DIFFERENTIAL EQUATION

→ A differential equation where the no of independent variable is one is called ordinary differential equation.

→ If the no. of independent variables is more than one then the differential equation is called partial differential equation.

→ A differential equation is called linear if

(i) every dependent variable and derivatives involved occurs to the 1st degree only.

(ii) products of derivatives and ~~depent~~ dependent variable donot occur.

D.E

ODE　　PDE

1) $\dfrac{d^2y}{dx^2} + x^2\dfrac{dy}{dx} + xy = \sin x$　(L)

2) $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + y = 0$　(N·L)

3) $\dfrac{dy}{dx} + y^2 = 0$　(N-L)

4) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x^2$ (N.L)

5) $\frac{dy}{dt} = c^2\frac{d^2y}{dx^2}$ (L)

6) $\frac{dy}{dt} - \frac{d^2y}{dx^2} = 0$ (L)

7) $\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x}\cdot\frac{du}{\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$ (N.L)

8) $\frac{\partial^2 u}{\partial t^2} = K^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ (L)

1) $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ (1-D heat equation)

2) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ (1-D wave equation)

3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (2-D laplace equation)

4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x)$ (poison's equation)

Q Find suitable solution for the given differential equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

1) $$u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$L \cdot H \cdot S = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= 2 - 2 = 0 = R \cdot H \cdot S$$

2) $$u = \ln(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot (2x)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2}$$

$$= \frac{-2x^2 + 2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \cdot (2y)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2(x^2+y^2) - 2y(2y)}{(x^2+y^2)^2}$$

$$= \frac{-2y^2 + 2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-2x^2 + 2y^2 - 2y^2 + 2x^2}{(x^2+y^2)^2}$$

$$= 0$$

3) $u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$L.H.S = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= e^x \cos y - e^x \cos y$$

$$= 0 = R.H.S$$

4) $u = \sin x \cosh y$

$$\frac{\partial u}{\partial x} = \cos x \cosh y$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y$$

$$\frac{\partial^2 u}{\partial y^2} = \sin x \cosh y$$

$$L.H.S = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= -\sin x \cosh y + \sin x \cosh y$$
$$= 0 = R.H.S$$

Q Find suitable solution for the given differential equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(i) $u = x^2 + t^2$
(ii) $u = \cos 4t \sin 2x$
(iii) $u = \sin kct \cos kx$
(iv) $u = \sin at \sin bt$

Sol (i)

$$\frac{\partial u}{\partial t} = 2t$$

$$\frac{\partial^2 u}{\partial t^2} = 2$$

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial^2 u}{\partial x^2} = 2$$

$$2 = c^2 \cdot 2$$
$$c = \pm 1$$

(ii)

$$\frac{\partial u}{\partial t} = -4 \sin 4t \sin 2x$$

$$\frac{\partial^2 u}{\partial t^2} = -16 \cos 4t \sin 2x$$

$$\frac{\partial u}{\partial x} = 2\cos 4t \cos 2x$$

$$\frac{\partial^2 u}{\partial x^2} = -4\cos 4t \sin 2x$$

$$-16 \cancel{\cos 4t} \cancel{\sin} 2x = c^2\{-4\cancel{\cos 4t} \cancel{\sin 2x}\}$$

$$c^2 = \pm 2$$

(iii) $u = \sin kct \cos kx$

$$\frac{\partial u}{\partial t} = kc \cos kct \cos kx$$

$$\frac{\partial^2 u}{\partial t^2} = -k^2c^2 \sin kct \cos kx$$

$$\frac{\partial u}{\partial x} = -k \sin kct \sin kx$$

$$\frac{\partial^2 u}{\partial x^2} = -k^2 \sin kct \cos kx$$

$$-\cancel{k^2}c^2 \sin \cancel{kct} \cancel{\cos kx} = c^2(-\cancel{k^2} \cancel{\sin kct} \cancel{\cos kx})$$

Independent of c.

(iv) $u = \sin at \sin bx$

$$\frac{\partial u}{\partial t} = a\cos at \sin bx$$

$$\frac{\partial^2 u}{\partial t^2} = -a^2 \sin at \sin bx$$

$$\frac{\partial u}{\partial x} = b \sin at \cos bx$$

$$\frac{\partial^2 u}{\partial x^2} = -b^2 \sin at \sin bx$$

$$\cancel{-} a^2 \cancel{\sin at \sin bx} = c^2 \{ \cancel{-} b^2 \cancel{\sin at \sin bx} \}$$

$$c = \pm \frac{a}{b}$$

Q Find suitable solution for the given differential equation.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(i) $u = e^{-t} \sin x$

$$\frac{\partial u}{\partial t} = -e^{-t} \sin x$$

$$\frac{\partial u}{\partial x} = e^{-t} \cos x$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-t} \sin x$$

$$-e^{-t}\sin x = c^2(-e^{-t}\sin x)$$

$$c^2 = 1$$

$$c = \pm 1$$

(ii)

$$u = e^{-\omega^2 c^2 t}\cos\omega x$$

$$\frac{\partial u}{\partial t} = -\omega^2 c^2 e^{-\omega^2 c^2 t}\cos\omega x$$

$$\frac{\partial u}{\partial x} = -\omega e^{-\omega^2 c^2 t}\sin\omega x$$

$$\frac{\partial^2 u}{\partial x^2} = -\omega^2 e^{-\omega^2 c^2 t}\cos\omega x$$

$$-\omega^2 c^2 e^{-\omega^2 c^2 t}\cos\omega x = c^2(-\omega^2 e^{-\omega^2 c^2 t}\cos\omega x)$$

~~c = ±1~~ Independent of c

(iii)

$$u = e^{-9t}\cos\omega x$$

$$\frac{\partial u}{\partial t} = -9e^{-9t}\cos\omega x$$

$$\frac{\partial u}{\partial x} = -\omega e^{-9t}\sin\omega x$$

$$\frac{\partial^2 u}{\partial x^2} = -\omega^2 e^{-9t}\cos\omega x$$

$$+9\cancel{e^{-9t}}\cos\cancel{\omega x} = c^2(\cancel{-}\omega^2\cancel{e^{-9t}}\cancel{\cos\omega x})$$

$$c = \pm\frac{3}{\omega}$$

(iv) $$u = e^{-\pi^2 t} \cos 25x$$

$$\frac{\partial u}{\partial t} = -\pi^2 e^{-\pi^2 t} \cos 25x$$

$$\frac{\partial u}{\partial x} = -25\, e^{-\pi^2 t} \sin 25x$$

$$\frac{\partial^2 u}{\partial x^2} = -(25)^2 e^{-\pi^2 t} \cos 25x$$

$$-\pi^2 \cancel{e^{-\pi^2 t}} \cancel{\cos 25x} = c^2 \{ -(25)^2 \cancel{e^{-\pi^2 t}} \cancel{\cos 25x} \}$$

$$c = \frac{\pm \pi}{25}$$

Fundamental Theorem on Superposition:

A solution of a partial differential equation in some region R of the space of independent variables is a function that has all the partial derivatives appearing in PDE in some domain ~~containg~~ containing R and satisfies the PDE everywhere in R.

If u_1 and u_2 are solutions of a homogeneous linear partial differential equation in some region R then $c_1 u_1$ and $c_2 u_2$ are also a solution of that PDE in region R.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ---(i)$$

Since u_1 and u_2 are solution of equation(1)

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$\frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2) + \frac{\partial^2}{\partial y^2}(c_1 u_1 + c_2 u_2)$$

$$= c_1 \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial^2 u_2}{\partial x^2} + c_1 \frac{\partial^2 u_1}{\partial y^2} + c_2 \frac{\partial^2 u_2}{\partial y^2}$$

$$= c_1(0) + c_2(0)$$
$$= 0 = R.H.S$$

Q Solve $\frac{\partial^2 u}{\partial x^2} = 4u$

$$\frac{d^2 y}{dx^2} - 4y = 0$$

$$m^2 - 4 = 0$$
$$m = +2, -2$$

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

$c_1 \frac{d^2 y}{dx^2} + c_2 \frac{dy}{dx} + c_3 y = 0$

Auxiliary form

$$c_1 m^2 + c_2 m + c_3 = 0$$

$$m = m_1, m_2$$

$$y_{CF} = a e^{m_1 x} + b e^{m_2 x}$$

$$y_{CF} = (a + bx) e^{m_1 x} \rightarrow \text{same roots}$$

$$y_{CF} = e^{\alpha x}(a \cos\beta x + b \sin\beta x) \rightarrow \text{complex roots}$$

Q1 $$u_{yy} = 0$$

$$\frac{d^2 u}{dy^2} = 0$$

$$m^2 = 0$$

$$m = 0$$

$$y = (a + by) e^0$$

$$y = a + by$$

Q2 $$u_{xy} = -u_x$$

$$\frac{d^2 u}{dx\, dy} = \frac{-\partial u}{\partial x}$$

$$\frac{\partial}{\partial y}\left(\underbrace{\frac{\partial u}{\partial x}}_{V}\right) = -\frac{\partial u}{\partial x}$$

Let $u_x = V$

$$V_y = -V$$

$$\frac{\partial v}{\partial y} = -v$$

$$\frac{\partial v}{v} = -\partial y$$

$$\ln v = -y + c(x)$$

$$v = e^{-y+c(x)}$$
$$= e^{-y} \cdot e^{c(x)}$$
$$= c_1(x) e^{-y}$$

$$\frac{\partial u}{\partial x} = c_1(x) e^{-y}$$

$$\frac{\partial u}{\partial x} = c_1(x) e^{-y}$$

$$u = e^{-y} \int c_1(x)\, dx$$

$$= e^{-y} \left[f(x) + g(y) \right]$$

Q $u_{yy} = \cancel{4}x\, u_y$

$$\frac{\partial^2 u}{\partial y^2} = \cancel{\frac{\partial u}{\partial x}} \quad \cancel{\frac{\partial u}{\partial y}} \quad \frac{4x \partial u}{\partial y}$$

$$\frac{\partial u}{\partial y} = v$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial v}{\partial y}$$

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$$\frac{\partial v}{\partial y} = 4xv$$

$$\int \frac{\partial v}{v} = 4x \int \partial y$$

$$\ln v = 4xy + \ln c$$

$$v = ce^{4xy}$$

$$\frac{\partial u}{\partial y} = ce^{4xy}.$$

$$u = \frac{ce^{4xy}}{4x} + c_1$$

1) Find the value of c such that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(i) $u = 4x^2 + t^2$

$$\frac{\partial u}{\partial t} = 2t$$

$$\frac{\partial^2 u}{\partial t^2} = 2$$

$$\frac{\partial u}{\partial x} = 8x$$

$$\frac{\partial^2 u}{\partial x^2} = 8$$

$$z = 8c^2$$
$$c = \pm\frac{1}{2}$$

2) Verify that u satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x)$

where (i) $u = \frac{y}{x}$, $f = \frac{2y}{x^3}$

(ii) $u = \sin xy$, $f = -(x^2 + y^2)\sin xy$

Sol (i)

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2y}{x^3}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x}$$

$$\frac{\partial^2 y}{\partial y} = 0$$

Hence, $f(x) = \frac{2y}{x^3}$

(ii)

$$\frac{\partial u}{\partial x} = y\cos xy$$

$$\frac{\partial^2 u}{\partial x^2} = -y^2 \sin xy$$

$$\frac{\partial u}{\partial y} = x\cos xy$$

$$\frac{\partial^2 u}{\partial y^2} = -x^2 \sin xy$$

Hence, $f(x) = -(x^2+y^2)\sin xy$

3) Verify that $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Sol

$$\frac{\partial u}{\partial x} = \frac{-1}{\cancel{2}\sqrt{x^2+y^2+z^2}} \cdot \cancel{2}x$$

$$\frac{\partial u}{\partial x} = \frac{-x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-\sqrt{x^2+y^2+z^2} + \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}} \cdot \cancel{2}x^2}{x^2+y^2+z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-\cancel{x^2}+y^2+z^2+\cancel{x^2}}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

Q Verify the function $u(x,y) = a\ln(x^2+y^2)+b$ satisfies laplace equation and determine a and b so that u satisfies the boundary conditions $u = 110$ on the circle $x^2+y^2=1$ and $u=0$ on the circle $x^2+y^2=100$.

Sol
$$u(x,y) = a\ln(x^2+y^2)+b$$
$$110 = a\ln(1)+b$$
$$b = 110$$

$$0 = a\ln(100)+b$$
$$a = -\frac{110}{\ln(100)}$$

$$u(x,y) = -\frac{110}{\ln(100)}\ln(x^2+y^2)+110$$

Q Solve

(i) $u_{xx} + 16\pi^2 u = 0$ $\qquad u_{xx} = \frac{\partial^2 u}{\partial x^2}$

(ii) $u_y + y^2 u = 0$ $\qquad u_x = \frac{\partial u}{\partial x}$

(iii) $u_{yy} + 6u_y + 13u = ue^{3y}$ $\qquad u_y = \frac{\partial u}{\partial y}$

(iv) $u_{xx} = 0$ and $u_{yy} = 0$. $\qquad u_{yy} = \frac{\partial^2 u}{\partial y^2}$

Sol
$$u_{xx} + 16\pi^2 u = 0$$
The auxiliary form is
$$\lambda^2 + 16\pi^2 = 0$$
$$\lambda = \pm 4\pi i$$

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$$u(x,y) = a(y)\cos 4\pi x + b(y)\sin 4\pi x$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$I.F = e^{\int P(x)dx}$$

Solution is $y \cdot IF = \int Q(x)\cdot(IF)\,dx$

(ii) $$u_y + y^2 u = 0$$

$$I.F = e^{\int y^2 dy}$$

$$= e^{\frac{y^3}{3}}$$

$$u \cdot e^{\frac{y^3}{3}} = \int 0 \cdot e^{\frac{y^3}{3}}$$

$$u \cdot e^{\frac{y^3}{3}} = c(x)$$

$$u = c(x)e^{-y^3/3}$$

(iii) $$u_{yy} + 6u_y + 13u = 4e^{3y}$$

$$(D^2 + 2D + 1)y = e^{ax}$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{ax}$$

$$y_{P.I} = \frac{e^{ax}}{f(a)}, \; f(a) \neq 0$$

$\rightarrow$ $$y_{P.I} = \frac{1}{D^2+2D+1} x^2$$

$$= \frac{1}{(1+D)^2} x^2$$

$$= (1+D)^{-2} x^2$$

$$= (1-D^2)x^2$$

$$y_{P.I} = \frac{1}{D^2+2D+1} \cos 2x$$

$$= \frac{1}{-4+2D+1} \cos 2x$$

$$= \frac{1}{2D-3} \cos 2x$$

$$= \frac{2D+3}{4D^2-9} \cos 2x$$

Auxiliary function,

$$\lambda^2 + 6\lambda + 13 = 0$$

$$\lambda = \frac{-6 \pm \sqrt{36-52}}{2}$$

$$= \frac{-6 \pm 4i}{2}$$

$$= -3 \pm 2i$$

$$u(x,y) = e^{-3y}(a(x)\cos 2x + b(x)\sin 2x)$$

<u>PI:-</u> $u_{P.I} = \frac{1}{D^2+6D+13} 4 \cdot e^{3y}$

$$= \frac{4 \cdot 1}{9+18+13} e^{3y}$$

$$= 0.1 e^{3y}$$

$$u = e^{-3y}(a(x)\cos 2y + b(x)\sin 2y) + 0.1e^{3y}$$

(iv) $u_{xx} = 0$

Auxiliary function

$$\lambda^2 = 0$$

$$u(x,y) = c_1(y) + c_2(y)x$$

$$u_{yy} = 0$$

$$c_1''(y) + c_2''(y)x = 0$$

$$c_1''(y) = 0 \text{ and } c_2''(y) = 0$$

$$c_1''(y) = 0$$

$$c_1'(y) = c_3$$

$$c_1(y) = c_3 y + c_4$$

$$u(x,y) = c_3 y + c_4 + (c_5 y + c_6)x$$

<u>Derivation of wave equation</u>

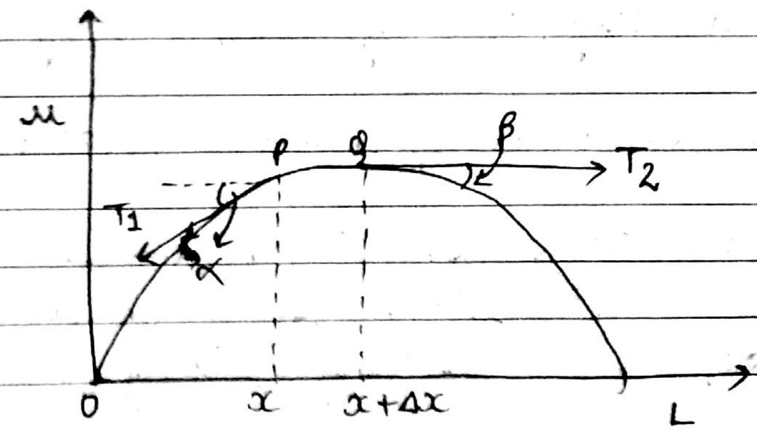
Consider a small transverse vibration of an elastic string such as a violon string. Replace the string along the x-axis, stretch it to length L and fix it at the length $x=0$ and $x=L$. We then destord it, and at some instant ~~set~~ say it equal to ~~so~~ zero releive it and allow to vibrate.

Physical assumptions:-

1> The mass of the string per unit length is constant. The string is perfectly elastic and does not offer any resistance to bending.

2> The tension caused by stretching the string before fastening it at the ends is too large that the action of gravitational force on the string can be neglected.

3> The string performs small transverse motion in a vertical plane that is every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remains small in absolute value.



Let the string be released from rest and about to vibrate. To obtain the differential equation, we consider the forces acting on a small ~~person~~ position portion ~~P and Q~~ PQ of the string. Let T_1 and T_2 be the tension at the end points which makes the angle α and β respectively. Since, the points of the string moves vertically, there is no motion in the horizontal direction. Hence, horizontal components of the tension must be constant,

i.e $T_1 \cdot \cos\alpha = T_2 \cos\beta = T = \text{constant}$ ----- (1)

In the vertical direction, we have two forces namely the vertical components $-T_1 \sin\alpha$ and $T_2 \sin\beta$ of T_1 and T_2. (The negative sign appears as the component at P is directed downward). Hence, the resultant force $= T_2 \sin\beta - T_1 \sin\alpha$.

According to Newton's second law, the net resultant force = mass × acceleration

$$\Rightarrow T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \cdot \frac{\partial^2 u}{\partial t^2} \quad \text{----- (2)}$$

$$\frac{T_2 \sin\beta}{T} - \frac{T_1 \sin\alpha}{T} = \frac{\rho \cdot \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

Now, $\tan\alpha$ and $\tan\beta$ are slopes of the curves of the string at x and $x+\Delta x$.

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\rho \Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{\rho} \quad \text{--- (3)}$$

Equation (3) is called 1-D wave equation.

<u>Solution of wave equation (Seperation of variables)</u>

(i) The model of a vibrating string consist of one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho} \quad \text{--- (1)}$$

for some unknown deflections $u(x,t)$ of the string.

(ii) <u>Boundary condition</u>

Since, the string is fastened at the ends $x=0$ and $x=L$, we have two boundary conditions:

$$u(0,t)=0 \;,\; u(L,t)=0 \;,\; t\geq 0 \quad \text{--} ②$$

(iii) Initial condition– The form of motion of the string will depend on its initial deflection $f(x)$ and initial velocity $g(x)$.

$$u(x,0)=f(x) \;,\; \frac{\partial u}{\partial t}(x,0)=g(x) \;,$$

$$0 \leq x \leq L. \quad \text{----} ③$$

Step 1: Method of seperation of variables
Let the solution of the wave equation be of the form $u(x,t)=F(x)\,G(t)$ ---- ④

$$u(x,y) = F(x)\,G(y)$$

Differentiating equation ④, we have

$$\frac{\partial^2 u}{\partial t^2} = F(x)\,\ddot{G}(t) \quad \text{----} ⑤$$

$$\frac{\partial^2 u}{\partial x^2} = F''(x)\,G(t) \quad \text{---} ⑥$$

where, $\cdot$ and $'$ denote the derivative w.r.t to t and x respectively.

~~Sbu~~ Substituting the ~~e~~ value of equation ⑤ and ⑥ in equation ①, we get

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$F(x)\,\ddot{G}(t) = c^2 F''(x)\,G(t)$$

$$\frac{\ddot{G}(t)}{G(t)} = c^2 \frac{F''(x)}{F(x)}$$

$$\Rightarrow \frac{\ddot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} \quad \text{---} (7)$$

Now the variables are seperated. The left hand side depends on t only and right hand side on x only. Hence, both the side remains constant because if they were variables then changing t or x will affect only one side leaving the other unchanged.

$$\frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)} = \frac{F''(x)}{F(x)} = K$$

$$\Rightarrow \quad F''(x) - K\,F(x) = 0$$

$$\ddot{G}(t) - c^2 K\,G(t) = 0$$

Case-1 : When $K = 0$

$F''(x) = 0$	$\ddot{G}(t) = 0$
$\Rightarrow F'(x) = a$	$\Rightarrow \dot{G}(t) = d$
$\Rightarrow F(x) = ax + b$	$\Rightarrow G(t) = dt + e$

Now solution is $u(x,t) = F(x)G(t)$

$$= (ax+b)(dt+e)$$

Boundary and Initial condition:

$u(0,t) = b(dt+e)$

$0 = b(dt+e) \quad \Rightarrow b = 0$

$u(L,t) = (aL+b)(dt+e)$

$0 = aL(dt+e)$

$a = 0$

$$F(x) = ax + b = 0$$
$$u(x,t) = F(x)G(t) = 0$$

case rejected because of zero solution.

Case 2 :- when $K = p^2$

$$F''(x) - KF(x) = 0$$
$$\ddot{G}(t) - KC^2 G(t) = 0$$

$$F''(x) - KF(x) = 0$$
$$\Rightarrow F''(x) - p^2 F(x) = 0$$
$$\Rightarrow \lambda^2 - p^2 = 0$$
$$\Rightarrow \lambda = \pm p$$
$$\Rightarrow F(x) = Ae^{px} + Be^{-px}$$
$$\cancel{u(x,t) = Ae^{px} + Be^{-px}}$$

$$\ddot{G}(t) - KC^2G(t) = 0$$
$$\Rightarrow \ddot{G}(t) - p^2c^2G(t) = 0$$
$$\Rightarrow \lambda^2 - p^2c^2 = 0$$
$$\Rightarrow \lambda = \pm pc$$
$$G(t) = De^{pct} + Ee^{-pct}$$

$$u(x,t) = (Ae^{px} + Be^{-px})(De^{pct} + Ee^{-pct})$$

Boundary and initial conditions :-

$$u(0,t) = 0$$
$$(A+B)(De^{pct} + Ee^{-pct}) = 0$$
$$A + B = 0$$

$$u(L,t) = 0$$
$$(Ae^{pL} + Be^{-pL})(De^{pct} + Ee^{-pct}) = 0$$

$$u(0,t) = 0$$
$$\therefore F(0) = 0$$
$$u(L,t) = 0 \Rightarrow F(L) = 0$$

$$u(0,t)=0 \qquad\qquad F(L)=0$$
$$F(0)=0$$
$$e^{PL}\times\big[\; A+B=0 \;\text{--- (8)} \qquad\qquad Ae^{PL}+Be^{-PL}=0 \;\text{--- (9)}$$

$$e^{PL}A + Be^{PL} = 0$$
$$Ae^{PL} + Be^{-PL} = 0$$
$$(-)\qquad (-)$$

$$B(e^{PL}-e^{-PL})=0$$
$$B=0$$

From (8), $A=0$

$$F=0$$
$$u(x,t)=0$$
case discarded.

<u>case 3:-</u> When $K=-P^2$.

$$F''+P^2F=0$$
$$\ddot{G}+P^2c^2G=0$$

$F''+P^2F=0$	$\ddot{G}+P^2c^2G=0$
$\lambda^2+P^2=0$	$\lambda^2+P^2c^2=0$
$\lambda=\pm Pi$	$\lambda=\pm Pci$
$F=A\cos Px+B\sin Px$	$G=D\cos Pct+E\sin Pct$

$$u(x,t)=(A\cos Px+B\sin Px)(D\cos Pct+E\sin Pct)$$

Boundary conditions $u(0,t)=0$

$$\therefore A(D\cos Pct+E\sin Pct)=0$$
$$A=0$$

$$u(L,t)=0$$
$$(A\cos PL+B\sin PL)(D\cos Pct+E\sin Pct)=$$
$$B\sin PL(D\cos Pct+E\sin Pct)=0$$

We must take $B \neq 0$ because if $B = 0$ then that will lead to trivial solution.

$$\sin PL = 0$$

$$\sin PL = \sin n\pi$$

$$PL = n\pi$$

$$P = \frac{n\pi}{L}$$

where, $n = 1, 2, 3, \ldots$

Setting $B = 1$,

$$u_n(x, t) = \sin \frac{n\pi}{L} x \left(D\cos \frac{n\pi c t}{L} + E\sin \frac{n\pi c t}{L} \right)$$

These ~~which~~ functions are called the eigen functions or characterstic functions and the value $cn\pi/L$ are called the eigen values or characterstics values of the vibrating string.

Setting $B = 1$, we obtain infinitely many solutions that is

$$F(x) = F_n(x) = \sin \frac{n\pi}{L} x \quad ; n = 1, 2, \ldots$$

Hence, the solution

$$u_n(x, t) = \sin \frac{n\pi}{L} x \left(D\cos \frac{cn\pi}{L} t + E\sin \frac{cn\pi}{L} t \right)$$

$$n = 1, 2, \ldots$$

Applying fundamental theorem, that the sum of finitely many solutions

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t)$$

$$= \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left(D\cos \frac{cn\pi}{L} t + E\sin \frac{cn\pi}{L} t \right)$$

To determine the unknown D and E, we apply the initial conditions $u(x,0) = f(x)$

$$\sum_{n=1}^{\infty} D \sin \frac{n\pi x}{L} = f(x)$$

Applying fourier series, we obtain

$$D = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \, ; \quad n = 1, 2, 3, \ldots$$

Applying ~~fourier~~ initial condition,

$$u_t(x,0) = g(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

$$\sum_{n=1}^{\infty} \left\{ -\frac{cn\pi}{L} D \sin \frac{cn\pi}{L} t + E \cos \frac{cn\pi}{L} t \left(\frac{cn\pi}{L} \right) \right.$$

$$\left. \times \right\} \sin \frac{n\pi}{L} x = g(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{cn\pi}{L} \right) E \sin \frac{n\pi}{L} x = g(x)$$

Applying fourier series, we obtain

$$E = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

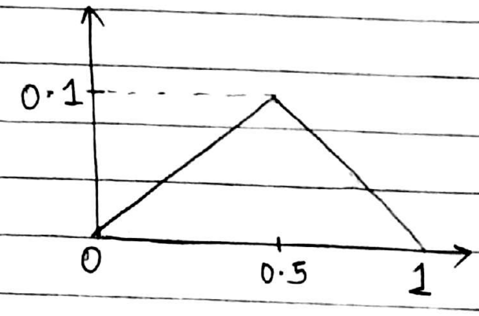
Finally, the solution is given by

$$u(x,t) = u_n(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(D \cos \frac{cn\pi}{L} t + E \sin \ldots \right.$$

~~$f-1)$, $u(x,0) = f(x)$~~

Q. Find $u(x,t)$ for the string of length $L=1$, $c^2=1$ when the initial velocity is zero and the initial deflection is given by the following:-

1) $K\sin 3\pi x$
2) $K\left(\sin \pi x - \frac{1}{2}\sin 2\pi x\right)$
3) $Kx(1-x)$
4) $Kx^2(1-x)$
5)



Sol-1) $u(x,t) = \sum_{n=1}^{\infty}\left(D_n \cos\frac{cn\pi t}{L} + E_n \sin\frac{cn\pi t}{L}\right)\sin\frac{n\pi x}{L}$

where, $D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

$E_n = \frac{2}{cn\pi}\int_0^L \cancel{g(x)}^{\,0} \sin\frac{n\pi}{L}x\,dx$

$u(x,t) = \sum_{n=1}^{\infty} D_n \cos\frac{cn\pi t}{L} \cdot \sin\frac{n\pi}{L}x$

$L = 1,\ c = 1,\ E_n = 0$

$u(x,0) = f(x) = K\sin 3\pi x$

$\sum_{n=1}^{\infty} D_n \cancel{\cos n\pi t}\,\sin n\pi x = K\sin 3\pi x$

$D_1 \sin \pi x + D_2 \sin 2\pi x + D_3 \sin 3\pi x + D_4 \sin 4\pi x$
$+ \cdots\cdots = K \sin 3\pi x$

comparing the coefficients,

$D_1 = 0, \; D_2 = 0, \; D_3 = K, \; D_4 = 0, \ldots$

$$u(x,t) = K \cos 3\pi t \sin 3\pi x$$

Sol - 2) $K(\sin \pi x - \frac{1}{2} \sin 2\pi x)$, $C = 1, L = 1, E_n = 0$

$$u(x,0) = K(\sin \pi x - \frac{1}{2} \sin 2\pi x)$$

$$\sum_{n=1}^{\infty} D_n \sin n\pi x = K \sin \pi x - \frac{K}{2} \sin 2\pi x$$

$$D_1 \sin \pi x + D_2 \sin 2\pi x + D_3 \sin 3\pi x + \cdots\cdots = K \sin \pi x - \frac{K}{2} \sin 2\pi x$$

$$D_1 = K, \; D_2 = -\frac{K}{2}, \; D_3 = 0$$

Hence, the solution becomes, $u(x,t) = \sum_{n=1}^{\infty} D_n \cos n\pi t \sin n\pi x$

$$u(x,t) = K \cos \pi t \sin \pi x - \frac{K}{2} \cos 2\pi t \sin 2\pi x$$

-3) $Kx(1-x)$

$$u(x,0) = Kx(1-x)$$

Here, $D_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

$$= 2K \int_0^1 \underset{\mathrm{I}}{(x - x^2)} \underset{\mathrm{II}}{\sin n\pi x} \, dx$$

$$= 2K \left[(x - x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) + \int_0^1 (1-2x)\left(\frac{\cos n\pi x}{n\pi}\right) dx \right]_0^1$$

$$= 2K\left[-(x-x^2)\frac{\cos n\pi x}{n\pi} + (1-2x)\frac{\sin n\pi x}{n^2\pi^2} + \frac{2\cos n\pi x}{n^3\pi^3}\right]_0^1$$

$$= 2K\left[\frac{2\cos n\pi}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]$$

$$= \frac{4K}{n^3\pi^3}\left[\cos n\pi - 1\right]$$

$$\cos n\pi = \begin{cases} -1 & ; n = \text{odd} \\ 1 & ; n = \text{even} \end{cases}$$

$$\cos n\pi - 1 = \begin{cases} -2 & ; n = \text{odd} \\ 0 & ; n = \text{even} \end{cases}$$

$$D_n = \begin{cases} \frac{-8K}{n^3\pi^3} & ; n = \text{odd} \\ 0 & ; n = \text{even} \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} D_n \cos n\pi t \sin n\pi x$$

$$= -\frac{8K}{\pi^3}\cos \pi t \sin \pi x - \frac{8K}{27K^3}\cos 3\pi t \sin 3\pi x - \frac{8K}{125\pi^3}\cos 5\pi t \sin 5\pi x + \cdots$$

Sol-4 $\quad u(x,0) = Kx^2(1-x)$

Here, $D_n = \frac{2}{L}\int_0^L f(x)\sin \frac{n\pi}{L} x \, dx$

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$$I_n = \frac{2}{1}K\int_0^1 (x^2-x^3)\sin n\pi x\,dx$$

$$I_n = 2K\left[(x^2-x^3)\left(-\frac{\cos n\pi x}{n\pi}\right)+\int_0^1(2x-3x^2)\left(\frac{\cos n\pi x}{n\pi}\right)dx\right]_0^1$$

$$I_n = 2K\left[-(x^2-x^3)\frac{\cos n\pi x}{n\pi}+(2x-3x^2)\frac{\sin n\pi x}{n^2\pi^2}-\int_0^1(2-6x)\frac{\sin n\pi x}{n^2\pi^2}dx\right]_0^1$$

$$I_n = 2K\left[-(x^2-x^3)\frac{\cos n\pi x}{n\pi}+(2x-3x^2)\frac{\sin n\pi x}{n^2\pi^2}-\frac{2\sin n\pi x}{n^2\pi^2}\right.$$

eq -> $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y - y_1}{x - x_1}$

$$f(x) = \begin{cases} \dfrac{x}{5}, & 0 \le x \le 0.5 \\ \dfrac{1}{5} - \dfrac{x}{5}, & 0.5 \le x \le 1 \end{cases}$$

$$D_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

$$D_n = 2 \int_0^1 f(x) \sin n\pi x \, dx$$

$$D_n = 2 \left[\int_0^{0.5} \frac{x}{5} \sin n\pi x \, dx + \int_{0.5}^1 \left(\frac{-x}{5} + \frac{1}{5} \right) \sin n\pi x \, dx \right]$$

$$D_n = 2 \left[0.2x \left(-\frac{\cos n\pi x}{n\pi} \right) + (0.2) \frac{\sin n\pi x}{n^2\pi^2} \right]_0^{0.5} +$$

$$2 \left[(0.2 - 0.2x) \left(-\frac{\cos n\pi x}{n\pi} \right) + (0.2) \left(\frac{\sin n\pi x}{n^2\pi^2} \right) \right]_{0.5}^1$$

$$= 2 \left[\left(-0.1 \frac{\cos n\pi/2}{n\pi} + 0.2 \frac{\sin n\pi/2}{n^2\pi^2} \right) \right] + 2 \left[0 + \right.$$

$$\left. 0.1 \left(\frac{\cos n\pi/2}{n\pi} \right) + 0.2 \frac{\sin n\pi/2}{n^2\pi^2} \right]$$

$$= -0.2 \frac{\cos n\pi/2}{n\pi} + 0.4 \frac{\sin n\pi/2}{n^2\pi^2} + 0.2 \frac{\cos n\pi/2}{n\pi}$$

$$+ 0.4 \frac{\sin n\pi/2}{n^2\pi^2}$$

Q Find the deflection $u(x,t)$ of the string of length $L=\pi$ and $c^2 = 1$ for zero initial displacement and triangular initial velocity.

$$g(x) = u_t(x,0) = \begin{cases} 0.01x & \text{if } 0 \le x \le \pi/2 \\ 0.01(\pi - x) & \text{if } \frac{\pi}{2} \le x \le \pi \end{cases}$$

sol

$$\because u(x,0) = 0$$

$$D_n = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{cn\pi t}{L} \sin\frac{n\pi x}{L}$$

$$\text{Here, } E_n = \frac{2}{cn\pi}\int_0^L g(x) \sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{n\pi}\int_0^{\pi} g(x) \sin nx\,dx$$

$$= \frac{2}{n\pi}\left[\int_0^{\pi/2} 0.01x \sin nx\,dx + \int_{\pi/2}^{\pi} 0.01(\pi - x)\sin nx\,dx\right]$$

$$= \frac{2}{n\pi}\left[0.01x\left(-\frac{\cos nx}{n}\right) + \frac{0.01 \sin nx}{n^2}\right]_0^{\pi/2} +$$

$$\frac{2}{n\pi}\left[0.01(\pi - x)\left(-\frac{\cos nx}{n}\right) + (-0.01)\frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi}$$

$$E_n = \frac{2}{n\pi}\left[-0.01\frac{\pi}{2}\cos\frac{n\pi}{2} + \frac{0.01 \sin\frac{n\pi}{2}}{n^2}\right] + \frac{2}{n\pi}\left[0 - \right.$$

$$\left. 0.01\left(\pi - \frac{\pi}{2}\right)\left(-\frac{\cos n\pi/2}{n}\right) + \frac{0.01 \sin n\pi/2}{n^2}\right]$$

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$$= -0.01\,\frac{2}{n\pi}\cdot\cancel{\frac{\pi}{2}\cdot\frac{\cos n\pi/2}{n}} + \frac{2}{n\pi}(0.01)\frac{\sin n\pi/2}{n^2}$$

$$+\,0.01\times\cancel{\frac{2}{n\pi}\,\pi/2\,\frac{\cos n\pi/2}{n}} + \frac{2}{n\pi}\,\frac{(0.01)}{n^2}\sin\frac{n\pi}{2}$$

$$= \frac{0.04}{n^2\pi}\sin\frac{n\pi}{2}$$

$$\sin\frac{n\pi}{2} = \begin{cases} 1, & n = 1, 5, 9, 13 \\ 0, & n = \text{even} \\ -1, & n = 3, 7, 11 \end{cases}$$

Hence, $u(x,t) = \sum_{n=1}^{\infty} E_n \sin nt \sin nx$

$$= \frac{0.04}{\pi}\sin t\sin x - \frac{0.04}{27\pi}\sin 3t\sin 3x + \frac{0.04}{125\pi}\sin 5t\sin 5x + \cdots$$

D'Alembert's solution of wave Equation

The solution of wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = \frac{T}{\rho} \quad \text{--- (1)}$$

can be obtained by transforming the new independent variables $v = x + ct$ and $w = x - ct$ ---- (2)

Now, u is a function of v and w.

so, $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial t}$$

$$= c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial w}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial w}\right)$$

$$= \frac{\partial}{\partial v}\left(c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial w}\right)\frac{\partial v}{\partial t} + \frac{\partial}{\partial w}\left(c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial w}\right)\frac{\partial w}{\partial t}$$

$$= c^2\frac{\partial^2 u}{\partial v^2} - c^2\frac{\partial^2 u}{\partial v\,\partial w} - c^2\frac{\partial^2 u}{\partial v\cdot\partial w} + c^2\frac{\partial^2 u}{\partial w^2}$$

$$= c^2\frac{\partial^2 u}{\partial v^2} - 2c^2\frac{\partial^2 u}{\partial v\,\partial w} + c^2\frac{\partial^2 u}{\partial w^2} \quad \text{--- } 3$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right)$$

$$= \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right)\cdot\frac{\partial w}{\partial x}$$

$$= \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v\cdot\partial w} + \frac{\partial^2 u}{\partial w^2} \quad \text{---- } 4$$

Putting equation 3 & 4 in equation 1

$$c^2\frac{\partial^2 u}{\partial v^2} - 2c^2\frac{\partial^2 u}{\partial v\cdot\partial w} + c^2\frac{\partial^2 u}{\partial w^2} = c^2\frac{\partial^2 u}{\partial x^2} + 2c^2\frac{\partial^2 u}{\partial v\cdot\partial w} +$$

$$4c^2 \frac{\partial^2 u}{\partial v \cdot \partial w} = 0$$

$$\frac{\partial^2 u}{\partial v \cdot \partial w} = 0$$

$$\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial w}\right) = 0$$

$$\frac{\partial u}{\partial w} = h(w)$$

$$u = \int h(w)\,\partial w + \phi(v)$$

$$= \psi(w) + \phi(v) \; ; \; \psi(w) = \int h(w)\,dw$$

$$\boxed{u(x,t) = \phi(x+ct) + \psi(x-ct)} \; --- \; (5)$$

Equation (5) is known as the d-Almbert solution of wave equation.

consider the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

$$\phi(x) + \psi(x) = f(x) \; ---- \; (6)$$

$$u_t(x,0) = g(x)$$

$$c\phi'(x) - c\psi'(x) = g(x)$$

$$\phi'(x) - \psi'(x) = \frac{1}{c} g(x)$$

Integrating both sides w.r.t x from x_0 to x, we have

$$\phi(x) - \phi(x_0) - \psi(x) - \psi(x_0) = \frac{1}{c}\int_{x_0}^{x} g(r)\,dr$$

$$\phi(x) - \psi(x) = \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x} g(r)\,dr$$

$$\phi(x) - \psi(x) = K(x_0) + \frac{1}{c}\int_{x_0}^{x} g(r)\,dr \quad \text{---- (7)}$$

$$\phi(x) + \cancel{\psi(x)} = f(x) \quad \text{--- (6)}$$

$$+\ \phi(x) - \cancel{\psi(x)} = K(x_0) + \frac{1}{c}\int_{x_0}^{x} g(r)\,dr \quad \text{— (7)}$$

$$2\phi(x) = f(x) + K(x_0) + \frac{1}{c}\int_{x_0}^{x} g(r)\,dr$$

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2}K(x_0) + \frac{1}{2c}\int_{x_0}^{x} g(r)\,dr \quad \text{— (8)}$$

~~Subst~~ subtracting equation (6) & (7), we have

$$2\psi(x) = f(x) - K(x_0) - \frac{1}{c}\int_{x_0}^{x} g(r)\,dr$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{K(x_0)}{2} - \frac{1}{2c}\int_{x_0}^{x} g(r)\,dr \quad \text{— (9)}$$

In equation (8), replace x be x + ct

$$\phi(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2}K(x_0) + \frac{1}{2c}\int_{x_0}^{x+ct} g(r)\,dr \text{—}$$

In equation (9), replace x by x − ct

$$\psi(x-ct) = \frac{1}{2}f(x-ct) - \frac{K(x_0)}{2} - \frac{1}{2c}\int_{x_0}^{x-ct} g(r)\,dr \quad — (11)$$

Hence, the solution becomes,

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

$$= \frac{1}{2}f(x+ct) + \cancel{\frac{1}{2}K(x_0)} + \frac{1}{2c}\int_{x_0}^{x+ct} g(r)\,dr$$

$$+ \frac{1}{2}f(x-ct) - \cancel{\frac{K(x_0)}{2}} + \frac{1}{2c}\int_{x-ct}^{x_0} g(r)\,dr$$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(r)\,dr}$$

Q Find the deflection $u(x,t)$ of a vibrating string of length $L = 1$, ends fixed, $c = 1$ with starting initial velocity 0 and initial deflection $K\sin\pi x$, $K = 0.01$.

Sol Here, $L = 1$, $c = 1$, $g(x) = 0$

$$f(x) = K\sin\pi x$$

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right]$$

$$= \frac{1}{2}\left[K\sin\pi(x+t) + K\sin(\pi(x-t))\right]$$

$$= \frac{K}{2}\left[\sin\pi(x+t) + \sin\pi(x-t)\right]$$

$$= \frac{K}{\cancel{2}}\cdot\cancel{2}\sin\pi x\cos\pi t$$

$$= K\sin\pi x\cos\pi t$$

$$= 0.01\sin\pi x\cos\pi t$$

Q Find deflection $u(x,t)$ of a vibrating string of length $L = 1$, $C = 1$, starting initial velocity zero and initial deflection.

1) $f(x) = K(1 - \cos x)$
2) $f(x) = Kx(1-x)$
3) $f(x) = K \sin 2\pi x$

Sol - 1) $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$

$= \frac{1}{2}[K(1 - \cos(x+t)) + K(1 - \cos(x-t))]$

$= \frac{1}{2}[2K - (K\cos(x+t) + K\cos(x-t))]$

$= \frac{K}{2}[2 - \cos x \cos t - \sin x \sin t + \cos x \cos t + \cancel{\sin x \sin t}]$

$= \frac{K}{2}[2 - 2\cos x \cos t]$

$= K[1 - \cos x \cos t]$

Sol - 2) $u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)]$

$= \frac{1}{2}[K(x+t)(1-x-t) + K(x-t)(1-x+t)]$

$= \frac{K}{2}[x - x^2 - \cancel{xt} + \cancel{t} - \cancel{xt} - t^2 + x - x^2 + \cancel{xt} - \cancel{t} + \cancel{xt} - t^2]$

$= \frac{K}{2}[2x - 2x^2 - 2t^2]$

$= 0.01[-x^2 + x - t^2]$

Differential Equation

- ODE
  - Linear
  - Non-linear
- PDE
  - Linear
  - Quasi linear
  - Non-linear

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = G$$

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + FU = G$$

If $B^2 - AC = 0 \rightarrow$ Parabolic

$B^2 - AC > 0 \rightarrow$ Hyperbolic

$B^2 - AC < 0 \rightarrow$ Elliptic

$AC - B^2 = 0 \rightarrow$ Parabolic

$AC - B^2 > 0 \rightarrow$ Elliptic

$AC - B^2 < 0 \rightarrow$ Hyperbolic

Characterstics : Types & Normal forms of PDE

Consider the partial differential equation of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = f(x, y, u, u_x, u_y) \quad \text{--- (1)}$$

The PDE is called quasi-linear because it is linear in the highest derivatives. Depending on the discriminant, the PDE's are classified as

Condition	Type
$AC - B^2 = 0$	Parabolic
$AC - B^2 > 0$	Elliptic
$AC - B^2 < 0$	Hyperbolic

Q $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

sol $A = 1,\ B = 0,\ C = -1$

$AC - B^2 = 1(-1) - 0 = -1 < 0$ Hyperbolic

Q $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

sol $A = 1,\ B = 0,\ C = 0$

$AC - B^2 = 0 - 0 = 0$ Parabolic PDE

Q $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

sol $A = 1,\ B = 0,\ C = 1$

$AC - B^2 = 1 - 0 = 1 > 0$ Elliptic PDE

Q $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$

$A = 1,\ B = 0,\ C = 1$

$AC - B^2 = 1 > 0$ Elliptic PDE

Transformation of Normal form

The normal form of equation ① and a corresponding transformations depend on the type of PDE. They are obtained by solving

the characteristic equation of equation ①
which is an ODE of the form
$$Ay'^2 - 2By' + C = 0 \quad \text{---} ②$$

The solution of equation ② are called
the characteristic of equation ①.

Q classify the PDE

1) $u_{xx} + 4u_{yy} = 0$

2) $u_{xx} - 16u_{yy} = 0$

3) $u_{xx} + 2u_{xy} + u_{yy} = 0$ Parabolic

4) $u_{xx} - 12u_{xy} + u_{yy} = 0$ Hyperbolic

5) $u_{xx} + 5u_{xy} + 4u_{yy} = 0$ Hyperbolic

Sol - 1) $A = 1, \; B = 0, \; C = 4$

$AC - B^2 = 1 \cdot 4 - 0 = 4 > 0$ Elliptic

Sol - 2) $A = 1, \; B = 0, \; C = -16$

$AC - B^2 = -16 - 0 = -16 < 0$ Hyperbolic

Sol - 4) $AC - B^2 = 1 - 36 = -35 < 0 \rightarrow$ Hyperbolic

Q $u_{xx} - 16u_{yy} = 0$

Sol $A = 1, \; B = 0, \; C = -16$

$AC - B^2 = -16 < 0$

Hence, it is hyperbolic

The characterstic equation of the given differential equation is given by

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F$$

$$Ay'^2 - 2By'y + C = 0$$

$$y'^2 - 16 = 0$$

$$(y'+4)(y'-4) = 0$$

$$y'+4 = 0, \quad y'-4 = 0$$

$dy = -4dx$ | $dy = 4dx$

$y = -4x + C_1$ | $y = 4x + c$

$4x + y = C_1$ | $4x - y = C_2$

~~Alternate Method~~

Here, $v = 4x + y$, $\omega = 4x - y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial \omega}\cdot\frac{\partial \omega}{\partial x}$$

$$= 4\frac{\partial u}{\partial v} + 4\frac{\partial u}{\partial \omega}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(4\frac{\partial u}{\partial v} + 4\frac{\partial u}{\partial \omega}\right)$$

$$= \frac{\partial}{\partial v}\left(4\frac{\partial u}{\partial v} + 4\frac{\partial u}{\partial \omega}\right)\cdot\frac{\partial v}{\partial t} + \frac{\partial}{\partial \omega}\left(4\frac{\partial u}{\partial \cancel{x}} + 4\frac{\partial u}{\partial \omega}\right)\cdot\frac{\partial \omega}{\partial x}$$

$$= 16\frac{\partial^2 u}{\partial v^2} + 32\frac{\partial^2 u}{\partial v\cdot\partial \omega} + 16\frac{\partial^2 u}{\partial \omega^2}$$

$$u_{xx} - 16u_{yy} = 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)$$

$$= \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right) \cdot \frac{\partial v}{\partial y} + \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right) \cdot \frac{\partial w}{\partial y}$$

$$= \frac{\partial^2 u}{\partial v^2} - \frac{2\partial^2 u}{\partial v \cdot \partial w} + \frac{\partial^2 u}{\partial w^2}$$

$$\cancel{16\frac{\partial^2 u}{\partial v^2}} + 32\frac{\partial^2 u}{\partial v \cdot \partial w} + \cancel{16\frac{\partial^2 u}{\partial w^2}} - \cancel{16\frac{\partial^2 u}{\partial v^2}} +$$

$$32\frac{\partial^2 u}{\partial v \cdot \partial w} + 32\frac{\partial^2 u}{\partial v \cdot \partial w} - \cancel{16\frac{\partial^2 u}{\partial w^2}} = 0$$

$$64\frac{\partial^2 u}{\partial v \cdot \partial w} = 0$$

$$\frac{\partial^2 u}{\partial v \cdot \partial w} = 0$$

$$\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial w}\right) = 0$$

$$\frac{\partial u}{\partial w} = h(w)$$

$$\partial u = h(w)\,dw$$

$$u = \int h(w)\,dw + \phi(v)$$

$$= \psi(w) + \phi(v)$$

$$= \psi(4x+y) + \psi(4x-y)$$

* NOTE : If we have a hyperbolic PDE then ~~a~~ we two solutions of the characterstic equation.
Find $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 y}{\partial x \partial y}$

Substitute in the equation and then we have $\frac{\partial^2 u}{\partial v \partial w} = 0$.

$$u(x,y) = \phi(v) + \psi(w)$$

$u_{xx} + 2u_{xy} + u_{yy} = 0$ --- (1)

$A = 1,\ B = 1,\ C = 1.$

$AC - B^2 = 0$ (Parabolic PDE)

The characterstic equation,

$$y'^2 - 2y + 1 = 0$$

$$(y' - 1)^2 = 0$$

$$y' - 1 = 0$$

$v = x$
$w = x - y$

$$dy = dx$$
$$y = x + c$$
$$x - y = c$$

Here, $v = x$, $w = x - y$, $\frac{\partial v}{\partial x} = 1$, $\frac{\partial v}{\partial y} = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x} \qquad \frac{\partial w}{\partial x} = 1, \frac{\partial w}{\partial y} = -1$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right)$$

$$= \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) \cdot \frac{\partial v}{\partial x} + \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) \cdot \frac{\partial w}{\partial x}$$

$$= \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v \cdot \partial w} + \frac{\partial^2 u}{\partial w^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y} = -\frac{\partial u}{\partial w}$$

$\to 0$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial w}\right) = \frac{\partial}{\partial v}\left(-\frac{\partial u}{\partial w}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial w}\left(-\frac{\partial u}{\partial w}\right)\frac{\partial w}{\partial x}$$

$$= -\frac{\partial^2 u}{\partial v \cdot \partial w} - \frac{\partial^2 u}{\partial w^2}$$

Equation (i) becomes

$$\frac{\partial^2 u}{\partial v^2} + \cancel{2\frac{\partial^2 u}{\partial v \cdot \partial w}} + \cancel{\frac{\partial^2 u}{\partial w^2}} - \cancel{2\frac{\partial^2 u}{\partial v \cdot \partial w}} - \cancel{2\frac{\partial^2 u}{\partial w^2}} + \cancel{\frac{\partial^2 u}{\partial w^2}} = 0$$

$$\frac{\partial^2 u}{\partial v^2} = 0$$

$$\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v}\right) = 0$$

$$\frac{\partial u}{\partial v} = c(\omega)$$

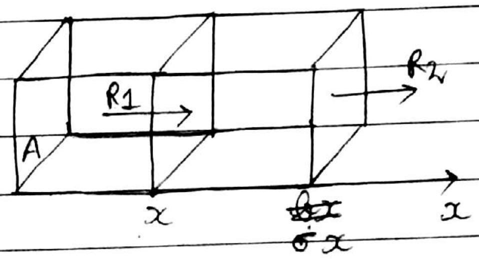
$$\partial u = c(\omega)\partial v$$

$$u = c(\omega)v + \phi(\omega)$$
$$= \phi(\omega) + vc(\omega)$$
$$= \phi(x-y) + xc(x-y)$$

Heat Equation :-

consider the temperature in a long thin metal bar or wire of constant cross-section and homogeneous material which is oriented along the x-axis and is perfectly insulated so that the heat flows in the direction of x only.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial T}{\partial t} = c^2 \frac{\partial^2 T}{\partial x^2}$$



The governing equation is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

with initial temperature in the bar at $t = 0$ is $f(x)$

i.e. $u(x,0) = f(x)$ — ②

and the boundary conditions

$u(0,t) = u(L,t) = 0 \quad \forall \; t \geq 0$ — ③

Solution of heat equation

Let the solution of equation ① be of the form $u(x,t) = F(t)G(t)$ — ④

$$\frac{\partial u}{\partial t} = F(x)\dot{G}(t)$$

$$\frac{\partial u}{\partial x} = F'(x)G(t)$$

$$\frac{\partial^2 u}{\partial x^2} = F''(x)G(t)$$

— ⑤

where · & ' denote the derinate w.r.t t & x respectively.

Substitute the equation ⑤ in equation (i)

$$F(x)\dot{G}(t) = c^2 F''(x)G(t)$$

$$\frac{\dot{G}(t)}{G(t)} = \frac{F''(x)}{F(x)}$$

$$\frac{\dot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = K$$

$$\dot{G}(t) - Kc^2 G(t) = 0 \quad \& \quad F''(x) - KF(x) = 0$$

Case I : When $K = 0$

$\dot{G}(t) = 0$ $\quad$ $F''(x) = 0$

$G(t) = A$ $\quad$ $F(x) = Bx + C$

$u(x,t) = F(x)\,G(t)$

$= (Bx + C)A$

$u(0,t) = 0$, $u(L,t) = 0$

$AC = 0$ $\quad$ $BL + CA = 0$

$A = 0$ or $C = 0$ $\quad$ $AB = 0$

$A = 0$ or $B = 0$

$G(t) = 0$

$u(x,t) = 0$ (trivial solution)

Case II : When $K = p^2$

$\dot{G} - p^2c^2G = 0$

$\frac{dG}{G} = p^2c^2\,dt$

$\ln G = p^2c^2t + c_1$

$G = e^{p^2c^2t} \cdot e^{c_1} = c_2\,e^{p^2c^2t}$

$u(x,t) = (Ae^{px} + Be^{-px})\,c_2\,e^{p^2c^2t}$

$u(0,t) = 0$	$u(L,t) = 0$
$\Rightarrow F(0)\,G(t) = 0$	$\Rightarrow F(L)\,G(t) = 0$
$\Rightarrow F(0) = 0$	$\Rightarrow F(L) = 0$
$A + B = 0$	$Ae^{pL} + Be^{-pL} = 0$

$F(x) = 0 \Rightarrow u(x,t) = 0$ (trivial solution)

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Case III :- When $K = -p^2$

$$\cancel{G + p^2c^2G = 0} \quad F''(x) + p^2 F(x) = 0$$

Auxiliary equation $\lambda^2 + p^2 = 0$

$$\lambda = \pm pi$$

$$F(x) = A\cos px + B\sin px$$

$$u(x,t) = (A\cos px + B\sin px)\, C_2\, e^{-p^2c^2t}$$

$$u(0,t) = 0 \Rightarrow F(0) = 0$$
$$\Rightarrow A = 0$$

$$u(L,t) = 0 \Rightarrow F(L) = 0$$
$$\Rightarrow A\cos pL + B\sin pL = 0$$
$$\Rightarrow B\sin pL = 0$$
$$\Rightarrow \sin pL = 0$$
$$\Rightarrow \sin pL = \sin n\pi$$
$$p = \frac{n\pi}{L}$$

$$u(x,t) = B\sin\frac{n\pi}{L}x \cdot C_2\, e^{-\frac{n^2\pi^2}{L^2}t} \quad ; n = 1,2,3,\ldots$$

$$u_n(x,t) = D_n \sin\frac{n\pi}{L}x\, e^{-\lambda_n^2 t} \quad \text{—— (6)}$$

$$\lambda_n = \frac{cn\pi}{L}$$

Now, to obtain a solution which satisfies the given initial condition, we consider a series of eigen function, i.e,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{L}x\, e^{-\lambda_n t} \quad \text{—— (7)}$$

$$u(x,0) = f(x)$$

$$\sum_{n=1}^{\infty} D_n \sin \frac{n\pi}{L} x = f(x) \quad — (8)$$

Hence, for equation (7) to satisfy eq^n (8) the coefficient D_n must be the cofficien of fourier sine series given by

$$D_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

Q To Find the temperature $u(x,t)$ in a bar of silver length = 10 cm and constant cross-section of area 1 cm^2, density = 10.6 g/c thermal conductivity = 1.04 cal/(cm-sec) specific heat = 0.056 cal/(g°C) that is perfectly insulated with ends kept at °C and initial temperature is given by

$f(x) = \sin 0.1\pi x$

$f(x) = x(10-x)$

$f(x) = 4 - 0.8|x-5|$

Sol-1) $$C^2 = \frac{K}{\sigma - \rho}$$

where, K = Thermal conductivity

ρ = density

σ = specific heat

$$c^2 = \frac{1.04}{0.056 \times 10.6} = 1.753$$

$c = 1.32$

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi}{L} x \, e^{-\lambda_n^2 t}$$

$$u(x,0) = f(x) = \sin 0.1\pi x$$

$$\sum_{n=1}^{\infty} D \sin \frac{n\pi}{10} x = \sin 0.1\pi x$$

$$D_1 \sin 0.1\pi x + D_2 \sin 0.2\pi x + D_3 \sin 0.3\pi x + \cdots = \sin 0.1\pi x$$

comparing both sides,

$D_1 = 1, \quad D_2 = 0, \quad D_3 = 0$

$$u(x,t) = \sin 0.1\pi x \, e^{-\lambda_n^2 t}$$

$$\lambda_n = \frac{cn\pi}{L}$$

$$\lambda_n^2 = \frac{1.753 \times \pi^2}{100} = 0.01753\pi^2$$

$$\therefore u(x,t) = \sin 0.1\pi x \, e^{-0.173t}$$

sol-2) $$u(x,t) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi}{L} x \, e^{-\lambda_n^2 t}$$

$$D_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

$$= \frac{2}{10}\int_0^{10} \underset{\mathrm{I}}{(10x - x^2)} \underset{\mathrm{II}}{\sin \frac{n\pi}{10} x} \, dx$$

$$= \frac{1}{5}\left[(10x - x^2)\left(\frac{-\cos \frac{n\pi}{10}}{\frac{n\pi}{10}}\right) - \int_0^{10} (10 - 2x)\left(\frac{-\cos \frac{n\pi}{10}}{\frac{n\pi}{10}}\right) dx \right.$$

$$= \frac{1}{5}\left[-\frac{(10x - x^2)\cos\frac{n\pi}{10}x}{\frac{n\pi}{10}} + \left\{ \frac{(10-2x)\sin\frac{n\pi}{10}x}{\frac{n^2\pi^2}{100}} - \frac{2\cos\frac{n\pi}{10}x}{\frac{n^3\pi^3}{1000}} \right\} \right]_0^{10}$$

$$= \frac{1}{5}\left[\frac{-2000}{n^3\pi^3}\cos n\pi + \frac{2000}{n^3\pi^3} \right]_0$$

$$= \frac{400}{n^3\pi^3}\left[1 - \cos n\pi\right]$$

$$\cos n\pi = \begin{cases} -1, & n = \text{odd} \\ 1, & n = \text{even} \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{L}x\, e^{-\lambda_n^2 t}$$

$$= \sum_{n=\text{odd}} \frac{800}{n^3\pi^3} \sin\frac{n\pi}{10}x\, e^{-\lambda_n^2 t}$$

Sol-3) $f(x) = 4 - 0.8|x-5|$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$|x-5| = \begin{cases} x-5, & x \geq 5 \\ -(x-5), & x < 5 \end{cases}$$

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$B_n = \frac{2}{10}\int_0^5 (4 + 0.8(x-5)) \sin \frac{n\pi}{L} x \, dx + \frac{2}{10}\int_5^{10} (4 - 0.8(x-5)) \sin \frac{n\pi}{10} x \, dx$$

$$= \frac{1}{5}\int_0^5 0.8x \sin \frac{n\pi}{10} x \, dx + \frac{1}{5}\int_5^{10} (8 - 0.8x) \sin \frac{n\pi}{10} x \, dx.$$

Ex - 12.6

Q 11) Show that for a completely insulated plate $u_x(0,t) = 0$ and $u_x(L,t) = 0$. $u(x,0) = f(x)$ and the seperation of variable gives the following solution

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{L} x \, e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

Sol

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The solution is of the form

$$u(x,t) = F(x)\, G(t)$$

$$\frac{\partial u}{\partial t} = F(x)\, \dot{G}(t)$$

$$\frac{\partial^2 u}{\partial x^2} = F''(x)\, G(t)$$

$$F(x)\, \dot{G}(t) = c^2 F''(x)\, G(t)$$

$$\frac{\dot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = -k^2 \text{ (constant)}$$

$$\dot{G}(t)\, c^2 k^2 G(t) = 0$$

$$F''(x) + k^2 F(x) = 0$$

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$$\frac{dG(t)}{G(t)} = -c^2 K^2 dt$$

$$\ln G(t) = -c^2 K^2 t$$

$$G(t) = A e^{-c^2 K^2 t}$$

$$F''(x) + K^2 F(x) = 0$$

Auxiliary equation :- $\lambda^2 + K^2 = 0$

$$\lambda = \pm K i$$

$$F(x) = B \cos Kx + D \sin Kx$$

$$u(x,t) = (B \cos Kx + D \sin Kx) A e^{-c^2 K^2 t}$$

$$\frac{\partial u}{\partial x} = (-BK \sin Kx + DK \cos Kx) A e^{-c^2 K^2 t}$$

$u_x(0,t) = 0$ $\qquad$ $u_x(L,t) = 0$

$F'(0) G(t) = 0$ $\qquad$ $F'(L) G(t) = 0$

$F'(0) = 0$ $\qquad$ $F'(L) = 0$

$$F'(x) = -BK \sin Kx + DK \cos Kx$$

$F'(0) = 0$ $\qquad$ $F'(L) = 0$

$DK = 0$ $\qquad$ $-BK \sin KL = 0$

$D = 0$ $\qquad$ $\sin KL = 0$

$$\sin KL = \sin m\pi$$

$$K = \frac{m\pi}{L} ; m = 0, 1, 2, \ldots$$

$$u(x,t) = (B \cos Kx) A e^{-\frac{c^2 m^2 \pi^2}{L^2} t}, m = 0, 1, 2, \ldots$$

$$u(x,t) = E_m \cos \frac{m\pi x}{L} e^{-\frac{c^2 m^2 \pi^2 t}{L^2}}$$

$$u(x,0) = f(x)$$
$$E_n \cos \frac{n\pi}{L} x = f(x)$$

Hence, E_n must be the cofficient of fourier cosine series

where, $E_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx$

The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} E_n \cos \frac{n\pi}{L} x e^{-\lambda_n^2 t}, \quad \lambda_n = \frac{cn\pi}{L}$$

$$= E_0 + \sum_{n=1}^{\infty} E_n \cos \frac{n\pi}{L} x e^{-\lambda_n^2 t}$$

Q Find the temperature in the completely insulated bar with $L = \pi$, $c = \pi$ and

(i) $f(x) = 1$

(ii) $f(x) = x$

(iii) $f(x) = \cos 2x$

(iv) $f(x) = \cancel{1 + |x|} \; 1 - \frac{x}{\pi}$

Sol

$$E_0 = \frac{1}{L} \int_0^L f(x) \, dx$$

$$E_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, dx$$

$$u(x,t) = ?$$

$$u_x(0,t) \cancel{= f(x)} = 0 = u_x(\pi,t)$$

(i) $u(x,0) = f(x) = 1$

$$\epsilon_0 = \frac{1}{\pi}\int_0^{\pi} dx = \frac{1}{\pi}(x-0) = 1$$

$$\epsilon_n = \frac{2}{\pi}\int_0^{\pi} \cos nx\, dx$$

$$= \frac{2}{\pi}\left(\frac{\sin nx}{n}\right)_0^{\pi}$$

$$= 0$$

$$u(x,t) = 1 + 0 = 1$$

(ii) $u(x,0) = x$

$$\epsilon_0 = \frac{1}{\pi}\int_0^{\pi} x\, dx = \frac{1}{\pi}\left[\frac{\pi^2}{2} - 0\right] = \frac{\pi}{2}$$

$$\epsilon_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}(\cos n\pi - 1)$$

$$u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}(\cos n\pi - 1)\cos nx\, e^{-\lambda^2 t}$$

(iii) $f(x) = \cos 2x$

$$\epsilon_0 = \frac{1}{\pi}\int_0^{\pi} \cos 2x\, dx = \frac{1}{\pi}\left(\frac{\sin 2x}{2}\right)_0^{\pi} = 0$$

$$\epsilon_n = \frac{2}{\pi}\int_0^{\pi} \cos 2x \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\cos 2x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \{\cos(2+n)x + \cos(2-n)x\} \, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(2+n)x}{2+n} + \frac{\sin(2-n)x}{2-n} \right]_0^{\pi}$$

$$= 0, \quad \forall n \neq 2$$

$$E_2 = \frac{2}{\pi}\int_0^{\pi} \cos^2 2x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1-\cos 4x}{2} \, dx$$

$$= \frac{2}{\pi}\left[\frac{1}{2}x - \frac{\sin 4x}{8} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\right] = 1$$

Boundary conditions

1) Dirichlet boundary condition
2) Neumann boundary condition
3) Mixed boundary condition / robin problem

Dirichlet Boundary condition

If the value of u is prescribed on the boundary of the region R then it is called dirichlet boundary condition.

Neumann boundary condition

If the normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on the boundary then it is called neumann boundary condition.

Mixed boundary condition

If the value of u is prescribed on some portions of the boundary and the normal derivate $u_n = \frac{\partial u}{\partial n}$ on the rest of the boundary then it is called mixed boundary condition.

2-D Heat Equation

The 2-D heat equation is given by

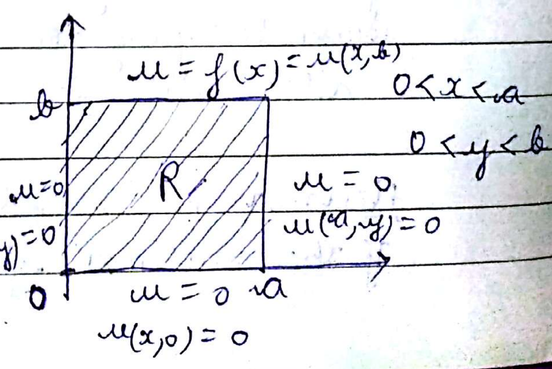
$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad ---- (1)$$

If the heat flow is steady (independent of time) then $\frac{\partial u}{\partial t} = 0$ and equation (1) reduces to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ --- (2) (laplace eqn.)

Solution of 2-D heat equation

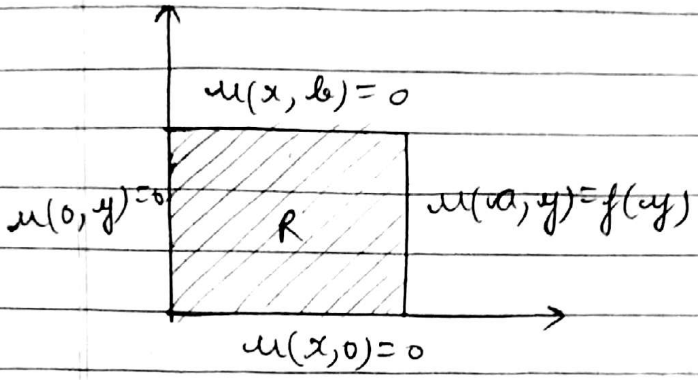
consider a rectangle in x-y plane satisfying the boundary conditions
$u(x,0) = 0$, $u(x,b) = f(x)$, $u(0,y) = 0$, $u(a,y) = 0$.

The solution of the heat equation is given by



$$u(x,y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \quad \text{--- (3)}$$

$$\text{where, } A_n^* = \frac{2}{a \sin\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin \frac{n\pi x}{a} \, dx \quad \text{--- (4)}$$



→ Replace a with b and b with a in the above formular

The laplace equation given in equation (ii) governs the electrostatic potential of electrical charges in any region. Thus, the steady state heat problem can also be interpreted as a electrostatic potential problem. In that case equation (3) and (4) is the potential in the rectangle R when the upper side of R is at potential f(x) and other three sides are grounded

Q Find the potential in the rectangle $0 \le x \le 20$, $0 \le y \le 40$ where upper side is kept at potential 110V and other side and grounded.

Sol $a = 20, \ b = 40$

$$A_n^* = \frac{2}{20 \sinh\left(\frac{n\pi \, 40}{20}\right)} \int_0^{20} 110 \sin \frac{n\pi x}{20} \, dx$$

$$= \frac{110}{10 \sinh(2m\pi)} \left(\frac{-\cos \frac{m\pi x}{20}}{m\pi/20} \right)_0^{20}$$

$$= \frac{11}{\sinh(2m\pi)} \frac{20}{m\pi} \left[-\cos m\pi + 1 \right]$$

$$= \frac{220}{m\pi \sinh(2m\pi)} \left[1 - \cos m\pi \right]$$

$$u(x,y) = \sum_{m=1}^{\infty} \frac{220}{m\pi \sinh(2m\pi)} (1 - \cos m\pi) \sin \frac{m\pi x}{20} \cdot \sinh \frac{m\pi y}{20}$$

Q Find the potential in the square $0 \le x \le 2$, $0 \le y \le 2$ if the upper side is kept at the potential $1000 \sin \frac{\pi x}{2}$ and the other sides are grounded.

Here, $a = 2$, $b = 2$, $f(x) = 1000 \sin \frac{\pi x}{2}$

$$A_m^* = \frac{2}{2 \sinh(m\pi)} \int_0^2 1000 \sin \frac{\pi x}{2} \sin \frac{m\pi x}{2} \, dx$$

$$= \frac{500}{\sinh(m\pi)} \int_0^2 2 \sin \frac{\pi x}{2} \sin \frac{m\pi x}{2} \, dx$$

$$= \frac{500}{\sinh(m\pi)} \int_0^2 \left[\cos (1-m)\frac{\pi}{2} x - \cos (1+m) \frac{\pi x}{2} \right] dx$$

$$= \frac{500}{\sinh(m\pi)} \left[\frac{\sin (1-m)\frac{\pi}{2} x}{(1-m)\frac{\pi}{2}} - \frac{\sin (1+m)\frac{\pi x}{2}}{(1+m)\frac{\pi}{2}} \right]_0^2$$

$$A_1^* = \frac{1000}{\sinh(m\pi)} \int_0^2 \sin^2 \frac{\pi x}{2} \, dx$$

$$= \frac{1000}{\sinh(n\pi)} \int_0^2 \left(\frac{1 - \cos \pi x}{2} \right) dx$$

$$= \frac{1000}{\sinh(n\pi)} \left[\frac{1}{2} x - \frac{\sin \pi x}{2\pi} \right]_0^2 = \frac{1000}{\sinh(n\pi)}$$

$$u(x, y) = \frac{1000}{\sinh \pi} \sin \frac{\pi x}{2} \sinh \frac{\pi y}{2}$$

Q The faces of a thick square plate with sides $a = 24$ is perfectly insulated. The upper ~~a~~ side is kept at 25° C and the other sides are kept at zero. Find the steady state temperature $u(x, y)$ from the plate.

Sol