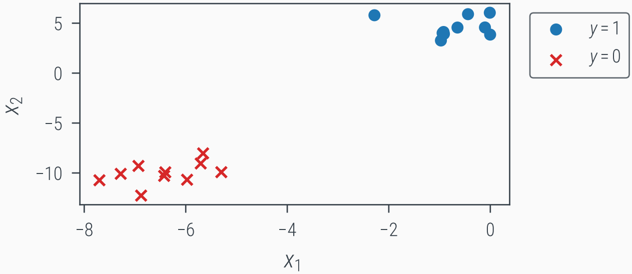
Laplace Approximation

# MLE

# Data



# Data




| x1 | x2 | y |
|---:|---:|---:|
| -5.97 | -10.68 | 0 |
| -0.44 | 5.90 | 1 |
| -0.97 | 3.27 | 1 |
| ... | ... | ... |

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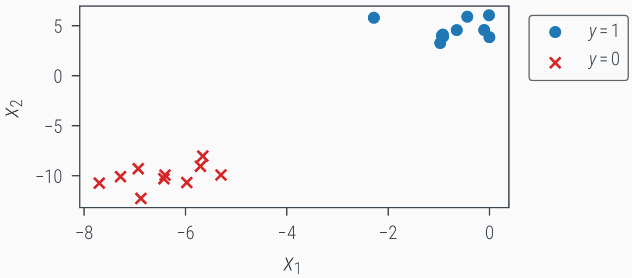


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$$
\begin{aligned}
\mathcal{D} &= \{(\boldsymbol{x}_i, y_i)\}_{i=1}^{N} \\
&= \{X, \boldsymbol{y}\}
\end{aligned}
$$

# Likelihood

$$p(\mathcal{D}|\boldsymbol{\theta}) = p(\boldsymbol{y}|X, \boldsymbol{\theta}) = \prod_{i=1}^{N} p(y_i|\boldsymbol{x}_i, \boldsymbol{\theta})$$

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# Likelihood

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&= \prod_{i=1}^{N} \boldsymbol{\sigma}\left(\boldsymbol{\theta}^T \boldsymbol{x}_i\right)^{y_i} \left(1 - \boldsymbol{\sigma}\left(\boldsymbol{\theta}^T \boldsymbol{x}_i\right)\right)^{1-y_i}
\end{aligned}
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## Likelihood

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$$\log p(\boldsymbol{y}|X,\boldsymbol{\theta}) = \sum_{i=1}^{N}\left[y_i \log \boldsymbol{\sigma}\left(\boldsymbol{\theta}^T\boldsymbol{x}_i\right) + (1-y_i)\log\left(1-\boldsymbol{\sigma}\left(\boldsymbol{\theta}^T\boldsymbol{x}_i\right)\right)\right]$$

# MLE

$$\frac{\partial}{\partial \boldsymbol{\theta}} - \log p(\mathcal{D}|\boldsymbol{\theta}) = X^T(\boldsymbol{\sigma}(X\boldsymbol{\theta}) - \boldsymbol{y})$$

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$$\frac{\partial}{\partial \boldsymbol{\theta}} - \log p(\mathcal{D}|\boldsymbol{\theta}) = X^T(\boldsymbol{\sigma}(X\boldsymbol{\theta}) - \boldsymbol{y})$$

However, $\log\left(\frac{\boldsymbol{y}}{1-\boldsymbol{y}}\right)$ is undefined when $y_i = 0$ or $y_i = 1$, which is always the case.

# MLE

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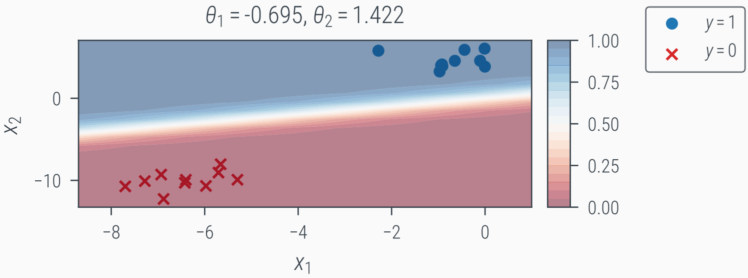


# MAP

# Prior

- We may use a Gaussian prior on $\boldsymbol{\theta}$.

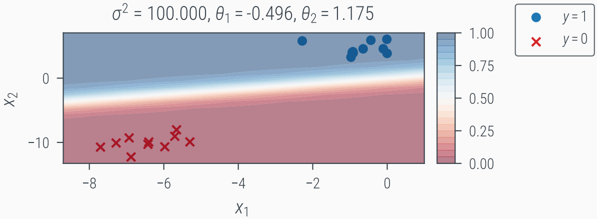
$$p(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta}|0, \sigma^2)$$

# Negative Log Joint

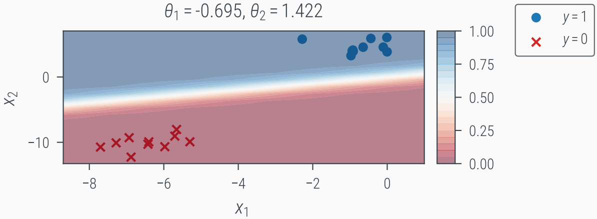
$$f(\theta) = -\log p(\mathcal{D}|\theta) - \log p(\theta)$$

# MAP with a weak prior

## MAP

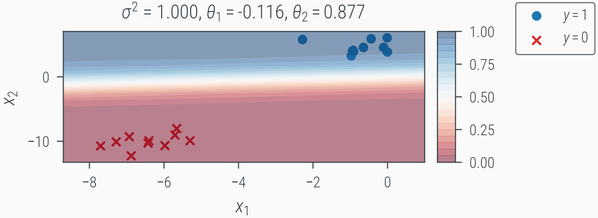



## MLE

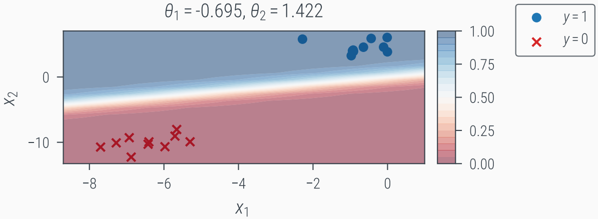


# MAP with a medium prior

## MAP

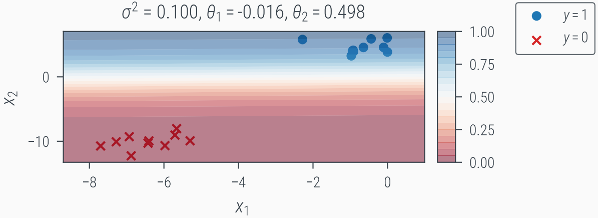



## MLE

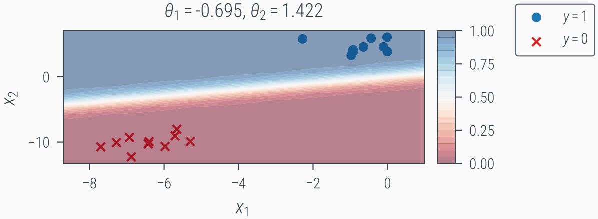


# MAP with a strong prior

## MAP




## MLE



# Fully Bayesian

# Posterior

$$p(\boldsymbol{\theta}|\mathcal{D}) = \frac{p(\mathcal{D}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{\int p(\mathcal{D}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}}$$

- Normal prior and Bernoulli likelihood do not form a conjugate pair. Thus, the denominator is intractable and we cannot find the posterior in closed form.

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- Normal prior and Bernoulli likelihood do not form a conjugate pair. Thus, the denominator is intractable and we cannot find the posterior in closed form.
- We need another method to find the posterior.
- Laplace approximation!

# Laplace Approximation

# A Quick Refresher

$$
\begin{aligned}
\text{Neg. Log Joint } f(\theta) &= -\log p(\mathcal{D}|\theta) - \log p(\theta) \\
&= -\sum_{i=1}^{N} \Big[ y_i \log \boldsymbol{\sigma} \left( \boldsymbol{\theta}^T \boldsymbol{x}_i \right) \\
&\quad + (1 - y_i) \log \left( 1 - \boldsymbol{\sigma} \left( \boldsymbol{\theta}^T \boldsymbol{x}_i \right) \right) \Big] \\
&\quad - \left( -\frac{1}{2} \frac{\boldsymbol{\theta}^T \boldsymbol{\theta}}{\sigma^2} - \frac{1}{2} \log(2\pi\sigma^2) \right) \\
\text{Laplace Posterior } q(\boldsymbol{\theta}) &= \mathcal{N}(\theta | \theta_{\text{MAP}}, \nabla^2 f(\theta_{\text{MAP}})^{-1})
\end{aligned}
$$

# Predictive distribution

$$p(y^* = 1|\boldsymbol{x}^*, \mathcal{D}) = \int p(y^* = 1|\boldsymbol{x}^*, \boldsymbol{\theta})p(\boldsymbol{\theta}|\mathcal{D})d\boldsymbol{\theta}$$

# Predictive distribution

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\begin{aligned}
p(y^* = 1 | \boldsymbol{x}^*, \mathcal{D}) &= \int p(y^* = 1 | \boldsymbol{x}^*, \boldsymbol{\theta}) p(\boldsymbol{\theta} | \mathcal{D}) d\boldsymbol{\theta} \\
&\approx \int p(y^* = 1 | \boldsymbol{x}^*, \boldsymbol{\theta}) q(\boldsymbol{\theta}) d\boldsymbol{\theta}
\end{aligned}
$$

# Predictive distribution

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&\approx \int p(y^* = 1|\boldsymbol{x}^*, \boldsymbol{\theta}) q(\boldsymbol{\theta}) d\boldsymbol{\theta} \\
&= \int \boldsymbol{\sigma}(\boldsymbol{\theta}^T \boldsymbol{x}^*) q(\boldsymbol{\theta}) d\boldsymbol{\theta}
\end{aligned}
$$

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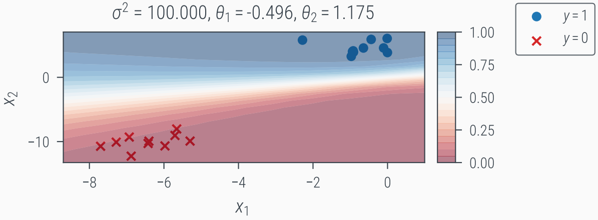
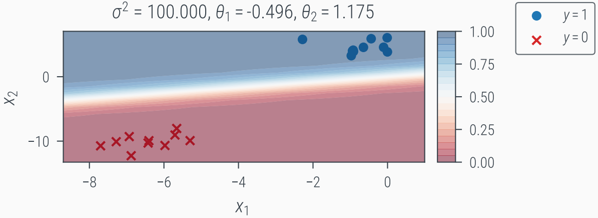
&= \mathbb{E}_{q(\boldsymbol{\theta})} \left( \boldsymbol{\sigma}(\boldsymbol{\theta}^T \boldsymbol{x}^*) \right) \\
&\approx \frac{1}{M} \sum_{i=1}^{M} \boldsymbol{\sigma}(\boldsymbol{\theta}_i^T \boldsymbol{x}^*)
\end{aligned}
$$

# Predictive Distribution with a Weak Prior

## Laplace

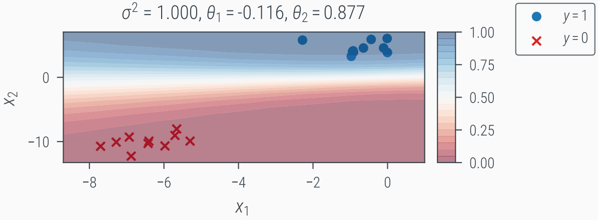



## MAP

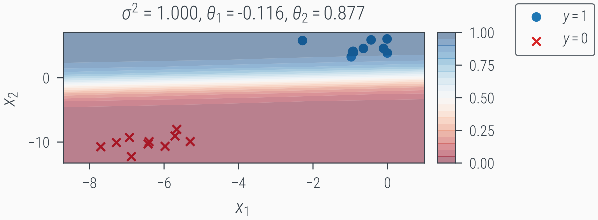


# Predictive Distribution with a Medium Prior

## Laplace

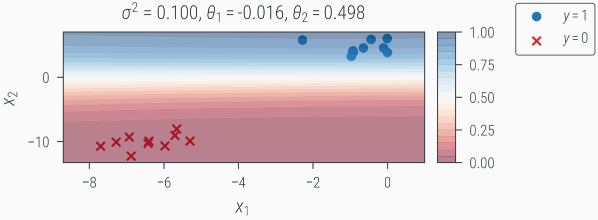



## MAP



# Predictive Distribution with a Strong Prior

## Laplace




## MAP