2. A random variable,

X belongs to a Gaussian mixture model (GMM)

if $X \sim \sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2)$,

where $\sum_{i=1}^{K} p_i = 1$.

$$\mathcal{N}(\mu_i, \sigma_i^2) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(x-\mu_i)^2}{2\sigma_i^2}\right)$$

Now, as per the questions,

$x \in GMM$

Hence,

$$\cancel{E(X) = \sum \int_{-\infty}^{\infty} x \cdot \mathcal{N}(\mu_i, \sigma_i^2)}$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot \sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2)\, dx$$

$$\Rightarrow E(X) = \sum_{i=1}^{K} \left[p_i \int_{-\infty}^{\infty} x \cdot \mathcal{N}(\mu_i, \sigma_i^2)\, dx \right]$$

$$\Rightarrow E(X) = \sum_{i=1}^{K} p_i \cdot \mu_i$$

Now, to determine the $Var(X)$,

we will find the $MGF(X)$.

$$f_X(x) = \sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2)$$

$$\therefore \cancel{MGF(x)}\ MGF(X) = \phi(t) = E(e^{tX})$$

$$\Rightarrow \phi(t) = \int_{-\infty}^{\infty} e^{tx} \left(\sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2) \right) \cdot dx.$$

$$= \sum_{i=1}^{K} \left[p_i \int_{-\infty}^{\infty} e^{tx} \mathcal{N}(\mu_i, \sigma_i^2) \cdot dx \right]$$

$$\cancel{= \sum_{i=1}^{K} \left[p_i \cdot \exp\left(\mu_i t + \tfrac{1}{2}\sigma_i t^2\right) \right]}.$$

We know that the moment generating function, $\phi_i(t)$ of $\mathcal{N}(\mu_i, \sigma_i^2)$,

$$\phi_i(t) = \exp\left(\mu_i t + \tfrac{1}{2}\sigma_i^2 t^2\right)$$

~~Thus, the moment~~

Hence, the moment generating function of the given Gaussian mixture model is given by,

$$\phi(t) = \sum_{i=1}^{K} p_i \exp\left(\mu_i t + \frac{1}{2}\sigma_i^2 t^2\right)$$

- We know that,

$$E(X) = \phi'(0)$$

and $E(X^2) = \phi''(0)$.

Hence, $\phi'(t) = \sum_{i=1}^{K} p_i (\mu_i + \sigma_i^2 t) \cdot \exp\left(\mu_i t + \frac{1}{2}\sigma_i^2 t^2\right)$

and, $\phi''(t) = \sum_{i=1}^{K} p_i \left\{\sigma_i^2 \exp\left(\mu_i t + \frac{1}{2}\sigma_i^2 t^2\right) + (\mu_i + \sigma_i^2 t)^2 \cdot \exp\left(\mu_i t + \frac{1}{2}\sigma_i^2 t^2\right)\right\}$

Hence,

$$E(X^2) = \phi''(0) = \sum_{i=1}^{K} p_i (\sigma_i^2 + \mu_i^2)$$

Thus,

$$Var(X) = E(X^2) - (E(X))^2$$

$$\Rightarrow \boxed{Var(X) = \sum_{i=1}^{K} p_i (\sigma_i^2 + \mu_i^2) - \left[\sum_{i=1}^{K} p_i \mu_i\right]^2}$$

Answers:

$$E(X) = \sum_{i=1}^{K} \mu_i p_i$$

$$Var(X) = \sum_{i=1}^{K} p_i (\sigma_i^2 + \mu_i^2) - \left[\sum_{i=1}^{K} p_i \mu_i\right]^2$$

$$\phi(t) = \sum_{i=1}^{K} p_i \exp\left(\mu_i t + \frac{1}{2}\sigma_i^2 t^2\right)$$

Given, $x_i \sim N(\mu_i, \sigma_i^2)$

and, $z = \sum_{i=1}^{K} p_i x_i$

Hence,

$$E(z) = E\left(\sum_{i=1}^{K} p_i x_i\right)$$

$$= \sum_{i=1}^{K} E(p_i x_i)$$

$$= \sum_{i=1}^{K} p_i E(x_i)$$

$$= \sum_{i=1}^{K} p_i \mu_i$$

~~and~~, Now, for Var(z) and PDF(z), we calculate ~~Var(z)~~ the MGF first -

$$\Rightarrow \phi(t) = E(e^{tz})$$

$$\cancel{= \int_{-\infty}^{\infty} e^{tz}}$$

$$\Rightarrow \phi(t) = E(e^{tp_1x_1} \cdot e^{tp_2x_2} \cdots e^{tp_Kx_K})$$

$$\cancel{= \prod_{i=1}^{K} p_i \exp\left(\mu_i t + \frac{1}{2}\sigma_i^2 t^2\right)}$$

$$\cancel{\Rightarrow \phi(t) = \left(\prod_{i=1}^{K} p_i\right) \exp\left(\left(\sum_{i=1}^{K} \mu_i\right)t + \frac{1}{2}\left(\sum_{i=1}^{K} \sigma_i^2\right)t^2\right)}$$

$$\phi(t) = \prod_{i=1}^{K} \exp\left(\mu_i p_i t + \frac{1}{2}\sigma_i^2 p_i^2 t^2\right)$$

Thus, MGF of Z:

~~Hence~~, $\phi(t) = \exp\left[(\Sigma \mu_i p_i)t + \frac{1}{2}(\Sigma \sigma_i^2 p_i^2)t^2\right]$

~~Thus~~ Now, comparing with MGF of general Gaussian expression,

$$\sigma_z^2 = \sum_{i=1}^{K} \sigma_i^2 p_i^2$$

$$\text{and, } \mu_z = \sum_{i=1}^{K} \mu_i p_i$$

Therefore,

the variance of z:

$$Var(z) = \Sigma_{i=1}^{K} p_i^2 \sigma_i^2$$

and, using the uniqueness property,

PDF of Z:

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_z^2}} \exp\left\{-\frac{(z-\mu_z)^2}{2\sigma_z^2}\right\}$$

where, $\mu_z = \sum_{i=1}^{K} \mu_i p_i$

$\sigma_z^2 = \Sigma_{i=1}^{K} p_i^2 \sigma_i^2$

Answers:

$$\mu_z = E(z) = \Sigma_{i=1}^{K} p_i \mu_i$$

$$\sigma_z^2 = Var(z) = \Sigma_{i=1}^{K} \sigma_i^2 p_i^2$$

$$PDF(z) = \frac{1}{\sqrt{2\pi\sigma_z^2}} \exp\left\{-\frac{(z-\mu_z)^2}{2\sigma_z^2}\right\}$$

$$\phi(t) = \exp\left[(\Sigma_{i=1}^{K} p_i \mu_i)t + \frac{1}{2}(\Sigma_{i=1}^{K} p_i^2 \sigma_i^2)t^2\right]$$

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