MATH 1554 QH, Written Assignment 3

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1. (5 points) Consider the matrix A below, where k is a real number.

$$A = \begin{bmatrix} -2 & 2 & 2 \\ 2 & k & -1 \\ -6 & 3 & 5 \end{bmatrix}$$

(a) The determinant of matrix A can be computed using cofactor expansion along the first row:

$$\det(A) = (-2) \cdot \det \begin{pmatrix} k & -1 \\ 3 & 5 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 2 & -1 \\ -6 & 5 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 2 & k \\ -6 & 3 \end{pmatrix}$$

Now, we compute each 2×2 determinant:

$$\det \begin{pmatrix} k & -1 \\ 3 & 5 \end{pmatrix} = k \cdot 5 - (-1) \cdot 3 = 5k + 3$$

$$\det \begin{pmatrix} 2 & -1 \\ -6 & 5 \end{pmatrix} = 2 \cdot 5 - (-1) \cdot (-6) = 10 - 6 = 4$$

$$\det \begin{pmatrix} 2 & k \\ -6 & 3 \end{pmatrix} = 2 \cdot 3 - k \cdot (-6) = 6 + 6k = 6k + 6$$

Substitute these into the cofactor expansion:

$$\det(A) = (-2) \cdot (5k + 3) - 2 \cdot 4 + 2 \cdot (6k + 6)$$

Simplifying:

$$\det(A) = -2(5k + 3) - 8 + 2(6k + 6)$$
$$\det(A) = -10k - 6 - 8 + 12k + 12$$
$$\det(A) = 2k - 2$$

Thus, the determinant of A is:

$$\det(A) = 2(k - 1)$$

(b) A matrix is singular if its determinant is zero. From part (a), we have:

$$\det(A) = 2(k - 1)$$

Setting this equal to zero:

$$2(k - 1) = 0$$
$$k - 1 = 0$$
$$k = 1$$

Thus, the matrix A is singular when $k = 1$.

(c) When $k = 1$, the matrix A becomes:

$$A = \begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & -1 \\ -6 & 3 & 5 \end{bmatrix}$$

To find the eigenvalues, we need to solve the characteristic equation:

$$\det(A - \lambda I) = 0$$

Substituting A and the identity matrix I, we have:

$$\det\left(\begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & -1 \\ -6 & 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

This simplifies to:

$$\det\begin{bmatrix} -2-\lambda & 2 & 2 \\ 2 & 1-\lambda & -1 \\ -6 & 3 & 5-\lambda \end{bmatrix} = 0$$

Now, we can calculate the determinant. Using the cofactor expansion along the first row, we have:

$$\begin{aligned}\det\begin{bmatrix} -2-\lambda & 2 & 2 \\ 2 & 1-\lambda & -1 \\ -6 & 3 & 5-\lambda \end{bmatrix} &= (-2-\lambda)\det\begin{bmatrix} 1-\lambda & -1 \\ 3 & 5-\lambda \end{bmatrix} - 2\det\begin{bmatrix} 2 & -1 \\ -6 & 5-\lambda \end{bmatrix} + 2\det\begin{bmatrix} 2 & 1-\lambda \\ -6 & 3 \end{bmatrix} \\ &= (-2-\lambda)\left((1-\lambda)(5-\lambda)+3\right) - 2\left(2(5-\lambda)+6\right) + 2\left(6-6(1-\lambda)\right).\end{aligned}$$

Calculating the determinants of the 2x2 matrices: 1.

$$\det\begin{bmatrix} 1-\lambda & -1 \\ 3 & 5-\lambda \end{bmatrix} = (1-\lambda)(5-\lambda)+3 = -\lambda^2+6\lambda+2$$

2.

$$\det\begin{bmatrix} 2 & -1 \\ -6 & 5-\lambda \end{bmatrix} = 2(5-\lambda)+6 = 16-2\lambda$$

3.

$$\det\begin{bmatrix} 2 & 1-\lambda \\ -6 & 3 \end{bmatrix} = 2\cdot 3 + 6(1-\lambda) = 6+6-6\lambda = 12-6\lambda$$

Substituting these back, we expand and simplify:

$$\begin{aligned} &= (-2-\lambda)(-\lambda^2+6\lambda+2) - 2(16-2\lambda) + 2(12-6\lambda) \\ &= (2+\lambda)(\lambda^2-6\lambda-2) - 32 + 4\lambda + 24 - 12\lambda \\ &= (2+\lambda)(\lambda^2-6\lambda-2) - 8\lambda - 8. \end{aligned}$$

Finally, we can factor the characteristic polynomial obtained:

$$\lambda(\lambda-2)^2 = 0$$

The solutions to this equation are:

$$\lambda_1 = 0, \quad \lambda_2 = 2$$

with $\lambda_2 = 2$ having an algebraic multiplicity of 2.

(d) Construct the eigenbasis for each eigenvalue when A is singular.

When A is singular, one of the eigenvalues is $\lambda_1 = 0$. To construct the eigenbasis for this eigenvalue, we solve the equation:

$$(A - 0I)\mathbf{v} = A\mathbf{v} = 0$$

where $\mathbf{v}$ is the eigenvector corresponding to $\lambda_1 = 0$.
Substituting the given matrix A:

$$A = \begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & -1 \\ -6 & 3 & 5 \end{bmatrix}$$

We solve the system:

$$\begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & -1 \\ -6 & 3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives the following system of equations:

$$-2v_1 + 2v_2 + 2v_3 = 0$$
$$2v_1 + v_2 - v_3 = 0$$
$$-6v_1 + 3v_2 + 5v_3 = 0$$

Solving the system:
From the first equation:

$$-2v_1 + 2v_2 + 2v_3 = 0 \implies v_1 = v_2 + v_3$$

Substitute $v_1 = v_2 + v_3$ into the second equation:

$$2(v_2 + v_3) + v_2 - v_3 = 0 \implies 3v_2 + v_3 = 0 \implies v_3 = -3v_2$$

Now substitute $v_3 = -3v_2$ into $v_1 = v_2 + v_3$:

$$v_1 = v_2 + (-3v_2) = -2v_2$$

Thus, the eigenvector corresponding to $\lambda_1 = 0$ is:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

Any scalar multiple of $\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 0$.
For the eigenvalue $\lambda_2 = 2$, we solve the equation:

$$(A - 2I)\mathbf{v} = 0$$

Substituting $A - 2I$:

$$A = \begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & -1 \\ -6 & 3 & 5 \end{bmatrix}, \quad 2I = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & -1 \\ -6 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -1 & -1 \\ -6 & 3 & 3 \end{bmatrix}$$

Now, we solve the system:

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -1 & -1 \\ -6 & 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This system gives the following equations:

$$-4v_1 + 2v_2 + 2v_3 = 0$$
$$2v_1 - v_2 - v_3 = 0$$
$$-6v_1 + 3v_2 + 3v_3 = 0$$

Solving the system:
From the second equation:

$$2v_1 - v_2 - v_3 = 0 \implies v_3 = 2v_1 - v_2$$

Substitute $v_3 = 2v_1 - v_2$ into the first equation:

$$-4v_1 + 2v_2 + 2(2v_1 - v_2) = 0 \implies -4v_1 + 2v_2 + 4v_1 - 2v_2 = 0 \implies 0 = 0$$

This equation simplifies to $0 = 0$, we proceed to the third equation:
Substitute $v_3 = 2v_1 - v_2$ into the third equation:

$$-6v_1 + 3v_2 + 3(2v_1 - v_2) = 0 \implies -6v_1 + 3v_2 + 6v_1 - 3v_2 = 0 \implies 0 = 0$$

This equation also simplifies to $0 = 0$.
The solution has free variables. Let $v_1 = 1$, and $v_2 = 0$. Then, $v_3 = 2$, which gives the eigenvector:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Similarly, if $v_1 = 1$, and $v_2 = 2$, then $v_3 = 0$, which gives the second eigenvector:

$$\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The set of independent vectors corresponding to $\lambda_2 = 2$ is given by:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

2. (5 points) Consider the Markov chain given.

(a) The transition matrix P can be expressed as:

$$P = \begin{bmatrix} 0.5 & 0.1 & 0 \\ 0.5 & 0.5 & 1 \\ 0 & 0.4 & 0 \end{bmatrix}$$

(b) A Markov chain is regular if, for some power P^k, all entries are positive. To verify this, let's compute P^2:

$$P^2 = P \times P = \begin{bmatrix} 0.5 & 0.1 & 0 \\ 0.5 & 0.5 & 1 \\ 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0.1 & 0 \\ 0.5 & 0.5 & 1 \\ 0 & 0.4 & 0 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 & 0.1 \\ 0.5 & 0.7 & 0.5 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$$

Since P^2 has all positive entries, the Markov chain is regular.

(c) To find the steady-state vector $\mathbf{q}$, we need to solve the equation:

$$(P - I)\mathbf{x} = 0$$

where I is the identity matrix. The matrix $P - I$ is given by:

$$P - I = \begin{bmatrix} 0.5 & 0.1 & 0 \\ 0.5 & 0.5 & 1 \\ 0 & 0.4 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.1 & 0 \\ 0.5 & -0.5 & 1 \\ 0 & 0.4 & -1 \end{bmatrix}$$

Now we set up the augmented matrix for the system:

$$\begin{bmatrix} -0.5 & 0.1 & 0 & | & 0 \\ 0.5 & -0.5 & 1 & | & 0 \\ 0 & 0.4 & -1 & | & 0 \end{bmatrix}$$

Next, we row-reduce this matrix. The reduced row echelon form is:

$$\begin{bmatrix} 1 & 0 & -0.5 & | & 0 \\ 0 & 1 & -2.5 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

From the matrix , we can express the variables in terms of x_3:

$$x_1 = 0.5x_3, \quad x_2 = 2.5x_3$$

Letting $x_3 = t$, we can write the general solution as:

$$\mathbf{x} = t \begin{bmatrix} 0.5 \\ 2.5 \\ 1 \end{bmatrix}$$

The basis for the solution space is given by the vector:

$$\begin{bmatrix} 0.5 \\ 2.5 \\ 1 \end{bmatrix}$$

To find the steady-state vector $\mathbf{q}$, divide the vector by the sum of its entries:

$$0.5 + 2.5 + 1 = 4$$

Thus, the steady-state vector $\mathbf{q}$ is:

$$\mathbf{q} = \frac{1}{4} \begin{bmatrix} 0.5 \\ 2.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.625 \\ 0.25 \end{bmatrix}$$