Q-3.1] - Data = $\{x^1, x^2 \ldots x^n\}$.

$C_1 = \{x^1, x^2 \ldots x^m\}$ $C_2 = \{x^{m+1}, x^{m+2}, \ldots x^n\}$.

Let the cluster centre be O_1, O_2 respectively s.t.

$$O_1 = \frac{\sum_{i=1}^{m} x^i}{m} \qquad O_2 = \frac{\sum_{i=m+1}^{n} x^i}{(n-m)}$$

~~Let there~~ i.e.

We want to prove existence for a plane $a^T x + b = 0$

s.t $a^T x + b < 0$ $\forall x \in C_1$ $a^T x + b > 0$ $\forall x \in C_2$. — ①

Now, we also know that for any $x \in C_1$ since our solution of O_1, O_2 is optimal,

$$\|x - O_1\|^2 < \|x - O_2\|^2 \quad — ②$$

and for any $x \in C_2$

$$\|x - O_1\|^2 > \|x - O_1\|^2. \quad — ③$$

expand ②:

$$\cancel{x^T x} - x^T O_1 - O_1^T x + O_1^T O_1 < \cancel{x^T x} - x^T O_2 - O_2^T x + O_2^T O_2.$$

Since $x^T y = y^T x$,

$$x^T (O_2 - O_1) - \left(\frac{\|O_1\|^2 - \|O_2\|^2}{2} \right) > 0 \quad — ④$$