MATH 1554 QH, Written Assignment 2

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1. (5 points) Consider the matrix A and vector b below.

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix}$$

The system $Ax = b$, where $x \in \mathbb{R}^3$, can be solved using the LU factorization.

(a) We first factor A into LU, where L is a lower triangular matrix, and U is an echelon form of matrix A. To obtain U, an echelon form of A, we can apply a series of row replacement operations:

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 = R_2 - (3/4)R_1} \begin{bmatrix} 4 & 3 & 0 \\ 0 & \frac{-1}{4} & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + (4)R_2} \begin{bmatrix} 4 & 3 & 0 \\ 0 & \frac{-1}{4} & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

From the row operations done above, the corresponding LU matrix product is:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \\ 0 & \frac{-1}{4} & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

(b) Now solve $Ly = b$. Substituting L and b, we have:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix}$$

Solving this, we get:

$$y_1 = 7, \quad y_2 = \frac{11}{4}, \quad y_3 = 14$$

(c) Now, solve for x using $Ux = y$. Substituting U and the solved values of y:

$$\begin{bmatrix} 4 & 3 & 0 \\ 0 & -\frac{1}{4} & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ \frac{11}{4} \\ 14 \end{bmatrix}$$

Solving this, we get:

$$x_1 = 3, \quad x_2 = -\frac{5}{3}, \quad x_3 = \frac{7}{3}$$

2. (5 points) Triangle S is determined by the points $P(1,1), Q(3,1), R(1,2)$. Transform T rotates points counterclockwise about the point $(0,1)$ by $\pi/2$ radians.

   (a) The points $P(1,1)$, $Q(3,1)$, and $R(1,2)$ can be represented in homogeneous coordinates as:

$$D = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

   (b) The transformation or standard matrix for rotating points counterclockwise by $\pi/2$ radians about the point $(0,1)$ is:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

   (c) The new coordinates of the transformed points can be calculated as $A \cdot D$:

$$A \cdot D = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

   (d) The coordinates of the triangle after the transformation are:
   $P' = (0,2)$
   $Q' = (0,4)$
   $R' = (-1,2)$