MATH 1554 QH, Written Assignment 4

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1. (5 points) Consider the set of vectors in $\mathbb{R}^3$:

$$W = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \\ 2 \end{pmatrix} \right\}$$

(a) Let's apply the Gram-Schmidt process to convert W into an orthogonal set U:
First vector:

$$\mathbf{u}_1 = \mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Second vector:

$$\begin{aligned}
\mathbf{u}_2 &= \mathbf{w}_2 - \text{proj}_{\mathbf{u}_1}(\mathbf{w}_2) \\
\text{proj}_{\mathbf{u}_1}(\mathbf{w}_2) &= \frac{\mathbf{w}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \\
&= \frac{9}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \\
\mathbf{u}_2 &= \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}
\end{aligned}$$

Third vector:

$$\begin{aligned}
\mathbf{u}_3 &= \mathbf{w}_3 - \text{proj}_{\mathbf{u}_1}(\mathbf{w}_3) - \text{proj}_{\mathbf{u}_2}(\mathbf{w}_3) \\
\text{proj}_{\mathbf{u}_1}(\mathbf{w}_3) &= \frac{\mathbf{w}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \\
\text{proj}_{\mathbf{u}_2}(\mathbf{w}_3) &= \frac{\mathbf{w}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} \\
\mathbf{u}_3 &= \begin{pmatrix} 7 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}
\end{aligned}$$

(b) Let's normalize the vectors in U:
For $\mathbf{u}_1$:

$$\|\mathbf{u}_1\| = \sqrt{1^2 + 2^2 + 2^2} = 3 \quad \Rightarrow \quad \hat{\mathbf{e}}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

For $\mathbf{u}_2$:

$$\|\mathbf{u}_2\| = \sqrt{2^2 + (-2)^2 + 1^2} = 3 \quad \Rightarrow \quad \hat{\mathbf{e}}_2 = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

For $\mathbf{u}_3$:

$$\|\mathbf{u}_3\| = \sqrt{2^2 + 1^2 + (-2)^2} = 3 \quad \Rightarrow \quad \hat{\mathbf{e}}_3 = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$$

(c) The Q matrix is:

$$Q = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

(d) The R matrix is given by $Q^T A$:
where $A =$

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$R = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

2. (5 points) Consider the points $P(-1,0,1)$, $Q(0,-1,0)$, $R(1,0,1)$, and $S(0,1,4)$.

(a) The over-determined system $A\vec{x} = \vec{z}$ is:

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 4 \end{pmatrix}$$

(b) The normal equations $(A^T A)\vec{x} = A^T \vec{z}$ give us:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$$

$$A^T A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^T z = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$$

Solving the system:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$$

gives us:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 2 \\ x_3 &= \frac{3}{2} \end{aligned}$$

(c) The equation of the best-fit plane is:

$$z = 2y + \frac{3}{2}$$

or equivalently:

$$z = 0x + 2y + 1.5$$