EIGENVALUES

$$\text{Let} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

AX=Y ....(1)

Where A is the matrix , X is the column vector and Y is also column vector.

Here column vector X is transformed into the column vector Y by means of the square matrix A.

Let X be a such vector which transforms into λX by means of the transformation (1). Suppose the linear transformation Y = AX transforms X into a scalar multiple of itself i.e. λX.

$AX = Y = \lambda X$

$AX - \lambda IX = 0$

$(A - \lambda I) X = 0$

Ex. Find the eigenvalues of the following matrices

(i) $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

Note:

Note:

1. Characteristic Polynomial
2. Characteristic Equation
3. Characteristic Roots or Eigenvalues

(ii) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 3 & 5 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

(vii) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 8 \end{bmatrix}$

Note1: Direct Characteristic equation for matrix A

Order 2: $\lambda^2 - trac(A)\lambda + \det(A) = 0$

Order 3:

$\lambda^3 - trac(A)\lambda^2 + \left(Minor(a_{11}) + Minor(a_{22}) + Minor(a_{33})\right)\lambda - det(A) = 0$

Note2: The eigenvalue of

(a) a symmetric/Hermitian matrix are real

(b) a skew-symmetric/skew-Hermitian matrix are zero or pure imaginary

(c)an orthogonal matrix are of magnitude 1 and are real or complex conjugate pairs

(d) an unitary matrix are of magnitude 1

Some Important Properties of Eigenvalues

(1) Any square matrix A and its transpose A' have the same eigenvalues.

(2) The sum of the eigenvalues of a matrix is equal to the trace of the matrix.

(3) The product of the eigenvalues of a matrix A is equal to the determinant of A.

(4) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then the eigen values of

(i) kA are $k\lambda_1, k\,\lambda_2, \ldots, k\lambda_n$.
(ii) A^m are $\lambda_1^m,\ \lambda_2^m, \ldots, \lambda_n^m$.
(iii) A^{-1} $are\, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$.

(5) $(A - kI)^{-1}$ has the eigenvalue $\frac{1}{\lambda - k}$.

(6) (A-kI) has the eigenvalue $\lambda - k$.

(7) For a real matrix A, if $\alpha + i\beta$ is an eigenvalue, then its conjugate $\alpha - i\beta$ is also an eigenvalue. When the matrix A is complex, this property does not hold.

Theorem:(Cayley-Hamilton Theorem)
Every square matrix A satisfies its own characteristic equation

Ex. Verify Cayley-Hamilton theorem for the following matrices. Also find the inverse of the matrix.

(i) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

CHARACTERISTIC VECTORS OR EIGEN VECTORS

A column vector X is transformed into column vector Y by means of a square matrix A.

Now we want to multiply the column vector X by a scalar quantity λ so that we can find the same transformed column vector Y. i.e., $AX = \lambda X$
X is known as eigenvector.

Show that the vector (1, 1, 2) is an eigen vector of the matrix

$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ *corresponding to the eigen value 2.*

Note: Corresponding to each characteristic root λ, we have a corresponding non-zero vector X
which satisfies the equation $[A - \lambda I]\, X = 0$. The non-zero vector X is called characteristic vector or Eigen vector.

Ex. Find the eigenvalues and eigenvectors of the following matrices

(i) $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(viii) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(ix) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

PROPERTIES OF EIGEN VECTORS:

1. The eigenvector X of a matrix A is not unique.

2. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues of an n × n matrix then corresponding eigenvectors $X_1, X_2, \ldots, X_n$ form a linearly independent set.

3. If two or more eigenvalues are equal it may or may not be possible to get linearly independent eigenvectors corresponding to the equal roots.

4. Two eigenvectors X_1 and X_2 are called orthogonal vectors if $X_1' X_2 = 0$.

5. Eigen vectors of a symmetric matrix corresponding to different eigenvalues are orthogonal.