

ADJOINT OF A SQUARE MATRIX

Let the determinant of the square matrix A be $|A|$.

$$\text{If } A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad \text{Then } |A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

The matrix formed by the co-factors of the elements in

$$|A| \text{ is } \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}.$$

$$\begin{aligned} \text{where } A_1 &= \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = b_2c_3 - b_3c_2, & A_2 &= -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -b_1c_3 + b_3c_1 \\ A_3 &= \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1, & B_1 &= -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = -a_2c_3 + a_3c_2 \\ B_2 &= \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1, & B_3 &= -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = -a_1c_2 + a_2c_1 \\ C_1 &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2, & C_2 &= -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -a_1b_3 + a_3b_1 \\ C_3 &= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \end{aligned}$$

Then the transpose of the matrix of co-factors

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}^T \text{ is called the adjoint of the matrix } A \text{ and is written as } \text{adj } A.$$

Ex. Find the adjoint of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

PROPERTY OF ADJOINT MATRIX

(1) $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = |A| \cdot I_n$ where, A is a square matrix, I is an identity matrix of same order.

(2) if A is invertible square matrix

$$|adj(A)| = |A|^{n-1}$$

(3) if A is invertible square matrix

$$adj(adj(A)) = |A|^{n-2} \cdot \underline{A}$$

if $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is adjoint of 3×3 matrix with $|A| = 4$ then the value of α is

- a. 2
- b. 11
- c. 13
- d. $1/3$

$$\begin{aligned} |adj A| &= |A|^n \\ 1(0) - \alpha(-2) + 3(-2) &= (4)^2 \\ 2\alpha - 6 &= 16 \Rightarrow 2\alpha = 22 \\ \alpha &= 11 \end{aligned}$$

$$(4) adj(AB) = adj(B) \cdot adj(A)$$

$$(5) (adj(A))^T = adj(A^T)$$

$$(6) adj(kA) = k^{n-1} adj(A)$$

INVERSE OF A MATRIX

If A and B are two square matrices of the same order, such that $AB = BA = I$ (I = unit matrix) then B is called the inverse of A i.e. $B = A^{-1}$ and A is the inverse of B .

To find the inverse of the Matrix A we use

$$A^{-1} = \frac{1}{|A|} (Adj(A)), \quad \text{if } |A| \neq 0$$

1. A is the **coefficient matrix**, b the **right hand side**, and x is the **solution vector**.
2. If b not **equal to zero** system is called **non-homogeneous**.
3. If b is **zero** its call **homogeneous**.
4. The system of equations is called **consistent** if it **has at least one solution**.
otherwise the system is **inconsistent**.

Homogeneous system of equations:

Consider the homogeneous system of equations $Ax = 0$

Trivial solution $x = 0$ is always a solution of this system.

If A is non singular, then $x = A^{-1}0 = 0$ is the solution.

Thus $Ax = 0$ is the **always consistent**.

We conclude that non-trivial solution for $Ax = 0$ **exist if and only if** A is singular, in this case this system has **infinite** solutions.

Ex. Solve the system of the equation using matrix method $x - y + z = 0$, $2x + y - 3z = 0$, $x + y + z = 0$

Ex. If the system of the equations $x - ky - z = 0$, $kx - y - z = 0$, $x + y - z = 0$ has non-zero solution then values of K are

- a. -1,2
- b. 0,1
- c. 1,1
- d. -1,1

Ex. If the system of the equations $kx + y + z = 0$, $-x + ky + z = 0$, $-x - y + kz = 0$ has non-zero solution then value of K is

- a. 0
- b. 1
- c. -1
- d. 2

Solution of Non-homogeneous system of equations

The non-homogeneous system of equations $Ax = b$ can be solved by the following methods

(i) Matrix method

(ii) Cramer's Rule

(i) Matrix method:

Let A be non-singular, then pre-multiplying $Ax = b$ by

A^{-1} , we obtain

$$x = A^{-1}b$$

Ex. Solve the system of the equation using matrix method $x - y + z = 4$, $2x + y - 3z = 0$, $x + y + z = 2$

Ex. Solve the system of the equation using matrix method $-x + y + 2z = 2$, $3x - y + z = 3$, $-x + 3y + 4z = 6$

Ex. Solve the system of the equation using matrix method $2x - z = 1$, $5x + y = 7$, $y + 3z = 5$

(ii) **Cramer's Rule:**

Let A be a non-singular matrix then by Cramer's rule solution of $Ax=b$ is given by

$$x_i = \frac{|A_i|}{|A|}, \quad i = 1, 2, 3, \dots, n$$

Where $|A_i|$ is the determinant of the matrix $|A_i|$ obtained by replacing the i th column of A by the right hand side column vector b .

Ex. Solve the system of the equation using $x - y + z = 4$,
 $2x + y - 3z = 0$, $x + y + z = 2$

Note: We have the following cases in this method

Case 1: when $|A| \neq 0$, the system is consistent and the unique solution is obtained by using the above method.

Case 2: When $|A| = 0$, and one or more of $|A_i|, i = 1, 2, 3, \dots, n$

are not zero then the system of the equations has no solution that is the system is inconsistent.

Case 3: When $|A| = 0$, and all $|A_i| = 0$, $i = 1, 2, 3, \dots, n$, then the system of equations is consistent and has infinite number of solutions. The system of equations has at least a one-parameter family of solutions.

Ex. Solve the system of the equation using $4x + 9y + 3z = 6$,
 $2x + 3y + z = 2$, $2x + 6y + 2z = 7$

Ex. Solve the system of the equation using matrix method

$$x - y + 3z = 3, \quad 2x + 3y + z = 2, \quad 3x + 2y + 4z = 5$$

Ex. The system of linear equations $x + y + z = 2$, $2x + 3y + 2z = 5$, $2x + 3y + (a^2 - 1)z = a + 1$

a. Is inconsistent for $a = 4$

b. Has unique solution for $a = \sqrt{3}$

c. Has infinite solution for $a = 4$

- b. Has unique solution for $a = \sqrt{3}$
- c. Has infinite solution for $a = 4$
- d. Inconsistent for $a = \sqrt{3}$

Ex.

If the system of linear equation $x - 4y + 7z = g$, $3y - 5z = h$, $-2x + 5y - 9z = k$ is consistent, then

- A $g + h + k = 0$
- B $2g + h + k = 0$
- C $g + h + 2k = 0$
- D $g + 2h + 2k = 0$

(iii) Gauss elimination Method for Non-homogeneous System

Let we have the non-homogeneous system $Ax = b$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Now we write the augmented matrix of order $m \times (n + 1)$

$$(A | b) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

Now we can reduce this matrix in to row echelon form using elementary operations

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1n} & b_1 \\ 0 & \bar{a}_{22} & \cdots & \bar{a}_{2r} & \cdots & \bar{a}_{2n} & \bar{b}_2 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & a_{rr}^* & \cdots & a_{rn}^* & b_r^* \\ 0 & 0 & \cdots & 0 & \cdots & 0 & b_{r+1}^* \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & b_m^* \end{array} \right].$$

Ex. Solve following system using gauss elimination

$$(i) \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix},$$

Note:

1. Let $r < m$ and one or more elements $b_{r+1}^*, b_{r+2}^*, \dots, b_m^*$ are not zero. Then $\text{rank}(A) \neq \text{rank}(A|b)$ and the system of equation has no solution .
2. Let $m \geq n$ and $r = n$ (the number of columns in A) and $b_{r+1}^*, b_{r+2}^*, \dots, b_m^*$ are all zero. In this case $\text{rank}(A) = \text{rank}(A|b) = n$ and the system of equations has unique solution.
3. Let $r < n$ and $b_{r+1}^*, b_{r+2}^*, \dots, b_m^*$ are all zero. In this case x_1, x_2, \dots, x_r can be determined in term of remaining $(n-r)$ unknowns $x_{r+1}, x_{r+2}, \dots, x_n$.

$$(ii) \quad \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix},$$

$$(iii) \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}.$$

(iv) Gauss-Jordan method

Ex. Using gauss-Jordan method solve the system of equations $Ax=b$, where

$$[A \mid b] \xrightarrow[\text{row operations}]{\text{Elementary}} [I \mid c]$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & -3 & 4 \\ 1 & 1 & 1 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -5 & 4 \\ 0 & 2 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2/3 \\ \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 2 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + R_2 \\ R_3 - 2R_2 \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 10/3 & -5/3 \end{array} \right] R_3 / (10/3)$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 1 & -1/2 \end{array} \right] \begin{array}{l} R_1 + 2R_3/3 \\ R_2 + 5R_3/3 \end{array} \approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array} \right]$$

$$\mathbf{x} = [1 \quad 1/2 \quad -1/2]^T.$$

Ex. Using Gauss-Jordan method find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$[\mathbf{A} \mid \mathbf{I}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \mathbf{A}^{-1}]$$

1. 1

2. 2