★ Method of Variation of Parameters

(i) We can apply this method to find the solution of LDE with variable Coefficients
Eg: $x^2y''+xy'+y=x, \; x\neq 0$

2) LDE with constant Coefficients, when we can't find P.I. using 5 special Cases
Eg: $y''+16y=32\sec 2x$

3) Also, to find solution of LDE with constant coeff. when special Cases can also be applied.

4) We can also use this method, when C.F is known or two L.I. solutions are given.

Method: Consider $a_0(x)y''+a_1(x)y'+a_2(x)y=r(x), \quad a_0(x)\neq 0$

Let C.F. is $\quad y_c(x)=Ay_1(x)+By_2(x)$; y_1, y_2 are two L.I. solutions

Assum that general solution is

$$y_g = A(x)y_1(x)+B(x)y_2(x) \quad \text{—①}$$

P.I.
$y_p = y_1A(x)+y_2B(x)$

Find $W(y_1,y_2)=W$

& $g(x)=\dfrac{r(x)}{a_0(x)}=\dfrac{r(x)}{a_0(x)}$

$$A(x)=-\int\frac{g(x)y_2(x)}{W}dx \; \cancel{+C_1}$$

$$B(x)=\int\frac{g(x)y_1(x)}{W}dx \; \cancel{+C_2}$$

∴ from eq ① $\quad y=\left[-\int\frac{g(x)y_2(x)}{W}dx+C_1\right]y_1(x)+\left[\int\frac{g(x)y_1(x)}{W}dx+C_2\right]y_2(x)$

$$y=-y_1(x)\int\frac{g(x)y_2(x)}{W}dx+C_1y_1(x)+y_2(x)\int\frac{g(x)y_1(x)}{W}dx+C_2y_2(x)$$

$$\therefore\; y=\underbrace{C_1y_1(x)+C_2y_2(x)}_{\text{C.F.}}\underbrace{-\,y_1(x)\int\frac{g(x)y_2(x)}{W}dx+y_2(x)\int\frac{g(x)y_1(x)}{W}dx}_{y_p\ \text{P.I.}}$$

$$y=Ay_1(x)+By_2(x)+y_1(x)A(x)+y_2(x)B(x)$$

Q1 $y''+3y'+2y=2e^x$

$m^2+3m+2=0$

$m=-1,-2$

$y_c=C_1e^{-x}+C_2e^{-2x}$

Let, $y_1(x)=e^{-x}$, $y_2(x)=e^{-2x}$,

$g(x)=\frac{r(x)}{a_0}=2e^x$

$$W(y_1,y_2)=\begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x}\end{vmatrix} = -2e^{-3x}+e^{-3x}=-e^{-3x}$$

$A(x)=-\frac{e^{2x}}{2}$

$B(x)=-\frac{e^{3x}}{3}$

$-y_1\left(-\frac{e^{2x}}{2}\right)+y_2\left(\frac{-e^{3x}}{3}\right)$

$+e^{-x}e^{2x}+e^{-2x}\left(-\frac{2}{3}e^{3x}\right)$

$e^x-\frac{2}{3}e^x$

$=y_1A(x)+y_2B(x)$

$$\therefore y_p(x)=-y_1\int\underbrace{\frac{g(x)y_2}{W}}\,dx+y_2\int\frac{g(x)y_1}{W}dx$$

$$=-e^{-x}\int\frac{2e^x\cdot e^{-2x}}{-e^{-3x}}dx+e^{-2x}\int\frac{2e^x\cdot e^{-x}}{-e^{-3x}}$$

$$=2e^{-x}\int e^{-x}\cdot e^{3x}dx+2e^{-2x}\int e^{3x}dx$$

$$=2e^{-x}\int e^{2x}dx+2e^{-2x}\int e^{3x}dx$$

$$=\cancel{2}e^{-x}\cdot\frac{e^{2x}}{\cancel{2}}+2e^{-2x}\cdot\frac{e^{3x}}{3}$$

$$=e^x-\frac{2}{3}e^x=\frac{1}{3}e^x$$

$\therefore$ g.s is

$y=y_c+y_p$

$=C_1e^{-x}+C_2e^{-2x}+\frac{1}{3}e^x$

Q2: $y''+16y=32\sec 2x$

AE $m^2+16=0$

$m=\pm4i$

$y_c(x)=C_1\cos4x+C_2\sin4x$

Let, $y_1(x)=\cos4x$, $y_2(x)=\sin4x$

$g(x)=\frac{r(x)}{a_0}=32\sec2x$

$$W=\begin{vmatrix}\cos4x & \sin4x\\ -4\sin4x & 4\cos4x\end{vmatrix}$$

$=4\cos^2 4x+4\sin^2 4x=4$

$y_p(x)=$

$$-y_1\int\frac{g(x)y_2}{W}dx+y_2\int\frac{g(x)y_1}{W}dx$$

$$=-\cos4x\int\frac{32\sec2x\cdot\sin4x}{4}dx$$

$$+\sin4x\int\frac{32\sec2x\cdot\cos4x}{4}dx$$

$$=-\frac{\cos4x\cdot32}{4}\int\sec2x\sin4x\,dx$$

$$+\frac{32\cdot\sin4x}{4}\int\sec2x\cdot\cos4x\,dx$$

$=-8\cos4x\,I_1+8\sin4x\,I_2$ (1)

Now

$I_1=\int\sec2x\sin4x\,dx$

$$=\int\frac{1}{\cos2x}\cdot2\sin2x\cos2x\,dx$$

$[\because \sin2\theta=2\sin\theta\cos\theta$

$\therefore \sin4\theta=2\sin2\theta\cos2\theta]$

$$=2\int\sin2x\,dx=2\left(-\frac{\cos2x}{2}\right)$$

$\boxed{I_1=-\cos2x}$

$I_2=\int\sec2x\cdot\cos4x\,dx$

$$=\int\frac{1}{\cos2x}\cdot(2\cos^2 2x-1)\,dx$$

$[\because \cos2\theta=2\cos^2\theta-1]$

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$\int(\cos 2x - \sec 2x)\,dx$

$= \int \cos 2x\,dx - \int \sec 2x\,dx$

$= \frac{\sin 2x}{2} - \frac{\ln|\sec 2x - \tan 2x|}{2}$

$I_2 = \frac{\sin 2x}{2} - \frac{1}{2}\ln|\sec 2x - \tan 2x|$

$\therefore$ from (1)

$y_p(x) = +8\cos 4x \cos 2x \cancel{+8\sin 4x}$

$+8\sin 4x\left[\frac{\sin 2x}{2} - \frac{\ln|\sec 2x - \tan 2x|}{2}\right]$

$\left\{ \cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)] \right.$

$\left. \sin A \sin B = \frac{1}{2}[\cos(a-b) - \cos(a+b)] \right\}$

$= +\frac{8}{2}[\cos 6x + \cos 2x] + \frac{8}{2}[\cos 2x - \cos 6x]$

$- 4\sin 4x \ln|\sec 2x - \tan 2x|$

$= +4\cos 6x + 4\cos 2x + 4\cos 2x - 4\cos 6x$

$-4\sin 4x \ln|\sec 2x - \tan 2x|$

$= 8\cos 2x - 4\sin 4x \ln|\sec 2x - \tan 2x|$

$\therefore y = C_1\cos 4x + C_2\sin 4x$

$+8\cos 2x - 4\sin 4x \ln|\sec 2x - \tan 2x|$

=0=

Q3 $y'' + y = \operatorname{cosec} x$

C.F. $m^2 + 1 = 0$

$m^2 = -1 \Rightarrow m = \pm i$

$y_c = C_1\cos x + C_2\sin x$

Let $y_1 = \cos x$, $y_2 = \sin x$, $g(x) = \operatorname{cosec} x$

$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x = 1$

$\cancel{y_p = -\cos x \int g(x)}$

$\cancel{y_p = -y_1 A(x) + y_2 B()}$

$y_p = -\cos x \int \frac{g(x) y_2}{W}\,dx + y_2 \int \frac{g(x) y_1}{W}\,dx$

$= -\cos x \int \frac{\operatorname{cosec} x \cdot \sin x}{1}\,dx$

$+ \sin x \int \operatorname{cosec} x \cdot \cos x\,dx$

$= -\cos x \int dx + \sin x \int \cot x\,dx$

$= -\cos x \cdot x + \sin x[\ln|\sin x|]$

$= -x\cos x + \sin x \cdot \ln|\sin x|$

$\therefore y(x) = y_c + y_p$

$= C_1\cos x + C_2\sin x$

$- x\cos x + \sin x \cdot \ln|\sin x|$

=0=

Q: It is given that $y_1 = x$ & $y_2 = \frac{1}{x}$ are two L.I. solutions of $x^2 y'' + xy' - y = x$, $x \neq 0$. Find G.S

* Method of undetermined Coefficients:

Method: we choose a particular integral depending on the form of X.

Cases	P.I.
(i) $X = e^{ax}$	$y_p(x) = Ce^{ax}$
(ii) $X = x^m$	$y_p(x) = C_0x^m + C_1x^{m-1} + \dots + C_{m-1}x + C_m$
(iii) $X = \cos\beta x$ or $\sin\beta(x)$	$y_p(x) = C_1\cos\beta x + C_2\sin\beta x$
(iv) $X = e^{ax}\sin bx$ or $e^{ax}\cos bx$	$y_p(x) = e^{ax}(C_1\cos bx + C_2\sin bx)$.
(v) $X = e^{ax}x^m$	$y_p(x) = e^{ax}(C_0x^m + C_1x^{m-1} + \dots + C_m)$

Note: If any term of trial solution appears in C.F., we multiply trial solution by x^m, m represents the number of times the term is repeated in C.F.

$-6C_1 - 9C_2x + 9xC_2 = 0 \Rightarrow \boxed{C_1 = 0}$

$6C_2 - 9xC_1 + 9xC_1 = 1$

$\boxed{C_2 = \frac{1}{6}}$

Q: $y'' - 2y' - 3y = 6e^{-x} - 8e^{x}$ →

C.F.) $y_c = C_1e^{-x} + C_2e^{3x}$

P.I. Let $y_p = axe^{-x} + be^{x}$

$y' = ae^{-x} - axe^{-x} + be^{x}$

$y'' = -ae^{-x} - ae^{-x} + axe^{-x} + be^{x}$

$= axe^{-x} - 2ae^{-x} + be^{x}$

Put in given eq

$a = -\frac{3}{2}, b = 2$

$y(x) = C_1e^{-x} + C_2e^{3x} - \frac{3}{2}xe^{-x} + 2e^{x}$

Q: $y'' + y = 32x^3$

$P.I. = C_1x^3 + C_2x^2 + C_3x + C_4$

Q: $y'' + y' - 2y = e^{-x}$

$m = 1, -2$

$y_c = C_1e^{x} + C_2e^{-2x}$

$y_p = ae^{-x}$

$y' = -ae^{-x}$

$y'' = ae^{-x}$

$ae^{-x} - ae^{-x} - 2ae^{-x} = e^{-x}$

$-2ae^{-x} = e^{-x}$

$-2a = 1$

$a = -\frac{1}{2}$

$y_p = -\frac{e^{-x}}{2}$

$y = C_1e^{x} + C_2e^{-2x} - \frac{e^{-x}}{2}$

Q: $y'' + 9y = \cos 3x$

$m^2 = -9$

$m = \pm 3i$

$y_c = A\cos 3x + B\sin 3x$

$y_p = \frac{1}{6}x\sin 3x$

We chose $y_p = x(C_1\cos 3x + C_2\sin 3x)$

$y_p' = C_1\cos 3x + C_2\sin 3x + x(-3C_1\sin 3x + 3C_2\cos 3x)$

$y_p'' = -3C_1\sin 3x + 3C_2\cos 3x$

$-3C_1\sin 3x + 3C_2\cos 3x$

$+ x(-9C_1\cos 3x - 9C_2\sin 3x)$

$= (-6C_1 - 9C_2x)\sin 3x + (6C_2 - 9xC_1)\cos 3x$

$+ 9x(C_1\cos 3x + C_2\sin 3x)$

Cauchy Euler Equation

Solution of Euler Cauchy Equation.

If x = log function

n^{th} order LDE

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_n y = X$$

or $a_0 x^n D^n + a_1 x^{n-1} y^{n-1} + \cdots + a_{n-1} x y' + a_n y = X$

where, $a_0, a_1, \ldots, a_n$ are constants.

Method

Step 1 Let $x = e^z$ or $z = \log x$.

Step 2 Replace $x\frac{dy}{dx} = xDy = D_1 y$ (θy) where, $D_1 = \frac{d}{dz}$ (θ)

$$x^2\frac{d^2y}{dx^2} = x^2D^2y = D_1(D_1-1)y \quad \text{where } D_1 = \frac{d}{dz}$$

$\theta(\theta-1)$

In general $x^n\frac{d^ny}{dx^n} = x^nD^ny = D_1(D_1-1)\cdots(D_1-(n-1))y$

Step 3: Use previous methods to find G.S

Step 4. Replace $z = \log x$, i.e $x = e^z$

Q: $x^2y'' - 5xy' + 13y = 30x^2 \Rightarrow (x^2D^2 - 5xD + 13)y = 30x^2$

Sol: Let $x = e^z$, $z = \log x$, $D_1 = \frac{d}{dz}$

$x^2D^2 = x^2D_1^2$

$xD = xD_1$

$\therefore$ given eq becomes

$$(\theta(\theta-1) - 5\theta + 13)y = 30e^{2z}$$

A.E

$\theta^2 - \theta - 5\theta + 13 = 0$

$\theta^2 - 6\theta + 13 = 0$

$$\theta = \frac{6\pm\sqrt{36-52}}{2} = \frac{6\pm\sqrt{-16}}{2} = \frac{6\pm 4i}{2} = 3\pm 2i$$

$$y_c(z) = e^{3z}[C_1\cos 2z + C_2\sin 2z]$$

P.I. $y_p(z) = \frac{1}{\theta^2 - 6\theta + 13} 30e^{2z}$

$$= 30\cdot\frac{1}{4-12+13}e^{2z} = 30\cdot\frac{1}{5}e^{2z} = 6e^{2z}$$

$y = y_c + y_p$

$= x^3[C_1\cos(2\log x) + C_2\sin(2\log x)] + 6x^2$

Q: $x^2y'' + 5xy' + 3y = \ln x$

$x = e^z$, $z = \log x$,

$(x^2D_1^2 + 5xD_1 + 3)y = z$

$(\theta(\theta-1) + 5\theta + 3)y = z$, $D_1 = \frac{d}{dz}$

$\theta^2 - \theta + 5\theta + 3 = 0$

$\theta^2 + 4\theta + 3 = 0$

$3 \quad 3\times1$

$+ 4 \quad 3+1$

$\theta^2 + 3\theta + \theta + 3 = 0$

$\theta(\theta+3) + 1(\theta+3) = 0$

$(\theta+1)(\theta+3) = 0$

$\theta = -1, -3$

$$y_c(z) = C_1e^{-z} + C_2e^{-3z}$$

$$y_p(z) = \frac{1}{\theta^2+4\theta+3} z$$

$$= \frac{1}{3\left[1 + \frac{\theta^2}{3} + \frac{4\theta}{3}\right]} z$$

$$= \frac{1}{3}\left[1 + \left(\frac{D_1^2}{3} + \frac{4D_1}{3}\right)\right]^{-1} z$$

$$= \frac{1}{3}\left[1 - \left(\frac{D_1^2}{3} + \frac{4D_1}{3}\right) + \cdots\right] z$$

$=\frac{1}{3}\left[z-\left(\frac{0}{3}+\frac{4}{3}\right)+0\right]$

$=\left(z-\frac{4}{3}\right)\frac{1}{3}=\frac{1}{3}z-\frac{4}{9}.$

$y=c_1e^{-z}+c_2e^{-3z}+\frac{1}{3}z-\frac{4}{9}$

$=c_1x^{-1}+c_2x^{-3}+\frac{1}{3}\ln x-\frac{4}{9}$

$\frac{1}{P(D)}xV=x\frac{1}{P(D)}V+\frac{d}{dD}\left(\frac{1}{P(D)}\right)V$

$y''+4y'+3y=x\sin 3x$