

Linear Differential Equations

Let y be dependent variable and x be the independent variable. We denote the derivatives as

$$\frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = y'', \quad \frac{d^3y}{dx^3} = y''' \text{ etc.}$$

\downarrow
rate of change of y w.r.t x .

Derivatives of higher order represents rate of rates.

Differential Equation : A differential equation contains derivatives of various orders and the variables.

Ex - (i) $y' = 6x^2$

(ii) $y'' + 16y = 2x$.

Linear Differential Equation

A linear ordinary differential equation of order n , is written as

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{dy}{dx} +$$

$$a_n(x) y = g(x)$$

or $a_0(x) y^n(x) + a_1(x) y^{n-1}(x) + \dots + a_{n-1}(x) y'(x) + a_n(x) y(x) = g(x)$,

where y is dependent variable and x is independent variable and $a_0(x) \neq 0$.

If $g(x) = 0$, then it is called a homogeneous equation,

otherwise it is called a non-homogeneous equation.

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad a_0(x) \neq 0$$

is a second order homogeneous equation

$$\text{and } a_0(x)y'' + a_1(x)y' + a_2(x)y = g(x), \quad a_0(x) \neq 0$$

is a second order non-homogeneous equation.

If $a_i(x)$, $i=0, 1, 2$ are constants, then the equations are linear second order constant coefficient equations.

E Ex $y'' + 4y' + 3y = x^2 e^x \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{constant coefficients}$

$$y'' + 2y' + y = \sin x$$

$$x^2 y'' + xy' + (x^2 - 4)y = 0$$

$$(1-x^2)y'' - 2xy' + 2y = 0$$

$\left. \begin{array}{l} \\ \end{array} \right\} \text{variable coefficients}$

(1) $y'' - a^2 y = 0 \rightarrow \text{constant coefficient}$

(2) $y' = \frac{y}{x} \Rightarrow xy' = y \rightarrow \text{variable coefficient}$

(3) $y''' + 3y'' + 6y' + 18y = x^2 \rightarrow \text{constant coefficient}$

(4) $x^3 y''' + 9x^2 y'' + 18xy' + 6y = 0 \rightarrow \text{variable coefficient}$

(5) $(1-x)y'' + xy' - y = 0 \rightarrow \text{variable coefficient}$

(6) $y'' - (1+x^2)y = 0 \rightarrow \text{variable coefficient}$

Solutions of Linear Differential Equations

Th^m If the functions $a_0(x), a_1(x), \dots, a_n(x)$ and $\varphi(x)$ are continuous over I and $a_0(x) \neq 0$ on I, then there exists a unique solution to the problem

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = \varphi(x) \quad (1)$$

$$y(x_0) = C_1, y'(x_0) = C_2, \dots, y^{(n-1)}(x_0) = C_n$$

where $x_0 \in I$ and C_1, C_2, \dots, C_n are n known constants.

If the conditions of the th^m are satisfied, then the diff. eq (1) is said to be normal on I.

Ex : find the intervals on which the following differential equations are normal.

(a) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, n is an integer.

Sol : Here $a_0(x) = 1-x^2$, $a_1(x) = -2x$, $a_2(x) = n(n+1)$.

Now a_0, a_1, a_2 are continuous everywhere in $(-\infty, \infty)$.

Also, $a_0(x) \neq 0$ for all x in $(-\infty, \infty)$ except at the points $-1, 1$.

Hence, the diff. eq is normal on every subinterval I of the open intervals $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$.

(b) $x^2y'' + xy' + (n^2 - x^2)y = 0$, n is real.

Sol : Here $a_0(x) = x^2$, $a_1(x) = x$, $a_2(x) = n^2 - x^2$

Now a_0, a_1, a_2 are continuous everywhere in $(-\infty, \infty)$.

Also $a_0(x) \neq 0$ for all $x \in (-\infty, \infty)$ except $x=0$.

Hence, the diff. eq. is normal on every subinterval

I of the open intervals $(-\infty, 0)$, $(0, \infty)$.

Find the intervals on which the following eqs are normals

$$(7) \quad y' = \frac{3y}{x} \Rightarrow xy' - 3y = 0$$

$$a_0(x) = x, \quad a_1(x) = -3.$$

Now a_0 and a_1 are continuous everywhere in $(-\infty, \infty)$.

Also, $a_0(x) = x \neq 0$ for all x in $(-\infty, \infty)$ except $x=0$.

\therefore The diff eq is normal on every subinterval of the open interval $(-\infty, 0)$ and $(0, \infty)$.

$$(8) \quad (1+x^2)y'' + 2xy' + y = 0$$

Sol :- $a_0(x) = 1+x^2$, $a_1(x) = 2x$, $a_2(x) = 1$

Now, $a_0(x)$, $a_1(x)$ and $a_2(x)$ are continuous everywhere in $(-\infty, \infty)$.

Also, $a_0(x) = 1+x^2 \neq 0$ everywhere in $(-\infty, \infty)$. ~~except~~

Hence, the diff eq is normal on every subinterval of $(-\infty, \infty)$.

$$(9) \quad x^2y'' - 4xy' + 6y = x.$$

Sol :- $a_0(x) = x^2$, $a_1(x) = -4x$, $a_2(x) = 6$, $r(x) = x$.

Now, a_0 , a_1 , a_2 and $r(x)$ are continuous everywhere in $(-\infty, \infty)$.

Also, $a_0(x) \neq 0$ everywhere in $(-\infty, \infty)$ except $x=0$.

Hence, the diff. eq is normal on every subinterval on $(-\infty, 0)$ and $(0, \infty)$.

$$(10) \quad y'' + 3y' + \sqrt{x}y = \sin x$$

Sol :- $a_0(x) = 1$, $a_1(x) = 3$, $a_2(x) = \sqrt{x}$, $r(x) = \sin x$.

Now $a_0(x)$, $a_1(x)$, $a_2(x)$, $r(x)$ are continuous everywhere in $[0, \infty)$.

Also, $a_0(x) \neq 0$ everywhere in $[0, \infty)$.

Hence, the diff eq is normal on every subinterval on $[0, \infty)$.

These sols does not contradict th^m 5.1 as
the point $x=0$, where eq is not normal is
included in the intervals.

So, th^m 5.1 is not applicable.

Linear Combination of functions

Let $f_1(x), f_2(x), \dots, f_m(x)$ be m functions. Then

$C_1 f_1(x) + C_2 f_2(x) + \dots + C_m f_m(x)$, where C_1, C_2, \dots, C_m are constants is called a linear combination of the given functions.

Th^m If $y_1(x), y_2(x), \dots, y_m(x)$ are m solutions of the linear homogeneous equation

$$a_0 y^{(m)} + a_1 y^{(m-1)} + \dots + a_{n-1} y' + a_n y = 0 \text{ on } I, \quad (1)$$

then the linear combination of the solutions

$C_1 y_1(x) + C_2 y_2(x) + \dots + C_m y_m(x)$, where C_1, C_2, \dots, C_m are constants is also a sol of eq (1) on I .

Remark : The above th^m does not hold for a non-homogeneous equation or a nonlinear equation.

Verify that the given functions are solution
of the associated differential equation. Verify
also that a linear combination of these functions
is also a solution.

$$(18) \quad 1, x, e^x; \quad y''' - y'' = 0.$$

For $y_1 = 1$

$$y_1' = y_1'' = y_1''' = 0.$$

$$y_1''' - y_1'' = 0 - 0 = 0.$$

For $y_2 = x$, $y_2' = 1$, $y_2'' = 0$, $y_2''' = 0$.

$$y_2''' - y_2'' = 0 - 0 = 0.$$

For $y_3 = e^x$, $y_3' = e^x$, $y_3'' = e^x$, $y_3''' = e^x$

$$y_3''' - y_3'' = e^x - e^x = 0.$$

Substituting $y = C_1 \cdot 1 + C_2 \cdot x + C_3 e^x$
 $= C_1 y_1 + C_2 y_2 + C_3 y_3$

$$\begin{aligned} y''' - y'' &= (C_1 y_1 + C_2 y_2 + C_3 y_3)''' - (C_1 y_1 + C_2 y_2 + C_3 y_3)'' \\ &= C_1 y_1''' + C_2 y_2''' + C_3 y_3''' - C_1 y_1'' - C_2 y_2'' - C_3 y_3'' \\ &= C_1 (y_1''' - y_1'') + C_2 (y_2''' - y_2'') + C_3 (y_3''' - y_3'') \\ &= C_1 \cdot 0 + C_2 \cdot 0 + C_3 \cdot 0 = 0. \end{aligned}$$

$$(19) \quad e^x, e^{-2x}; y'' + y' - 2y = 0.$$

$$\text{or } y_1 = e^x, y_1' = e^x, y_1'' = e^x$$

$$y_1'' + y_1' - 2y_1 = e^x + e^x - 2e^x = 0e^x - 2e^x = 0.$$

$$\text{for } y_2 = e^{-2x}, y_2' = -2e^{-2x}, y_2'' = 4e^{-2x}.$$

$$y_2'' + y_2' - 2y_2 = 4e^{-2x} + (-2e^{-2x}) - 2e^{-2x} = 0.$$

$$\text{Substituting } y = c_1 e^x + c_2 e^{-2x} = c_1 y_1 + c_2 y_2$$

$$\begin{aligned} y'' + y' - 2y &= (c_1 y_1 + c_2 y_2)'' + (c_1 y_1 + c_2 y_2)' - 2(c_1 y_1 + c_2 y_2) \\ &= c_1 y_1'' + c_2 y_2'' + c_1 y_1' + c_2 y_2' - 2c_1 y_1 - 2c_2 y_2 \\ &= c_1(y_1'' + y_1' - 2y_1) + c_2(y_2'' + y_2' - 2y_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0. \end{aligned}$$

$$(20) \quad e^{-x} \cos 2x, e^{-x} \sin 2x; y'' + 2y' + 5y = 0.$$

$$\begin{aligned} \text{for } y_1 &= e^{-x} \cos 2x \Rightarrow y_1' = -e^{-x} \cos 2x + e^{-x}(-2 \sin 2x) \\ &= -e^{-x} \cos 2x - 2e^{-x} \sin 2x \\ y_1'' &= e^{-x} \cos 2x + 2e^{-x} \sin 2x + 2e^{-x} \sin 2x \\ &\quad - 4e^{-x} \cos 2x \\ &= -3e^{-x} \cos 2x + 4e^{-x} \sin 2x. \end{aligned}$$

$$\begin{aligned} y'' + 2y' + 5y &= (-3e^{-x} \cos 2x + 4e^{-x} \sin 2x) + 2(-e^{-x} \cos 2x - 2e^{-x} \sin 2x) \\ &\quad + 5e^{-x} \cos 2x \\ &= 0. \end{aligned}$$

Wronskian

Let $f_1(x), f_2(x), \dots, f_n(x)$ be n functions.

$$W(b_1, b_2, \dots, b_n) = \begin{vmatrix} b_1 & b_2 & \cdots & b_n \\ b'_1 & b'_2 & \cdots & b'_n \\ \vdots & \vdots & \ddots & \vdots \\ b_1^{(n-1)} & b_2^{(n-1)} & \cdots & b_n^{(n-1)} \end{vmatrix} = W(x).$$

Th^m If the coefficients $a_0(x), a_1(x) \dots a_n(x)$ in the linear homogeneous equations

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0, a_0 \neq 0 \quad \text{--- (1)}$$

are continuous on I and $y_1(x), \dots, y_n(x)$ are n sols of this eq, then

$$(i) \quad W(x) = W(y_1, y_2, \dots, y_n) \neq 0 \text{ for all } x \in I,$$

$\Leftrightarrow y_1(x), y_2(x), \dots, y_n(x)$ are LI on I.

$$(ii) \quad W(x_0) = 0 \text{ where } x_0 \in I \text{ is any fixed point, implies } W(x) = 0 \text{ for all } x \in I \text{ and the functions } y_1(x), y_2(x), \dots, y_n(x) \text{ are LD.}$$

Ex 5.6

Show that the functions x, x^2, x^3 are LI
on any interval I, not containing zero.

Sol :-

$$\begin{aligned} W(x) &= \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} \\ &= x(12x^2 - 6x^2) - x^2(6x) + x^3(2) \\ &= 6x^3 - 6x^3 + 2x^3 \\ &= 2x^3. \end{aligned}$$

Therefore, $W(x) \neq 0$ on any interval not containing zero. Hence, the functions are LI in $(-\infty, 0), (0, \infty)$.

Ex 5.7 Show that the functions 1, $\sin x$, $\cos x$ are LI.

Sol:

$$W(x) = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}$$

$$= -\cos^2 x - \sin^2 x = -1 \neq 0.$$

Hence, the given functions are LI on any interval I.

Ex 5.8 Show that e^x, e^{2x}, e^{3x} are the fundamental solutions of $y''' - 6y'' + 11y' - 6y = 0$, on any interval I.

Sol: $y_1(x) = e^x, y_1' = e^x, y_1'' = e^x, y_1''' = e^x$

$$y_1''' - 6y_1'' + 11y_1' - 6y_1 = e^x - 6e^x + 11e^x - 6e^x = 0.$$

$$y_2(x) = e^{2x}, y_2'(x) = 2e^{2x}, y_2'' = 4e^{2x}, y_2''' = 8e^{2x}.$$

$$y_2''' - 6y_2'' + 11y_2' - 6y_2 = 8e^{2x} - 24e^{2x} + 22e^{2x} - 6e^{2x} = 0.$$

$$y_3(x) = e^{3x}, y_3'(x) = 3e^{3x}, y_3''(x) = 9e^{3x}, y_3'''(x) = 27e^{3x}.$$

$$y_3''' - 6y_3'' + 11y_3' - 6y_3 = 27e^{3x} - 54e^{3x} + 33e^{3x} - 6e^{3x} = 0.$$

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ 2e^{2x} & 4e^{2x} & 9e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x \begin{vmatrix} 1 & e^x & e^{2x} \\ 1 & 2e^x & 3e^{2x} \\ 1 & 4e^x & 9e^{2x} \end{vmatrix}$$

$$= e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x}(18 - 12 - 9 + 3 + 4 - 2)$$

$$= 9e^{6x} \neq 0.$$

∴ Solutions are LI and they form a set of fundamental solutions on any interval I.

Solution of Linear differential equation - Operator

Method

ART NO. 1600
ERASER

Operators :- $\frac{d}{dx}, \frac{d^2}{dx^2}, \dots, \frac{d^n}{dx^n}, \dots$

For sake of convenience, the operators

$\frac{d}{dx}, \frac{d^2}{dx^2}, \dots, \frac{d^n}{dx^n}$ are denoted by D, D^2, D^3, \dots, D^n .

$$\frac{d^3y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \quad - \textcircled{2}$$

$$\Rightarrow D^2y + 5Dy + 6y = 0$$

$$\Rightarrow (D^2 + 5D + 6)y = 0.$$

Solution of ^{second order}n Homogeneous Linear Equations (Complementary function)

Eq in operator form :- $(D^2 + 5D + 6)y = 0 \Rightarrow f(D)y = 0$

Auxiliary eq :- $m^2 + 5m + 6 = 0$ (Replace D by m to get A.E.)
 $(m+2)(m+3) = 0 \Rightarrow m = -2, -3$

Solution of eq (2) depends on the nature of the roots.

Case 1/2 The G.S. is $y = C_1 e^{-2x} + C_2 e^{-3x}$

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0. \rightarrow \text{2nd order homogeneous linear differential equation with constant coefficients.}$$

Eq in operator form

$$(a_0 D^2 + a_1 D + a_2) y = 0.$$

$$\text{A.E. is } a_0 m^2 + a_1 m + a_2 = 0. \quad -(3)$$

Case 1 The roots are real and distinct, say, m_1, m_2 are the roots of the equation (3).

The general solution is

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}, \text{ where } C_1 \text{ and } C_2 \text{ are constants.}$$

$$\underline{\text{Ex}} \rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$$

$$\text{In operator form, } (D^2 - D - 6)y = 0.$$

$$\text{A.E. is } m^2 - m - 6 = 0$$

$$\Rightarrow m^2 - 3m + 2m - 6 = 0$$

$$\Rightarrow m(m-3) + 2(m-3) = 0$$

$$\Rightarrow (m+2)(m-3) = 0.$$

$$\Rightarrow m = -2, 3.$$

$$\text{The general sol is } y(x) = C_1 e^{-2x} + C_2 e^{3x}.$$

Ex $4y'' - 8y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 3.$

Eq in operator form: $(4D^2 - 8D + 3)y = 0.$

A.E. is $4m^2 - 8m + 3 = 0$

$$\Rightarrow 4m^2 - 6m - 2m + 3 = 0$$

$$\Rightarrow 2m(2m-3) - 1(2m-3) = 0$$

$$\Rightarrow (2m-1)(2m-3) = 0$$

$$\Rightarrow m = \frac{1}{2}, \frac{3}{2}$$

The general sol is $y = Ce^{\frac{1}{2}x} + C_2 e^{\frac{3}{2}x}, \quad y' = \frac{1}{2}Ce^{\frac{1}{2}x} + \frac{3}{2}C_2 e^{\frac{3}{2}x}$

$$y(0) = 1 \Rightarrow Ce^0 + C_2 e^0 = 1$$

$$\Rightarrow C + C_2 = 1.$$

$$y'(0) = 3 \Rightarrow \frac{1}{2}C + \frac{3}{2}C_2 = 3$$

$$\Rightarrow C + 3C_2 = 6$$

$$2C_2 = 5 \Rightarrow C_2 = \frac{5}{2}$$

$$C = 1 - \frac{5}{2} = -\frac{3}{2}$$

\therefore The sol of the given problem is

$$y(x) = -\frac{3}{2}e^{\frac{1}{2}x} + \frac{5}{2}e^{\frac{3}{2}x}$$

$$\underline{\text{Case 2}} \quad m^2 + a_1 m + a_2 = 0. \quad \text{---(3)}$$

Roots are real and equal, say, m, m are the roots of the eq (3).

$$\text{The G.S. is } y = (C_1 + x C_2) e^{mx}.$$

$$\underline{\text{Ex}} \quad 4y'' + 4y' + y = 0.$$

$$(4D^2 + 4D + 1)y = 0 \quad (\text{Eq in operator form})$$

$$\Rightarrow 4m^2 + 4m + 1 = 0 \quad (\text{A.E.})$$

$$\Rightarrow 4m^2 + 2m + 2m + 1 = 0$$

$$\Rightarrow 2m(2m+1) + 1(2m+1) = 0$$

$$\Rightarrow (2m+1)(2m+1) = 0$$

$$\Rightarrow m = -\frac{1}{2}, -\frac{1}{2}$$

$$\text{The general sol. is } y(x) = C_1 e^{-\frac{1}{2}x} + x C_2 e^{-\frac{1}{2}x}$$

$$\underline{\text{Ex}} \quad y'' + 6y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = 3.$$

$$\text{Eq in operator form, } (D^2 + 6D + 9)y = 0$$

$$\text{A.E. is } m^2 + 6m + 9 = 0$$

$$m^2 + 3m + 3m + 9 = 0$$

$$m(m+3) + 3(m+3) = 0$$

$$\Rightarrow (m+3)(m+3) = 0$$

$$\Rightarrow m = -3, -3.$$

$$\text{The general sol is } y(x) = (C_1 + x C_2) e^{-3x}$$

$$y'(x) = C_2 e^{-3x} + (-3)(C_1 + x C_2) e^{-3x}$$

$$y(0) = 2 \Rightarrow C_1 = 2$$

$$y'(0) = 3 \Rightarrow C_2 - 3(C_1) = 3$$

$$\Rightarrow C_2 - 3(2) = 3$$

$$\Rightarrow C_2 = 9.$$

$y(x) = (2 + 9x)e^{-3x}$ is the sol of given problem.

Case 3 $m^2 + 2m + 2 = 0$.

$$d+i\beta, d-i\beta$$

When the roots are complex, say, ~~$p+iq, p-iq$~~ .

The general sol is

$$y(x) = C_1 e^{px} [C_2 \cos \beta x]$$

$$y(x) = e^{px} [C_1 \cos \beta x + C_2 \sin \beta x].$$

Ex $y'' + 2y' + 2y = 0$.

Eq in operator method, $(D^2 + 2D + 2)y = 0$

$$\text{A.E is } m^2 + 2m + 2 = 0$$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4-4(2)}}{2(1)} = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

The general solution is

$$y(x) = e^{-x} [C_1 \cos x + C_2 \sin x].$$

Ex $y'' + 4y' + 13y = 0, y(0) = 0, y'(0) = 1.$

$$(D^2 + 4D + 13)y = 0$$

$$\Rightarrow m^2 + 4m + 13 = 0$$

$$\Rightarrow m = \frac{-4 \pm \sqrt{16 - 4(13)}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

$$y(x) = e^{-2x} [C_1 \cos 3x + C_2 \sin 3x].$$

$$\begin{aligned} y'(x) &= e^{-2x} [C_1(-3 \sin 3x) + C_2(3 \cos 3x)] \\ &= e^{-2x} [-3C_1 \sin 3x + 3C_2 \cos 3x] \end{aligned}$$

$$y(0) = 0 \Rightarrow C_1 = 0.$$

$$y'(0) = 1 \Rightarrow 3C_2 = 1 \Rightarrow C_2 = \frac{1}{3}$$

$$\therefore y(x) = e^{-2x} \left[\frac{1}{3} \sin 3x \right] = \frac{1}{3} e^{-2x} \sin 3x.$$

Solution of Higher order Homogeneous linear equations with constant coefficients.

Consider $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = 0$.

Auxiliary or characteristic eq is

$$a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0. \quad (4)$$

Case 1 The roots of (4) are real and distinct, say, m_1, m_2, \dots, m_n .

Then, general solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$$

Ex $3y''' - 2y'' - 3y' + 2y = 0.$

Eq in operator form, $(3D^3 - 2D^2 - 3D + 2)y = 0.$

A.E. is $3m^3 - 2m^2 - 3m + 2 = 0.$

$$\Rightarrow m^2(3m-2) - 1(3m-2) = 0$$

$$\Rightarrow (m^2-1)(3m-2) = 0$$

$$\Rightarrow (m-1)(m+1)(3m-2) = 0$$

$$\Rightarrow m=1, -1, \frac{2}{3}.$$

The sol is $y = C_1 e^x + C_2 e^{-x} + C_3 e^{\frac{2}{3}x}.$

$$2 \rightarrow 1, -1, 2, -2$$

$$3 \rightarrow 1, -1, 3, -3$$

$$\text{Roots} = 1, -1, \frac{1}{3}, \frac{1}{3}, 2, -2, \frac{2}{3}, -\frac{2}{3}.$$

$$\begin{array}{c|ccccc} 1 & 3 & -2 & -3 & 2 \\ & & 3 & 1 & -2 \\ \hline -1 & 3 & 1 & -2 & 0 \\ & & -3 & 2 & \\ \hline & 3 & -2 & 0 & \end{array}$$

Ex Solve the IVP

$$4y''' - 4y'' - 9y' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0.$$

Eq in operator form,

$$(4D^3 - 4D^2 - 9D + 9)y = 0$$

A.E. is $4m^3 - 4m^2 - 9m + 9 = 0$

$$4m^2(m-1) - 9(m-1) = 0$$

$$\Rightarrow (4m^2 - 9)(m-1) = 0$$

$$\Rightarrow ((2m)^2 - 3^2)(m-1) = 0$$

$$\Rightarrow (2m-3)(2m+3)(m-1) = 0$$

$$\Rightarrow m = 1, \frac{3}{2}, -\frac{3}{2}$$

The general sol is $y = C_1 e^x + C_2 e^{\frac{3}{2}x} + C_3 e^{-\frac{3}{2}x}$

$$y' = C_1 e^x + \frac{3}{2} C_2 e^{\frac{3}{2}x} - \frac{3}{2} C_3 e^{-\frac{3}{2}x}$$

$$y'' = C_1 e^x + \frac{9}{4} C_2 e^{\frac{3}{2}x} + \frac{9}{4} C_3 e^{-\frac{3}{2}x}$$

$$y(0) = 1 \Rightarrow C_1 + C_2 + C_3 = 1 \quad \text{--- (1)}$$

$$y'(0) = 0 \Rightarrow C_1 + \frac{3}{2} C_2 - \frac{3}{2} C_3 = 0 \Rightarrow 2C_1 + 3C_2 - 3C_3 = 0 \quad \text{--- (2)}$$

$$y''(0) = 0 \Rightarrow C_1 + \frac{9}{4} C_2 + \frac{9}{4} C_3 = 0 \Rightarrow 4C_1 + 9C_2 + 9C_3 = 0 \quad \text{--- (3)}$$

$$\textcircled{1} \times 3 \quad 3G + 3C_2 + 3C_3 = 3$$

$$2G + 3C_2 - 3C_3 = 0$$

$$\Rightarrow 5G + 6C_2 = 3$$

$$\textcircled{1} \times 9 \Rightarrow 9G + 9C_2 + 9C_3 = 9$$

$$4G + 9C_2 + 9C_3 = 0$$

$$\Rightarrow 5G = 9 \Rightarrow \boxed{G = \frac{9}{5}}$$

$$\Rightarrow 5 \cdot \frac{9}{5} + 6C_2 = 3$$

$$\Rightarrow 6C_2 = -6 \Rightarrow \boxed{C_2 = -1}$$

$$G + C_2 + C_3 = 1 \Rightarrow \frac{9}{5} - 1 + C_3 = 1$$

$$\Rightarrow C_3 = 1 - \frac{4}{5} = \frac{1}{5}$$

$$\boxed{C_3 = \frac{1}{5}}$$

$$y = \frac{9}{5}e^x - e^{\frac{3}{2}x} + \frac{1}{5}e^{-\frac{3}{2}x}$$

Case 2

When ^{two} roots are real and equal, say,

$$\underline{m_1, m_1, m_2, m_3, \dots}$$

(Two roots are equal)

$$m_1 = m, m_2 = m, m_3, \dots, m_n$$

The general sol is

$$y = (C_1 + x C_2) e^{mx} + C_3 e^{m_3 x} + \dots + C_n e^{m_n x}$$

(Three roots are equal) $m_1 = m_2 = m_3 = m, m_4, \dots, m_n$

The general sol is

$$y = (C_1 + x C_2 + x^2 C_3) e^{mx} + C_4 e^{m_4 x} + \dots + C_n e^{m_n x}$$

Case 3 Ex.

$$y''' - 3y' - 2y = 0$$

Eq in operator form, $(D^3 - 3D - 2)y = 0$
A.E. is $m^3 - 3m - 2 = 0 \dots \textcircled{1}$

int solutions of this eq are somewhere in the

list: 1, -1, 2, -2.

$$2x^3 + 9x^2 + 13x + 6 = 0$$

↓ factors ↓
1, 2 ↴ ↴ 1, 2, 3, 6

Cubic eq's integer sols
are somewhere in the
this list :-

$$\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, 1, -1, 2, -2, 3, -3, 6, -6.$$

$$m=1, 1-3-2 \neq 0$$

$$m=-1, -1+3-2=0 \Rightarrow m=-1 \text{ is a root of } \textcircled{1}.$$

$$(m+1)(m^2-m-2)=0$$

$$\Rightarrow (m+1)(m^2-8m+m-2)=0$$

$$\Rightarrow (m+1)(m(m-2)+1(m-2))=0$$

$$\Rightarrow (m+1)(m+1)(m-2)=0$$

$$\Rightarrow m = -1, -1, 2.$$

$$\begin{array}{c} m^3 - m - 2 \\ m+1 \sqrt{m^3 - 3m^2 - 2} \\ \underline{+ m^3 + m^2} \\ -m^2 - 3m \\ -m^2 - m \\ \underline{+ \quad +} \\ -2m - 2 \\ -2m - 2 \\ \hline X \end{array}$$

$$\begin{array}{r|rrr} -1 & 1 & 0 & -3 & -2 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & 0 \\ & & -1 & 2 \\ \hline & 1 & -2 & 0 \end{array}$$

The general sol is

$$y = C_1 e^{-x} + x C_2 e^{-x}$$

$$y = (C_1 + x C_2) e^{-x} + C_3 e^{2x}$$

Ex $8y''' - 12y'' + 6y' - y = 0$

Eq in operator form, $(8D^3 - 12D^2 + 6D - 1) y = 0$. factors of 8 = 1, 2, 4

AE is $8m^3 - 12m^2 + 6m - 1 = 0$.

$m=1, 8-12+6-1 \neq 0$

$m=-1, -8-12-6-1 \neq 0$

~~$m=2, 64-48+12-1$~~

-1, 1, -2, 2, -4, 4.

factors of 1 = 1, -1

factors of given eq =

-1, 1, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$.

$$\begin{array}{c|cccc} m = -\frac{1}{2}, \frac{1}{2} & 8 & -12 & 6 & -1 \\ \hline & -4 & 8 & -7 & \\ 8 & -16 & 14 & -8 & \end{array}$$

$$\begin{array}{c|cccc} m = \frac{1}{2} & \frac{1}{2} & 8 & -12 & -6 & -1 \\ \hline & -4 & -8 & 2 & 0 & \end{array}$$

$$\begin{array}{c|cccc} & 8 & -12 & 6 & -1 \\ \hline -\frac{1}{2} & & -4 & -8 & 1 \\ \hline \frac{1}{2} & 8 & -16 & -2 & 0 \\ \hline & 8 & -12 & 2 & 0 \end{array}$$

$$\left(m - \frac{1}{2}\right)(8m^2 - 8m + 2) = 0$$

$$\Rightarrow \left(m - \frac{1}{2}\right)(8m^2 - 4m - 4m + 2) = 0$$

$$\Rightarrow \left(m - \frac{1}{2}\right)(4m(2m-1) - 2(2m-1)) = 0$$

$$\Rightarrow \left(m - \frac{1}{2}\right)(4m-2)(2m-1) = 0 \Rightarrow m = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}.$$

The general sol is

$$y = (C_1 + xC_2 + x^2 C_3) e^{\frac{x}{2}}$$

Ex: $y''' + 3y'' - 4y = 0, y(0)=1, y'(0)=0, y''(0) = \frac{1}{2}$

$$(D^3 + 3D^2 - 4D)y = 0 \Rightarrow (D^3 + 3D^2 - 4)y = 0$$

$$m^3 + 3m^2 - 4m = 0 \Rightarrow m^3 + 3m^2 - 4 = 0 \quad 4 = 1, 2, 4$$

$$\cancel{m(m^2 + 3m - 4)} = 0$$

$$\text{factors } 4 = -1, 1, 2, -2, \\ 4, -4$$

$$m=1, 1+3-4=0$$

$$\begin{array}{c|cccc} \cancel{m=1} & 1 & 3 & \cancel{1} & -4 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

$$(m-1)(m^2 + 4m + 4) = 0$$

$$\Rightarrow (m-1)(m+2)^2 = 0.$$

$$\Rightarrow m=1, -2, -2.$$

The G.S. is $y = C_1 e^x + (C_2 + xC_3) e^{-2x}$

$$y' = C_1 e^x + (C_2 + xC_3)(-2e^{-2x}) + C_3 e^{-2x}$$
$$= C_1 e^x - 2C_2 e^{-2x} - 2xC_3 e^{-2x} + C_3 e^{-2x}$$

$$y'' = C_1 e^x + 4C_2 e^{-2x} - 2C_3(1 \cdot e^{-2x} + x(-2e^{-2x}))$$
$$+ C_3 (-2e^{-2x})$$

$$= C_1 e^x + 4C_2 e^{-2x} - 2C_3 e^{-2x} + 1 \cancel{4C_3 e^{-2x}} - \cancel{2C_3 e^{-2x}}$$

$$= C_1 e^x + 4C_2 e^{-2x}$$

$$y''' = C_1 e^x + 4C_2 e^{-2x} - 2C_3 e^{-2x} + 4C_3 x e^{-2x} - 9C_3 e^{-2x}$$

$$y = C_1 e^x + 4C_2 e^{-2x} - 4C_3 e^{-3x} + 4C_3 x e^{-2x}$$

$$y(0) = 1 \Rightarrow C_1 + C_2 = 1$$

$$y'(0) = 0 \Rightarrow C_1 - 2C_2 + C_3 = 0$$

$$y''(0) = \frac{1}{2} \Rightarrow C_1 + 4C_2 - 4C_3 = \frac{1}{2}$$

$$4C_1 - 8C_2 + 4C_3 = 0$$

$$\Rightarrow 5C_1 - 4C_2 = \frac{1}{2}$$

$$5C_1 + 5C_2 = 5 \Rightarrow 9C_2 = 5 - \frac{1}{2} = \frac{9}{2}$$

$$C_2 = \cancel{\frac{5+1}{2}} = \frac{1}{2}$$

$$\boxed{C_2 = \frac{1}{2}}$$

$$C_1 = 1 - \frac{1}{2} \Rightarrow \boxed{C_1 = \frac{1}{2}}$$

$$\frac{1}{2} - \cancel{2 \cdot \frac{1}{2}} + C_3 = 0$$

$$\boxed{C_3 = \frac{1}{2}}$$

$$\therefore y = \frac{1}{2} e^x + \left(\frac{1}{2} + \frac{x}{2}\right) e^{-2x}$$

Case 3 (i) Roots are complex and non-repeated. When one pair of roots is complex.

Say, $\alpha \pm i\beta, \alpha_3, \dots, \alpha_n$.

The general sol is

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) + C_3 e^{\alpha_3 x} + \dots + C_n e^{\alpha_n x}$$

(ii) Roots are complex and repeated. (When one pair of roots is equal and imaginary)
Say, $\alpha \pm i\beta, \alpha \pm i\beta, \alpha_5, \alpha_6, \dots, \alpha_n$.

The G.S. is

$$y = e^{\alpha x} [(C_1 + xC_2) \cos \beta x + (C_3 + xC_4) \sin \beta x] + C_5 e^{\alpha_5 x} + \dots + C_n e^{\alpha_n x}$$

Ex

$$y'''' + 5y'' + 4y = 0$$

$$(D^4 + 5D^2 + 4) \Rightarrow y = 0$$

$$m^4 + 5m^2 + 4 = 0$$

$$\Rightarrow m^4 + 4m^2 + m^2 + 4 = 0$$

$$\Rightarrow m^2(m^2 + 4) + 1(m^2 + 4) = 0$$

$$\Rightarrow (m^2 + 1)(m^2 + 4) = 0$$

$$\Rightarrow m = \pm i, \pm 2i$$

$$m^2 = \frac{-5 \pm \sqrt{25 - 16}}{2}$$

$$= \frac{-5 \pm 3}{2} = -1, -4$$

$$m = \pm i, \pm 2i$$

$$m^2 + 1 = 0$$

$$\Rightarrow m = -0 \pm \sqrt{0 - 4(1)}$$

$$y = C_1 \cos x + C_2 \sin x + C_3 \cos 2x + C_4 \sin 2x. \quad \begin{matrix} 2 \\ = \pm i \end{matrix}$$

Ex

$$y^{iv} + 32y'' + 256y = 0$$

$$(D^4 + 32D^2 + 256)y = 0$$

$$m^4 + 32m^2 + 256 = 0$$

$$\Rightarrow (m^2)^2 + (16)^2 + 2(m^2)(16) = 0$$

$$\Rightarrow (m^2 + 16)^2 = 0$$

$$\Rightarrow m = \pm 4i, \pm 4i.$$

The G.S is

$$y = (C_1 + xC_2)\cos 4x + (C_3 + xC_4)\sin 4x.$$

Ex

$$y^{iv} + y'' = 0, y(0) = 1, y'(0) = 2, y''(0) = -1, y'''(0) = -1$$

$$(D^4 + D^2)y = 0$$

$$\Rightarrow m^4 + m^2 = 0$$

$$\Rightarrow m^2(m^2 + 1) = 0$$

$$\Rightarrow m = 0, 0, \pm i$$

$$y = C_1 + xC_2 + C_3 \cos x + C_4 \sin x$$

$$y' = \cancel{C_1} + C_2 + C_3(-\sin x) + C_4 \cos x$$

$$= \cancel{C_1} + C_2 - C_3 \sin x + C_4 \cos x$$

$$y'' = -C_3 \cos x - C_4 \sin x$$

$$y''' = +C_3 \sin x - C_4 \cos x$$

$$y(0)=1 \Rightarrow C_1 + C_3 = 1$$

$$y'(0)=2 \Rightarrow C_2 + C_4 = 2$$

$$y''(0)=-1 \Rightarrow -C_3 = -1 \Rightarrow C_3 = 1$$

$$y'''(0)=-1 \Rightarrow -C_4 = -1 \Rightarrow C_4 = 1$$

$$\cancel{C_1} \neq 1, C_2 = 1, C_3 = 0, C_4 = 1$$

$$\therefore y = x + \cos x + \sin x.$$

Ex

$$\frac{d^3y}{dx^3} + y = 0$$

$$(D^3 + 1)y = 0$$

$$m^3 + 1 = 0$$

$$(m+1)(m^2 + 1 - m) = 0 \Rightarrow (m+1)(m^2 - m + 1) = 0$$

$$m = -1, \frac{1 \pm \sqrt{1-4}}{2} = -1, \frac{1 \pm \sqrt{3}i}{2}$$

$$m = -1, \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

$$a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$$

$$\therefore y = C_1 e^{-x} + e^{\frac{x}{2}} \left[C_2 \cos \frac{\sqrt{3}}{2}x + C_3 \sin \frac{\sqrt{3}}{2}x \right].$$

Ex 5.3

- (29) Find a homogeneous LDE which has following particular integral solution.

$$xe^{-x} + e^{2x}$$

Sol: $xe^{-x} \rightarrow$ Repeated roots. $m = -1, -1$
 $e^{2x} \rightarrow$ Root is $m = 2$.

$$(m+1)^2(m-2) = 0$$

$$(m^2+1+2m)(m-2) = 0$$

$$m^3 - 2m^2 + m - 2 + 2m^2 - 4m = 0$$

$$\Rightarrow m^3 - 3m - 2 = 0$$

$$\frac{d^3y}{dx^3} - \frac{3dy}{dx} - 2y = 0$$

Ex $(D^2 + D + 5)y = 3$

Sol: C.P $m^2 + m + 5 = 0$

$$m = \frac{-1 \pm \sqrt{1-20}}{2}$$

$$= \frac{-1 \pm \sqrt{19}i}{2}$$

$$y(x) = e^{-x/2} [C_1 \cos \sqrt{19}x + C_2 \sin \sqrt{19}x]$$

$$\text{P.I. } y_p(x) = \frac{1}{D^2 + D + 5} 3e^{0x}$$

$$= 3 \frac{1}{0+0+5} e^{0x}$$

$$= \frac{3}{5}.$$

$$y(x) = e^{-\sqrt{19}x} [C_1 \cos \sqrt{19}x + C_2 \sin \sqrt{19}x] + \frac{3}{5}.$$

$$\text{Ex } (D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0$$

$$m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$$

$$m=1, \quad 1-2+5-8+4=0$$

$$(m-1)(m^3 - m^2 + 4m - 4) = 0$$

$$(m-1)[m^2(m-1) + 4(m-1)] = 0$$

$$(m-1)(m^2 + 4)(m-1) = 0$$

$$m=1, 1, \pm 2i$$

$$y(x) = (C_1 + xC_2)e^x + C_3 \cos 2x + C_4 \sin 2x.$$

$$\begin{array}{r|rrrrr} & 1 & 1 & -2 & 5 & -8 & 4 \\ & & 1 & -1 & 4 & -4 & \\ \hline & 1 & -1 & 4 & -4 & 0 \end{array}$$