$\Rightarrow$ Now, we need to prove

$$K(x,y) = \exp\left(\frac{-1}{2\sigma^2}\|x-y\|^2\right) = \exp\left(\frac{-1}{2\sigma^2} x^T x\right).$$

$$\cdot \exp\left(\frac{x^T y}{\sigma^2}\right)$$

$$\cdot \exp\left(\frac{-1}{2\sigma^2} y^T y\right).$$

In the RHS, just consider the middle term.

$$K'(x,y) = \exp\left(\frac{x^T y}{\sigma^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(x^T y)^n}{n! \cdot \sigma^{2n}}.$$

Now, since $K(x,y) = x^T y$ is a valid kernel, using property 1 and property 3, we get $K'(x,y)$ is also a valid kernel.

Hence, there exist a $\phi(x) : R^n \to H$.

s.t $K'(x,y) = \phi(x)^T \phi(y)$.

Let us define $\phi_{new} = \phi(x) \cdot \exp\left(\frac{-1}{2\sigma^2} x^T x\right)$.

$\hookrightarrow$ scalar.

$\therefore$ using this ϕ_{new}, we can create a kernel