Solution of Non-homogeneous Linear Differential Equations

Operator Method for finding Particular Integrals

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X, \quad a_0 \frac{d^ny}{dx^n} + \cdots + a_n y = X \quad \text{—(1)}$$

$\Rightarrow \ (a_0 D^2 + a_1 D + a_2)\, y = X$

$\Rightarrow \ F(D)\, y = X$, where X is a function of x.

General solution = Complementary function + Particular Integral

Complementary function

It is the general solution of the homogeneous equation

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0.$$

It contains n arbitrary constants if the given differential equations is

$$a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n y = 0.$$

Particular Integral-

If v is any particular sol which satisfies eq (1).

$$a_0 \frac{d^nv}{dx^n} + \cdots + a_n v = X.$$

P.I. is free from arbitrary constants.

Operator Method for finding particular Integral

$D \to$ differential operator

$D^{-1} \to$ Integral operator.

$$f(D)\, y = X$$

P.I. is $y(x) = \frac{1}{f(D)} X$.

Case 1 When $X = e^{ax}$.

P.I is $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$, provided $f(a) \neq 0$.

Ex. $y'' - 2y' - 3y = 3e^{2x}$.

Sol: C.F: $(D^2 - 2D - 3)\, y = 0$

$m^2 - 2m - 3 = 0 \Rightarrow m^2 - 3m + m - 3 = 0 \Rightarrow (m+1)(m-3) = 0$

$m = -1, 3.$

$y_c(x) = c_1 e^{-x} + c_2 e^{3x}$.

P.I. $y_p(x) = \frac{1}{f(D)} 3e^{2x} = 3 \frac{1}{D^2 - 2D - 3} e^{2x}$

$$= 3 \frac{1}{2^2 - 2(2) - 3} e^{2x}$$

$$= \frac{3}{-3} e^{2x}$$

$$= -e^{2x}.$$

The general sol is

$$y = y_c(x) + y_p(x)$$

$$\Rightarrow y(x) = c_1 e^{-x} + c_2 e^{3x} - e^{2x}.$$

<u>Ex</u> $\quad y''' - 2y'' - 5y' + 6y = 4e^{-x} - e^{2x}$

<u>Sol</u>: $(D^3 - 2D^2 - 5D + 6)y = 4e^{-x} - e^{2x}$

<u>C.F.</u>: $m^3 - 2m^2 - 5m + 6 = 0$

$6 = 1, -1, 2, -2, 3, -3$

1	1	-2	-5	6
		1	-1	-6
	1	-1	-6	0

$\Rightarrow (m-1)(m^2 - m - 6) = 0$

$\Rightarrow (m-1)(m^2 - 3m + 2m - 6) = 0$

$\Rightarrow (m-1)(m+2)(m-3) = 0$

$\Rightarrow m = 1, -2, 3.$

$$y_c(x) = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x}.$$

<u>P.I</u>: $y_p(x) = \dfrac{1}{D^3 - 2D^2 - 5D + 6}\, 4e^{-x} - e^{2x}$

$$= 4 \cdot \frac{1}{D^3 - 2D^2 - 5D + 6} e^{-x} - \frac{1}{D^3 - 2D^2 - 5D + 6} e^{2x}$$

$$= \frac{4}{-1-2+5+6} e^{-x} - \frac{1}{8-8-10+6} e^{2x}$$

$$= \frac{4}{8} e^{-x} - \frac{e^{2x}}{\cancel{2}\,-4}$$

$$= \frac{e^{-x}}{2} + \frac{e^{2x}}{4}$$

G.S. is

$$y = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x} + \frac{e^{-x}}{2} + \frac{e^{2x}}{4}.$$

Ex Find P.I. of $(D^3 - D^2 - D + 1)y = e^x$

Sol :

$$y_p(x) = \frac{1}{D^3 - D^2 - D + 1} e^x$$

$$= \frac{1}{1 - 1 - 1 + 1} e^x$$

$$= \frac{1}{0} e^x$$

$$y_p(x) = x \frac{1}{3D^2 - 2D - 1} e^x$$

$$= x \frac{1}{3 - 2 - 1} e^x$$

$$= \frac{x}{0} e^x$$

$$y_p(x) = x^2 \frac{1}{6D - 2} e^x$$

$$= x^2 \frac{1}{6 - 2} e^x = \frac{x^2 e^x}{4}.$$

Case of failure, if $f(a) = 0$.

Then we proceed as

$$\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(D)} e^{ax}$$

$$= x \frac{1}{f'(a)} e^{ax}, \text{ provided } \cancel{f'(D) \neq} \; f'(a) \neq 0.$$

Further, if $f'(a) = 0$, then again its is a case of failure.

We further proceed as

$$\frac{1}{f(D)} e^{ax} = x^2 \cdot \frac{1}{f''(D)} e^{ax}$$

$$= \frac{x^2}{f''(a)} e^{ax}, \quad f''(a) \neq 0.$$

Ex : Find the general solution of
$$9y''' + 3y'' - 5y' + y = 42e^{x} + 64e^{x/3}$$

Sol : $(9D^3 + 3D^2 - 5D + 1)\,y = 42e^{x} + 64e^{x/3}$

C.F

$9m^3 + 3m^2 - 5m + 1 = 0$

$$\begin{array}{r|rrrr} -1 & 9 & 3 & -5 & 1 \\ & & -9 & 6 & -1 \\ \hline & 9 & -6 & 1 & 0 \end{array}$$

$(m+1)(9m^2 - 6m + 1) = 0$

$(m+1)(3m-1)^2 = 0$

$m = -1, \frac{1}{3}, \frac{1}{3}$

$$y_c(x) = c_1 e^{-x} + (c_2 + x c_3) e^{x/3}$$

P.I. $y_p(x) = \dfrac{1}{9D^3 + 3D^2 - 5D + 1}\; 42e^{x} + 64e^{x/3}$

$$= 42 \frac{1}{9D^3 + 3D^2 - 5D + 1} e^{x} + 64 \frac{1}{9D^3 + 3D^2 - 5D + 1} e^{x/3}$$

$$= 42 \frac{1}{9 + 3 - 5 + 1} e^{x} + 64 \frac{1}{\cancel{9}\cdot\frac{1}{\cancel{27}_{3}} + \cancel{3}\cdot\frac{1}{\cancel{9}_{3}} - \frac{5}{3} + 1} e^{x/3}$$

$\frac{1}{3} + \frac{1}{3} + 1 = \frac{5}{3}$

$$= \frac{42}{8} e^{x} + \frac{64}{0} e^{x/3}$$

$\downarrow$

Case of failure

$$= \frac{21}{4} e^{x} + x \frac{64}{27D^2 + 6D - 5} e^{x/3}$$

$$= \frac{21e^{x}}{4} + x \frac{64}{27\cdot\frac{1}{9} + \frac{6}{3} - 5} e^{x/3}$$

$$= \frac{21e^{x}}{4} + \frac{x(64)}{3 + 2 - 5} e^{x/3} = \frac{21e^{x}}{4} + \underbrace{\frac{64x}{0}\, e^{x/3}}$$

Case of failure

$$= \frac{21e^x}{4} + x^2 \frac{64}{54D+6} e^{x/3}$$

$$= \frac{21e^x}{4} + \frac{x^2(64)}{18+6} e^{x/3}$$

$$= \frac{21e^x}{4} + \frac{64x^2 e^{x/3}}{24} \cancel{e^{x/3}}$$

$$= \frac{21e^x}{4} + \frac{8x^2 e^{x/3}}{3}$$

G.S. is

$$y(x) = C_1 e^{-x} + (C_2 + xC_3) e^{x/3} + \frac{21e^x}{4} + \frac{8x^2 e^{x/3}}{3}.$$

Case 2 When $X = \sin(ax+b)$ or $\cos(ax+b)$

$$y_p(x) = \frac{1}{f(D^2)} \sin(ax+b)$$

$$= \frac{\sin(ax+b)}{f(-a^2)}, \quad f(-a^2) \neq 0.$$

Ex :- Find G.S. of $y'' + 4y = 6\cos x$

Sol :- $(D^2+4)y = 6\cos x$

C.F :- $m^2 + 4 = 0$

$m = \pm 2i$

$$\frac{-0 \pm \sqrt{0 - 4(4)}}{2}$$

$y_c(x) = C_1 \cos 2x + C_2 \sin 2x$

$$= \pm \frac{4i}{2} = \pm 2i.$$

P.I $\Rightarrow$ $y_p(x) = \dfrac{1}{D^2+4} 6\cos x$

$= \dfrac{1}{-1^2+4} 6\cos x = \dfrac{6\cos x}{3} = 2\cos x.$

G.S. is $y = C_1\cos 2x + C_2\sin 2x + 2\cos x.$

Ex Find general sol of $y''' - y'' + 4y' - 4y = \sin 3x$

Sol: $(D^3 - D^2 + 4D - 4)y = \sin 3x$

C.F $\cancel{\text{is}}$ $\cancel{y_c(x) =}$ $m^3 - m^2 + 4m - 4 = 0$

$m^2(m-1) + 4(m-1) = 0$

$(m^2+4)(m-1) = 0$

$m = 1, \pm 2i$

$y_c(x) = C_1 e^x + C_2\cos 2x + C_3\sin 2x.$

P.I $y_p(x) = \dfrac{1}{D^3 - D^2 + 4D - 4} \sin 3x$

$= \dfrac{1}{D^2 \cdot D - (-3^2) + 4D - 4} \sin 3x$

$= \dfrac{1}{D(-3^2) + 9 + 4D - 4} \sin 3x$

$= \dfrac{1}{-9D + 5 + 4D} \sin 3x$

$= \dfrac{1}{5 - 5D} \sin 3x$

$= \dfrac{5+5D}{25 - 25D^2} \sin 3x = \dfrac{5+5D}{25 - 25(-9)} \sin 3x = \dfrac{5+5D}{250} \sin 3x$

$$= \frac{1}{50}\sin 3x + \frac{1}{50}\frac{d}{dx}(\sin 3x)$$

$$= \frac{\sin 3x}{50} + \frac{1}{50}(3\cos 3x)$$

$$= \frac{\sin 3x}{50} + \frac{3\cos 3x}{50}$$

G.S. is

$$y = C_1 e^{x} + C_2\cos 2x + C_3 \sin 2x + \frac{\sin 3x}{50} + \frac{3\cos 3x}{50}$$

Ex :- Find the P.I. of $(D^2+4)y = \cos 2x$

Sol :- $$\text{P.I} = \frac{1}{D^2+4}\cos 2x$$

$$= \frac{1}{-2^2+4}\cos 2x$$

$$= \frac{1}{0}\cos 2x \rightarrow \text{Case of failure}$$

$$\text{P.I.} = \frac{x}{2D}\cos 2x$$

$$= \frac{x}{2}\int \cos 2x \, dx$$

$$= \frac{x}{2}\,\frac{\sin 2x}{2}$$

$$= \frac{x\sin 2x}{4}$$

In Case of failure when $f(-a^2)=0$, we proceed as

$$\frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(D^2)}\sin(ax+b)$$

$$= \frac{x}{f'(-a^2)}\sin(ax+b), \quad f'(-a^2)\neq 0.$$

If $f'(a^2)=0$, then we further proceed as

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{x^2\sin(ax+b)}{f''(-a^2)}, \quad f''(-a^2)\neq 0.$$

and so on.

Ex. 5.7

37 $(D^4+5D^2+4)y = 16\sin x + 64\cos 2x$

Sol: C.F.

$$m^4+5m^2+4=0$$

$$m^4+4m^2+m^2+4=0$$

$$(m^2+1)(m^2+4)=0$$

$$m = \pm i, \pm 2i$$

$$y_c(x) = C_1\cos x + C_2\sin x + C_3\cos 2x + C_4\sin 2x.$$

P.I.

$$y_p(x) = \frac{16}{D^4+5D^2+4}\sin x + \frac{64}{D^4+5D^2+4}\cos 2x$$

$$= 16\frac{1}{(-1^2)(-1^2)+5(-1^2)+4}\sin x + \frac{64}{(-2^2)(-2^2)+5(-2^2)+4}\cos 2x$$

$$= \frac{16\sin x}{1-5+4} + \frac{64}{16-20+4}\cos 2x \rightarrow \text{Case of failure}$$

$$y_p(x) = \frac{16x}{4D^3+10D}\sin x + \frac{64x}{4D^3+10D}\cos 2x$$

$$= \frac{16x}{4D(-1)+10D}\sin x + \frac{64x}{4D(-4)+10D}\cos 2x$$

$$= \frac{16x}{6D}\sin x + \frac{64x}{-6D}\cos 2x$$

$$= \frac{8x}{3}\int \sin x\,dx - \frac{32x}{3}\int \cos 2x\,dx$$

$$= \frac{8x}{3}(-\cos x) - \frac{32x}{3}\frac{\sin 2x}{2}$$

$$= \frac{-8x\cos x - 16x\sin 2x}{3}$$

G.S. is

$$y = C_1\cos x + C_2\sin x + C_3\cos 2x + C_4\sin 2x$$
$$-8x(\cos x + 2\sin 2x)/3.$$

Case 3

When $X = x^m$, a polynomial of degree m, m is positive integer.

From $f(D)$, take the lowest degree term outside so that the remaining expression in $f(D)$ becomes $[1 \pm \phi(D)]$. Take it to numerator and expand it.

Useful Results

1. $(1-D)^{-1} = 1+D+D^2+ - - -$
2. $(1+D)^{-1} = 1-D+D^2- - - - - -$
3. $(1-D)^{-2} = 1+2D+3D^2+ - - -$
4. $(1-D)^{-3} = 1+3D+6D^2+ - - - -$
5. $(1+D)^{2} = 1-2D+3D^2- - - - -$

Ex Find the G.S. of ~~$y''+4y'$~~ =

$$y''+16y = 64x^2$$

Sol: C.F: $y''+16y=0$

$$y_c(x) = C_1 \cos 4x + C_2 \sin 4x$$

P.I $y_p = \dfrac{1}{D^2+16} 64x^2$

$$= \frac{64}{D^2+16} x^2$$

$$= \frac{64}{D^2\left(1+\frac{16}{D^2}\right)} x^2 = \frac{64}{D^2}\left[1+\frac{16}{D^2}\right]^{-1} x^2$$

$$= \cancel{\frac{64}{D^2}\Big[}$$

$$= \frac{64}{16\left(1+\frac{D^2}{16}\right)} x^2$$

$$= \frac{64}{16}\left(1+\frac{D^2}{16}\right)^{-1} x^2$$

$$= 4\left(1-\frac{D^2}{16}+\frac{D^4}{(16)^2} ----\right) x^2$$

$$= 4\left[x^2 - \frac{1}{16}(2) + 0 ---\right]$$

$$= 4x^2 - \frac{1}{2}$$

G.S. is $y = C_1 \cos 4x + C_2 \sin 4x + 4x^2 - \frac{1}{2}$

Ex. 5.7

38) $(D^2+25)y = 9x^3 + 4x^2$

C.F. $\quad m^2 + 25 = 0$

$m = \pm 5i$

$y_c(x) = C_1 \cos 5x + C_2 \sin 5x.$

P.I. $\quad y_p(x) = \dfrac{1}{D^2+25}(9x^3 + 4x^2)$

$$= 9\frac{1}{D^2+25}x^3 + 4\frac{1}{D^2+25}x^2$$

$$= 9\frac{1}{25\left(1+\frac{D^2}{25}\right)}x^3 + 4\frac{1}{25\left(1+\frac{D^2}{25}\right)}x^2$$

$$= \frac{9}{25}\left(1+\frac{D^2}{25}\right)^{-1}x^3 + \frac{4}{25}\left(1+\frac{D^2}{25}\right)^{-1}x^2$$

$$= \frac{9}{25}\left[1-\frac{D^2}{25}+\frac{D^4}{625}-\cdots\right]x^3 + \frac{4}{25}\left[1-\frac{D^2}{25}+\frac{D^4}{25}--\right]x^2$$

$$= \frac{9}{25}\left[x^3-\frac{1}{25}(6x)\right] + \frac{4}{25}\left[x^2-\frac{1}{25}(2)\right]$$

$$D^2(x^3) = D(3x^2) = 6x.$$
$$D^2(x^2) = D(2x) = 2$$

$$= \frac{9x^3}{25} - \frac{54x}{625} + \frac{4x^2}{25} - \frac{8}{625}$$

G.S. is $y = C_1\cos 5x + C_2\sin 5x + \frac{9x^3}{25} + \frac{4x^2}{25} - \frac{54x}{625} - \frac{8}{625}$

(39) $(D^2+6D+9)y = 4x^2-1$

<u>C.F.</u> $m^2+6m+9=0$

$(m+3)^2=0$

$m=-3,-3.$

$$y = (C_1 + xC_2)e^{-3x}$$

$$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \text{---}$$

$$\underline{P.I.}\ y_p(x) = \frac{1}{(D+3)^2}\ \cancel{e}\ 4x^2 - 1$$

$$= \frac{1}{9\left(1+\frac{D}{3}\right)^2}(4x^2-1)$$

$$= \frac{1}{9}\left(1+\frac{D}{3}\right)^{-2}(4x^2-1)$$

$$= \frac{1}{9}\left[1 - \frac{2D}{3} + \frac{3D^2}{9} - \frac{4D^3}{27} + \text{--}\right](4x^2-1)$$

$$= \frac{1}{9}\left[4x^2 - 1 - \frac{2}{3}(8x) + \frac{3}{9}(8) + 0\right]$$

$$= \frac{1}{9}\left[4x^2 - 1 - \frac{16x}{3} + \frac{24}{9}\right]$$

$$= \frac{1}{9}\left[4x^2 - 1 - \frac{16x}{3\cancel{9}} + \frac{8}{3}\right]$$

$$= \frac{1}{27}\left[12x^2 - 3 - 16x + 8\right]$$

$$= \frac{12x^2 - 16x + 5}{27}$$

G.S is

$$y = (C_1 + xC_2)e^{-3x} + \frac{12x^2 - 16x + 5}{27}$$

(40) $(D^2-2D-3)y = 2x^2+6x$

A.E. : $m^2-2m-3=0$

$m^2-3m+m-3=0$

$(m+1)(m-3)=0$

C.F. $m=-1, 3.$

$y_c(x) = C_1 e^{-x} + C_2 e^{3x}$

P.I. $y_p(x) = \dfrac{1}{D^2-2D-3}\, 2x^2+6x$

$$= \frac{1}{(-3)\left(1-\frac{D^2}{3}+\frac{2D}{3}\right)}(2x^2+6x)$$

$$= \frac{-1}{3}\left(1-\left(\frac{D^2}{3}-\frac{2D}{3}\right)\right)^{-1}(2x^2+6x)$$

$$= \frac{-1}{3}\left[1+\frac{D^2}{3}-\frac{2D}{3}+\left(\frac{D^2}{3}-\frac{2D}{3}\right)^2\right](2x^2+6x)$$

$$= \frac{-1}{3}\left[2x^2+6x+\frac{1}{3}(4)-\frac{2}{3}(4x+6)+\frac{4}{9}(4)\right]$$

$$= \frac{-1}{3}\left[2x^2+6x+\frac{4}{3}-\frac{8x}{3}-\frac{12}{3}+\frac{16}{9}\right]$$

$$= \frac{-1}{3}\left[2x^2+\frac{10x}{3}-\frac{\overset{8}{\cancel{40}}}{9}\right] = \frac{-(18x^2+30x-8)}{27}$$

$$\cancel{= \frac{-18x^2+30x-40}{27}}$$

G.S. is

$$y(x) = c_1 e^{-x} + c_2 e^{3x} - \frac{(18x^2 + 30x - 8)}{27}$$

Case 4 When $X = e^{ax} V(x)$,

then, $\frac{1}{f(D)} e^{ax} V(x) = e^{ax} \frac{1}{f(D+a)} V(x)$

Ex $y'' + 4y' + 3y = x \sin 2x$

Sol:- C.F. $m^2 + 4m + 3 = 0$

$(m+1)(m+3) = 0$

$m = -1, -3$

$$y_c(x) = c_1 e^{-x} + c_2 e^{-3x}$$

P.I. $y_p(x) = \frac{1}{D^2 + 4D + 3} x \sin 2x$

Ex → $16y'' + 8y' + y = 48xe^{-x/4}$

C.F. :- $16m^2 + 8m + 1 = 0$

$(4m+1)^2 = 0$

$m = \frac{-1}{4}, \frac{-1}{4}$

$y(x) = (C_1 + xC_2)e^{-x/4}$.

P.I. $\Rightarrow y_p(x) = \frac{1}{(4D+1)^2} 48xe^{-x/4}$

$= e^{-x/4} \frac{48}{\left[4\left(D-\frac{1}{4}\right)+1\right]^2} x$

$= 48e^{-x/4} \frac{1}{(4D-1+1)^2} x$

$= 48 e^{-x/4} \cdot \frac{1}{16D^2} x$

$= 3e^{-x/4} \frac{1}{D^2} x$

$= 3e^{-x/4} \frac{1}{D} \frac{x^2}{2}$

$= 3e^{-x/4} \frac{x^3}{6}$

$= \frac{x^3 e^{-x/4}}{2}$

∴ Gen Sol is $y = (C_1 + xC_2) e^{-x/4} + \frac{x^3 e^{-x/4}}{2}$.

Ex $\quad y'' - 4y' + 13y = 18e^{2x}\sin 3x$

C.F. $\quad m^2 - 4m + 13 = 0$

$$m = \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$y_c(x) = e^{2x}\left[C_1 \cos 3x + C_2 \sin 3x\right]$$

P.I. $\quad y_p(x) = \frac{1}{(D^2 - 4D + 13)\cancel{y}} 18e^{2x}\sin 3x$

$$= 18e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 13} \sin 3x$$

$$\cancel{= 18e^{2x} \frac{1}{D^2 + 4 + 4D - 4D + 8 + 13} \sin 3x}$$

$$\cancel{= \frac{18e^{2x}}{D^2 - 8D}}$$

$$= 18e^{2x} \frac{1}{D^2 + 4 + \cancel{4D} - \cancel{4D} - 8 + 13} \sin 3x$$

$$= 18e^{2x} \frac{1}{D^2 + 9} \sin 3x$$

$$= 18e^{2x} \frac{1}{-9+9} \sin 3x = 18xe^{2x} \frac{1}{2D} \sin 3x$$

$$= 9x e^{2x}\left(-\frac{\cos 3x}{3}\right)$$

$$= -3x e^{2x} \cos 3x$$

G.S. is

$$y(x) = e^{2x}\left[c_1 \cos 3x + c_2 \sin 3x\right] - 3x e^{2x} \cos 3x.$$

Case 5 When $x = xV$, V being a function of x.

$$\frac{1}{f(D)} xV = x \frac{1}{f(D)} V + \left(\frac{d}{dD} \frac{1}{f(D)}\right) V.$$

Ex: $y'' + 4y' + 3y = x \sin 2x$

Sol: C.F.: $m^2 + 4m + 3 = 0$

$$(m+1)(m+3) = 0$$

$$m = -1, -3$$

$$y_c(x) = c_1 e^{-x} + c_2 e^{-3x}$$

P.I. $\frac{1}{f(D)} x \sin 2x$

$$= \frac{1}{D^2 + 4D + 3} x \sin 2x$$

$$= x \frac{1}{D^2 + 4D + 3} \sin 2x + \left[\frac{d}{dD}\left(\frac{1}{D^2 + 4D + 3}\right)\right] \sin 2x$$

$$\left[\frac{d}{dD}\left[\frac{1}{D^2+4D+3}\right] = \frac{d}{dD}\left[(D^2+4D+3)^{-1}\right]\right.$$

$$= -(D^2+4D+3)^{-2}(2D+4)$$

$$\left.= \frac{-(2D+4)}{(D^2+4D+3)^2}\right]$$

$$y_p(x) = x\,\frac{1}{-4+4D+3}\sin 2x + \left[\frac{-(2D+4)}{(D^2+4D+3)^2}\right]\sin 2x$$

$$= x\,\frac{1}{4D-1}\sin 2x + \left[\frac{-(2D+4)}{(-4+4D+3)^2}\sin 2x\right]$$

$$= x\,\frac{4D+1}{16D^2-1}\sin 2x - \frac{(2D+4)}{(4D-1)^2}\sin 2x$$

$$= x\,\frac{(4D+1)}{-65}\sin 2x - \frac{(2D+4)}{16D^2+1-8D}\sin 2x$$

$$= -\frac{x}{65}\left[4(2\cos 2x)+\sin 2x\right] - \frac{(2D+4)}{-64+1-8D}\sin 2x$$

$$= \frac{-x}{65}\left[8\cos 2x+\sin 2x\right] - \frac{(2D+4)}{-(8D+63)}\sin 2x.$$

$$= x \frac{4D+1}{16D^2-1} \sin 2x - \frac{(2D+4)(4D+1)^2}{[(4D-1)(4D+1)]^2} \sin 2x$$

$$= x \frac{4D+1}{-65} \sin 2x - \frac{(2D+4)[16D^2+1+8D]}{[16D^2-1]^2} \sin 2x$$

$$= -\frac{x}{65}\left[8\cos 2x + \sin 2x\right] - \frac{1}{(65)^2}\left[32D^3 + 2D + 16D^2 + 64D^2 + 4 + 32D\right] \sin 2x$$

$$= \frac{-x}{65}\left[8\cos 2x + \sin 2x\right] - \frac{1}{4225}\left[32(-8\cos 2x) + 2(2\cos 2x) + 16(-4\sin 2x) + 64(-4\sin 2x) + 4\sin 2x + 32(2\cos 2x)\right]$$

$$\left[D(\sin 2x) = 2\cos 2x,\ D^2(\sin 2x) = -4\sin 2x,\ D^3(\sin 2x) = -8\cos 2x\right]$$

$$= -\frac{x}{65}\left[8\cos 2x + \sin 2x\right] - \frac{1}{4225}\left\{(-256+4+64)\cos 2x + (-64-256+4)\sin 2x\right)$$

$$= \frac{-x}{65}\left[8\cos 2x + \sin 2x\right] - \frac{1}{4225}\left[-188\cos 2x - 316\sin 2x\right]$$