3. Markov's inequality states that,

if X is a random variable that takes ~~as~~ only non negative values, then for any value $a>0$,

$$P(X \geq a) \leq \frac{E(X)}{a}$$

To prove: With mean μ, and variance σ^2:

$$P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}, \quad \tau > 0$$

and,

$$P(X-\mu \geq \tau) \geq 1-\frac{\sigma^2}{\sigma^2+\tau^2}, \quad \tau < 0$$

Proof: ~~We have that~~

$$\cancel{P[X \geq a] \leq \frac{E(X)}{a}}$$

~~For~~ Case: $\tau > 0$

Clearly, for

$$P\left[(X-\mu)^2 \geq \tau^2\right] = P\left[|X-\mu| \geq \tau\right], \quad -(1)$$

$$\forall \tau > 0$$

Let Y be a random variable such that,

$$Y = X-\mu.$$

Now,

$$P[X-\mu \geq \tau] = P[Y \geq \tau]$$

$$\Rightarrow P[X-\mu \geq \tau] = P[Y+\alpha \geq \tau+\alpha], \text{ for some } \alpha > 0 \quad -(2)$$

Now, clearly, ~~by~~ using (1),

$$P[Y+\alpha \geq \tau+\alpha] \leq P.\left[(Y+\alpha)^2 \geq (\tau+\alpha)^2\right] \quad -(3)$$

Now, as per Markov's Inequality,

$$P[X \geq a] \leq \frac{E(X)}{a}$$

$$\Rightarrow P[(Y+\alpha)^2 \leq (\tau+\alpha)^2] \leq \frac{E\left[(Y+\alpha)^2\right]}{(\tau+\alpha)^2} \quad -(4)$$

using ②, ③, and ④,

$$P(X-\mu \geq \tau) \leq \frac{E((Y+\alpha)^2)}{(\alpha+\tau)^2} \quad \text{— ⑤}$$

~~As~~ We have that,

$$E[(Y+\alpha)^2] = E(Y^2 + 2\alpha Y + \alpha^2)$$

$$\Rightarrow E[(Y+\alpha)^2] = E(Y^2) + 2\alpha E(Y) + \alpha^2$$

Since, $Y = X - \mu$,

$$E(Y^2) = \sigma^2$$

and, $E(Y) = 0$.

$$\Rightarrow E[(Y+\alpha)^2] = \sigma^2 + \alpha^2 \quad \text{— ⑥}$$

using ⑤ and ⑥

$$P(X-\mu \geq \tau) \leq \frac{\sigma^2+\alpha^2}{(\tau+\alpha)^2} \quad \text{— ⑦}$$

Since, the above inequality is true $\forall \alpha \geq 0$;

we ~~cho~~ select an α, such that RHS is minimum.

Let $$y = \frac{\sigma^2+\alpha^2}{(\tau+\alpha)^2}$$

For minimum,

$$\therefore \frac{dy}{d\alpha} = \frac{2\alpha}{(\tau+\alpha)^2} - \frac{2(\sigma^2+\alpha^2)}{(\alpha+\tau)^3} = 0$$

$$\Rightarrow \alpha(\alpha+\tau) - (\sigma^2+\alpha^2) = 0.$$

$$\therefore \boxed{\alpha = \frac{\sigma^2}{\tau}}$$

Substituting, $\alpha = \frac{\sigma^2}{\tau}$ in ⑦,

$$\boxed{P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\tau^2+\sigma^2}}$$

- Now,

Case: $\tau < 0$

We proceed as in the previous, with $\beta = -\tau > 0$, and $\alpha \geq 0$.

$$\Rightarrow P(X-\mu < \tau) = P(Y < -\beta)$$
$$= P(-Y > \beta) \leq \frac{\sigma^2}{\beta^2+\sigma^2}$$

Hence, $P(X-\mu<\tau) \leq \frac{\sigma^2}{\tau^2+\sigma^2}$ $[\because \beta = -\tau]$

- Taking compliment,

$$P(X-\mu \geq \tau) \geq 1 - \frac{\sigma^2}{\tau^2+\sigma^2}$$

Hence, proved.