Method of Variation of parameter

1

Second Order LDE (non-homo)

$a_0(x) y'' + a_1(x) y' + a_2(x) y = r(x)$, $a_0(x) \neq 0$.

First find auxiliary eqn

$$a_0(x) m^2 + a_1(x) m + a_2(x) = 0$$

then find roots and write the soln

CF $\leftarrow y_c(x) = A y_1(x) + B y_2(x)$, where A, B are arbitrary constants.

Now assume the soln

$$y(x) = A(x) y_1(x) + B(x) y_2(x)$$

& find $A(x) = -\int \frac{g(x) y_2(x)}{W(x)} dx + c_1$, where $g(x) = \frac{r(x)}{a_0(x)}$

$c_1 \to$ const

$$B(x) = \int \frac{g(x) y_1(x)}{W(x)} dx + c_2$$

$c_2 \to$ const

Wronskian $\leftarrow W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \to$ always.

Now put the values of $A(x)$ & $B(x)$, we have

G.S

$$y(x) = A(x) y_1(x) + B(x) y_2(x)$$

Qus 2 $y'' + 3y' + 2y = 2e^x$, using the method of variation of parameter

* Simultaneous differential equations using operator method.

$$y'' + 3y' + 3y = 2e^{x}$$

$$m^2 + 3m + 2 = 0$$

$$m = -1, -2$$

Case 1 → Case 2 → Case 3 → Case 4 → Case 5

$$y_c = Ae^{-x} + Be^{-2x} \quad ; \quad y_1(x) = e^{-x} \quad y_2(x) = e^{-2x}$$

Assume G.S is $y(x) = A(x)e^{-x} + B(x)e^{-2x}$

Now $g(x) = \dfrac{r(x)}{a_0(x)} = 2e^{x}$

$$W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$A(x) = -\int \frac{g(x)y_2(x)}{W} + c_1 = -\int \frac{2e^{x}\cdot(e^{-2x})}{-e^{-3x}} + c_1 = 2\int e^{-x} \times e^{3x}\,dx + c_1$$

$$= 2\int e^{2x}\,dx + c_1 = \frac{2e^{2x}}{2} = e^{2x} + c_1$$

$$B(x) = \int \frac{g(x)y_1(x)}{W}\,dx + c_2 = \int \frac{2e^{x}\cdot e^{-x}}{-e^{-3x}}\,dx + c_2$$

$$= -2\int e^{3x}\,dx + c_2 = -2\frac{e^{3x}}{3} + c_2$$

General solution is

$$y(x) = A(x)e^{-x} + B(x)e^{-2x}$$

$$= (e^{2x} + c_1)e^{-x} + \left(-\frac{2}{3}e^{3x} + c_2\right)e^{-2x} \qquad 1 - \frac{2}{3} = \frac{3-2}{3} = \frac{1}{3}$$

$$= c_1e^{-x} + c_2e^{-2x} + e^{x} - \frac{2}{3}e^{x}$$

$$= c_1e^{-x} + c_2e^{-2x} + \frac{1}{3}e^{x}$$

Q → $y'' + 16y = 32\sec 2x$; C.F $= A\cos 4x + B\sin 4x$, $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 4$

$$A(x) = -\int \frac{g(x)\,y_2(x)}{W}\,dx + c_1 = -\int \frac{32\sec 2x \cdot \sin 4x\,dx}{4} + c_1 \qquad g(x) = r(x)/a_0(x) = 32\sec 2x$$

$\therefore \sin 2x = 2\sin x\cos x$

$$= -16\int \sin 2x\,dx + c_1$$

$\therefore \sin 4x = 4\sin 2x\cos 2x$

$$B(x) = \int \frac{g(x)\,y_1(x)}{W}\,dx + c_2$$

$$= \frac{1}{4}\int 32\sec 2x \cos 4x\,dx + c_2$$

$$= 8\int \frac{2\cos^2 2x - 1}{\cos 2x}\,dx + c_2$$

$$= 8\int (2\cos 2x - \sec 2x)\,dx + c_2$$

$$= 8\sin 2x - 4\ln|\sec 2x + \tan 2x| + c_2$$

$$y(x) = A(x)\cos 4x + B(x)\sin 4x$$

$$= c_1\cos 4x + c_2\sin 4x + 8\cos 2x\cos 4x + 8\sin 2x\sin 4x - 4\sin 4x\ln|\sec 2x + \tan 2x|$$

$$= c_1\cos 4x + c_2\sin 4x + 8\cos 2x - 4\sin 4x\ln|\sec 2x + \tan 2x|$$

... of system of equations ...

od of Reduction of order for Variable coefficients] now need

DE Second Linear Homogeneous Second Order Equations.

- Second order homogeneous LDE is given by

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad a_0(x) \neq 0 \text{ on } I. \quad (1)$$

Let $y = y_1(x)$ be a non-trivial sol^n of eq^n (1),

Now we need to find $y_2(x)$ be another sol^n of (A).

Consider $y_2 = u(x) \cdot y_1(x)$

where $u(x) = \int v(x)\,dx$

and $v(x) = \dfrac{1}{y_1^2} e^{-\int p(x)\,dx} = e^{-\int \frac{a_1(x)}{a_0(x)}dx}$, $p(x) = \dfrac{a_1(x)}{a_0(x)}$.

Ques: Let $\frac{1}{x}$ is a sol^n of the DE $x^2y'' + 4xy' + 2y = 0$. Find the second linearly independent sol^n & write the G.S.

Sol^n Let $y_1(x) = \dfrac{1}{x}$ (given)

$W(x) = ce^{-\int \frac{a_1(x)}{a_0(x)}dx}$

Wronskian.

Now, $y_2(x) = u(x) \cdot y_1(x)$

$u(x) = \int v(x)$

& $v(x) = \dfrac{1}{y_1^2} e^{-\int p(x)dx} = \dfrac{1}{y_1^2} e^{-\int \frac{4x}{x^2}dx}$

$= \dfrac{1}{\cancel{y}x^2} e^{-\int \frac{4}{x}dx} = \dfrac{1}{x^2} e^{-4\log x} = \dfrac{1}{x^2} e^{-\log x^4}$

$= \dfrac{1}{\cancel{y}x^2} \cdot \left(\dfrac{1}{x^4}\right) = x^2 \cdot \dfrac{1}{x^4} = \dfrac{1}{x^2}$

$v(x) = \dfrac{1}{x^2}$

$u(x) = \int v(x)\, dx = \int \frac{1}{x^2} dx = \frac{-1}{x}$

$y_2(x) = u(x) \cdot y_1(x) = \frac{-1}{x} \cdot \frac{1}{x} = \frac{-1}{x^2}$

$G.S \Rightarrow y(x) = c_1 y_1(x) + c_2 y_2(x)$

$= \boxed{c_1 \cdot \frac{A}{x} + \cancel{-} c_2 \cdot \frac{1}{x^2}}$ sol^n

② $(x-2)y'' - xy' + 2y = 0, \quad x \neq 2, \quad y_1(x) = e^x$ is given

Let $y_2(x) = u(x) \cdot y_1(x)$

& $u(x) = \int v(x) dx$ $\qquad p(x) = \frac{a_1(x)}{a_0(x)} = \frac{-x}{x-2}$

& $v(x) = \frac{1}{y_1^2} e^{-\int p(x) dx}$

$\frac{x}{x-2} = \frac{x-2+2}{x-2}$

$v(x) = \frac{1}{e^{2x}} e^{+\int \frac{x}{x-2} dx}$

$= 1 + \frac{2}{x-2}$

$= \frac{1}{e^{2x}} e^{\int \left[1 + \frac{2}{x-2}\right] dx}$

$= \frac{1}{e^{2x}} e^{\left[x + 2\log(x-2)\right]}$

$= \frac{1}{e^{2x}} \left[e^{x} \cdot e^{2\log(x-2)}\right]$

$= \frac{1}{e^{2x}} \left[e^{x} \cdot e^{\log(x-2)^2}\right]$

$= \frac{1}{e^{x}} \cdot (x-2)^2 = (x-2)^2 \cdot e^{-x}$

$e^{-x}(x-2)^2 + \int 2(x-2) e^{-x} dx$

$e^{-x}(x-2)^2 + 2\left[\int x e^{-x} dx - 2\int e^{-x}\right]$

$= 2\left[x e^{-x} - e^{-x}\right.$

$u(x) = \int e^{-x}(x-2)^2 dx = -e^{-x}\left[x^2 - 2x + 2\right]$

$y_2(x) = \cancel{e^{x}(x-2)^2 \cdot e^{x}}$

$= -(x^2 - 2x + 2)$

Ques 2: $y'' + 3y' - 4y = 0, \quad y_1 = e^x$