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Question 3

Part a:

Given a m x n matrix $\mathbf{A}$,

- $\mathbf{P} = \mathbf{A}^T\mathbf{A}$ is a n x n matrix
  Consider a vector $\mathbf{y}$ with n elements
  $$\mathbf{y}^t\mathbf{P}\mathbf{y} = \mathbf{y}^t\mathbf{A}^T\mathbf{A}\mathbf{y} = (\mathbf{A}\mathbf{y})^t\mathbf{A}\mathbf{y} = ||\mathbf{A}\mathbf{y}||_2^2 \geq 0$$
  Note: $||\mathbf{x}||_2^2$ is the square of 2-norm of $\mathbf{x}$
- $\mathbf{Q} = \mathbf{A}\mathbf{A}^T$ is a m x m matrix
  Consider a vector $\mathbf{z}$ with m elements
  $$\mathbf{z}^t\mathbf{Q}\mathbf{z} = \mathbf{z}^t\mathbf{A}\mathbf{A}^T\mathbf{z} = (\mathbf{A}^T\mathbf{z})^t\mathbf{A}^T\mathbf{z} = ||\mathbf{A}^T\mathbf{z}||_2^2 \geq 0$$
- Consider an eigenvector $\mathbf{y}$, with n elements and eigenvalue λ_P, of $\mathbf{P}$
  $$\mathbf{P}\mathbf{y} = \lambda_P\mathbf{y}$$
  Pre-multiplication of the above equation with $\mathbf{y}^t$
  $$\mathbf{y}^t\mathbf{P}\mathbf{y} = \lambda_P\mathbf{y}^t\mathbf{y}$$
  Replacing $\mathbf{y}^t\mathbf{P}\mathbf{y} = ||\mathbf{A}\mathbf{y}||_2^2$ from 1^{st} point
  $$\lambda_P = \frac{||\mathbf{A}\mathbf{y}||_2^2}{||\mathbf{y}||_2^2} \geq 0$$
  $\therefore$ The eigenvalues of P are non-negative
- Consider an eigenvector $\mathbf{z}$, with m elements and eigenvalue λ_Q, of $\mathbf{Q}$
  $$\mathbf{Q}\mathbf{z} = \lambda_Q\mathbf{z}$$
  Pre-multiplication of the above equation with $\mathbf{z}^t$
  $$\mathbf{z}^t\mathbf{Q}\mathbf{z} = \lambda_Q\mathbf{z}^t\mathbf{z}$$
  Replacing $\mathbf{z}^t\mathbf{Q}\mathbf{z} = ||\mathbf{A}^T\mathbf{z}||_2^2$ from 2^{nd} point
  $$\lambda_Q = \frac{||\mathbf{A}^T\mathbf{z}||_2^2}{||\mathbf{z}||_2^2} \geq 0$$
  $\therefore$ The eigenvalues of Q are non-negative

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Part b:

- Consider an eigenvector $\mathbf{u}$, with n elements and eigenvalue λ, of $\mathbf{P}$

  $$\mathbf{Pu} = \lambda\mathbf{u}$$

  $$\mathbf{A}^T\mathbf{Au} = \lambda\mathbf{u}$$

  Pre-multiplication of the above equation with $\mathbf{A}$

  $$\mathbf{AA}^T(\mathbf{Au}) = \lambda(\mathbf{Au})$$

  $\therefore$ $\mathbf{Au}$ is an eigenvector, with m elements and eigenvalue λ, of $\mathbf{AA}^T$

- Consider an eigenvector $\mathbf{v}$, with m elements and eigenvalue μ, of $\mathbf{Q}$

  $$\mathbf{Qv} = \mu\mathbf{v}$$

  $$\mathbf{AA}^T\mathbf{v} = \mu\mathbf{v}$$

  Pre-multiplication of the above equation with $\mathbf{A}^T$

  $$\mathbf{A}^T\mathbf{A}(\mathbf{A}^T\mathbf{v}) = \mu(\mathbf{A}^T\mathbf{v})$$

  $\therefore$ $\mathbf{A}^T\mathbf{v}$ is an eigenvector, with n elements and eigenvalue μ, of $\mathbf{A}^T\mathbf{A}$

Part c:

Consider an eigenvector $\mathbf{v}_i$, with m elements and eigenvalue λ_i, of $\mathbf{Q}$

$$\mathbf{Qv}_i = \lambda_i\mathbf{v}_i$$

$$\mathbf{AA}^T\mathbf{v}_i = \lambda_i\mathbf{v}_i$$

Consider a vector $\mathbf{u}_i = \dfrac{\mathbf{A}^T\mathbf{v}_i}{||\mathbf{A}^T\mathbf{v}_i||_2}$

$$\mathbf{Au}_i = \frac{\lambda_i}{||\mathbf{A}^T\mathbf{v}_i||_2}\mathbf{v}_i$$

Replacing $\gamma_i = \dfrac{\lambda_i}{||\mathbf{A}^T\mathbf{v}_i||_2}$ in the above equation

$$\mathbf{Au}_i = \gamma_i\mathbf{v}_i$$

γ_i is non-negative because $\lambda_i \geq 0$ (proved in Part a 3^{rd} point)
γ_i is real because $\mathbf{A}$ and $\mathbf{v}_i$ are real-valued (given in question)
$\therefore$ There exist a real, non-negative γ_i such that $\mathbf{Au}_i = \gamma_i\mathbf{v}_i$

Part d:

Given in the question, $\mathbf{u}_i^t\mathbf{u}_j = 0$ for $i \neq j$
As defined in the previous part, $\mathbf{u}_i^t\mathbf{u}_i = \frac{(\mathbf{A}^T\mathbf{v}_i)^t\mathbf{A}^T\mathbf{v}_i}{||\mathbf{A}^T\mathbf{v}||_2^2} = 1$
$\mathbf{V} = [\mathbf{u}_1|\mathbf{u}_2|...|\mathbf{u}_n]$ is a n x m matrix
Columns of $\mathbf{V}$ are orthonormal $\Rightarrow \mathbf{V}^T\mathbf{V} = \mathbf{I}_m$

Similarily we can say, $\mathbf{U} = [\mathbf{v}_1|\mathbf{v}_2|...|\mathbf{v}_m]$ also has orthonormal columns
Here we are assuming $\mathbf{v}_i$ to be a unit vector for consistency
$\therefore \mathbf{U}^T\mathbf{U} = \mathbf{I}_m$

$$\mathbf{A} = \mathbf{U\Gamma V}^T$$

Pre-multiply with $\mathbf{U}^T$ and post-multiply with $\mathbf{V}$

$$\mathbf{U}^T\mathbf{AV} = \mathbf{U}^T\mathbf{U\Gamma V}^T\mathbf{V} = \mathbf{\Gamma}$$

Thus, $\mathbf{A} = \mathbf{U\Gamma V}^T \Leftrightarrow \mathbf{U}^T\mathbf{AV} = \mathbf{\Gamma}$

$$\begin{aligned}
\mathbf{U}^T\mathbf{AV} &= \begin{bmatrix} \mathbf{v}_1^t \\ \mathbf{v}_2^t \\ ... \\ \mathbf{v}_m^t \end{bmatrix} \mathbf{A}[\mathbf{u}_1|\mathbf{u}_2|...|\mathbf{u}_m] \\
&= \begin{bmatrix} \mathbf{v}_1^t \\ \mathbf{v}_2^t \\ ... \\ \mathbf{v}_m^t \end{bmatrix} [\mathbf{Au}_1|\mathbf{Au}_2|...|\mathbf{Au}_m] \\
&= \begin{bmatrix} \mathbf{v}_1^t \\ \mathbf{v}_2^t \\ ... \\ \mathbf{v}_m^t \end{bmatrix} [\gamma_1\mathbf{v}_1|\gamma_2\mathbf{v}_2|...|\gamma_m\mathbf{v}_m]
\end{aligned}$$

Using result of part c: $\mathbf{Au}_i = \gamma_i\mathbf{v}_i$ above
$(\mathbf{U}^T\mathbf{AV})_{ij} = \gamma_j\mathbf{v}_i^t\mathbf{v}_j = \begin{cases} \gamma_j & i = j \\ 0 & i \neq j \end{cases}$
As defined, $\mathbf{\Gamma}$ is a m x n diagonal matrix with i^{th} diagonal entry equal to γ_i

$$\therefore \mathbf{U}^T\mathbf{AV} = \mathbf{\Gamma} \Rightarrow \mathbf{A} = \mathbf{U\Gamma V}^T$$

Hence proved.