* Simultaneous differential equations using operator method

Eg: Find the solution of the system of equations

$$\frac{dy_1}{dt} + 2\frac{dy_2}{dt} - 2y_1 - y_2 = e^{2t}$$

$$\frac{dy_2}{dt} + y_1 - 2y_2 = 0 \quad (*)$$

Sol: ~~Sol~~ Equations in operator form

$$Dy_1 + 2Dy_2 - 2y_1 - y_2 = e^{2t}$$

$$Dy_2 + y_1 - 2y_2 = 0$$

OR

$$(D-2)y_1 + (2D-1)y_2 = e^{2t} \quad (1)$$

$$y_1 + (D-2)y_2 = 0 \quad (2)$$

Multiply eq (2) by $(D-2)$,

$$(D-2)y_1 + (D-2)^2 y_2 = 0 \quad (3)$$

Subtracting eq (3) from (1), we get

$$\cancel{(D-2)y_1} + (2D-1)y_2 = e^{2t}$$

$$\underline{\cancel{(D-2)y_1} + (D-2)^2 y_2 = 0}$$

$$(2D-1)y_2 - (D-2)^2 y_2 = e^{2t}$$

or $[2D-1-(D-2)^2]y_2 = e^{2t}$

$$(2D-1-D^2+4+4D)y_2 = e^{2t}$$

$$(-D^2+6D-5)y_2 = e^{2t}$$

$$(D^2-6D+5)y_2 = -e^{2t}$$

$$m^2-6m+5=0$$

$$(m-1)(m-5)=0$$

$$m = 1, 5$$

$$(y_2)_c = c_1 e^{t} + c_2 e^{5t}$$

$$(y_2)_p = -\frac{1}{D^2-6D+5} e^{2t}$$

$$= -\frac{1}{4-12+5} e^{2t}$$

$$= -\frac{1}{-3} e^{2t} = \frac{e^{2t}}{3}$$

$$y_2(t) = y_2(c) + y_2(p)$$

$$= c_1 e^{t} + c_2 e^{5t} + \frac{e^{2t}}{3}$$

from (*)

$$y_1 = 2y_2 - \frac{dy_2}{dt}$$

$$= 2\left(c_1 e^{t} + c_2 e^{5t} + \frac{e^{2t}}{3}\right) - \left(c_1 e^{t} + 5c_2 e^{5t} + \frac{1}{3}e^{2t}\right)$$

$$\Rightarrow y_1(t) = 2c_1 e^{t} + 2c_2 e^{5t} + \frac{2}{3}e^{2t} - c_1 t - 5c_2 e^{5t} - \frac{2}{3}e^{2t}$$

$$\boxed{y_1(t) = c_1 t - 3c_2 e^{5t}}$$

Eg:

Find the solution of $3\frac{dy_1}{dt} + 3y_1 + 3\frac{dy_2}{dt}$ $\frac{dy_1}{dt} + \frac{dy_2}{dt}$

$$(3D+1)y_1 + 3Dy_2 = 3t+1 \quad (1)$$

$$(D-3)y_1 + Dy_2 = 2t \quad (2)$$

Multiply (2) by '3', we get

$$(3D-9)y_1 + 3Dy_2 = 6t \quad (3)$$

Subtracting (3) from (1)

$$(3D+1)y_1 + \cancel{3Dy_2} = 3t+1$$

$$\underline{(3D-9)y_1 + \cancel{3Dy_2} = 6t}$$

$$(3D+1-3D+9)y_1 = -3t+1$$

$$10y_1 = -3t+1$$

$$\Rightarrow y_1 = \frac{1}{10}(1-3t)$$

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$$Dy_1 + 2Dy_2 - 2y_1 - y_2 = e^{2t}$$

$$Dy_2 + y_1 - 2y_2 = 0$$

OR $(D-2)y_1 + (2D-1)y_2 = e^{2t}$ — (1)

$y_1 + (D-2)y_2 = 0$ — (2)

Multiply eq (2) by $(D-2)$,

$(D-2)y_1 + (D-2)^2 y_2 = 0$ — (3)

Subtracting eq (3) from (1), we get

$$\cancel{(D-2)y_1} + (2D-1)y_2 = e^{2t}$$

$$\underline{-\cancel{(D-2)y_1} - (D-2)^2 y_2 = 0}$$

$$(2D-1)y_2 - (D-2)^2 y_2 = e^{2t}$$

or $[2D-1-(D-2)^2]y_2 = e^{2t}$

$$(2D-1-D^2+4+4D)y_2 = e^{2t}$$

$$(-D^2+6D-5)y_2 = e^{2t}$$

$$(D^2-6D+5)y_2 = -e^{2t}$$

$m^2-6m+5=0$

$(m-1)(m-5)=0$

$m=1,5$

$$(y_2)_c = c_1e^{t} + c_2e^{5t}$$

$$(y_2)_p = -\frac{1}{D^2-6D+5}e^{2t}$$

$$= -\frac{1}{4-12+5}e^{2t}$$

$$= -\frac{1}{-3}e^{2t} = \frac{e^{2t}}{3}$$

$$y_2(t) = y_2(c) + y_2(p)$$

$$= c_1e^{t} + c_2e^{5t} + \frac{e^{2t}}{3}$$

from (*)

$$y_1 = 2y_2 - \frac{dy_2}{dt}$$

$$= 2\left(c_1e^{t} + c_2e^{5t} + \frac{e^{2t}}{3}\right) - \left(c_1e^{t} + 5c_2e^{5t} + \frac{1}{3}e^{2t}\right)$$

$$\Rightarrow y_1(t) = 2c_1e^{t} + 2c_2e^{5t} + \frac{2}{3}e^{2t} - c_1t - 5c_2e^{5t} - \frac{2}{3}e^{2t}$$

$$\boxed{y_1(t) = c_1t - 3c_2e^{5t}}$$

Eg:

Find the solution of $3\frac{dy_1}{dt} + 3y_1 + 3\frac{dy_2}{dt}$ $\frac{dy_1}{dt} + \frac{dy_2}{dt}$

$(3D+1)y_1 + 3Dy_2 = 3t+1$ — (1)

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$(3D-9)y_1 + 3Dy_2 = 6t$ — (3)

Subtracting (3) from (1)

$$(3D+1)y_1 + \cancel{3Dy_2} = 3t+1$$

$$\underline{-(3D-9)y_1 - \cancel{3Dy_2} = -6t}$$

$$(3D+1-3D+9)y_1 = -3t+1$$

$$10y_1 = -3t+1$$

$$\Rightarrow y_1 = \frac{1}{10}(1-3t)$$

$$\Rightarrow y_1(t) = \frac{1}{10}(1-3t)$$

from ②,

$$\mathcal{D}y_2 = 2t - (\mathcal{D}-3)y_1$$

$$= 2t - (\mathcal{D}-3)\left(\frac{1}{10}(1-3t)\right)$$

$$= 2t - \mathcal{D}\left(\frac{1-3t}{10}\right) + 3\left(\frac{1-3t}{10}\right)$$

$$= 2t - \left(\frac{-3}{10}\right) + \frac{3}{10} - \frac{9}{10}t$$

$$= 2t - \frac{9}{10}t + \frac{3}{10} + \frac{3}{10}$$

$$\mathcal{D}y_2 = \frac{11}{10}t + \frac{6}{10}$$

$$y_2 = \frac{1}{\mathcal{D}}\left(\frac{11}{10}t + \frac{6}{10}\right)$$

$$= \frac{11}{10} \times \frac{t^2}{2} + \frac{6}{10}t + C$$

$$y_2(t) = \frac{11}{20}t^2 + \frac{6}{10}t + C$$

=o=

Eg: $(2\mathcal{D}-4)y_1 + (3\mathcal{D}+5)y_2 = 3t+2$ — ①

$(\mathcal{D}-2)y_1 + (\mathcal{D}+1)y_2 = t$ — ②

Multiply eq ② by '2' & subtract from ①

$$\cancel{(2\mathcal{D}-4)}y_1 + (3\mathcal{D}+5)y_2 = 3t+2$$

$$\cancel{(2\mathcal{D}-4)}y_1 + (2\mathcal{D}+2)y_2 = 2t$$

$$(\mathcal{D}+3)y_2 = t+2.$$

$$(\mathcal{D}+3)y_2 = t+2$$

C.F. $m+3=0$

$m=-3$

$$(y_2)_c = C_1 e^{-3t}$$

$$(y_2)_p = \frac{1}{\mathcal{D}+3}\, t+2$$

$$= \frac{1}{3\left[1+\frac{\mathcal{D}}{3}\right]}(t+2)$$

$$= \frac{1}{3}\left[1+\frac{\mathcal{D}}{3}\right]^{-1}(t+2)$$

$$= \frac{1}{3}\left[1-\frac{\mathcal{D}}{3}+\frac{\mathcal{D}^2}{9}+\cdots\right](t+2)$$

$$= \frac{1}{3}\left[(t+2)-\frac{1}{3}(1)+\frac{1}{9}(0)+0\right]$$

$$= \frac{1}{3}\left[t+2-\frac{1}{3}\right] = \frac{1}{3}\left[t+\frac{5}{3}\right]$$

$\therefore y_2(t) = (y_2)_c + (y_2)_p$

$$\boxed{y_2(t) = C_1 e^{-3t} + \frac{1}{9}[3t+5]}$$

Substituting in ②

$$(\mathcal{D}-2)y_1 + \mathcal{D}y_2 + y_2 = t$$

$$\Rightarrow (\mathcal{D}-2)y_1 + \left(-3C_1e^{-3t} + \frac{1}{9}(3)\right) + C_1e^{-3t} + \frac{1}{9}(3t+5) = t$$

$$\Rightarrow (\mathcal{D}-2)y_1 = 2C_1e^{-3t} - \frac{8}{9} + \frac{2}{3}t$$

$$\boxed{y_1(c) = C_2 e^{2t}} \qquad \left[\because \begin{matrix} m-2=0 \\ m=2 \end{matrix}\right]$$

$$y_1(p) = \frac{1}{\mathcal{D}-2}\left[2C_1e^{-3t} + \frac{2}{3}t - \frac{8}{9}\right]$$

$$= \frac{2C_1}{-5}e^{-3t} + \frac{1}{0-2}\left(\frac{-8}{9}e^{0x}\right) + \frac{2}{3}\frac{1}{-2}\left[1-\frac{\mathcal{D}}{2}\right]^{-1}t$$

$$= -\frac{2}{5}C_1e^{-3t} + \frac{4}{9} - \frac{1}{3}\left[1+\frac{\mathcal{D}}{2}+\frac{\mathcal{D}^2}{2}\right]t$$

$$= -\frac{2}{5}C_1e^{-3t} + \frac{4}{9} - \frac{1}{3}t - \frac{1}{6}$$

$$\therefore y_1(t) = C_2e^{2t} - \frac{2}{5}C_1e^{-3t} + \frac{(5-6t)}{18}$$

Method of Reduction of order

Linear Homogeneous Second Order Equations

Second order homogeneous LDE is given by

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad , \quad a_0(x) \neq 0 \text{ on } I$$

Let $y = y_1(x)$ be a non-trivial solution of equation ①,

Now, we need to find $y_2(x)$ (another solution of eq ①)

Consider $y_2(x) = u(x).y_1(x)$

where $u(x) = \int v(x)dx$

and $v(x) = \frac{1}{y_1^2} e^{-\int p(x)dx}$, where $p(x) = \frac{a_1(x)}{a_0(x)}$

Q: Let $\frac{1}{x}$ is a solution of the DE $x^2y'' + 4xy' + 2y = 0$. Find the second linearly independent solution and write the General Solution.

Sol: Given $y_1(x) = \frac{1}{x}$ given eq: $x^2y'' + 4xy' + 2y = 0$ —— ①

Let $y_2(x)$ is the second sol. of eq. ①

Now, $y_2(x) = u(x).y_1(x)$

& $u(x) = \int v(x)dx$

where $v(x) = \frac{1}{y_1^2} . e^{-\int p(x)dx}$

In given eq: $a_1(x) = 4x$, $a_0(x) = x^2$ $\Rightarrow$ $p(x) = \frac{a_1(x)}{a_0(x)} = \frac{4x}{x^2} = \frac{4}{x}$

$\therefore v(x) = \frac{1}{y_1^2} . e^{-\int \frac{4}{x}dx} = \frac{1}{\frac{1}{x^2}} . e^{-4\int \frac{1}{x}dx}$

~~Sole Regularity (In Lawrence)~~.

$= x^2 \cdot e^{-4\log x}$

$= x^2 \cdot e^{\log x^{-4}} = x^2 \cdot x^{-4} = x^2 \cdot \frac{1}{x^4} = \frac{1}{x^2}$

$\Rightarrow v(x) = \frac{1}{x^2}$

Now $u(x) = \int v(x)\,dx = \int \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1} = \frac{-1}{x}$

& $y_2(x) = u(x) \cdot y_1(x)$

$= \frac{-1}{x} \cdot \frac{1}{x} = -\frac{1}{x^2} \Rightarrow \boxed{y_2(x) = -\frac{1}{x^2}}$

$\therefore$ G.S is

$y(x) = C_1 y_1(x) + C_2 y_2(x)$

$= C_1 \cdot \frac{1}{x} + C_2\left(-\frac{1}{x^2}\right)$

Q2: $(x-2)y'' - xy' + 2y = 0$, $x \neq 2$; $y_1(x) = e^x$ is given.

~~Let~~ Given

$(x-2)y'' - xy' + 2y = 0$ — ① ; $y_1(x) = e^x$

$\Rightarrow a_0(x) = x-2,\quad a_1(x) = -x \Rightarrow p(x) = \frac{a_1(x)}{a_0(x)} = \frac{-x}{x-2}$

Let

$y_2(x) = u(x) \cdot y_1(x)$

$u(x) = \int v(x)$

& $v(x) = \frac{1}{y_1^2} \cdot e^{-\int p(x)\,dx} = \frac{1}{y_1^2}\, e^{+\int \frac{x}{x-2}\,dx}$

$\frac{x}{x-2} = \frac{x-2+2}{x-2} = 1 + \frac{2}{x-2}$

$= \frac{1}{e^{2x}}\, e^{\int\left(1+\frac{2}{x-2}\right)dx}$

$= \frac{1}{e^{2x}}\, e^{[x + 2\log(x-2)]}$

★ Method of Variat...
(i) ...

$$\frac{e^{x}\cdot e^{2\log(x-2)}}{e^{2x}}$$

$$= \frac{e^{\log(x-2)^2}}{e^{x}}$$

$$v(x) = \frac{(x-2)^2}{e^x}$$

$$\therefore u(x) = \int v(x)\,dx$$

$$= \int \frac{(x-2)^2}{e^x}\,dx$$

$$= \int (x-2)^2 \cdot e^{-x}\,dx$$

$$= (x-2)^2 \cdot \frac{e^{-x}}{-x} - \int 2(x-2)\cdot\frac{e^{-x}}{-1}\,dx$$

$$= -(x-2)^2 e^{-x} + 2\int (x-2)e^{-x}\,dx$$

$$= -(x-2)^2 e^{-x} + 2\left\{(x-2)\frac{e^{-x}}{-1} - \int 1\cdot\frac{e^{-x}}{-1}\,dx\right\}$$

$$= -(x-2)^2 e^{-x} + 2\left\{-(x-2)e^{-x} + \int e^{-x}\,dx\right\}$$

$$= -(x-2)^2 e^{-x} + 2\left\{-(x-2)e^{-x} + \frac{e^{-x}}{-1}\right\}$$

$$= -(x-2)^2 e^{-x} - 2(x-2)e^{-x} - 2e^{-x}$$

$$= -e^{-x}\left[(x-2)^2 + 2(x-2) + 2\right]$$

$$= -e^{-x}\left[x^2 + 4 - 4x + 2x - 4 + 2\right]$$

$$u(x) = -e^{-x}\left[x^2 - 2x + 2\right]$$

Now, $y_2(x) = u(x)\cdot y_1(x)$

$$= -e^{-x}\left[x^2 - 2x + 2\right] \times e^{x}$$

$$= -(x^2 - 2x + 2)$$

$\therefore$ g.s. is

$$y = c_1 y_1(x) + c_2 y_2(x)$$

$$\Rightarrow \boxed{y = c_1 e^{x} + c_2(-(x^2 - 2x + 2))}$$

Q3 $y'' + 3y' - 4y = 0$, $y_1 = e^x$
find general sol.

Sol: Given $y'' + 3y' - 4y = 0$ — ①

; $y_1(x) = e^x$

from ①, $a_0 = 1$, $a_1 = 3$

$$\Rightarrow p(x) = \frac{a_1}{a_0} = \frac{3}{1} = 3$$

Now, Let

$$y_2(x) = u(x)\cdot y_1(x)$$

$$u(x) = \int v(x)\,dx$$

$$v(x) = \frac{1}{y_1^2}\cdot e^{-\int p(x)\,dx}$$

$$= \frac{1}{e^{2x}}\, e^{-\int 3\,dx}$$

$$= \frac{1}{e^{2x}}\, e^{-3x}$$

$$\boxed{v(x) = e^{-5x}}$$

Now

$$u(x) = \int v(x)\,dx$$

$$= \int e^{-5x}\,dx$$

$$= \frac{e^{-5x}}{-5} = -5e^{-5x}$$

...efficients
...pecial Cases
...es can
...n.

Now

$y_2(x) = u(x) \cdot y_1(x)$

$= -5e^{-5x} \cdot e^{x}$

$= -5e^{-4x}$

$\therefore$ G.S is

$y = C_1 y_1(x) + C_2 y_2(x)$

$= C_1 e^{x} + C_2 (-5e^{-4x})$

or

$= C_1 e^{x} + C_3 e^{-4x}$; where $\boxed{C_3 = -5C_2}$