4. Let X be the random variable, such that the PDF of X is $f_X(x)$.

The moment generating function is given by,

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x) \cdot dx$$

Part 1:

~~For t>0;~~

a) We have that,

$$P[X \geq x] = \int_{x}^{\infty} f_X(z) \cdot dz$$

$$\Rightarrow P[X \geq x] = e^{-tx} \cdot e^{tx} \int_{x}^{\infty} f_X(z) \cdot dz$$

$$= e^{-tx} \int_{x}^{\infty} e^{tx} f_X(z) \cdot dz \quad — (1)$$

Now, as $t>0$,

$$\int_{x}^{\infty} e^{tx} f_X(z) \cdot dz \leq \int_{-\infty}^{\infty} e^{tx} f_X(z) \cdot dz$$

$$\leq \int_{-\infty}^{\infty} e^{tz} f_X(z) \cdot dz \quad — (2)$$

(as $t>0$)

$\Rightarrow$ using (1) and (2)

$$P[X \geq x] \leq e^{-tx} \int_{-\infty}^{\infty} e^{tz} f_X(z) \cdot dz$$

$$\Rightarrow \boxed{P[X \geq x] \leq e^{-tx} \cdot \phi_X(t)}$$

b) Here,

$$P[X \leq x] = \int_{-\infty}^{x} f_X(z) dz$$

$$\Rightarrow P[X \leq x] = e^{-tx} \cdot e^{tx} \int_{-\infty}^{x} f_X(z) \cdot dz$$

$$= e^{-tx} \int_{-\infty}^{x} e^{tx} f_X(z) \cdot dz \quad — (3)$$

As, $t<0$,

$$\int_{-\infty}^{x} e^{tx} f_X(z) \cdot dz \leq \int_{-\infty}^{x} e^{tz} f_X(z) dz$$

$$\leq \int_{-\infty}^{\infty} e^{tz} f_X(z) \cdot dz = \phi_X(t). \quad — (4)$$

(as $t<0$)

Hence, using ③ and ④

$$\boxed{P[X \le x] \le e^{-tx}\phi_x(t)}, \text{ for } t<0$$

Part 2:

$$X = \sum_{i=1}^{n} X_i, \quad E(X_i) = p_i$$

$$\mu = \sum_{i=1}^{n} p_i = E(x)$$

To prove:

$$P[x > (1+\delta)\mu] \le \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)t\mu}} \quad \forall t \ge 0, \ \forall \delta > 0$$

Proof: As proved in the previous question, for $t \ge 0$

$$P[x > (1+\delta)\mu] \le e^{-t\mu(\delta+1)} \cdot \phi_x(t) \quad \text{—(i)}$$

~~Now,~~, $\cancel{\phi_x(t)} = \cancel{B}$

As $X = \sum_{i=1}^{K} X_i$, and are Bernoulli variables, that are indepent.

Hence,

$$\phi_X(t) = \prod_{i=1}^{K} \phi_{X_i}(t) \quad \text{—(ii)}$$

For a Bernoulli variable, the moment generating function,

$$\phi_{X_i}(t) = 1 - p_i + p_i e^t$$
$$= 1 + p_i(e^t - 1)$$

Now, $1 + x \le e^x$

$$\Rightarrow \phi_{X_i}(t) \le e^{p_i(e^t-1)} \quad \text{—(iii)}$$

Hence, using (ii) and (iii)

$$\phi_X(t) \le \exp\left(\sum_{i=1}^{K} p_i(e^t - 1)\right) \quad \text{—(iv)}$$

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$$\phi_x(t) \leq \exp(\mu(e^t - 1)) \quad \text{— (iv)}$$

Hence, by using (i) and (iv),

$$P[X > (1+\delta)\mu] \leq \frac{e^{\mu(e^t - 1)}}{e^{(1+\delta)t\mu}}$$

Now, to tighten the bounds, we minimise the RHS.

RHS : $\exp[\mu e^t - \mu - t\mu - \delta t\mu]$

• Taking logarithm with base e,

$$\log_e(\text{RHS}) = \mu e^t - \mu - t\mu - \delta t\mu.$$

→ Differentiating the above expression with respect to t, and equating to zero.

$$\mu e^t = \cancel{\mu(1+t)}\ \mu(1+\delta).$$

$$\Rightarrow \boxed{t = \ln(1+\delta)}$$

Hence, to tighten the bounds, the value of

$$t = \ln(1+\delta).$$

Thus, the inequality converts to,

$$P[X > (1+\delta)\mu] \leq \frac{e^{\mu\delta}}{(1+\delta)^{\mu\delta}} = \left(\frac{e}{1+\delta}\right)^{\mu\delta}$$

QED.