Matrices

Definition: An m x n matrix is an arrangement of mn objects (not necessarily distinct) in m rows and n columns in the form

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

We say that the matrix is of order m x n (m by n). The objects $a_{11}, a_{12}, \ldots, a_{mn}$, are called the elements of the matrix.

Each element of the matrix can be a real or a complex number or a function of one more variables or any other object. The element a; which is common to the i th row and the j th column is called its general element. The matrices are usually denoted by boldface uppercase letters , C, ... etc.

When the order of the matrix is understood, we can simply write $A = [a_{ij}]$. If all elements of a matrix are real, it is called a real matrix, whereas if one or more elements of a are complex it is called a complex matrix.

Types of Matrices

1. Row Vector: A matrix of order 1 x n that is, it has one row and n column is called row matrix or row vector. And it can be written as $[a_{11}, a_{12}, \ldots, a_{1n}]$ in which a_{1j} is the j th element.

Q. What is the order of row vector ?

1. Column vector: A matrix of order mx1, that is, it has m row and one column is called column vector or column matrix of order m and is written as $\begin{bmatrix} \\ a_{m1} \end{bmatrix}$

Q. What is the order of column vector ?

3. Rectangular matrix: A matrix A of order m x n, $m \neq n$ is called a rectangular matrix.

4. Square matrices: A matrix A of order m x n in which m = n, that is number of rows is equal to the number of columns is called a square matrix of order n.

. diagonal elements

. principal diagonal

. off-diagonal elements.

. Trace of the matrix.

5. Null matrix: A matrix A of order mxn in which all the elements are zero is called a null matrix or a zero matrix and is denoted by 0

A. What is order of the null matrix

1. Diagonal Matrix: A square matrix A in which all the off-diagonal elements $a_{ij},\ i \neq j$ are zero is called a diagonal matrix. For example

2. Unit Matrix
3. Equal matrix
4. Sub Matrix
5. Scalar Matrix

Example 1. *Find the values of x, y, z and 'a' which satisfy the matrix equation.*

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

-3, -2, 4, -3
-2,-3,4,3
-3. -2. 4. 3

-3, -2, 4, 3

Matrix Algebra

(i) Multiplication of a matrix by a scalar,
If a matrix is multiplied by a scalar quantity k, then each element is multiplied by k, i.e.

https://www.geogebra.org/m/jaJwgaar

(ii) Addition/subtraction of two matrices,

$$A = \begin{bmatrix} 4 & 2 & 5 \\ 1 & 3 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 4+1 & 2+0 & 5+2 \\ 1+3 & 3+1 & -6+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ 4 & 4 & -2 \end{bmatrix}$$

Note: Only matrices of the same order can be added or subtracted.
(i) Commutative Law: A + B = B + A.
(ii) Associative law: A + (B + C) = (A + B) + C.

Given $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$

Find x, y, z and w.

a. 2,4,1,3
b. 2,1,3,4

c. 4,2,3,4
d. 1,3,2,4

(iii) Multiplication of two matrices.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$AB = \begin{array}{c} \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \times \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} \end{array} = \begin{bmatrix} R_1\,C_1 & R_1\,C_2 \\ R_2\,C_1 & R_2\,C_2 \\ R_3\,C_1 & R_3\,C_2 \end{bmatrix}$$

https://www.geogebra.org/m/Thnmhe5T

PROPERTIES OF MATRIX MULTIPLICATION

1. Multiplication of matrices is not commutative: $AB \neq BA$
2. Matrix multiplication is associative, if conformability is assured. A (BC) = (AB) C
3. Matrix multiplication is distributive with respect to addition. A (B + C) = AB + AC
4. Multiplication of matrix A by unit matrix. AI = IA = A

5. Multiplicative inverse of a matrix exists if $|A| \neq 0$.
$A\,.\,A^{-1} = A^{-1}\,.\,A = I$

6. If A is a square then $A \times A = A^2, A \times A \times A = A^3$.

7. $A^0 = I$
8. $I^n = I$, where n is positive integer.

Some special Matrices

Transpose of the matrix:

If in a given matrix A, we interchange the rows and the corresponding columns, the new matrix obtained is called the transpose of the matrix A and
is denoted by $A' or\ A^T$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}.$$

Symmetric Matrix: A square matrix will be called symmetric, if for all values of i and j,
$a_{ij} = a_{ji}$ i.e., $A = A^T$

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

Skew symmetric matrix: A square matrix is called skew symmetric matrix, if
(1) $a_{ij} = - a_{ji}$ for all values of i and j, or $A^T = - A$
(2) All diagonal elements are zero,

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

Triangular matrix: (Echelon form) A square matrix, all of whose elements below the leading diagonal are zero, is called an upper triangular matrix. A square matrix, all of whose elements above the leading diagonal are zero, is called a lower triangular matrix

e.g.,

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

Upper triangular matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$

Lower triangular matrix

Orthogonal Matrix: A square matrix A is called an orthogonal matrix if the product of the matrix A and the transpose matrix A′ is an identity matrix

e.g. $A.A^T = I$

Ex...???

Note: if | A | = 1, matrix A is proper.

Conjugate matrix:

$$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

Hermitian Matrix:

A square matrix $A = (a_{ij})$ is called Hermitian matrix, if every i-jth element of A is equal to conjugate complex j-ith element of A. That means

$$a_{ij} = \bar{a}_{ji}$$

$$\begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

Skew-Hermitian Matrix:

A square matrix $A = (a_{ij})$ will be called a Skew Hermitian matrix if every i-jth element of A is equal to negative conjugate complex of j-ith element of A. That means

$$a_{ij} = -\bar{a}_{ji}$$

$$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$$

Note: All the diagonal elements of a Skew Hermitian Matrix are either zeros or pure imaginary

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Matrix A^{θ}: Transpose of the conjugate of a matrix A is denoted by A^{θ}.

Note: Necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^{\theta}$ i.e. conjugate transpose of A

$$\Longrightarrow \qquad A = (\bar{A})^{T}$$

Unitary Matrix: A square matrix A is said to be unitary if A^{θ} A = I

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

Idempotent Matrix: A matrix, such that $A^2 = A$ is called

Idempotent Matrix.

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}.$$

Periodic Matrix: A matrix A will be called a Periodic Matrix, if $A^{k+1} = A$ where k is a +ve integer. If k is the least + ve integer, for which $A^{k+1} = A$, then k is said to be the period of A. If we choose k = 1, we get $A^2 = A$ and we call it to be idempotent matrix. For ex.

$$\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

Nilpotent Matrix: A matrix will be called a Nilpotent matrix, if $A^k = 0$ (null matrix) where
k is a +ve integer ; if however k is the least +ve integer for which $A^k = 0$, then k is the index
of the nilpotent matrix.

$$, A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

Involuntary Matrix: A matrix A will be called an Involuntary matrix, if $A^2 = I$ (unit matrix). Since $I^2 = I$ always

So we can say: Unit matrix is involuntary.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$A = \begin{bmatrix} 4 & -1 \\ 15 & -4 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

Singular Matrix: If the determinant of the matrix is zero, then the matrix is known as singular matrix