MATH 2551, Written Assignment 2

Niraj Khatri

February 2025

1. Consider the function
$$f(x, y) = x^2 + y^2 - 3x - xy.$$

   (a) Identify the locations of all critical points of f.

   To find the critical points, compute the partial derivatives of f with respect to x and y, and set them equal to zero:
$$f_x = \frac{\partial f}{\partial x} = 2x - 3 - y = 0,$$
$$f_y = \frac{\partial f}{\partial y} = 2y - x = 0.$$
   Solve the system of equations: 1. From $f_y = 0$, we get $x = 2y$. 2. Substitute $x = 2y$ into $f_x = 0$:
$$2(2y) - 3 - y = 0 \implies 4y - 3 - y = 0 \implies 3y - 3 = 0 \implies y = 1.$$
   3. Substitute $y = 1$ into $x = 2y$:
$$x = 2(1) = 2.$$
   Thus, the only critical point is $(2, 1)$.

   (b) Use the second derivative test to classify the critical point.

   Compute the second partial derivatives:
$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2,$$
$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2,$$
$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = -1.$$
   The discriminant D is given by:
$$D = f_{xx} f_{yy} - (f_{xy})^2 = (2)(2) - (-1)^2 = 4 - 1 = 3.$$
   Since $D > 0$ and $f_{xx} > 0$, the critical point $(2, 1)$ is a **local minimum**.

   (c) Determine the absolute maximum and minimum values of f on the curve $x^2 + y^2 = 9$.

   Use the method of Lagrange multipliers. Define the constraint:
$$g(x, y) = x^2 + y^2 - 9 = 0.$$
   The gradients of f and g must satisfy:
$$\nabla f = \lambda \nabla g.$$

Compute the gradients:

$$\nabla f = (2x - 3 - y, 2y - x),$$
$$\nabla g = (2x, 2y).$$

Set up the equations:

$$2x - 3 - y = \lambda(2x),$$
$$2y - x = \lambda(2y).$$

Solve for λ: From the first equation:

$$\lambda = \frac{2x - 3 - y}{2x}.$$

From the second equation:

$$\lambda = \frac{2y - x}{2y}.$$

Set the two expressions for λ equal:

$$\frac{2x - 3 - y}{2x} = \frac{2y - x}{2y}.$$

Cross-multiply and simplify:

$$(2x - 3 - y)(2y) = (2y - x)(2x),$$
$$4xy - 6y - 2y^2 = 4xy - 2x^2,$$
$$-6y - 2y^2 = -2x^2,$$
$$2x^2 - 2y^2 - 6y = 0,$$
$$x^2 - y^2 - 3y = 0.$$

Use the constraint $x^2 + y^2 = 9$ to substitute $x^2 = 9 - y^2$:

$$(9 - y^2) - y^2 - 3y = 0,$$
$$9 - 2y^2 - 3y = 0,$$
$$2y^2 + 3y - 9 = 0.$$

Solve the quadratic equation:

$$y = \frac{-3 \pm \sqrt{9 + 72}}{4} = \frac{-3 \pm \sqrt{81}}{4} = \frac{-3 \pm 9}{4}.$$

Thus:

$$y = \frac{6}{4} = \frac{3}{2} \quad \text{or} \quad y = \frac{-12}{4} = -3.$$

For $y = \frac{3}{2}$:

$$x^2 = 9 - \left(\frac{3}{2}\right)^2 = 9 - \frac{9}{4} = \frac{27}{4} \implies x = \pm\frac{3\sqrt{3}}{2}.$$

For $y = -3$:

$$x^2 = 9 - (-3)^2 = 0 \implies x = 0.$$

Evaluate f at these points: 1. $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$:

$$f = \left(\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - 3\left(\frac{3\sqrt{3}}{2}\right) - \left(\frac{3\sqrt{3}}{2}\right)\left(\frac{3}{2}\right) = -2.691$$

2. $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$:

$$f = \left(-\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - 3\left(-\frac{3\sqrt{3}}{2}\right) - \left(-\frac{3\sqrt{3}}{2}\right)\left(\frac{3}{2}\right) = 20.691$$

3. $(0, -3)$:

$$f = 0^2 + (-3)^2 - 3(0) - 0(-3) = 9.$$

4. $(2, 1)$:

$$f = 2^2 + 1^2 - 3(2) - (2)(1) = 4 + 1 - 6 - 2 = -3.$$

Since we are taking absolute values:

$$|-2.691| = 2.691, \quad |20.691| = 20.691, \quad |-3| = 3, \quad |9| = 9.$$

Thus, the absolute maximum is:

$$20.691$$

and the absolute minimum is:

$$2.691$$

2. Consider the parabola

$$y^2 = 4x,$$

and let $P = (2, 1)$ be a fixed point. The goal is to find the point on the parabola, Q, that is closest to P using the method of Lagrange multipliers.

(a) Define the squared distance function from a point (x, y) on the parabola to P.

The squared distance function is:

$$D(x, y) = \sqrt{(x - 2)^2 + (y - 1)^2}$$

(b) Use the method of Lagrange multipliers to determine the candidate point(s) on the parabola that minimizes the distance between P and the parabola.

The constraint is $y^2 = 4x$. Define:

$$g(x, y) = y^2 - 4x = 0.$$

The gradients of D and g must satisfy:

$$\nabla D = \lambda \nabla g.$$

Compute the gradients:

$$\nabla D = (2(x - 2), 2(y - 1)),$$
$$\nabla g = (-4, 2y).$$

Set up the equations:

$$2(x - 2) = \lambda(-4),$$
$$2(y - 1) = \lambda(2y).$$

Solve for λ: From the first equation:

$$\lambda = \frac{2(x - 2)}{-4} = -\frac{x - 2}{2}.$$

From the second equation:

$$\lambda = \frac{2(y - 1)}{2y} = \frac{y - 1}{y}.$$

Set the two expressions for λ equal:

$$-\frac{x - 2}{2} = \frac{y - 1}{y}.$$

Cross-multiply and simplify:

$$-(x - 2)y = 2(y - 1),$$
$$-xy + 2y = 2y - 2,$$
$$-xy = -2,$$
$$xy = 2.$$

Use the constraint $y^2 = 4x$ to substitute $x = \frac{y^2}{4}$:

$$\left(\frac{y^2}{4}\right) y = 2,$$

$$\frac{y^3}{4} = 2,$$
$$y^3 = 8,$$
$$y = 2.$$

Substitute $y = 2$ into $y^2 = 4x$:

$$4 = 4x \implies x = 1.$$

Thus, the candidate point is $Q = (1, 2)$.

(c) Show that the line joining P to Q is normal to the parabola.

The slope of the line joining $P = (2, 1)$ and $Q = (1, 2)$ is:

$$m_{\text{line}} = \frac{2 - 1}{1 - 2} = -1.$$

The slope of the tangent to the parabola $y^2 = 4x$ at $Q = (1, 2)$ is found by differentiating implicitly:

$$2y\frac{dy}{dx} = 4 \implies \frac{dy}{dx} = \frac{2}{y}.$$

At $Q = (1, 2)$, the slope is:

$$m_{\text{tangent}} = \frac{2}{2} = 1.$$

Since $m_{\text{line}} \cdot m_{\text{tangent}} = (-1)(1) = -1$, the line is normal to the parabola.

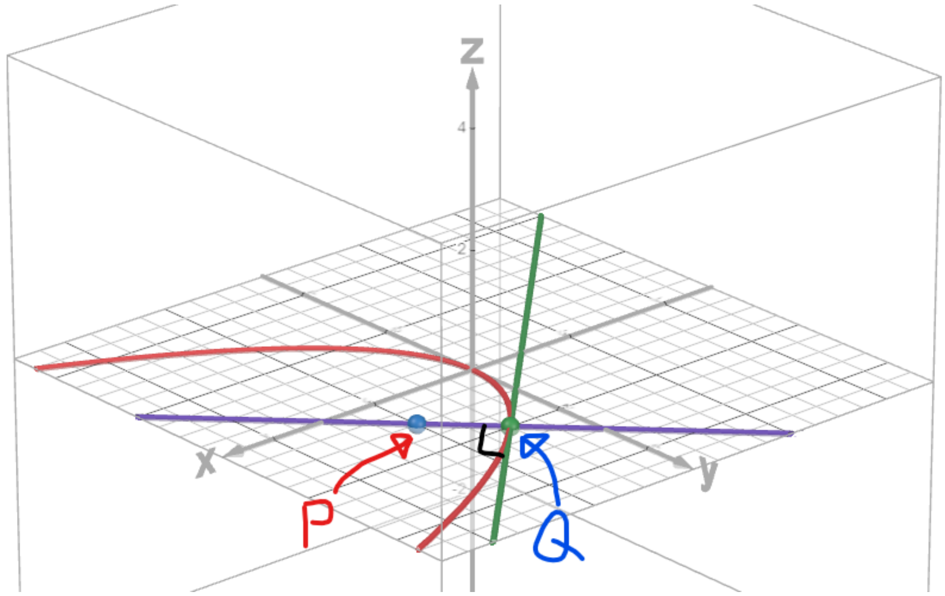

Figure 1: Sketch

(d) Sketch of the parabola with points Q, P, the tangent line to the parabola at Q, and the line connecting P and Q is shown in Figure 1: