(R18A0021) Mathematics -I

Course Objectives: To learn

1. The concept of rank of a matrix which is used to know the consistency of system of linear equations and also to find the eigen vectors of a given matrix.
2. Finding maxima and minima of functions of several variables.
3. Applications of first order ordinary differential equations. (Newton's law of cooling, Natural growth and decay)
4. How to solve first order linear, non linear partial differential equations and also method of separation of variables technique to solve typical second order partial differential equations.
5. Solving differential equations using Laplace Transforms.

UNIT I: Matrices

Introduction, types of matrices-symmetric, skew-symmetric, Hermitian, skew-Hermitian, orthogonal, unitary matrices. Rank of a matrix - echelon form, normal form, consistency of system of linear equations (Homogeneous and Non-Homogeneous). Eigen values and Eigen vectors and their properties (without proof), Cayley-Hamilton theorem (without proof), Diagonalisation.

UNIT II:Functions of Several Variables

Limit continuity, partial derivatives and total derivative. Jacobian-Functional dependence and independence. Maxima and minima and saddle points, method of Lagrange multipliers, Taylor's theorem for two variables.

UNIT III: Ordinary Differential Equations

First order ordinary differential equations: Exact, equations reducible to exact form. Applications of first order differential equations - Newton's law of cooling, law of natural growth and decay.

Linear differential equations of second and higher order with constant coefficients: Non-homogeneous term of the type $f(x) = e^{ax}$, sinax, cosax, x^n, e^{ax} V and x^n V. Method of variation of parameters.

UNIT IV: Partial Differential Equations

Introduction, formation of partial differential equation by elimination of arbitrary constants and arbitrary functions, solutions of first order Lagrange's linear equation and non-linear equations, Charpit's method, Method of separation of variables for second order equations and applications of PDE to one dimensional (Heat equation).

UNIT V: Laplace Transforms

Definition of Laplace transform, domain of the function and Kernel for the Laplace transforms, Existence of Laplace transform, Laplace transform of standard functions, first shifting Theorem, Laplace transform of functions when they are multiplied or divided by "t", Laplace transforms of derivatives and integrals of functions, Unit step function, Periodic function.

Inverse Laplace transform by Partial fractions, Inverse Laplace transforms of functions when they are multiplied or divided by "s", Inverse Laplace Transforms of derivatives and integrals of functions, Convolution theorem, Solving ordinary differential equations by Laplace transforms.

TEXT BOOKS:

i) Higher Engineering Mathematics by B V Ramana ., Tata McGraw Hill.
ii) Higher Engineering Mathematics by B.S. Grewal, Khanna Publishers.
iii) Advanced Engineering Mathematics by Kreyszig, John Wiley & Sons.

REFERENCE BOOKS:

i)Advanced Engineering Mathematics by R.K Jain & S R K Iyenger, Narosa Publishers.
ii)Advanced Engineering Mathematics by Michael Green Berg, Pearson Publishers .
iii)Engineering Mathematics by N.P Bali and Manish Goyal.

Course Outcomes: After learning the concepts of this paper the student will be able to
1.Analyze the solution of the system of linear equations and to find the Eigen values and Eigen vectors of a matrix.
2.Find the extreme values of functions of two variables with / without constraints.
3.Solve first and higher order differential equations.
4.Solve first order linear and non-linear partial differential equations.
5.Solve differential equations with initial conditions using Laplac

UNIT-I

MATRICES

Matrix : A system of mn numbers real (or) complex arranged in the form of an ordered set of 'm' rows, each row consisting of an ordered set of 'n' numbers between [ ] (or) ( ) (or) || || is called a matrix of order m xn.

Eg: $\begin{bmatrix} a_{11} & a_{12}.........a_{1n} \\ a_{21} & a_{12}.........a_{2n} \\ .... & \\ .... & \\ a_{m1} & a_{m2}.......a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{mxn}$ where $1 \le i \le m$, $1 \le j \le n$.

Order of the Matrix: The number of rows and columns represents the order of the matrix. It is denoted by mxn, where m is number of rows and n is number of columns.

Types of Matrices:

Row Matrix: A Matrix having only one row is called a "Row Matrix".

Eg: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1x3}$

Column Matrix: A Matrix having only one column is called a "Column Matrix" .

Eg: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3x1}$

Null Matrix: A= $\left[a_{ij}\right]_{mxn}$ such that $a_{ij}=0$ $\forall$ i and j. Then A is called a "Zero Matrix". It is denoted by Omxn.

Eg: $O_{2x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Rectangular Matrix: If A=$\left[a_{ij}\right]_{mxn}$, and m$\neq$n then the matrix A is called a "Rectangular Matrix"

Eg : $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2x3 matrix

Square Matrix: If A=$\left[a_{ij}\right]_{mxn}$ and m = n then A is called a "Square Matrix".

Eg : $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is a 2x2 matrix

Lower Triangular Matrix: A square Matrix $A_{nXn} = \left[a_{ij}\right]_{nxn}$ is said to be lower Triangular of $a_{ij} = 0$ if i <j i.e. if all the elements above the principle diagonal are zeros.

Eg: $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 6 \end{bmatrix}$ is a Lower triangular matrix

Upper Triangular Matrix: A square Matrix $A = \left[a_{ij}\right]_{nxn}$ is said to be upper triangular of $a_{ij} = 0$ if i.>j. i.e. all the elements below the principle diagonal are zeros.

Eg: $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$ is an Upper triangular matrix

Triangle Matrix: A square matrix which is either lower triangular or upper triangular is called a triangle matrix.

Principal Diagonal of a Matrix: In a square matrix, the set of all a_{ij}, for which i = j are called principal diagonal elements. The line joining the principal diagonal elements is called principal diagonal.

Note: Principal diagonal exists only in a square matrix.

Diagonal elements in a matrix: A= $[a_{ij}]_{nxn}$, the elements a_{ij} of A for which i = j.

i.e. a_{11}, a_{22}….a_{nn} are called the diagonal elements of A

Eg: A= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ diagonal elements are 1, 5, 9

Diagonal Matrix: A Square Matrix is said to be diagonal matrix, if $a_{ij} = 0$ for $i \neq j$ i.e. all the elements except the principal diagonal elements are zeros.

Note: 1. Diagonal matrix is both lower and upper triangular.

2. If d_1, d_2…….d_n are the diagonal elements in a diagonal matrix it can be represented as **diag** $[d_1, d_2, \ldots\ldots, d_n]$

Eg : A = diag (3,1,-2)= $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Scalar Matrix: A diagonal matrix whose leading diagonal elements are equal is called a "Scalar Matrix". Eg : A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Unit/Identity Matrix: If A = $\left[a_{ij}\right]_{nxn}$ such that a_{ij}=1 for i = j, and a_{ij}=0 for $i \neq j$ then A is called a "Identity Matrix" or Unit matrix. It is denoted by I_n

Eg: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Trace of Matrix: The sum of all the diagonal elements of a square matrix A is called Trace of a matrix A, and is denoted by Trace A or tr A.

Eg : A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then tr A = a+b+c

Singular & Non Singular Matrices: A square matrix A is said to be “Singular” if the determinant of $|A| = 0$, Otherwise A is said to be “Non-singular”.

Note: 1. Only non-singular matrices possess inverse.

2. The product of non-singular matrices is also non-singular.

Inverse of a Matrix: Let A be a non-singular matrix of order n if there exist a matrix B such that AB=BA=I then B is called the inverse of A and is denoted by A^{-1}.

If inverse of a matrix exist, it is said to be invertible.

Note: 1. The necessary and sufficient condition for a square matrix to posses inverse is that $|A| \neq 0$.

2 .Every Invertible matrix has unique inverse.

3. If A, B are two invertible square matrices then AB is also invertible and

$(AB)^{-1} = B^{-1}A^{-1}$

4. $A^{-1} = \dfrac{AdjA}{\det A}$ $\quad where\ detA \neq 0$,

Theorem: The inverse of a Matrix if exists is Unique.

Note: 1. $(A^{-1})^{-1} = A$ 2. $I^{-1} = I$

Theorem: If A, B are invertible matrices of the same order, then

(i). $(AB)^{-1} = B^{-1}A^{-1}$

(ii). $(A^1)^{-1} = (A^{-1})^1$

Sub Matrix: - A matrix obtained by deleting some of the rows or columns or both from the given matrix is called a sub matrix of the given matrix.

Eg: Let A = $\begin{bmatrix} 1 & 5 & 6 & 7 \\ 8 & 9 & 10 & 5 \\ 3 & 4 & 5 & -1 \end{bmatrix}$. Then $\begin{bmatrix} 8 & 9 & 10 \\ 3 & 4 & 5 \end{bmatrix}_{2x3}$ is a sub matrix of A obtained by deleting first row and 4th column of A.

Minor of a Matrix: Let A be an mxn matrix. The determinant of a square sub matrix of A is called a minor of the matrix.

Note: If the order of the square sub matrix is ‘t’ then its determinant is called a minor of order ‘t’.

Eg: A = $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$ be a 4x3 matrix

Here B = $\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is a sub-matrix of order '2'

$|B|$ = 2-3 = -1 is a minor of order '2

And C = $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}$ is a sub-matrix of order '3'

det C = 2(7-12)-1(21-10)+(18-5) = -9

Properties of trace of a matrix: Let A and B be two square matrices and λ be any scalar

1) tr (λ A) = λ (tr A) ; 2) tr(A+B) = trA + trB ; 3) tr (AB) = tr(BA)

Idempotent Matrix: A square matrix A Such that A^2=A then A is called "Idempotent Matrix".

Eg: A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Involutary Matrix: A square matrix A such that A^2 = I then A is called an Involutary Matrix.

Eg: A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Nilpotent Matrix: A square matrix A is said to be Nilpotent if there exists a + ve integer n such that A^n = 0 here the least n is called the Index of the Nilpotent Matrix.

Eg: A = $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Transpose of a Matrix: The matrix obtained by interchanging rows and columns of the given matrix A is called as transpose of the given matrix A. It is denoted by A^T or A^1

Eg: A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Then A^T = $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Properties of transpose of a matrix: If A and B are two matrices and A^T , B^T are their transposes then

1) $\left(A^T\right)^T = A$; 2) $\left(A+B\right)^T = A^T + B^T$; 3) $\left(KA\right)^T = KA^T$; 4) $\left(AB\right)^T = B^T A^T$

Symmetric Matrix: A square matrix A is said to be symmetric if $A^T = A$

If $A = \left[a_{ij}\right]_{nxn}$ $then\ A^T = \left[a_{ji}\right]nxn$ where $a_{ij} = a_{ji}$

Eg: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

Skew-Symmetric Matrix: A square matrix A is said to be Skew symmetric If $A^T = -A$.

If $A = \left[a_{ij}\right]_{nxn}$ *then* $A^T = \left[a_{ji}\right]_{nxn}$ *where* $a_{ij} = -a_{ji}$.

Eg : $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

Note: All the principle diagonal elements of a skew symmetric matrix are always zero.

Since $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

Theorem: Every square matrix can be expressed uniquely as the sum of symmetric and skew symmetric matrices.

Proof: Let A be a square matrix, A = $\frac{1}{2}(A+A) = \frac{1}{2}\left(A+A^T+A-A^T\right) =$

$\frac{1}{2}\left(A+A^T\right)+\frac{1}{2}\left(A-A^T\right)$ = P + Q, where P = $\frac{1}{2}\left(A+A^T\right)$; Q = $\frac{1}{2}\left(A-A^T\right)$

Thus every square matrix can be expressed as a sum of two matrices.

Consider $P^T = \left[\frac{1}{2}\left(A+A^T\right)\right]^T = \frac{1}{2}\left(A+A^T\right)^T = \frac{1}{2}\left(A^T+\left(A^T\right)^T\right) = \frac{1}{2}\left(A+A^T\right)$ =P, since $P^T = P$,

P is symmetric

Consider $Q^T = \left[\frac{1}{2}\left(A-A^T\right)\right]^T = \frac{1}{2}\left(A-A^T\right)^T = \frac{1}{2}\left(A^T-\left(A^T\right)^T\right) = -\frac{1}{2}\left(A-A^T\right) = -Q$

Since $Q^T = -Q$, Q is Skew-symmetric.

To prove the representation is unique: Let A= R+S $\rightarrow (1)$ be the representation, where R is symmetric and S is skew symmetric. i.e. $R^T = R, S^T = -S$

Consider $A^T = (R+S)^T = R^T + S^T = R - S \rightarrow (2)$

$(1)-(2) \Rightarrow A - A^T = 2S \Rightarrow S = \frac{1}{2}\left(A-A^T\right) = Q$

Therefore every square matrix can be expressed as a sum of a symmetric and a skew symmetric matrix

Ex. Express the given matrix A as a sum of a symmetric and skew symmetric matrices

where **A=**$\begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 9 & 5 & 11 \end{bmatrix}$

Solution: $A^T = \begin{bmatrix} 2 & 14 & 3 \\ -4 & 7 & 5 \\ 9 & 3 & 11 \end{bmatrix}$

$$A + A^T = \begin{bmatrix} 4 & 10 & 12 \\ 10 & 14 & 18 \\ 12 & 18 & 22 \end{bmatrix} \Rightarrow P = \frac{1}{2}\left(A + A^T\right) = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix}; P \text{ is symmetric}$$

$$A - A^T = \begin{bmatrix} 0 & -18 & 6 \\ 18 & 0 & 8 \\ -6 & -8 & 0 \end{bmatrix} \Rightarrow Q = \frac{1}{2}\left(A - A^T\right) = \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}; Q \text{ is skew-symmetric}$$

Now A =P+Q = $\begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix} + \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

Orthogonal Matrix: A square matrix A is said to be an Orthogonal Matrix if $AA^T = A^TA = I$, Similarly we can prove that $A = A^{-1}$; Hence A is an orthogonal matrix.

Note: 1. If A, B are orthogonal matrices, then AB and BA are orthogonal matrices.

2. Inverse and transpose of an orthogonal matrix is also an orthogonal matrix.

Result: If A, B are orthogonal matrices, each of order n then AB and BA are orthogonal matrices.

Result: The inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal

Solved Problems :

1. Show that A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **is orthogonal.**

Sol: Given A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then $A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Consider $A.A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal matrix.

2. Prove that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ **is orthogonal.**

Sol: Given A = $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ Then $A^T = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Consider $A.A^T = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\Rightarrow A.A^T = I$

Similarly $A^T.A = I$

Hence A is orthogonal matrix

3. Determine the values of a, b, c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

Sol: - For orthogonal matrix $AA^T = I$

So, $AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = I$

$$\begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving $2b^2-c^2=0$, $a^2-b^2-c^2=0$

We get $c = \pm\sqrt{2}b$ $a^2 = b^2+2b^2 = 3b^2$

$\Rightarrow a = \pm\sqrt{3}b$

From the diagonal elements of I

$4b^2+c^2 = 1 \Rightarrow 4b^2+2b^2=1$ (since $c^2=2b^2$) $\Rightarrow b = \pm\frac{1}{\sqrt{6}}$

$a = \pm\sqrt{3}b = \pm\frac{1}{\sqrt{2}}$; $b = \pm\frac{1}{\sqrt{6}}$; $c = \pm\sqrt{2}b = \pm\frac{1}{\sqrt{3}}$

4. Is matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ Orthogonal?

Sol:- Given A=$\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

$\Rightarrow AA^T = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}\begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix} \neq I_3$

$AA^T \neq A^TA \neq I_3$

$\therefore$ Matrix is not orthogonal.

Complex matrix: A matrix whose elements are complex numbers is called a complex matrix.

Conjugate of a complex matrix: A matrix obtained from A on replacing its elements by the corresponding conjugate complex numbers is called conjugate of a complex matrix. It is denoted by $\overline{A}$

If $A=\left[a_{ij}\right]_{mxn}$, $\overline{A}=\left[\overline{a_{ij}}\right]_{mxn}$, where $\overline{a_{ij}}$ is the conjugate of a_{ij}.

Eg: If A=$\begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$ then $\overline{A}=\begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$

Note: If $\overline{A}$ and $\overline{B}$ be the conjugate matrices of A and B respectively, then

(i) $\overline{\left(\overline{A}\right)}$=A (ii) $\overline{A+B}=\overline{A}+\overline{B}$ (iii) $\overline{(KA)}=\overline{K}\ \overline{A}$

Transpose conjugate of a complex matrix: Transpose of conjugate of complex matrix is called transposed conjugate of complex matrix. It is denoted by A^{θ} *or* A^*.

Note: If A^{θ} and B^{θ} be the transposed conjugates of A and B respectively, then

(i) $\left(A^{\theta}\right)^{\theta}$ = A (ii) $\left(A \pm B\right)^{\theta} = A^{\theta} \pm B^{\theta}$

(iii) $\left(KA\right)^{\theta} = \overline{K}A^{\theta}$ (iv) $\left(AB\right)^{\theta} = A^{\theta}B^{\theta}$

Hermitian Matrix: A square matrix A is said to be Hermitian Matrix iff $A^{\theta} = A$.

Eg: A=$\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ then $\overline{A}=\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ and A$^{\theta}$=$\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

Note: 1. In Hermitian matrix the principal diagonal elements are real.

2. The Hermitian matrix over the field of Real numbers is nothing but real symmetric matrix.

3. In Hermitian matrix A= $\left[a_{ij}\right]_{nxn}$, $a_{ij}=\overline{a_{ji}}\ \forall i, j$.

Skew-Hermitian Matrix: A square matrix A is said to be Skew-Hermitian Matrix iff $A^{\theta}=-A$.

Eg: Let A=$\begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$ then $\overline{A}=\begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$ and $\left(\overline{A}\right)^T=\begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix}$

$\therefore$ $\left(\overline{A}\right)^T$=-A $\therefore$ A is skew-Hermitian matrix.

Note: 1. In Skew-Hermitian matrix the principal diagonal elements are either Zero or Purely Imaginary.

2. The Skew- Hermitian matrix over the field of Real numbers is nothing but real Skew - Symmetric matrix.

3. In Skew-Hermitian matrix A= $\left[a_{ij}\right]_{nxn}$, $a_{ij} = -\overline{a_{ji}}\ \forall i, j$.

Unitary Matrix: A Square matrix A is said to be unitary matrix iff

$AA^{\theta} = A^{\theta}A = I \text{ or } A^{\theta} = A^{-1}$

Eg: $B = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$

Theorem1: Every square matrix can be uniquely expressed as a sum of Hermitian and skew – Hermitian Matrices.

Proof: - Let A be a square matrix write

$$A = \frac{1}{2}(2A) = \frac{1}{2}(A+A) = \frac{1}{2}(A + A^{\theta} + A - A^{\theta})$$
$$A = \frac{1}{2}(A+A^{\theta}) + \frac{1}{2}(A - A^{\theta}) i.e A = P + Q$$

Let $P = \frac{1}{2}\left(A + A^{\theta}\right); Q = \frac{1}{2}\left(A - A^{\theta}\right)$

Consider $P^{\theta} = \left[\frac{1}{2}(A+A^{\theta})^{\theta}\right] = \frac{1}{2}(A+A^{\theta})^{\theta} = (A+A^{\theta}) = P$

I.e. $P^{\theta} = P, P$ is Hermitian matrix.

$Q^{\theta} = \left[\frac{1}{2}(A - A^{\theta})\right]^{\theta} = \frac{1}{2}\left(A^{\theta} - A\right) = -\frac{1}{2}(A - A^{\theta}) = -Q$

Ie $Q^{\theta} = -Q, Q$ is skew – Hermitian matrix.

Thus every square matrix can be expressed as a sum of Hermitian & Skew Hermitian matrices.

To prove such representation is unique:

Let A = R+S------- (1) be another representation of A where R is Hermitian matrix & S is skew – Hermitian matrix.

$\therefore R = R^{\theta}; S^{\theta} = -S$

Consider $A^{\theta} = (R+S)^{\theta} = R^{\theta} + S^{\theta} = R - S -$. Ie $A^{\theta} = R - A - - - - - (2)$

$$(1)+(2) \Rightarrow A + A^{\theta} = 2R \; ie \; R = \frac{1}{2}\left(A + A^{\theta}\right) = P$$

$$(1)-(2) \Rightarrow A - A^{\theta} = 2S \; ie \; S = \frac{1}{2}\left(A - A^{\theta}\right) = Q$$

Thus every square matrix can be uniquely expressed as a sum of Hermitian & skew Hermitian matrices.

Solved Problems :

1) If A= $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ **then show that A is Hermitian and iA is skew-Hermitian.**

Sol: Given A= $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then

$\overline{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$ And $\left(\overline{A}\right)^T = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

$\therefore A = \left(\overline{A}\right)^T$ Hence A is Hermitian matrix.

Let B= iA

i.e B= $\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$ then

$$\overline{B} = \begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$$

$$\left(\overline{B}\right)^T = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix} = (-1)\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix} = -B$$

$\therefore \left(\overline{B}\right)^T$ =-B

$\therefore$ B= iA is a skew Hermitian matrix.

2). If A and B are Hermitian matrices, prove that AB-BA is a skew-Hermitian matrix.

Sol: Given A and B are Hermitan matrices

$\therefore \left(\overline{A}\right)^T = A$ And $\left(\overline{B}\right)^T = B$ ------------- (1)

Now $\overline{(AB-BA)}^T = \left(\overline{AB}-\overline{BA}\right)^T$

$$= \left(\overline{A}\,\overline{B}-\overline{B}\,\overline{A}\right)^T$$

$$= \left(\overline{A}\overline{B}\right)^T - \left(\overline{B}\overline{A}\right)^T = \left(\overline{B}\right)^T\left(\overline{A}\right)^T - \left(\overline{A}\right)^T\left(\overline{B}\right)^T$$

$$= BA - AB \text{ (By (1))}$$

$$= -\left(AB-BA\right)$$

Hence AB-BA is a skew-Hemitian matrix.

3). Show that A=$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if $a^2+b^2+c^2+d^2=1$

Sol: Given A=$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$

Then $\overline{A} = \begin{bmatrix} a-ic & -b-id \\ b-id & a+ic \end{bmatrix}$

Hence $A^\theta = \left(\overline{A}\right)^T = \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$

$$\therefore AA^\theta = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}\begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$$

$$= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$$

$\therefore AA^\theta = I$ if and only if $a^2+b^2+c^2+d^2=1$

4)Given that A=$\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I-A)(I+A)^{-1}$ is a unitary matrix.

Sol: we have $I-A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \text{ And}$$

$$I+A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$\therefore (I+A)^{-1} = \frac{1}{1-(4i^2-1)}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$=\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Let $B=(I-A)(I+A)^{-1}$

$$B=\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}=\frac{1}{6}\begin{bmatrix} 1+(1-2i)(-1-2i) & -1-2i-1-2i \\ 1-2i+1-2i & (-1-2i)(1-2i)+1 \end{bmatrix}$$

$$B=\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

Now $\overline{B}=\frac{1}{6}\begin{bmatrix} -4 & -2+4i \\ 2+4i & -4 \end{bmatrix}$ and $\left(\overline{B}\right)^T=\frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$$B\left(\overline{B}\right)^T=\frac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$=\frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

$$\left(\overline{B}\right)^T=B^{-1}$$

i.e. B is unitary matrix.

$\therefore (I-A)(I+A)^{-1}$ is a unitary matrix.

5) Show that the inverse of a unitary matrix is unitary.

Sol: Let A be a unitary matrix. Then $AA^{\theta}=I$

i.e $\left(AA^{\theta}\right)^{-1}=I^{-1}$

$$\Rightarrow \left(A^{\theta}\right)^{-1} A^{-1} = I$$

$$\Rightarrow \left(A^{-1}\right)^{\theta} A^{-1} = I$$

Thus A^{-1} is unitary.

Rank of a Matrix:

Let A be mxn matrix. If A is a null matrix, we define its rank to be ‘0’. If A is a non-zero matrix, we say that ‘**r**’ is the rank of A if

i. Every $(r+1)^{th}$ order minor of A is ‘0’ (zero) &

ii. At least one r^{th} order minor of A which is not zero.

It is denoted by ρ (A) and read as rank of A.

Note: 1. Rank of a matrix is unique.

2. Every matrix will have a rank.

3. If A is a matrix of order mxn, then Rank of A $\leq$ min (m,n)

4. If ρ (A) = r then every minor of A of order r+1, or minor is zero.

5. Rank of the Identity matrix I_n is n.

6. If A is a matrix of order n and A is non-singular then ρ (A) = n

7. If A is a singular matrix of order n then ρ (A) < n

Important Note:

1. The rank of a matrix is $\leq$ r if all minors of $(r+1)^{th}$ order are zero.

2. The rank of a matrix is $\geq$ r, if there is at least one minor of order ‘r’ which is not equal to zero.

1. Find the rank of the given matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Sol: Given matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

det A = 1(48-40)-2(36-28)+3(30-28) = 8-16+6 = -2 $\neq$ 0

We have minor of order 3 $\therefore \rho$ (A) =3

2. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Given the matrix is of order 3x4

Its Rank $\leq$ min(3,4) = 3

Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

Determinant of minor is 1(-49)-2(-56) + 3(35-48) = -49+112-39 = 24 $\neq$ 0.

Hence rank of the given matrix is '3'.

Elementary Transformations on a Matrix:

i). Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$

(ii). If i^{th} row is multiplied with k then it is denoted by $R_i \rightarrow kR_i$

(iii). If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by $R_j \rightarrow R_j + kR_i$

Note: 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$c_i \leftrightarrow c_j$, $c_i \rightarrow k\, c_j$ $c_j \rightarrow c_j + kc_i$

2. The elementary operations on a matrix do not change its rank.

Equivalance of Matrices: If B is obtained from A after a finite number of elementary transformations on A, then B is said to be equivalent to A.It is denoted as B~A.

Note : 1. If A and B are two equivalent matrices, then rank A = rank B.

2. If A and B have the same size and the same rank, then the two matrices are equivalent.

Elementary Matrix or E-Matrix: A matrix is obtained from a unit matrix by a single elementary transformation is called elementary matrix or E-matrix.

Notations: We use the following notations to denote the E-Matrices.

1) E_{ij} $\rightarrow$ Matrix obtained by interchange of i^{th} and j^{th} rows (columns).

2) $E_{i(k)}$ $\rightarrow$ Matrix obtained by multiplying ith row (column) by a non- zero number k.

3) $E_{ij(k)}$ $\rightarrow$ Matrix obtained by adding k times of j^{th} row (column) to ith row (column).

Echelon form of a matrix:

A matrix is said to be in Echelon form, if

(i) Zero rows, if any exists, they should be below the non-zero row.

(ii) The first non-zero entry in each non-zero row is equal to '1'.

(iii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note : 1. The number of non-zero rows in echelon form of A is the rank of 'A'.

1. The rank of the transpose of a matrix is the same as that of original matrix.

2. The condition (ii) is optional.

Eg: 1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

2. $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

Solved Problems :

1. Find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ **by reducing it to Echelon form.**

Sol: Given A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ Applying row transformations on A.

$R_1 \leftrightarrow R_3$ $\quad A \sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$

$R_2 \to R_2 - 3R_1;\ R_3 \to R_3 - 2R_1$ $\quad \sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$

$R_2 \to R_2/7, R_3 \to R_3/9$ $\quad \sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\quad \sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

This is the Echelon form of matrix A.

The rank of a matrix A= Number of non – zero rows =2

2. For what values of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$ **has rank '3'.**

Sol: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

A = $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$

Applying $R_2 \to 4R_2 - R_1$, $R_3 \to 4R_3 - kR_1$, $R_4 \to 4R_4 - 9R_1$

We get A ~ $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$

Since Rank A = 3 ⇨ det A =0

⇨ 4 $\begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$

⇨ 1[(8-4k) 3]-1(8-4k) (4k+27)] = 0

⇨ (8-4k) (3-4k-27) = 0

⇨ (8-4k)(-24-4k) =0

⇨ (2-k)(6+k)=0

⇨ k =2 or k = -6

3). Find the rank of the matrix using echelon form $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

By applying $R_2 \to R_2 - 2R_1\,; R_3 \to R_3 - 4R_1\,; R_4 \to R_4 - 4R_1$ A ~ $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$

$R_1 \to \frac{R_1}{-1}, R_2 \to \frac{R_2}{-1}, R_3 \to \frac{R_3}{-3}$ A ~ $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \end{bmatrix}$

$R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$ A ~ $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

⇒ A is in echelon form ∴ Rank of A = 2

4). Find the rank of the matrix A = $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$ **by reducing into echelon form.**

Sol: By applying $R_2 \to R_2 - 2R_1; R_3 \to R_3 - 3R_1; R_4 \to R_4 + R_1$ $A \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $A \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -1 & 2 \end{bmatrix}$

$R_3 \leftrightarrow R_4$ $A \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 \to R_3 + R_2$ $A \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clear it is in echelon form, rank of A = 2

Normal form/Canonical form of a Matrix:

Every non-zero Matrix can be reduced to any one of the following forms. $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}; [I_r \quad 0]; \begin{bmatrix} I_r \\ 0 \end{bmatrix}; [I_r]$ Known as normal forms or canonical forms by using Elementary row or column or both transformations where I_r is the unit matrix of order ‘r’ and ‘O’ is the null matrix.

Note: 1.In this form “the rank of a matrix is equal to the order of an identity matrix.

2. Normal form another name is “canonical form”

Solved Problems :

1. By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ **into normal form, find its rank.**

Sol: Given A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

$R_2 \rightarrow R_2 - 2R_1;\ R_3 \rightarrow R_3 - 3R_1$ $\quad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$

$R_3 \rightarrow R_3/\text{-}2$ $\quad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix}$

$R_3 \rightarrow R_3 + R_2$ $\quad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

$c_2 \rightarrow c_2 - 2c_1,\ c_3 \rightarrow c_3 - 3c_1,\ c_4 \rightarrow c_4 - 4c_1$ $\quad A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

$c_3 \rightarrow 3c_3 - 2c_2,\ c_4 \rightarrow 3c_4 - 5c_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix}$

$c_2 \rightarrow c_2/\text{-}3,\ c_4 \rightarrow c_4/18$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$c_4 \leftrightarrow c_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

This is in normal form $[I_3\ 0]$, $\quad \therefore$ Hence Rank of A is '3'.

2). Find the rank of the matrix A = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$ **by reducing into canonical form or normal form.**

Sol: Given A = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$

By applying $R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 3R_1$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$

$R_3 \to R_3 - R_2, R_4 \to R_4 + 7R_2$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix}$

$R_4 \to R_4 - 6R_3$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix}$

$R_4 \to \dfrac{R_4}{-18}$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Apply $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - C_1$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_3 \to C_3 - 2C_2; C_4 \to C_4 + 5C_2$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_4 \to C_4 + 2C_3$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Clearly it is in the normal form $[I_4]$ $\therefore$ Rank of A = 4

3). Define the rank of the matrix and find the rank of the following matrix $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Sol: Let A= $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$
$R_4 \to R_4 - 4R_1$ $A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - 3R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in echelon form. So, rank of matrix = no. of non zero rows in echelon form. $\therefore Rank \quad \rho(A) = 2$

4). Reduce the matrix A to normal form and hence find its rank $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

$C_1 \to \frac{1}{2} C_1$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 1 & 3 & 7 & 5 \\ 1 & 5 & 11 & 6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 4R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 8 & 2 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to 4C_4 - C_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to \frac{1}{3}C_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

This is in normal form. Thus Rank of matrix = Order of identify matrix. $\therefore$ Rank $\rho(A) = 3$

5). Reduce the matrix A = $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ **into canonical form and then find its rank.**

Sol: Apply $C_1 \leftrightarrow C_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

$R_3 \to R_3 - R_1$ $\quad A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$C_3 \to C_3 - 2C_1; C_4 \to C_4 + 2C_1$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$R_2 \to \frac{R_2}{2}$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_2 \leftrightarrow C_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 \to C_3 - 2C_2; C_4 \to C_4 - 3C_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Which is in the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\quad \therefore \rho$ (A) = 2

Note: .If A is an mxn matrix of rank r, there exists non-singular matrices P and Q such that

PAQ = $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

Suppose we want to find P and Q we have procedure.

Let order of matrix 'A' is '3 $\quad$ i.e. A = I_3 A I_3

A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ A $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now we go on applying elementary row operations and column operations on the matrix A (L.H.S) until it is reduced to the normal form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

Every row operations will also be applied to the pre-factor of on R.H.S

Every column operation will also be applied to the post –factor of on R.H.S.

Solved Problems :

1. Find the non-singular matrices P and Q is of the normal form where A = $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$

Sol: Write A = I_3 A I_3

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$ $\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_3 \to R_3 - R_2,\ R_2 \to {}^{1}/_{3} R_2$ $\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/3 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$c_3 \to c_3 - 2c_1$ $\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/3 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$ where $P = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/3 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. Find the non-singular matrices P and Q such that the normal form of A is P A Q.

Where A = $\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$**. Hence find its rank.**

Sol: we write A = I_3A I_4

$$\sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$ $\quad \sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $\quad \sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Applying $c_2 \to c_2 - 3c_1, c_3 \to c_3 - 6c_1$, and $c_4 \to c_4 + c_1$, we get.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $c_3 \to c_3 + c_2$ and $c_4 \to c_4 - 2c_2$, we get.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⇨ $\begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} = P\,A\,Q$ Where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Here $A \sim \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$, $\quad \therefore$ Hence $\rho(A) = 2$

System of linear equations: In this chapter we shall apply the theory of matrices to study the existence and nature of solutions for a system of m linear equations in 'n' unknowns.

The system of m linear equations in 'n' unknowns $x_{1,} x_2, x_3 ----x_n$ given by

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m \end{array}\right\} \text{---------------------------- (1)}$$

The above set of equations can be written in the Matrix form as A X = B

$$\begin{bmatrix} a_{11} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{n1} & a_{n2} & - & - & - & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix} \to (2)$$

A-Coefficient Matrix; X-Set of unknowns; B-Constant Matrix

Homogeneous Linear Equations: If $b_1 = b_2 = \ldots\ldots\ldots\ldots = b_m = 0$ then B = 0

Hence eqn (2) Reduces to AX = 0 which are known as homogeneous linear equations

Non-Homogeneous Linear equations:

It at least one of b_1, b_2----b_m is non zero. Then $B \neq 0$, the system Reduces to AX = B is known as Non-Homogeneous Linear equations.

Solutions: A set of numbers $x_1, x_2 --x_n$ which satisfy all the equations in the system is known as solution of the system.

Consistent: If the system possesses a solution then the system of equations is said to be consistent.

Inconsistent: If the system has no solution then the system of equations is said to be Inconsistent.

Augmented Matrix: A matrix which is obtained by attaching the elements of B as the last column in the coefficient matrix A is called Augmented Matrix. It is denoted by [A|B]

$$[A/B] = C = \begin{bmatrix} a_{11} & a_{12} & a_{13} & - & - & a_{1n} & : & b_1 \\ a_{21} & a_{22} & a_{23} & - & - & a_{2n} & : & b_2 \\ a_{m1} & a_{m2} & a_{m3} & - & - & a_{mn} & : & b_3 \end{bmatrix}$$

1. If $\rho(A/B) = \rho(\mathrm{A})$, then the system of equations AX = B is consistent (solution exits).

a). If $\rho(A/B) = \rho(\mathrm{A}) = \mathrm{r} = \mathrm{n}$ (no. of unknowns) system is consistent and have **a unique solution**

b). If $\rho(A/B) = \rho(\mathrm{A}) = \mathrm{r} < \mathrm{n}$ (no. of unknowns) then the system of equations AX = B will have an **infinite no. of solutions**. In this case (n-r) variables can be assigned arbitrary values.

2. If $\rho(A/B) \neq \rho(\mathrm{A})$ then the system of equations AX=B is **inconsistent (no solution).**

In case of homogeneous system AX = 0, the system is always consistent.

(or) $x_1 = 0$, $x_2 = 0$, --------, $x_n = 0$ is always the solution of the system known as the" zero solution ".

Non-trivial solution:

If $\rho(A/\mathrm{B}) = \rho(\mathrm{A}) = \mathrm{r} < \mathrm{n}$ (no. of unknowns) then the system of equations AX = 0 will have an infinite no. of non zero (non trivial) solutions. In this case (n-r) variables can be assigned arbitrary values.

Also we use some direct methods for solving the system of equations.

Note: The direct methods are Cramer's rule, Matrix Inversion, Gaussian Elimination, Gauss Jordan, Factorization Tridiagonal system. These methods will give a unique solution.

Procedure to solve AX = B (Non Homogeneous equations)

Let us first consider n equations in n unknowns ie. m=n then the system will be of the form

$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots..+a_{1n}x_n = b_1$

$a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots..+a_{2n}x_n = b_2$

……………………………………..

$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\ldots..+a_{nn}x_n = b_n$

The above system can be written as AX = B --------- (1)

Where A is an $n \times n$ matrix.

Solving AX = B using Echelon form:

Consider the system of m equations in n unknowns given by

$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots..+a_{1n}x_n = b_1$

$a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots..+a_{2n}x_n = b_2$

……………………………………..

$a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+\ldots..+a_{mn}x_n = b_m$

We know this system can be we write as AX = B

The augmented matrix of the above system is $[A / B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$

The system AX = B is consistent if ρ (A) = ρ[A/B]

i). ρ (A) = ρ [A/B] = r < n (no. of unknowns).Then there is infinite no. of solutions.

ii). ρ (A) = ρ [A/B] = number of unknowns then the system will have unique solution.

iii). ρ (A) ≠ ρ [A/B] the system has no solution.

Solved Problems :

1). Show that the equations x+y+z = 4, 2x+5y-2z =3, x+7y-7z =5 are not consistent.

Sol: Write given equations is of the form AX = B i.e; $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

Consider the Augment matrix is [A /B] ⇨ $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

Applying $R_2 \to R_2-2R_1$ and $R_3 \to R_3-R_1$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$

Applying $R_3 \to R_3 - 2R_2$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$

$\therefore \rho(A) = 2$ and $\rho(A/B) = 3$

The given system is inconsistent as $\rho(A) \neq \rho[A/B]$.

2). Show that the equations given below are consistent and hence solve them

x-3y-8z = -10, 3x+y-4z =0, 2x+5y+6z =3

Sol: Matrix notation is $\begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 3 \end{bmatrix}$

Augmented matrix [A/B] is $[A/B] = \begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 3 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$ $\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 23 \end{bmatrix}$

$R_2 \to 1/10\, R_2$ $\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 11 & 22 & 23 \end{bmatrix}$

$R_3 \to R_3 - 11R_2$ $\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix}$

This is the Echelon form of [A/B] $\therefore \rho(A) = 2$, $\rho(A/B) = 3$

$\rho(A) \neq \rho[A/B]$.

The given system is inconsistent.

3). Find whether the following equations are consistent, if so solve them. x+y+2z=4, 2x-y+3z=9, 3x-y-z=2

Sol: We write the given equations in the form AX=B i.e; $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

The Augmented matrix $[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$

Applying $R_3 \rightarrow 3R_3 - 4R_2$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$

this matrix is in Echelon form. $\rho(A) = 3$ and $\rho(A/B) = 3$

Since $\rho(A) = \rho[A/B]$. ∴ The system of equations is consistent.

Here the number of unknowns is 3

Since $\rho(A) = \rho[A/B]$ = number of unknowns

∴ The system of equations has a unique solution

We have $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$

⇨ $-17z = -34$ ⇨ $z = 2$

$-3y - z = 1$ ⇨ $-3y = z+1$ ⇨ $-3y = 3$ ⇨ $y = -1$

and $x+y+2z = 4$ ⇨ $x = 4-y-2z = 4+1-4 = 1$

∴ $x=1, y=-1, z=2$ is the solution.

4). Show that the equations $x+y+z=6$, $x+2y+3z=14$, $x+4y+7z=30$ are consistent and solve them.

Sol: We write the given equations in the form AX=B i.e. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$

The Augmented matrix $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$

Applying $R_3 \rightarrow R_3 - 3R_2$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This matrix is in Echelon form. $\rho(A) = 2$ and $\rho(A/B) = 2$

Since $\rho(A) = \rho[A/B]$.

The system of equations is consistent. Here the no. of unknowns are 3

Since rank of A is less than the no. of unknowns, the system of equations will have infinite number of solutions in terms of n-r=3-2=1 arbitrary constant.

The given system of equations reduced form is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$

⇨x+y+z=6……… (1), y+2z=8…….(2)

Let z=k, put z=k in (2) we get y=8-2k

Put z=k y=8-2k in (1), we get

x=6-y-z=6-8+2k=-2+k

∴ x=-2+k, y=8-2k, z=k is the solution, where k is an arbitrary constant.

5). Show that $x+2y-z=3$ **;** $3x-y+2z=1$**;** $2x-2y+3z=2; x-y+z=-1$ **are consistent and solve them**

Sol: The above system in matrix notation is $\underset{A}{\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}_{4\times 3}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1}} = \underset{=B}{\begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}_{4\times 1}}$

The Augmented matrix is $[AB] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

$$\begin{matrix} R_2 \to R_2 \to 3R_1 \\ R_3 \to R_3 \to 2R_1 \\ R_4 \to R_4 \to R_1 \end{matrix} \quad \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -0 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$$R_2 \to R_2 - R_3 \quad \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$$\begin{matrix} R_3 \to R_3 \to 3R_1 \\ R_4 \to R_4 \to 3R_2 \end{matrix} \quad \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & +20 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{5} \quad \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

$$R_4 \to R_4 - 2R_3 \qquad \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \rho$ (A) = 3 = ρ (A/B

$\therefore \rho$ (A) =ρ (A/B) = No. of unknowns = 3

$\therefore$ The given system has unique solution.

The systems of equations equivalent to given system are

$x + 2y - z = 3 \qquad -y = -4; z = 4$

$x + 8 - 4 = 3 \qquad y = 4; z = 4$

$x = 3 - 4 = -1$

$\therefore x = -1, y = 4, z = 4.$

6). Solve $x + y + z = 3; 3x - 5y + 2z = 8; 5x - 3y + 4z = 14$

Sol: - $\begin{bmatrix} 1 & 1 & 1 \\ 3 & -5 & 2 \\ 5 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 14 \end{bmatrix}$

Augmented Matrix is $\qquad [AB] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & -5 & 2 & 8 \\ 5 & -3 & 4 & 14 \end{bmatrix}$

$$\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 5R_1 \end{array} \qquad \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & -8 & -1 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \qquad \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho[A] = \rho[AB] = 2 <$ Number of unknowns (3)

$\therefore$ The system has infinite number of solutions.

$x + y + z = -3, -8y - z = -1 \quad \Rightarrow 8y + z - 1$

$$Let\ z = k \quad \Rightarrow y = \frac{1-k}{8} \quad and \quad x = 3 - \frac{(1-k)}{8} - k = \frac{24 - 1 + k - 8k}{8} = \frac{23 - 7k}{8}$$

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{23}{8} - \frac{7}{8}k \\ \frac{1}{8} - \frac{k}{8} \\ 0 + k \end{bmatrix} \quad \Rightarrow \quad X = \begin{bmatrix} \frac{23}{8} \\ \frac{1}{8} \\ 1 \end{bmatrix}$$ where k is any real number.

7). Find whether the following system of equations is consistent. If so solve them.

$x + 2y + 2z = 2, \quad 3x - 2y + z = 5, \quad 2x - 5y + 3z = -4, \quad x + 4y + 6z = 0.$

Sol: In Matrix form it is $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$

$AX = B$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -5 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$

$R_2 \to R_2 - R_3$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$

$R_3 \to R_3 + 9R_2$
$R_4 \to R_4 - 2R_2$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & 0 & -37 & 55 \\ 0 & 0 & 12 & -16 \end{bmatrix}$

$R_4 \to \frac{1}{4} R_4$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & 0 & -37 & 55 \\ 0 & 0 & 3 & -4 \end{bmatrix}$

$R_4 \to 37R_4 + 3R_3$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & 0 & -37 & 55 \\ 0 & 0 & 0 & 17 \end{bmatrix}$ is in echelon form

$\rho[A] = 3 \text{ and } \rho[AB] = 4 \Rightarrow \rho[A] \neq \rho[AB]$.∴The given system is in consistent.

8). Discuss for what values of λ, μ the simultaneous equations x+y+z = 6, x+2y+3z =10, x+2y+λz = μ have

(i). No solution

(ii). A unique solution

(iii). An infinite number of solutions.

Sol: The matrix form of given system of Equations is A X = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$ = B

The augmented matrix is [A/B] = $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$ $\quad [A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\quad \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$

Case (i): let $\lambda \neq 3$ the rank of A = 3 and rank [A/B] = 3

Here the no. of unknowns is '3' $\quad \therefore \rho(A) = \rho(A/B) =$ No. of unknowns

The system has unique solution if $\lambda \neq 3$ and for any value of 'μ'.

Case (ii): Suppose $\lambda = 3$ and $\mu \neq 10$.

We have $\rho(A) = 2$, $\rho(A/B) = 3$

The system has no solution.

Case (iii): Let $\lambda = 3$ and $\mu = 10$.

We have $\rho(A) = 2$ $\rho(A/B) = 2$

Here $\rho(A) = \rho(A/B) \neq$ No. of unknowns =3

The system has infinitely many solutions.

9). Find the values of a and b for which the equations x+y+z=3; x+2y+2z=6;x+ay+3z=b have (i) No solution

(ii) A unique solution

(iii) Infinite no of solutions.

Sol: The above system in matrix notation is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & a & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ b \end{bmatrix}$

Augmented matrix $\quad [AB] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 1 & a & 3 & b \end{bmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$ $\quad \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-1 & 2 & b-2 \end{bmatrix}$

$R_3 \to R_3 - R_2$
$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-3 & 0 & b-9 \end{bmatrix}$$

• For $a = 3$ & $b = 9$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \rho[A] = \rho[AB] = 2 < 3 \Rightarrow$ It has infinite no of solutions.

• *For* $a \neq 3$ & $b = any\ value$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-3 & 0 & b-9 \end{array}\right]$$

$\therefore \rho[A] = \rho[AB] = 3 \Rightarrow$ It has a unique solution.

• *For* $a = 3$ & $b \neq 9$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & b-9 \end{array}\right]$$

$\therefore \rho[A] = 2 \neq \rho[AB] = 3 \Rightarrow$ Inconsistent $\Rightarrow$ no solution

10). Solve the following system completely. $x+y+z=1; x+2y+4x=\alpha; x+4y+10z=\alpha^2$

Sol: The above system in matrix notation is
$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}}\underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}}$$

Augmented Matrix is
$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \alpha \\ 1 & 4 & 10 & \alpha^2 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \alpha-1 \\ 0 & 3 & 9 & \alpha^2-1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \alpha-1 \\ 0 & 0 & 0 & \alpha^2-3\alpha+2 \end{array}\right]$$

Here $\rho[A] = 2\ and\ \rho[AB] = 3 \Rightarrow$ The given system of equations is consistent if

$\alpha^2 - 3\alpha + 2 = 0 \Rightarrow \alpha^2 - 2\alpha - \alpha + 2 = 0 \Rightarrow (\alpha-2)(\alpha-1) = 0 \Rightarrow \alpha = 2, \alpha = 1$

Case (i): When $\alpha = 1$

$\rho[A]=\rho[AB]=2<$ Number of unknowns.

$\therefore$ The system has infinite number of solutions.

The equivalent matrix is $\begin{bmatrix}1&1&1&1\\0&1&3&0\\0&0&0&0\end{bmatrix}$

The equivalent systems of equations are $x+y+z=1; y+3z=0$

$\Rightarrow Let\ z=k \quad \Rightarrow y=-3k \quad and \quad x+(-3k)+K=1 \Rightarrow x-2k=1 \Rightarrow x=1+2k$

$X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1+2k\\0-3k\\0+k\end{bmatrix}\Rightarrow X=\begin{bmatrix}1\\0\\0\end{bmatrix}+k\begin{bmatrix}2\\-3\\1\end{bmatrix}$ where k is any arbitrary constant.

Case (ii): When $\alpha=2$

$\rho[A]=\rho[AB]=2<$ no. of unknowns.

$\therefore$ The system has infinite number of solutions.

The equivalent matrix is $\sim\begin{bmatrix}1&1&1&1\\0&1&3&1\\0&0&0&0\end{bmatrix}$

The system of equations equivalent to the given system is x + y + z = 1; y + 3z = 1

$Let\ z=k \quad \Rightarrow y=1-3k\ and\ x+(1-3K)+k=1 \Rightarrow x=2k$

$\Rightarrow X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0+2k\\1-3k\\0+k\end{bmatrix}\Leftrightarrow X=\begin{bmatrix}0\\1\\0\end{bmatrix}+k\begin{bmatrix}2\\-3\\1\end{bmatrix}$ where k is any arbitrary constant.

11). Show that the equations $3x+4y+5z=a; 4x+5y+6z=b; 5x+6y+7z=c$ don't have a solution unless $a+c=2b$. solve equations when a=b=c= -1.

Sol: The Matrix notation is $\begin{bmatrix}3&4&5\\4&5&6\\5&6&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$

$A \quad X \quad = \quad B$

Augment Matrix is $[AB]=\begin{bmatrix}3&4&5&a\\4&5&6&b\\5&6&7&c\end{bmatrix}$

$R_2 \to 3R_2-4R_1$
$R_3 \to 3R_3-5R_1$

$\sim\begin{bmatrix}3&4&5&a\\0&-1&-2&3b-4a\\0&-2&-4&3c-5a\end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $\sim\begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & 0 & 0 & 3a-6b+3c \end{bmatrix}$

Here $\rho[A]=2 \; and \; \rho[AB]=3$

$\therefore$ The given system of equations is consistent if $3a-6b+3c=0 \Rightarrow 3a+3c=6b \Rightarrow a+c=2b$

Thus the equations don't have a solution unless $a+c=2b, when\ a=b=c=-1$

The equivalent matrix is $\begin{bmatrix} 3 & 4 & 5 & -1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho[A]=\rho[(AB)]=2<$ No. of unknowns.

$\therefore$ The system has infinite number of solutions. The system of equations equivalent to the given system $3x+4y+5z=-1; -y-2z=1 \Rightarrow y+2z=-1$

$Let\ z=k \Rightarrow y=-1-2k\ and\ 3x-4-8k+5k=-1 \Rightarrow x=1+k$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1+k \\ -1-2k \\ 0+k \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}+k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Linearly dependent set of vectors: A set { x_1,x_2,--------------x_r} of r vectors is said to be a linearly dependent set, if there exist r scalars k_1,k_2---k_r not all zero, such that $k_1x_1+k_2x_2$+--------+$k_rx_r = 0$

Linearly independent set of vectors: A set { x_1, x_2,-------------x_r} of r vectors is said to be a linearly independent set, if $k_1x_1+k_2x_2$+--------------+k_rx_r =0 then k_1 =0, k_2 = 0------k_r =0.

Linear combination of vectors:

A vector x which can be expressed in the form x = $k_1x_1+k_2x_2$+--------------+k_nx_n is said to be a linear combination of x_1,x_2------x_n here k_1,k_2-----------k_n are any scalars.

Linear dependence and independence of Vectors:

Solved Problems :

1). Show that the vectors (1, 2, 3), (3,-2, 1), (1,-6,-5) from a linearly dependent set.

Sol: The Given Vector $X_1=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} X_2=\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} X_3=\begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}$

The Vectors X_1, X_2, X_3 from a square matrix.

$$Let\ A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix} \quad Then\ |A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{vmatrix}$$

= 1(10+6)-2(15-1) + 3(-18+2)

= 16+32-48 = 0

The given vectors are linearly dependent $\because |A| = 0$

2). Show that the Vector X_1=(2,2,1), X_2=(1,4,-1) and X_3=(4,6,-3) are linearly dependent.

Sol: Given Vectors X_1=(2,-2,1) X_2=(1,4,-1) and X_3=(4,6,-3) The Vectors X_1, X_2, X_3 form a square matrix.

$$Let\ A = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{bmatrix} \quad Then\ |A| = \begin{vmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{vmatrix}$$

= 2(-12+6) + 2(-3+4)+1(6-16) = -20 $\neq$ 0

$\therefore$ The given vectors are linearly dependent $\because |A| \neq 0$

Consistency of system of Homogeneous linear equations:

A system of m homogeneous linear equations in n unknowns, namely

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots.. + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots.. + a_{2n}x_n = 0 \\ -------------------- \\ -------------------- \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots.. + a_{mn}x_n = 0 \end{array}\right\} ------(1)$$

i.e. $\begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots.a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots.a_{2n} \\ a_{m1} & a_{m2} & a_{m3}\ldots.a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ .. \\ .0 \end{bmatrix}$ Here A is called Co –efficient matrix.

Note: 1. Here $x_1 = x_2$ = --------- x_n = 0 is called trivial solution or zero solution of AX = 0

2. A zero solution always linearly dependent.

Theorem: The number of linearly independent solutions of the linear system AX = 0 is (n-r), r being the rank of the matrix A and n being the number of variables.

Note: 1.if A is a non-singular matrix then the linear system AX = 0 has only the zero solution.

2. The system AX =0 possesses a non-zero solution if and only if A is a singular matrix.

Working rule for finding the solutions of the equation AX = 0

(i). Rank of A = No. of unknowns i.e. r = n

∴ The given system has zero solution.

(ii). Rank of A < No of unknowns (r<n) and No. of equations < No. of unknowns (m<n) then the system has **infinite no. of solutions**.

Note: If AX =0 has more unknowns than equations the system always has infinite solutions.

Solved Problems :

1). Solve the system of equations x+3y-2z = 0, 2x-y+4z = 0, x-11y+14z = 0

Sol: We write the given system is AX = 0 i.e. $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & -4 \\ 1 & -11 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - R_1$ $\quad A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $\quad A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$

The Rank of the A = 2 i.e. ρ (A) = 2 < No. of unknowns = 3

We have infinite No. of solution

Above matrix can we write as x+3y-2z = 0 -7y+8z = 0, 0 = 0

Let z = k then y=8/7k & x= -10/7 k

Giving different values to k, we get infinite no. of values of x,y,z.

2). Find all the non-trivial solution $2x - y + 3z = 0; 3x + 2y + z = 0; x - 4y + 5z = 0.$

Sol: In Matrix form it is $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$AX = 0$

$$The\ Augumented\ matrix\left[A/O\right]=\begin{bmatrix}2 & -1 & 3 & 0\\ 3 & 2 & 1 & 0\\ 1 & -4 & 5 & 0\end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \qquad \sim\begin{bmatrix}1 & -4 & 5 & 0\\ 3 & 2 & 1 & 0\\ 2 & -1 & 3 & 0\end{bmatrix}$$

$$\begin{matrix}R_2 \to R_2 - 3R_1\\ R_3 \to R_3 - 2R_1\end{matrix} \qquad \sim\begin{bmatrix}1 & -4 & 5 & 0\\ 0 & 14 & -14 & 0\\ 0 & 7 & -7 & 0\end{bmatrix}$$

$$R_3 \to 2R_3 - R_2 \qquad \sim\begin{bmatrix}1 & -4 & 5 & 0\\ 0 & 14 & -14 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \quad it\ is\ echelon\ form.$$

The Rank of the A = 2 i.e. ρ(A) = 2 < No. of unknowns = 3

Hence the system has non trivial solutions. From echelon form, reduced equations are

$x-4y+5z=0\ and\ 14y-14z=0$

$Let\ z=k\ then\ y=k\ and\ x-4k+5k=0 \Rightarrow x=-k.$

Thus, the solution set is $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-k\\ k\\ k\end{bmatrix}=k\begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}\forall K.$

3). Show that the only real number λ for which the system x+2y+3z = λx, 3x+y+2z= λy, 2x+3y+z = λz, has non-zero solution is 6 and solve them.

Sol: Above system can we expressed as AX = 0 i.e. $\begin{bmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

Given system of equations possess a non –zero solution i.e. ρ (A) < no. of unknowns.

⇨ For this we must have det A = 0

$$\Rightarrow\begin{vmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$$R_1 \to R_1+R_2+R_3 \quad \Rightarrow\begin{vmatrix}6-\lambda & 6-\lambda & 6-\lambda\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$$\Rightarrow(6-\lambda)\begin{vmatrix}1 & 1 & 1\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1 \Rightarrow (6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$

⇨ $(6-\lambda)[(-2-\lambda)(-1-\lambda)+1] = 0$

⇨ $(6-\lambda)(\lambda^2+3\lambda+3) = 0$

⇨ $\lambda = 6$ only real values.

When $\lambda = 6$, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 5R_2+3R_1, R_3 \to 5R_3+2R_1$ $\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3+R_2$ $\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

⇨ $-5x+2y+3z = 0$ and $-19y+19z = 0$ ⇨ $y = z$

Let $z = k$ ⇨ $y = k$ and $x = k$.

∴ The solution is $x = y = z = k$.

Eigen Values and Eigen vectors:

Let A= $\left[a_{ij}\right]_{nxn}$ be a square Matrix. Suppose the linear transformation Y = AX transforms X into a scalar multiple of itself i.e. AX = Y = λ X, Then the unknown scalar λ is known as an "Eigen value" of the Matrix A and the corresponding non-zero vector X is known as "Eigen Vector" of A. Corresponding to Eigen value λ. Thus the Eigen values (or) characteristic values (or) proper values (or) latent roots are scalars λ which satisfy the equation.

AX = λ X for X $\neq$ 0, $AX - \lambda IX = 0 \Rightarrow (A-\lambda I)X = 0$

Which represents a system of 'n' homogeneous equations in 'n' variables x_1, x_2, ----, x_n this system of equations has non-trivial solutions If the coefficient matrix (A-λ I) is singular i.e.

$$|A-\lambda I| = 0 \quad \Rightarrow \begin{vmatrix} a_{11-\lambda} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22-\lambda} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{n1} & a_{n2} & - & - & - & a_{nn-\lambda} \end{vmatrix} = 0$$

Expansion of the determinant is $(-1)^n\lambda^n + K_1\lambda^{n-1} + K_2\lambda^{n-2} + ----+K_n$ is the n^{th} degree of a polynomial $P_n(\lambda)$ which is known as **"Characteristic Polynomial"**. Of A

$(-1)^n \lambda^n + K_1\lambda^{n-1} + K_2\lambda^{n-2} + - - - - - + K_n = 0$ is known as **"Characteristic Equation"**. Thus the Eigen values of a square Matrix A are the roots of the characteristic equation.

Eg: Let A= $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ X = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

AX = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ = $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.X$

Here Characteristic vector of A is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and Characteristic root of A is "1".

Eigen Value: The roots of the characteristic equation are called Eigen values or characteristic roots or latent roots or proper values.

Eigen Vector: Let A=$\left[a_{ij}\right]_{nxn}$ be a Matrix of order n. A non-zero vector X is said to be a characteristic vector (or) Eigen vector of A if there exists a scalar λ such that AX = λ X.

Method of finding the Eigen vectors of a matrix.

Let A=[a_{ij}] be a nxn matrix. Let X be an eigen vector of A corresponding to the eigen value λ.

Then by definition AX = λX.

⇒ AX = λIX

⇒ AX –λIX = 0

⇒ (A-λI)X = 0 ------- (1)

This is a homogeneous system of n equations in n unknowns.

Will have a non-zero solution X if and only if |A-λI| = 0

- A-λI is called characteristic matrix of A
- |A-λI| is a polynomial in λ of degree n and is called the characteristic polynomial of A
- |A-λI|=0 is called the characteristic equation
- Solving characteristic equation of A, we get the roots , $\lambda_{1,}\lambda_{2,}\lambda_{3,} \dots \dots . \lambda_{n,}$ These are called the characteristic roots or eigen values of the matrix.
- Corresponding to each one of these n eigen values, we can find the characteristic vectors.

Procedure to find Eigen values and Eigen vectors

Let A = $\begin{bmatrix} a_{11} & a_{12} \dots \dots & a_{1n} \\ a_{21} & a_{22} \dots . . & a_{2n} \\ \dots \dots . & \dots \dots . & \dots \dots . \\ a_{n1} & a_{n2} \dots \dots . & a_{nn} \end{bmatrix}$ $be\ a\ given\ matrix$

$Characteristic\ matrix\ of\ A\ is\ A - \lambda I$

$$i.e., A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Then the characteristic polynomial is $|A-\lambda I|$

$$say\,\phi(\lambda) = |A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix}$$

The characteristic equation is $|A-\lambda I| = 0$ we solve the $\emptyset(\lambda) = |A-\lambda I| = 0$, we get n roots, these are called eigen values or latent values or proper values.

Let each one of these eigen values say λ their eigen vector X corresponding the given value λ is obtained by solving Homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}$$ and determining the non-trivial solution.

Solved Problems

1. **Find the eigen values and the corresponding eigen vectors of** $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Sol: $Let\ A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Characteristic matrix = $[A-\lambda I] = \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$

Characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)(2-\lambda)+8=0$$

$$\Rightarrow 16+\lambda^2-10\lambda+8=0$$

$$\Rightarrow \lambda^2-10\,\lambda+24=0$$

$$\Rightarrow (\lambda-6)(\lambda-4) = 0$$

$$\Rightarrow \lambda = 6, 4 \text{ are eigen values of A}$$

Consider the system $\begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

Eigen vector corresponding to $\lambda = 4$

Put $\lambda = 4$ in the above system, we get $\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Longrightarrow 4x_1 - 4x_2 = 0 - - - (1) \qquad 2x_1 - 2x_2 = 0 - - - (2)$

$from\ (1) and\ (2) we\ have\ x_1 = x_2$

Let $x_1 = \alpha$

$$Eigen\ vector\ is \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $is\ a\ Eigen\ vector\ of\ matrix\ A, corresponding\ eigen\ value\ \lambda = 4$

Eigen Vector corresponding to $\boldsymbol{\lambda = 6}$

$put\ \lambda = 6$ in the above system, we get $\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Longrightarrow 2x_1 - 4x_2 = 0 - - - (1) \qquad 2x_1 - 4x_2 = 0 - - - (2)$

from (1) and (2) we have $x_1 = 2x_2$

Let $x_2 = \alpha \Rightarrow x_1 = 2\alpha$

$$Eigen\ vector = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $is\ eigen\ vector\ of\ matrix\ A\ corresponding\ eigen\ value\ \lambda = 6$

2. **Find the eigen values and the corresponding eigen vectors of matrix** $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

The characteristic equation is $|A-\lambda I|=0$

i.e. $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$$\Longrightarrow (2-\lambda)(2-\lambda)^2 - 0 + [-(2-\lambda)] = 0$$

$$\Longrightarrow (2-\lambda)^3 — (\lambda - 2) = 0$$

$$\Longrightarrow \lambda - 2\, [-(\lambda-2)^2 - 1] = 0$$

$$\Longrightarrow \lambda - 2\ [-\lambda^2 + 4\lambda - 3] = 0$$

$$\Longrightarrow (\lambda - 2)(\lambda - 3)(\lambda - 1) = 0$$

⇨ λ=1, 2, 3

The eigen values of A is 1, 2, 3.

$For\ finding\ eigen\ vector\ the\ system\ is\ (A - \lambda I)X = 0$

$$\Longrightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boldsymbol{Eigen\ vector\ corresponding\ to\ \lambda = 1}$

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_1 + x_3 = 0$

$x_2 = 0$

$x_1 + x_3 = 0$

$x_1 = -x_3, x_2 = 0$

$Let\ x_3 = \alpha$

$\Rightarrow x_1 = -\alpha \quad x_2 = 0, \quad x_3 = \alpha$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-\alpha\\0\\\alpha\end{bmatrix} = \alpha\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$\begin{bmatrix}-1\\0\\1\end{bmatrix}$ $is\ Eigen\ vector$

$\boldsymbol{Eigen\ vector\ corresponding\ to\ \lambda = 2}$

$$\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$Here\ x_1 = 0\ and\ x_3 = 0\ and\ we\ can\ take\ any\ arbitary\ value\ x_2\ i.e\ x_2 = \alpha\ (say)$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\\alpha\\0\end{bmatrix} = \alpha\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$Eigen\ vector\ is$ $\begin{bmatrix}0\\1\\0\end{bmatrix}$

$\boldsymbol{Eigen\ vector\ corresponding\ to\ \lambda = 3}$

$$\begin{bmatrix}-1 & 0 & 1\\0 & -1 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-x_1 + x_3 = 0$

$-x_2 = 0$

$x_1 - x_3 = 0$

$here\ by\ solving\ we\ get\ x_1 = x_3, x_2 = 0\ say\ x_3 = \propto$

$x_1 = \propto \ , \quad x_2 = 0 \quad , x_3 = \propto$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}\alpha\\0\\\alpha\end{bmatrix} = \alpha\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3. Find the Eigen values and Eigen vectors of the matrix is $\begin{bmatrix} 3 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Consider characteristic equation is $|A - \lambda I| = 0$ i.e. $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (8-\lambda)\left[(7-\lambda)(3-\lambda)-(16)\right]+6\left[(-6)(3-\lambda)+8\right]+2\left[24-2(7-\lambda)\right]=0$$
$$\Rightarrow (8-\lambda)\left[21-7\lambda-3\lambda+\lambda^2-16\right]+6\left[-18+6\lambda+8\right]+2\left[24-14+2\lambda\right]=0$$
$$\Rightarrow (8-\lambda)\left[\lambda^2-10\lambda-5\right]+6\left[6\lambda-10\right]+2\left[10+2\lambda\right]=0$$
$$\Rightarrow 8\lambda^2-80\lambda-40-\lambda^3+10\lambda^3+5\lambda+36\lambda-60+20+4\lambda=0$$
$$\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0$$
$$\Rightarrow \lambda\left[-\lambda^2+18\lambda-45\right]=0$$
$$\Rightarrow \lambda=0 \quad (OR) \quad -\lambda^2+18\lambda-45=0$$
$$\Rightarrow \lambda=0, \quad \lambda=3, \quad \lambda=15$$

Eigen Values $\lambda = 0, 3, 15$

Case (i): If $\lambda = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} X = 0$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 8x_1 - 6x_2 + 2x_3 = 0 ----(1)$$
$$-6x_1 + 7x_2 - 4x_3 = 0 ----(2)$$
$$2x_1 - 4x_2 + 3x_3 = 0 ----(3)$$

Consider (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{matrix}$$

$$\Rightarrow \frac{x_1}{21-16} = \frac{-x_2}{-18+8} = \frac{x_3}{24-14} = k$$

$$\Rightarrow \frac{x_1}{5} = \frac{-x_2}{-10} = \frac{x_3}{10} = k$$

$$\Rightarrow x_1 = k, \quad x_2 = 2k, \quad x_3 = 2k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case (ii): If $\lambda = 3$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5x_1 - 6x_2 + 2x_3 = 0 ------(1)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 ------(2)$$

$$2x_1 - 4x_2 + 0 = 0 --------(3)$$

Consider (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{matrix}$$

$$\Rightarrow \frac{x_1}{0-16} = \frac{-x_2}{0+8} = \frac{x_3}{24-8} = k$$

$$\Rightarrow \frac{x_1}{-16} = \frac{-x_2}{8} = \frac{x_3}{16} = k$$

$$\Rightarrow \frac{x_1}{-2} = \frac{-x_2}{1} = \frac{x_3}{2} = k$$

$$\Rightarrow \frac{x_1}{-2} = k, \quad -x_2 = k, \quad x_3 = 2k$$

$$\Rightarrow x_1 = -2k, \quad x_2 = -k, \quad x_3 = 2k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k \\ -k \\ 2k \end{bmatrix} = k\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Case (iii): If $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 + (-6x_2) + 2x_3 = 0 ----(1)$$

$$-6x_1 - 8x_2 - 4x_3 = 0 ------(2)$$

$$2x_1 - 4x_2 - 12x_3 = 0 ------(3)$$

Consider (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{matrix}$$

$$\Rightarrow \frac{x_1}{96-16} = \frac{-x_2}{72+8} = \frac{x_3}{24+16} = k$$

$$\Rightarrow \frac{x_1}{80} = \frac{-x_2}{80} = \frac{x_3}{40} = k$$

$$\Rightarrow \frac{x_1}{2} = k, \quad \frac{x_2}{2} = k, \quad \frac{x_3}{1} = k$$

$$\Rightarrow x_1 = 2k, \quad x_2 = 2k, \quad x_3 = k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} k$

4. Find the Eigen values and the corresponding Eigen vectors of the matrix.

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & -0 \end{bmatrix}$$

Sol: Let $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & -0\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$ i.e. $\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & -\lambda\end{vmatrix}=0$

$$\Rightarrow(-2-\lambda)\left[-\lambda(1-\lambda)-12\right]-2[-2\lambda-6]-3\left[2(-2)+(1-\lambda)\right]=0$$

$$\Rightarrow\lambda^3+\lambda^2-21\lambda-45=0$$
$$\Rightarrow(\lambda+3)(\lambda+3)(\lambda-5)=0$$
$$\Rightarrow\lambda=-3,-3,5$$

The Eigen values are -3,-3, and 5

Case (i): If $\lambda = -3$

We get $\begin{bmatrix}-2+3 & 2 & -3\\ 2 & 1+3 & -6\\ -1 & -2 & 0+3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

The augment matrix of the system is $\left[\begin{array}{ccc|c}1 & 2 & -3 & 0\\ 2 & 4 & -6 & 0\\ -1 & -2 & 3 & 0\end{array}\right]$

Performing R_2-2R_1, R_3+R_1, we get $\left[\begin{array}{ccc|c}1 & 2 & -3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$

Hence we have $x_1+2x_2-3xy=0\Rightarrow x_1=-2x_2+3x_3$

Thus taking $x_2=k_1$ *and* $x_3=k_2$, we get $x_1=-2k_1+3k_2;x_2=k_1;x_3=k_2$

Hence $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=k_1\begin{bmatrix}-2\\1\\0\end{bmatrix}+k_2\begin{bmatrix}3\\0\\1\end{bmatrix}$

So $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ and $\begin{bmatrix}3\\0\\1\end{bmatrix}$ are the Eigen vectors corresponding to $\lambda=-3$

Case (ii): If $\lambda = 5$

We get $\begin{pmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\Rightarrow -7x_1 + 2x_2 - 3x_3 = 0 ----(1)$$
$$2x_1 - 4x_2 - 6x_3 = 0 ----(2)$$
$$-x_1 - 2x_2 - 5x_3 = 0 ----(3)$$

Consider (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{array}$$

$$\Rightarrow \frac{x_1}{20-12} = \frac{-x_2}{-10-6} = \frac{x_3}{-4-4} = k_3$$

$$\Rightarrow \frac{x_1}{8} = \frac{-x_2}{-16} = \frac{-x_3}{-8} = k_3$$

$$\Rightarrow \frac{x_1}{1} = \frac{-x_2}{-2} = \frac{-x_3}{-1} = k_3$$

$$Eigen\ vector\ is \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} k_3$$

5. Find the Eigen values and Eigen vectors of the matrix A and it's inverse where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of "A" is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$
$$\Rightarrow \lambda = 1, 2, 3 \quad i.e.\ Eigen\ Values\ are\ 1, 2, 3$$

Note: In upper Δ^{le} (or) Lower Δ^{lar} of a square matrix the Eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Case (i): If $\lambda = 1$

$$\therefore (A-\lambda I)x=0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_2+4x_3=0; x_2+5x_3=0; 2x_3=0 \quad \Rightarrow x_1=k_1; x_2=0; x_3=0$$

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k_1$$

Case (ii): If $\lambda = 2$

$$\Rightarrow \begin{bmatrix} +1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1+3x_2+4x_3=0; 5x_2=0; x_3=0$$

$$\Rightarrow -x_1+3k+4(0)=0 \Rightarrow -x_1+3k=0 \Rightarrow x_1=3k$$

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Case (iii): If $\lambda = 3$

$$\Rightarrow\begin{bmatrix}1-\lambda & 3 & 4\\ 0 & 2-\lambda & 5\\ 0 & 0 & 3-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}-2 & 3 & 4\\ 0 & -1 & 5\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow -2x_1+3x_2+4x_3=0;-x_2+5x_3=0;\ x_3=0$$

$$Let\quad x_3=k$$

$$\Rightarrow -x_2+5x_3=0\Rightarrow x_2=5k$$

$$and -2x_1+3x_2+4k=0\Rightarrow -2x_1+15k+4k=0$$

$$\Rightarrow -2x_1+19x=0\Rightarrow x_1=\frac{19}{2}k$$

$$X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}\frac{19}{2}k\\ 5k\\ k\end{bmatrix}=k\begin{bmatrix}\frac{19}{2}\\ 5\\ 1\end{bmatrix}$$

Note: Eigen Values of A^{-1} are $\frac{1}{\lambda_1},\frac{1}{\lambda_2},\frac{1}{\lambda_3}\, i.e 1,\frac{1}{2},\frac{1}{3}$ and the Eigen vectors of A^{-1} are same as Eigen vectors of the matrix A

6. Determine the Eigen values and Eigen vectors of

$$B=2A^2-\frac{1}{2}A+3I\ where\ A=\begin{bmatrix}8 & -4\\ 2 & 2\end{bmatrix}$$

Sol $:-$ *Given that* $B=2A-\frac{1}{2}A+3\quad\Rightarrow A=\begin{bmatrix}8 & -4\\ 2 & 2\end{bmatrix}$

$$we\ have\ A^2=A.A=\begin{bmatrix}8 & -4\\ 2 & 2\end{bmatrix}\begin{bmatrix}8 & -4\\ 2 & 2\end{bmatrix}=\begin{bmatrix}56 & -40\\ 20 & -4\end{bmatrix}$$

$$B=2A^2-\tfrac{1}{2}A+3I$$

$$=2\begin{bmatrix}56 & -40\\ 20 & -4\end{bmatrix}-\frac{1}{2}\begin{bmatrix}8 & -4\\ 2 & 2\end{bmatrix}+3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}112 & -80\\ 40 & -8\end{bmatrix}-\begin{bmatrix}4 & -2\\ 1 & 1\end{bmatrix}+\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}111 & -78\\ 39 & -6\end{bmatrix}$$

Characteristic equation of B is $|B-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{vmatrix} = 0$$

i.e. $\Rightarrow \lambda^2 + 105\lambda - 2376 = 0$

$$\Rightarrow (\lambda - 33)(\lambda - 72) = 0$$

$$\Rightarrow \lambda = 33 \quad or \quad 72$$

Eigen Values of B are 33 and 72.

Case (i): If $\lambda = 33$

$$\Rightarrow \begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 78 & -78 \\ 39 & -39 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 39x_1 - 78x_2 = 0 \Rightarrow x_1 = 2x_2$$

$$\frac{x_1}{2} = \frac{x_2}{1} = k(say)$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} k$$

Case (ii): If $\lambda = 72$

$$\Rightarrow \begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 111-72 & -78 \\ 39 & -6-72 \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 39x_1 - 78x_2 = 0 \Rightarrow x_1 = 2x_2$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = k(say)$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} k$$

Properties of Eigen Values:

Theorem 1: The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

Proof: Characteristic equation of A is $|A-\lambda I|=0$

i.e. $\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$ *expanding this we get*

$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)-a_{12}(a\ polynomial\ of\ degree\ n-2)+a_{13}\ (a\ polynomial\ of\ degree\ n$ $-2)+\ldots=0$

$$\Rightarrow(-1)^n\left[\lambda^n-(a_{11}+a_{22}+\ldots.+a_{nn})\lambda^{n-1}+a\ polynomial\ \ of\ \deg ree\,(n-2)\right]=0$$

$$(-1)^n\lambda^n+(-1)^{n+1}(Trace\ A)\lambda^{n-1}+a\ polynomial\ of\ \deg ree\,(n-2)\,in\ \lambda=0$$

If $\lambda_1,\lambda_2\ \ldots..\lambda_n$ *are the roots of this equation*

sum of the roots $=\frac{(-1)^{n+1}Tr(A)}{(-1)^n}=Tr(A)$s

Further $|A-\lambda I|=(-1)^n\lambda^n+\cdots.+a_0$

put $\lambda=0$ *then* $|A|=a_0$

$(-1)^n\lambda^n+a_{n-1}\lambda^{n-1}+a_{n-2}\lambda^{n-2}+\ \ldots.+a_0=0$

Product of the roots $=\dfrac{(-1)^n\,a_0}{(-1)^n}=a_0$

but $a_0=\ |A|=\det A$

Hence the result

Theorem 2: If λ is an eigen value of A corresponding to the eigen vector X, then λ^n is eigen value A^n corresponding to the eigen vector X.

Proof: Since λ is an eigen value of A corresponding to the eigen value X, we have

AX= λX ----------(1)

Pre multiply (1) by A, A(AX) = A(λX)

(AA)X = λ(AX)

A^2X= λ(λX)

A^2X= λ^2X

λ^2 is eigen value of A^2 with X itself as the corresponding eigen vector. Thus the theorm is true for n=2

Let we assume it is true for n = k

i.e. $A^K X=\lambda^K X$------------(2)

Premultiplying (2) by A, we get

$A(A^kX)=A(\lambda^KX)$

$(AA^K)X=\lambda^K(AX)=\lambda^K(\lambda X)$

$A^{K+1}X = \lambda^{K+1}X$

λ^{K+1} is eigen value of A^{K+1} with X itself as the corresponding eigen vector.

Thus, by Mathematical induction. λ^n is an eigen value of A^n.

Theorem 3: A Square matrix A and its transpose A^T have the same eigen values.

Proof: We have $(A-\lambda I)^T = A^T - \lambda I^T$

$= A^T - \lambda I$

$|(A-\lambda I)^T| = |A^T - \lambda I|$ (or)

$|A-\lambda I| = |A^T - \lambda I|$ $\left[\because |A^T| = |A|\right]$

$|A-\lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$

Hence the theorem.

Theorem 4: If A and B are n-rowed square matrices and If A is invertible show that $A^{-1}B$ and BA^{-1} have same eigen values.

Proof: Given A is invertble i.e, A^{-1} exist

We know that if A and P are the square matrices of order n such that P is non-singular then A and $P^{-1}AP$ have the same eigen values.

Taking $A = BA^{-1}$ and P=A, we have

BA^{-1} and $A^{-1}(BA^{-1})A$ have the same eigen values

ie.,BA^{-1} and $(A^{-1}B)(A^{-1}A)$ have the same eigen values

ie.,BA^{-1} and $(A^{-1}B)I$ have the same eigen values

ie.,BA^{-1} and $A^{-1}B$ have the same eigen values

Theorem 5: If $\lambda_1, \lambda_2, \ldots\ldots\ldots\lambda_n$ are the eigen values of a matrix A then $k\lambda_1$, $k\lambda_2$, ….. $k\lambda_n$ are the eigen value of the matrix KA, where K is a non-zero scalar.

Proof: Let A be a square matrix of order n. Then $|KA - \lambda KI| = |K(A-\lambda I)| = K^n|A-\lambda I|$

Since $K \neq 0$, therefore $|KA - \lambda KI| = 0$ if and only if $|A-\lambda I| = 0$

$i.e., K\lambda$ *is an eigen value of* $KA \Leftrightarrow$ *if* λ *is an eigen value of* A

Thus $k\lambda_1, k\lambda_2 \ldots k\lambda_n$ *are the eigen values of the matrix KA if*

$\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of the matrix A

Theorem 6: If λ is an eigen values of the matrix A then λ+k is an eigen value of the matrix A+KI

Proof: Let λ be an eigen value of A and X the corresponding eigen vector. Then by definition

$AX = \lambda X$

Now $(A+KI)X$

$= AX + IKX = \lambda X + KX$

$= (\lambda+K)\,X$

$\lambda+k$ is an eigen value of the matrix A+KI.

Theorem 7: If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A, then $\lambda_1 - K, \lambda_2 - K, \ldots \lambda_n - K,$ *are the eigen values of the matrix* $(A - KI)$, *where K is a non – zero scalar*

Proof: Since $\lambda_1, \lambda_2, \ldots .\lambda_n$ are the eigen values of A.

The characteristic polynomial of A is

$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$-----------------------1

Thus the characteristic polynomial of A-KI is

$|(A - KI) - \lambda I| = |A - (k+\lambda)I|$

$= [\lambda_1 - (\lambda+K)][\lambda_2 - (\lambda+K)] \ldots\ldots [\lambda_n - (\lambda+K)].$

$= [(\lambda_1 - K) - \lambda][(\lambda_2 - K) - \lambda] \ldots\ldots [(\lambda_n - K) - \lambda].$

Which shows that the eigen values of A-KI are $\lambda_1 - K, \lambda_2 - K, \ldots \ldots \lambda_n - K$

Theorem 8: If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A, find the eigen values of the matrix $(A - \lambda I)^2$

Proof: First we will find the eigen values of the matrix A- λI

Since $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A

The characteristics polynomial is

$|A - \lambda I| = (\lambda_1 - K)(\lambda_2 - K) \ldots .(\lambda_n - K)$ ------(1) *where K is scalar*

The characteristic polynomial of the matrix (A- λI) is

$|A - \lambda I - KI| = |A - (\lambda+K)I|$

$= [\lambda_1 - (\lambda + K)]\,[\lambda_2 - (\lambda + K)] \ldots [\lambda_n - (\lambda+K)]$

$= [(\lambda_1 - \lambda) - K)]\,[(\lambda_2 - \lambda) - K] \ldots [(\lambda_n - \lambda) - K)]$

Which shows that eigen values of (A- λI) are $\lambda_1 - \lambda, (\lambda_2 - \lambda) \ldots .\lambda_n - \lambda$

We know that if the eigen values of A are $\lambda_1, \lambda_2 \ldots \lambda_n$ then the eigen values of A^2 are $\lambda_1^2, \lambda_2^2 \ldots .\lambda_n^2$ Thus eigen values of $(A - \lambda I)^2$ *are* $(\lambda_1 - \lambda)^2, (\lambda_2 - \lambda)^2, \ldots .(\lambda_n - \lambda)^2$

Theorem 9: If λ is an eigen value of a non-singular matrix A corresponding to the eigen vector X, then λ^{-1} *is an eigen value of* A^{-1} *and corresponding eigen vector X itself.*

Proof: Since A is non-singular and product of the eigen values is equal to |A|, it follows that none of the eigen values of A is 0.

If λ is an eigen value of the non-singular matrix A and X is the corresponding eigen vector $\lambda \neq 0$ and $AX = \lambda X$.

premultiplying this with A^{-1}, we get $A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow (A^{-1}A)X = \lambda A^{-1}X \Rightarrow IX = \lambda A^{-1}X$

$\therefore X = \lambda A^{-1}X \Rightarrow A^{-1}X = \lambda^{-1}X \ (\lambda \neq 0)$

Hence λ^{-1} is an eigen value of A^{-1}

Theorem 10: If λ is an eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an Eigen value of the matrix AdjA.

Proof: Since λ is an eigen value of a non-singular matrix, therefore $\lambda \neq 0$

Also λ is an eigen value of A implies that there exists a non-zero vector X such that

$AX = \lambda X$ -----(1)

$\Rightarrow (adj\ A)AX = (Adj\ A)(\lambda X)$

$\Rightarrow [(adj\ A)A]X = \lambda(\text{adj A})X$

$\Rightarrow |A|IX = \lambda\,(\text{adj A})X \left[\because (adjA)A = |A|I\right]$

$$\Rightarrow \frac{|A|}{\lambda}X = (adj\ A)X \ or\ (adj\ A)X = \frac{|A|}{\lambda}X$$

Since X is a non – zero vector, therefore the relation (1)

it is clear that $\frac{|A|}{\lambda}$ is an eigen value of the matrix Adj A

Theorem 11: If λ is an eigen value of an orthogonal matrix A, then $\frac{1}{\lambda}$ is also an Eigen value A

Proof: We know that if λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}

Since A is an orthogonal matrix, therefore $A^{-1} = A^{1}$

$\therefore$ $\frac{1}{\lambda}$ is an eigen value of A^{1}

But the matrices A and A^{1} have the same eigen values, since the determinants $|A - \lambda I|$ and $|A^{1} - \lambda I|$ are same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

Theorem 12: If λ is eigen value of A then prove that the eigen value of $B = a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$

Proof: If X be the eigen vector corresponding to the eigen valueλ, then $AX = \lambda X$ --- (1)

Premultiplying by A on both sides

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$

This shows that λ^2 is an eigen value of A^2

We have $B = a_0A^2+a_1A+a_2I$

$\therefore BX = a_0A^2+a_1A+a_2I) X$

$= a_0A^2 X+a_1AX+a_2 X$

$= a_0\lambda^2 X+a_1 \lambda X+a_2X = (a_0\lambda^2 +a_1 \lambda+a_2) X$

$(a_0\lambda^2 +a_1 \lambda+a_2)$ is an eigen value of B and the corresponding eigen vector of B is X.

Theorem 13: Suppose that A and P be square matrices of order n such that P is non singular. Then A and $P^{-1}AP$ have the same eigen values.

Proof: Consider the characteristic equation of $P^{-1}AP$

It is $|(P^{-1}AP)-\lambda I| = |P^{-1}AP-\lambda P^{-1}IP|$ $(\because I = P^{-1}P)$

$= |P^{-1}(A-\lambda I) P| = |P^{-1}|\,|A-\lambda I|\,|P|$

$= |A-\lambda I|$ since $|P^{-1}|\,|P| = 1$

Thus the characteristic polynomials of $P^{-1}AP$ and A are same. Hence the eigen values of $P^{-1}AP$ and A are same.

Corollary 1: If A and B are square matrices such that A is non-singular, then $A^{-1}B$ and BA^{-1} have the same eigen values.

Corollary 2: If A and B are non-singular matrices of the same order, then AB and BA have the same eigen values.

Theorem 14: The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof: Let $A = \begin{vmatrix} a_{11} & a_{12}\ldots\ldots & a_{1n} \\ 0 & a_{22}\ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0\ldots\ldots & a_{nn} \end{vmatrix}$ be a triangular matrix of order n

The characteristic equation of A is $|A-\lambda I|=0$

i.e., $\begin{vmatrix} a_{11-\lambda} & a_{12}\ldots\ldots & a_{1n} \\ 0 & a_{22-\lambda}\ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0\ldots\ldots & a_{nn-\lambda} \end{vmatrix} = 0$

i.e, $(a_{11}-\lambda)(a_{22}-\lambda)\ldots\ldots(a_{nn}-\lambda)=0$

$\Longrightarrow \lambda = a_{11}, a_{22}, \ldots, a_{nn}$

Hence the eigen values of A are $a_{11}, a_{22}, \ldots, a_{nn}$ which are just the diagonal elements of A.

Note: Similarly we can show that the eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Theorem 15: The eigen values of a real symmetric matrix are always real.

Proof: Let λ be an eigen value of a real symmetric matrix A and Let X be the corresponding eigen vector then AX= $\lambda X----(1)$

Take the conjugate $\bar{A}\bar{X} = \bar{\lambda}\,\bar{X}$

Taking the transpose $\bar{X}^T(\bar{A})^T = \bar{\lambda}\,\bar{X}^T$

Since $\bar{A} = A \text{ and } A^T = A, we\ have\ \bar{X}^T\ A = \bar{\lambda}\,\bar{X}^T$

Post multiplying by X, we get $\bar{X}^T\ AX = \bar{\lambda}\,\bar{X}^T X$------- (2)

Premultiplying (1) with $\bar{X}^T$, we get $\bar{X}^T AX = \lambda \bar{X}^T X$ ------ (3)

(1) – (3) gives $(\lambda - \bar{\lambda})\bar{X}^T X = 0$ but $\bar{X}^T X \neq 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda - \bar{\lambda} \Rightarrow \lambda$ is real. Hence the result follows

Theorem 16: For a real symmetric matrix, the eigen vectors corresponding to two distinct eigen values are orthogonal.

Proof: Let λ_1, λ_2 be eigen values of a symmetric matrix A and let X_1, X_2 be the corresponding eigen vectors.

Let $\lambda_1 \neq \lambda_2$. We want to show that X_1 is orthogonal to X_2 (i.e., $X_1^T X_2 = 0$)

Sice X_1, X_2 are eigen values of A corresponding to the eigen values λ_1, λ_2 we have

$AX_1 = \lambda_1 X_1$ ----- (1) $AX_2 = \lambda_2 X_2$ ------- (2)

Premultiply (1) by X_2^T

$\Longrightarrow\ X_2^T A X_1 = \lambda_1 X_2^T X_1$

Taking transpose to above, we have

$\Rightarrow X_2^T A^T (X_2^T)^T = \lambda_1 X_1^T (X_2^T)^T$

$i.e, X_1^T A X_2 = \lambda_1 X_1^T X_2$-------- (3) s

Premultiplying (2) by $X_1^T, we\ get\ X_1^T A X_2 = \lambda_2 X_1^T X_2 - - - - - (4)$

Hence from (3) and (4) we get

$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$

$\Longrightarrow\ X_1^T X_2 = 0$

$(\because \lambda_1 \neq \lambda_2)$

$X_1\ is\ orthogonal\ to\ X_2$

Note: If λ is an eigen value of A and f(A) is any polynomial in A, then the eigen value of f(A) is f(λ) .

Theorem17: The Eigen values of a Hermitian matrix are real.

Proof: Let A be Hermitian matrix. If X be the Eigen vector corresponding to the eigen value λ of A, then AX = λ X -------------------- (1)

Pre multiplying both sides of (1) by X^θ ,we get

$X^\theta AX = \lambda X^\theta X$ ----------------------- (2)

Taking conjugate transpose of both sides of (2)

We get $\left(X^\theta AX\right)^\theta = \left(\lambda X^\theta X\right)^\theta$

i.e $X^\theta A^\theta \left(X^\theta\right)^\theta = \bar{\lambda} X^\theta \left(X^\theta\right)^\theta \left[\because (ABC)^\theta = C^\theta B^\theta A^\theta \; and \; (KA)^\theta = \bar{K} A^\theta\right]$

(or) $X^\theta A^\theta X = \bar{\lambda} X^\theta X \left[\because \left(X^\theta\right)^\theta = X, \left(A^\theta\right)^\theta = A\right]$-------------------- (3)

From (2) and (3), we have

$$\lambda X^\theta X = \bar{\lambda} X^\theta X$$

i.e $\left(\lambda - \bar{\lambda}\right) X^\theta X = 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$$\Rightarrow \lambda = \bar{\lambda} \left(\because X^\theta X \neq 0\right)$$

$\therefore$ Hence λ is real.

Note: The Eigen values of a real symmetric are all real

Corollary: The Eigen values of a skew-Hermitian matrix are either purely imaginary (or) Zero

Proof: Let A be the skew-Hermitian matrix

If X be the Eigen vector corresponding to the Eigen value λ of A, then

$AX = \lambda X (or) (iA) X = (i\lambda) X$

From this it follows that $i\lambda$ is an Eigen value of iA

Which is Hermitian (since A is skew-hermitian)

$\therefore A^\theta = -A$

Now $(iA)^\theta = \bar{i} A^\theta = -iA^\theta = -i(-A) = iA$

Hence $i\lambda$ is real. Therefore λ must be either

Zero or purely imaginary.

Hence the Eigen values of skew-Hermitian matrix are purely imaginary or zero

Theorem 18: The Eigen values of an unitary matrix have absolute value l.

Proof: Let A be a square unitary matrix whose Eigen value is λ with corresponding eigen vector X

$$\Rightarrow \quad AX = \lambda X \rightarrow (1)$$

$$\Rightarrow \quad \overline{A}\overline{X} = \bar{\lambda}\overline{X} \Rightarrow \overline{X}^T \overline{A}^T = \bar{\lambda} X^T \rightarrow (2)$$

Since A is unitary, we have $\left(\overline{A}\right)^{T} A = I \rightarrow (3)$

(1) and (2) given $\overline{X}^{T}\overline{A}^{T} \ (AX) = \lambda\overline{\lambda}\overline{X}^{T} X$

i.e $\overline{X}^{T} X = \lambda\overline{\lambda}\overline{X}^{T} X$ From (3)

$\Rightarrow \overline{X}^{T} X\left(1-\lambda\overline{\lambda}\right) = 0$

Since $\overline{X}^{T} X \neq 0$,we must have $1-\lambda\overline{\lambda} = 0$

$$\Rightarrow \lambda\overline{\lambda} = 1$$

Since $|\lambda| = \left|\overline{\lambda}\right|$

We must have $|\lambda|=1$

Note 1: From the above theorem, we have "The characteristic root of an orthogonal matrix is of unit modulus".

2. The only real eigen values of unitary matrix and orthogonal matrix can be $\pm$ 1

Theorem 19: Prove that transpose of a unitary matrix is unitary.

Proof: Let A be a unitary matrix, then $A.A^{\theta} = A^{\theta}.A = I$

where A^{θ} is the transposed conjugate of A.

$$\therefore \left(AA^{\theta}\right)^{T} = \left(A^{\theta}A\right)^{T} = (I)^{T}$$

$$\therefore \left(AA^{\theta}\right)^{T} = \left(A^{\theta}A\right)^{T} = (I)^{T}$$

$$\Rightarrow \left(A^{\theta}\right)^{T} A^{T} = A^{T}\left(A^{\theta}\right)^{T} = I$$

$$\Rightarrow \left(A^{T}\right)^{\theta} A^{T} = A^{T}\left(A^{T}\right)^{\theta} = I$$

Hence A^{T} is a unitary matrix.

Solved Problems:

1. For the matrix $A=\begin{bmatrix}1 & 2 & -3\\0 & 3 & 2\\0 & 0 & -2\end{bmatrix}$ find the Eigen values of $3A^3+5A^2-6A+2I$

Sol: The Characteristic equation of A is $|A-\lambda I|=0$ i.e. $\begin{vmatrix}1-\lambda & 2 & -3\\0 & 3-\lambda & 2\\0 & 0 & -2-\lambda\end{vmatrix}=0$

$\Rightarrow (1-\lambda)(3-\lambda)(-2-\lambda)=0$

$\therefore$ Eigen values are 1, 3,-2.

If λ is the Eigen value of A. and F (A) is the polynomial in A then the Eigen value of f (A) is f (λ)

Let $f(A)=3A^3+5A^2-6A+2I$

$\therefore$ Eigen Value of f (A) are f (1), f (-2), f (3)

f (1) = 3+5-6+2 = 4

f (-2) = 3(-8)+5(4)-6(-2)+2 = -24+20+12+2 = 10

f (3) = 3(27)+5(9)+6(3)+2 = 81+45-18+2 = 110

The Eigen values of f (a) are f (λ) = 4, 10,110

2. Find the eigen values and eigen vectors of the matrix A and its inverse, where

A = $\begin{bmatrix}1 & 3 & 4\\0 & 2 & 5\\0 & 0 & 3\end{bmatrix}$

Sol: Given A = $\begin{bmatrix}1 & 3 & 4\\0 & 2 & 5\\0 & 0 & 3\end{bmatrix}$

The characteristic equation of A is given by |A-λI| = 0

$\Longrightarrow \begin{vmatrix}1-\lambda & 3 & 4\\0 & 2-\lambda & 5\\0 & 0 & 3-\lambda\end{vmatrix}=0$

$\Longrightarrow (1-\lambda)[(2-\lambda)(3-\lambda)]=0$

$\Longrightarrow \lambda=1,2,3$

Charecetstic roots are 1, 2, 3.

Case (i): If $\lambda=1$

$$For\ \lambda = 1, becomes \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 3x_2 + 4x_3 = 0$

$x_2 + 5x_3 = 0$

$2x_3 = 0$

$x_2 = 0, x_3 = 0$ and $x_1 = \alpha$

$$X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is the solution where α is arbitrary constant

$$\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is the eigen vector corresponding to $\lambda = 1$

Case (i): If $\lambda = 2$

$$For\ \lambda = 2, becomes \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 + 3x_2 + 4x_3 = 0$

$5x_3 = 0 \Rightarrow x_3 = 0$

$-x_1 + 3x_2 = 0 \Rightarrow x_1 = 3x_2$

$Let\, x_2 = k$

$x_1 = 3k$

$$X = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

is the solution where k is arbitrary constant

$$\therefore X = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$ is the eigen vector corresponding to $\lambda = 2$

Case (iii): If $\lambda = 3$

$$For\ \lambda = 3, becomes \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \ -2x_1 + 3x_2 + 4x_3 = 0$

$-x_2 + 5x_3 = 0$

$Say\ x_3 = K \implies x_2 = 5K$

$x_1 = \frac{19}{2}K$

X = $\begin{bmatrix} \frac{19}{2}K \\ 5K \\ K \end{bmatrix} = \frac{K}{2}\begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$ is the solution, where k/2 is arbitrary constant.

$\therefore X = \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$

Eigen values of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$.

We know Eigen vectors of A^{-1} are same as eigen vectors of A.

3. Find the eigen values of A=$\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

Sol: we have A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

So $\overline{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$ and $A^T = \begin{bmatrix} 3i & -2+i \\ 2+i & -i \end{bmatrix}$

$\Rightarrow \overline{A} = -A^T$

Thus A is a skew-Hermitian matrix.

$\therefore$ The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow A^T = \begin{vmatrix} 3i-\lambda & -2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 2i\lambda + 8 = 0$

$\Rightarrow \lambda = 4i, -2i$ are the Eigen values of A

4.Find the eigen values of $A = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

$Now\ \overline{A} = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$ and $(\overline{A})^T = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$

We can see that $\overline{A}^T . A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus A is a unitary matrix

$\therefore$ The characterstic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} \frac{1}{2}i-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i-\lambda \end{vmatrix}=0$$

Which gives $\lambda=\frac{\sqrt{3}}{2}+i\frac{1}{2}\, and\, \frac{-\sqrt{3}}{2}+\frac{1}{2}i$ and

Hence above λ values are Eigen values of A.

Cayley-Hamilton Theorem: Every Square Matrix satisfies its own characteristic equation

To find Inverse of matrix: If A is non-singular Matrix, then A^{-1} exists, Pre multiplying (1) above by A^{-1} we have $a_0A^{n-1}+a_1A^{n-2}+---+a_{n-1}I+a_nA^{-1}=0$,

$$A^{-1}=\frac{1}{a_n}\left[a_0A^{n-1}+a_1A^{n-2}+---a_{n-1}I\right]$$

To find the powers of A: - Let K be a +ve integer such that $K\geq n$

Pre multiplying (1) by A^{k-n} we get $a_0A^K+a_1A^{K-1}+---+a_nA^{k-n}=0$,

$$A^K=\frac{-1}{a_0}\left[a_1A^{k-1}+a_2A^{K-1}---+a_nA^{K-n}\right]$$

Solved Problems :

1. S.T the matrix A = $\begin{bmatrix}1 & -2 & 1\\ 1 & -2 & 3\\ 0 & -1 & 2\end{bmatrix}$ **satisfies its characteristic equation and hence find A^{-1}**

Sol: Characteristic equation of A is det (A-λI) = 0

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix}=0$$

$C_2 \rightarrow C_2+C_3$ $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix}=0$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2-\lambda \end{vmatrix}=0$$

$$\Rightarrow \lambda^3-\lambda^2+\lambda-1=0$$

By Cayley – Hamilton theorem, we have $A^3-A^2+A-I=0$

$$A=\begin{bmatrix}1 & -2 & 2\\ 1 & -2 & 3\\ 0 & -1 & 2\end{bmatrix} \quad A^2=\begin{bmatrix}-1 & 0 & 0\\ -1 & -1 & 2\\ -1 & 0 & 1\end{bmatrix} \quad A^3=\begin{bmatrix}-1 & 2 & -2\\ -2 & 2 & -1\\ -1 & 1 & 0\end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiplying with A^{-1} we get $A^2 - A + I = A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

2. Using Cayley - Hamilton Theorem find the inverse and A^4 of the matrix

A $= \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by $|A-\lambda I|=0$ *i.e.,* $\begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley – Hamilton theorem we have $A^3-5A^2+7A-3I=0$…..(1)

Multiply with A^{-1} we get

$$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

$$A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \quad A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

multiplying (1) with A,we get,

$A^4-5A^3+7A^2-3A= 0$

$A^4 = 5A^3-7A^2+3A$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

3. *If* $A=\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}$ **Verify Cayley-Hamilton theorem hence find A^{-1}**

Sol: - Given that $A=\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

i.e. $\begin{vmatrix}2-\lambda & 1 & 2\\ 5 & 3-\lambda & 3\\ -1 & 0 & -2-\lambda\end{vmatrix}=0$

$$\Rightarrow(2-\lambda)\left[-6-3\lambda+2\lambda+\lambda^2\right]-1\left[-10-5\lambda+3\right]+2\left[0+(3-\lambda)\right]$$

$$\Rightarrow(2-\lambda)\left[\lambda^2-\lambda-6\right]-1\left[-5\lambda-7\right]+2\left[3-x\right]=0$$

$$\Rightarrow 2\lambda^2-2\lambda-12-\lambda^3+\lambda^2+6\lambda+5\lambda+7+6-2\lambda=0$$

$$\Rightarrow -\lambda^3+3\lambda^2+7\lambda+1=0$$

$$\Rightarrow \lambda^3-3\lambda^2-7\lambda-1=0----(1)$$

According to Cayley Hamilton theorem. Square matrix 'A' satisfies equation (1)

Substitute A in place of λ

Now $A=\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}$

$$A^2=\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}=\begin{bmatrix}7 & 5 & 3\\ 22 & 14 & 13\\ 0 & -1 & 2\end{bmatrix}$$

$$A^3=A^2.A=\begin{bmatrix}7 & 5 & 3\\ 22 & 14 & 13\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}=\begin{bmatrix}36 & 22 & 23\\ 101 & 64 & 60\\ -7 & -3 & -7\end{bmatrix}$$

Now $A^3-3A^2-7A-I=0$

$$\Rightarrow\begin{bmatrix}36 & 22 & 23\\ 101 & 64 & 60\\ -7 & 3 & -7\end{bmatrix}+\begin{bmatrix}21 & -15 & -9\\ -66 & -42 & -39\\ 0 & 3 & -6\end{bmatrix}+\begin{bmatrix}14 & -7 & -14\\ -35 & -21 & -21\\ 7 & 0 & 14\end{bmatrix}+\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0$$

Cayley –Hamilton theorem is verified.

To find A^{-1}

$$\Rightarrow A^3-3A^2-7A-I=0$$

Multiply A^{-1}, we get

$$A^{-1}\left(A^3-3A^2-7A-I\right)=0$$

$$\Rightarrow A^2-3A-7I-A^{-1}=0$$

$$\Rightarrow A^{-1}=A^2-3A-7I$$

$$\therefore A^{-1}=\begin{bmatrix}7 & 5 & 3\\ 22 & 14 & 13\\ 0 & -1 & 2\end{bmatrix}+\begin{bmatrix}-6 & -3 & -6\\ 15 & -9 & 9\\ 3 & 0 & 6\end{bmatrix}+\begin{bmatrix}-7 & 0 & 0\\ 0 & -7 & 0\\ 0 & 0 & -7\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-6 & 2 & -3\\ 7 & -2 & 4\\ 3 & -1 & 1\end{bmatrix}$$

Check $A.A^{-1}=I$

$$A.A^{-1}=\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}\begin{bmatrix}-6 & 2 & -3\\ 7 & -2 & 4\\ 3 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

4. Using Cayley – Hamilton theorem, find A^8, if $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$

Sol: Given $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$

Characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow (1-\lambda)(-1-\lambda)-4=0$$

$$\Rightarrow \lambda^2-5=0 ----(1)$$

Substitute A in place of λ

$$A^2-5I=0\Rightarrow A^2=5I$$

$$find\ A^8$$

$$\therefore A^8=5A^6=5(A^2)(A^2)(A^2)$$

$$=5(5I)(5I)(5I)$$

$$=625I$$

$$\Rightarrow A^8=625I$$