Solution, HW3

Q1. Given, $2x^2 + 5y^2 = 14$.

Possible values of x and y

x	y
0	0
1	1
2	2
$\vdots$	$\vdots$

As x^2 and y^2 are always non-negative conclusion will hold for negative values of x, y

Case 1: $x = 0, y = 1$

$L.H.S = 2 \times 0 + 5 \times 1 = 5$
$\neq R.H.S$ No solution

Case 2: $x = 0, y = 0$

$L.H.S = 2 \times 0 + 5 \times 0 = 0 \neq R.H.S$ No solution

Case 3: $x = 1, y = 1$

$L.H.S = 2 \times 1 + 5 \times 1 = 2 + 5 = 7 \neq R.H.S$ No solution

case 4: $x = 2, y = 1$

$L.H.S = 2 \times 4 + 5 \times 1 = 8 + 5 = 13 \neq R.H.S$ No solution

case 5: $x = 2, y = 0$

$L.H.S = 2 \times 2^2 + 5 \times 0 = 8 + 0 = 8 \neq R.H.S$ No solution

Proved

Q2

If $\underbrace{x^2 - 2a + 7 \text{ is even}}_{p}$, then $\underbrace{x \text{ is odd}}_{q}$,

So, $p \rightarrow q \equiv \neg q \rightarrow \neg p$ [contrapositive

$\neg q$: x is not odd $\equiv$ x is even

$\neg p$: $x^2 - 2a + 7$ is not even

$\equiv x^2 - 2a + 7$ is odd

Let's consider $x = 2k$, k is an integer

So, $x^2 - 2a + 7 = 4k^2 - 4k + 7$

$= 2\underbrace{(2k^2 - 2k + 3)}_{\text{integer: } L} + 1$

$= 2L + 1 \equiv$ ODD

The statement is proved in contrapositive sense.

Q3. Given

If $3n^2+4n+1$ is even, then $3n+1$ is even or $n+1$ is even

p : $3n^2+4n+1$ is even
q : $3n+1$ is even
r : $n+1$ is even

$p \rightarrow (q \vee r)$

$\neg(q \vee r) \rightarrow \neg p$

$\Rightarrow \neg q \wedge \neg r \rightarrow \neg p$

So, $\neg p$: $3n^2+4n+1$ is odd
$\neg q$: $3n+1$ is odd $= 2j+1$
$\neg r$: $n+1$ is odd $= 2k+1$

So,
$$3n^2+4n+1 = 3n^2+3n+n+1$$
$$= 3n(n+1)+1(n+1)$$
$$= (3n+1)(n+1)$$

Replacing the value :
$$3n^2+4n+1 = (2j+1)(2k+1)$$
$$= 4jk+2j+2k+1$$
$$= 2\underbrace{(jk+j+k)}_{\text{Integer}: L}+1$$
$$= 2L+1 \equiv \text{ODD}$$

Proved

Q4. We apply proof by contradiction. Given,

no natural number n is both even and o

Let's assume that, the opposite/negation is true

That is: n is both even and odd

↳ natural number

So, $n = 2k$, when n is even

$n = 2j+1$, when n is odd

We know that even + odd is always

So, $2k = 2j+1$, as $n \begin{cases} \to 2k \\ \to 2j+1 \end{cases}$

$\Rightarrow 2(k-j) = 1$

$\Rightarrow \underbrace{2 \times \text{Integer}}_{\text{Even}} = \underbrace{1}_{\text{odd}}$

contradiction

Thus, "natural number n is both even and odd," i a wrong statemen

So, no natural number n is both even and odd

Proved

Q5. Part a) L.H.S $= 1 + 3 + \cdots + (2n-1)$ | R.H.S $= n^2$

$= 1$ $n=1$ | $= 1$

So, Basic step: for $n=1$, the given statemen is true.

Inductive step: Let's assume that for $n=k$ the given statement is tru

That is, $1+3+5+\cdots+(2k-1)=k^2 \quad \cdots \quad \cdots (2)$

Now, we have to show that the given stateme[nt] is true for $n=k+1$ as well. That is, we have to show that

$$1+3+5+\cdots\cdots+(2(k+1)-1)=(k+1)^2$$

$$\Rightarrow 1+3+5+\cdots\cdots+(2k+1)=(k+1)^2$$

So, $L.H.S = \underbrace{1+3+5+\cdots+(2(k)-1)}_{k^2}+(2(k+1)-1)$

$$= k^2+2k+1=(k+1)^2=R.H.S$$

Proved

Part b: Basic step: $n=1$, $4^1-1=3$, so, 3 is divisible by 3 multiple 3

Inductive step: Let's assume that for $n=k$ the given statement is true, that is, 4^{k-1} is a multiple of 3. $4^k-1=3m$, where m is an integer

Now, we have to show that 4^n-1 is divisib[le] multiple of 3 for $n=k+1$. So,

$$4^{k+1}-1 = 4^k.4-1 = 4^k(3+1)-1$$

$$= 3.4^k+4^k-1 = 3.4^k+\underbrace{3m}_{\text{for } n= \text{ par}}$$

$$= 3.\underbrace{(4^k+m)}_{\text{Integer}: L}$$

$$= 3L \quad \text{multiple of } 3$$

Part c: Given. $2+4+6+\cdots\cdots+2n = n(n+1)$

Basic step: For $n=1$, $L.H.S = 2\times 1 = 2$

$R.H.S = 1(1+1) = 2$

So, $L.H.S = R.H.S$.

Inductive step: Let's assume that for $n=K$ the given statement is true. That is,

$$2+4+6+\cdots\cdots+2K = K(K+1)$$

Now, we have to show that for $n=K+1$ the given statement is true as well.

$$L.H.S = \underbrace{2+4+6+\cdots\cdots+2K}_{K(K+1)} + 2(K+1)$$

$$= K(K+1) + 2(K+1) = (K+1)(K+2)$$

$$R.H.S = n(n+1)$$

$$= (K+1)(K+1+1), \text{ for } n=K+1$$

$$= (K+1)(K+2) = L.H.S$$

That is, the given statement is true for $n=K+1$ as well.

Proved

Part d: $L.H.S = -1+2+5+\cdots+(3n-4)$

$= -1$, for $n=1$

$R.H.S = (n/2)(3n-5) = \frac{1}{2}(3-5) = \frac{1}{2}\times($

$= -1 = L.H.S$

Assume that the given statement is true for $n=K$. That is

$$-1+2+5+\cdots\cdots+(3K-4) = (K/2)(3K-5)$$

Now, we have to show that the given statement is true for $n = k+1$. That is,

$$-1+2+5+\cdots\cdots+(3(k+1)-4) = \frac{(k+1)}{2}(3(k+1)-$$

$$= \frac{(k+1)(3k-2)}{2}$$

$$\text{L.H.S} = \underbrace{-1+2+5+\cdots+(3k-4)}_{(k/2)(3k-5)} + (3(k+1)-4)$$

$$= \frac{k}{2}(3k-5) + (3k-1)$$

$$= \frac{k(3k-5)+2(3k-1)}{2}$$

$$= \frac{3k^2-5k+6k-2}{2} = \frac{3k^2+k-2}{2}$$

$$= \frac{3k^2+3k-2k-2}{2} = \frac{3k(k+1)-2(k+1}{2}$$

$$= \frac{(k+1)(3k-2)}{2} = \text{R.H.S}$$

So, the given statement is proved.