

# Written Assignment 0

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## Exercise 2.1

Proving  $\chi = V - E + F = 1$  for a polygonal disk:

If we start with a polygonal disk which has non-triangular faces, we can triangulate it by adding diagonals. Each diagonal increases the number of edges and faces by 1. This process of triangulation leaves  $\chi$  invariant. Now that all the faces are triangular, the remainder of the proof must show that  $\chi = 1$ . We start by removing triangles from the boundary which involves either:

- Removing 1 face and 1 edge
- Removing 1 vertex, 2 edges and 1 face

Both would leave  $\chi$  the same. We are left with the base case which is just a triangle which has  $\chi = 3 - 3 + 1 = 1$ .

Proving  $\chi = V - E + F = 2$  for a polygonal sphere:

We project the polyhedron to the 2D plane to re-use the previous result. To do this imagine shining a light from the top and casting a shadow on a surface placed on the bottom. This would yield a projection which has the same number of edges and vertices as before. If this is not the case, then we are allowed to reposition the vertices so that it casts a proper shadow. We are allowed to do this since  $\chi$  is a topological property. The shadow we cast will have one less face than the original because that face is now the boundary of our shadow. So  $\chi = 1 + 1 = 2$  for the polygonal sphere.

## Exercise 2.2

Angles argument:

- **Triangles.** The interior angle of an equilateral triangle is 60 degrees. Thus on a regular polyhedron, only 3, 4, or 5 triangles can meet a vertex. If there were more than 6 their angles would add up to at least 360 degrees which they can't. Consider the possibilities:
  - 3 triangles meet at each vertex, giving rise to a Tetrahedron
  - 4 triangles meet at each vertex, giving rise to an Octahedron
  - 5 triangles meet at each vertex, giving rise to an Icosahedron
- **Squares.** Since the interior angle of a square is 90 degrees, at most three squares can meet at a vertex. This is indeed possible and it gives rise to a hexahedron or cube.
- **Pentagons.** As in the case of cubes, the only possibility is that three pentagons meet at a vertex. This gives rise to a Dodecahedron.
- **Hexagons** or regular polygons with more than six sides cannot form the faces of a regular polyhedron since their interior angles are at least 120 degrees.

We end up with 5 platonic solids.

Connectivity argument:

Because this is a regular polytope/mesh, the valence of each vertex is equal, so we argue each face is an identical  $n$ -gon, for some positive  $n$ .

Being regular implies  $n \geq 3$ .

With the same argument, each vertex is identical, so let  $d$  be the degree of vertices.

Being regular implies  $d \geq 3$ .

As usual  $V$  is number of vertices,  $E$  is number of edges and  $F$  is number of faces.

Each edge touches two faces, so  $\frac{nF}{2} = E \implies F = \frac{2E}{n}$ .

Each edge touches two vertices, so  $\frac{dV}{2} = E \implies V = \frac{2E}{d}$ .

Using Euler's formula:

$$\begin{aligned}\chi = 2 &= V - E + F \\ &= \frac{2E}{d} - E + \frac{2E}{n} \\ &= E \left( \frac{2}{d} - 1 + \frac{2}{n} \right)\end{aligned}$$

From earlier  $n \geq 3$  and  $d \geq 3$ , so we get  $\frac{1}{n} \leq \frac{1}{3}$  and  $\frac{1}{d} \leq \frac{1}{3}$ .

$E$  must be positive so:


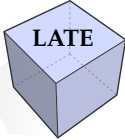
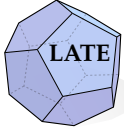
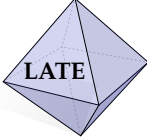

$$\begin{aligned}\frac{2}{d} - 1 + \frac{2}{n} &> 0 \\ \frac{1}{d} &> \frac{1}{2} - \frac{1}{n} > \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \\ 3 &\leq d < 6\end{aligned}$$

When  $d = 3$ ,  $\frac{1}{n} > \frac{1}{6}$ , so  $n = 3, 4$  or  $5$ .

When  $d = 4$ ,  $\frac{1}{n} > \frac{1}{4}$ , so  $n = 3$ .

When  $d = 5$ ,  $\frac{1}{n} > \frac{3}{10}$ , so  $n = 3$ .

Overall, this gives us the following table of platonic solids:

$d$	$n$	$V$	$E$	$F$	Solid	Mesh
3	3	4	6	4	Tetrahedron	
3	4	8	12	6	Cube	
3	5	20	30	12	Dodecahedron	
4	3	6	12	8	Octahedron	
5	3	12	30	20	Icosahedron	

The whole 'LATE' thing is a joke that can be found here.

Exercise 2.3

Previous formulas apply here, with  $n = 3$  and  $d = 6$ . We apply the Euler-Poincaré formula:

$$\begin{aligned}\chi &= 2 - 2g = V - E + F \\ &= \frac{2E}{d} - E + \frac{2E}{n} \\ &= \frac{2E}{6} - E + \frac{2E}{3} \\ &= 0\end{aligned}$$

So we get  $2(1 - g) = 0 \implies g = 1$ , which is a torus.

Exercise 2.4

Let the number of vertices with irregular valence be  $n$ .

The valences of these  $n$  vertices are  $v_1, v_2, \dots, v_n$ , and we assume  $v_i \geq 3$ . Using the previous formula for regular triangle mesh with degree  $d$ :  $dV = 2E = 3F$ . This degree for irregular mesh is not uniformly  $d$  so we now have:

$$\begin{aligned}6(V - n) + \sum_i^n v_i &= 2E = 3F \\ \implies F &= \frac{6(V - n) + \sum_i^n v_i}{3}\end{aligned}$$

We apply the Euler-Poincaré formula, and express in terms of  $V$ :

$$\begin{aligned}\chi &= 2 - 2g = V - E + F \\ &= V - \frac{3}{2}F + F \\ &= V - \frac{1}{2}F \\ &= V - \frac{6(V - n) + \sum_i^n v_i}{6} \\ &= n - \frac{\sum_i^n v_i}{6}\end{aligned}$$

So  $n = 2 - 2g + \frac{\sum_i^n v_i}{6}$ . Because  $v_i \geq 3$ :

$$\sum_i^n v_i \geq 3n \implies \frac{\sum_i^n v_i}{6} \geq \frac{n}{2} \implies n - 2 + 2g \geq \frac{n}{2} \implies n \geq 4 - 4g$$

When  $g = 0$ : we have  $n \geq 4$ .

When  $g = 1$ : we have  $n = 0$  from Exercise 2.3

When  $g \geq 2$ : we have  $n \leq -4$ .

$n$  is non-negative, and if  $n = 0 = 2\chi$ , implies  $\chi = 0$  for genus  $g \geq 2$  which is invalid. So the valid values start from  $n \geq 1$ .

Which gives us our result:

$$m(K) = \begin{cases} 4, & g = 0 \\ 0, & g = 1 \\ 1, & g \geq 2 \end{cases}$$

Exercise 2.5

Triangle mesh:

Each edge has 2 faces on either side, each face is bounded by 3 edges. So  $3F = 2E$  or  $E : F = 3 : 2$ .

We apply the Euler-Poincaré formula:

$$\begin{aligned}\chi &= 2 - 2g = V - E + F \\ &= V - E + \frac{2}{3}E \\ \implies E &= 3(V - 2 + 2g)\end{aligned}$$

An edge connects two vertices, but we can say that the edge belongs to only one of the vertices. So, mean valence for a triangle mesh of large  $V$ :

$$\lim_{V \rightarrow \infty} \frac{2E}{V} = \lim_{V \rightarrow \infty} \frac{6(V - 2 + 2g)}{V} = 6$$

Exercise 2.6

Quad mesh:

Very similar to the previous calculation, except each face is bounded by 4 edges, so  $E : F = 4 : 2 = 2 : 1$ .

We apply the Euler-Poincaré formula and get:  $E = 2(V - 2 + 2g)$

Mean valence for a quad mesh of large  $V$ :

$$\lim_{V \rightarrow \infty} \frac{2E}{V} = 4$$

Exercise 2.7

Tet mesh:

TODO: find the right answer I found an explanation from Stack Exchange which gives us the following result:

$$V : E : F : T = 1 : 4 : 6 : 3$$

whereas the data included in the problem gives us something like:

$$V : E : F : T = 2 : 14 : 3 : 1$$

So, I don't know what to make of this proof.

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Let  $d$  be a positive integer and  $M$  a  $d$ -dimensional triangularizable geometric object. Let  $F_j$  denote the number of  $j$ -dimensional faces of a triangularization  $T$  of  $M$ . (For example,  $F_0$  is the number of vertices and  $F_1$  is the number of edges.) Each time a new vertex is added into the interior of an  $n$ -simplex in  $T$ , we see that the new triangularization  $T'$  satisfies

$$F'_j = \begin{cases} F_j + \binom{d+1}{j}, & \text{if } j = 0, 1, 2, \dots, d-1, \\ F_d + d, & \text{if } j = d, \end{cases}$$

where  $F'_j$  is the number of  $j$ -dimensional faces in  $T'$  for each  $j = 0, 1, 2, \dots, d$ . Hence, if you refine the mesh on  $M$  nicely (i.e., avoid fiddling with boundaries of all  $n$ -simplices), then the ratio

$$F_0 : F_1 : \dots : F_{d-1} : F_d$$

should tend to

$$\binom{d+1}{0} : \binom{d+1}{1} : \dots : \binom{d+1}{d-1} : d,$$

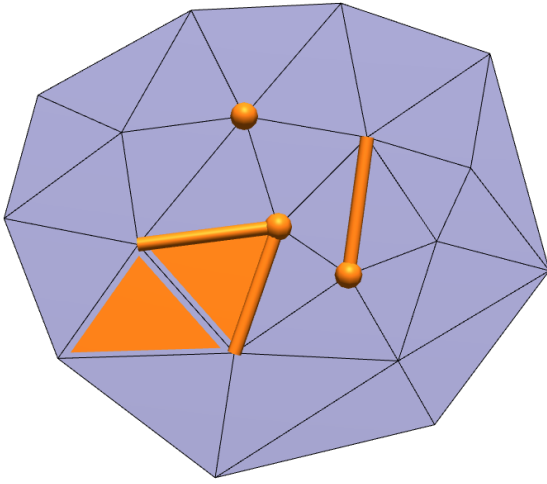
as the number of vertices increases. In particular, for  $d = 3$ , one would expect

$$\frac{F_1}{F_0} \approx \frac{\binom{3+1}{1}}{\binom{3+1}{0}} = 4.$$

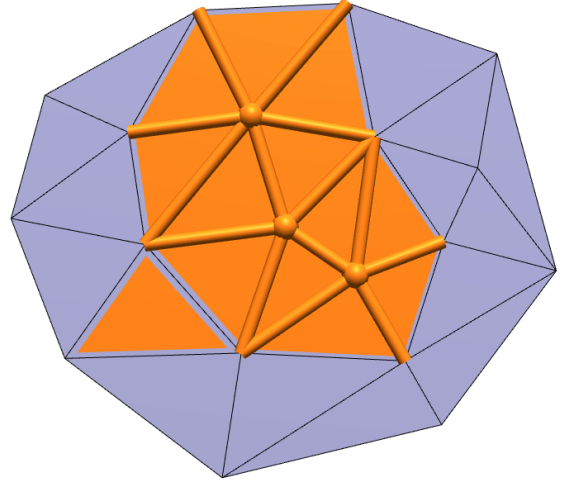
If we are allowed to play with the boundaries of  $n$ -simplices, then the ratios  $\frac{F_j}{F_{j-1}}$  for  $j = 1, 2, \dots, d$  may not have limits, or can tend to arbitrarily large values, provided that  $d \geq 3$ .

Exercise 2.8

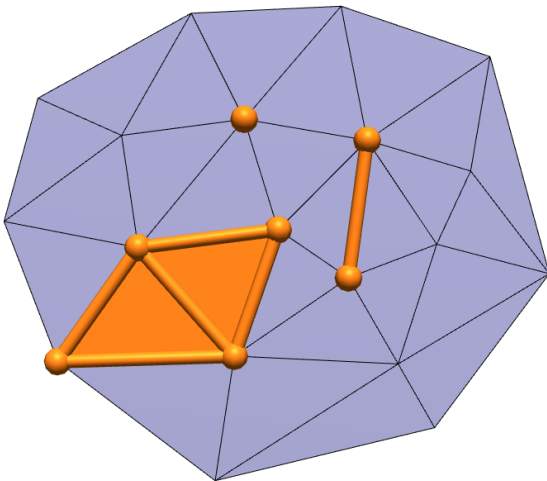
Luckily the mesh in this problem matches the one in the coding exercise!



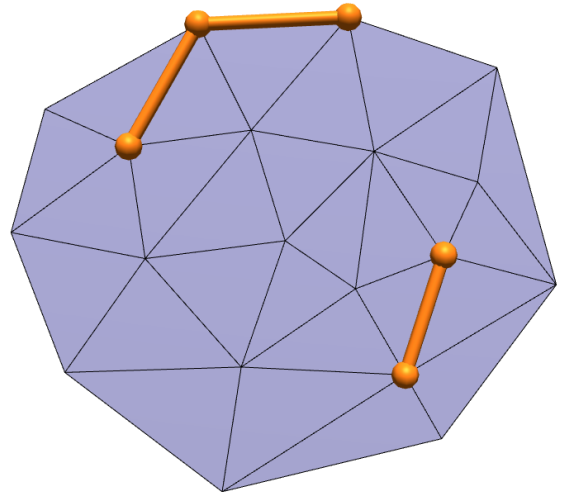
(a) Original subset  $\mathcal{S}$



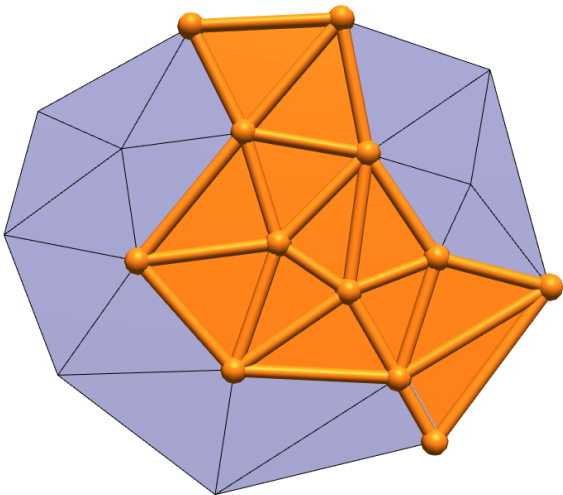
(b) Star  $\text{St}(\mathcal{S})$



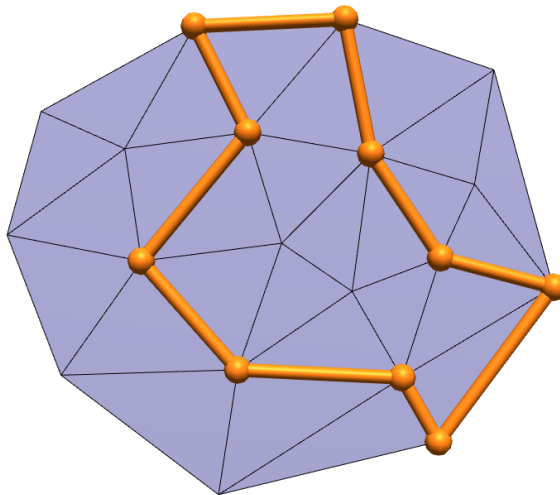
(c) Closure  $\text{Cl}(\mathcal{S})$



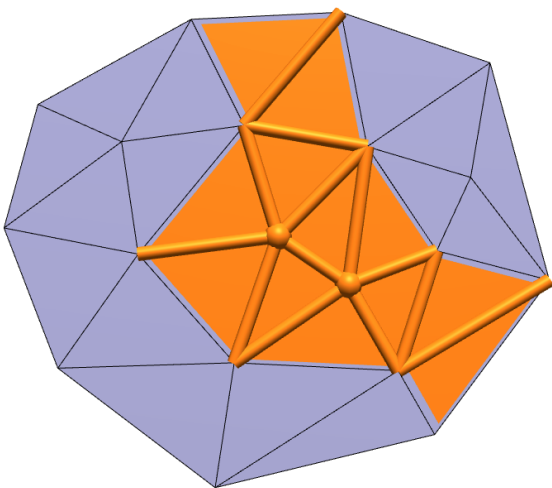
(d) Link  $\text{Lk}(\mathcal{S})$



(a) Original subset  $\mathcal{K}'$



(b) Boundary  $\text{bd}(\mathcal{K}')$



(c) Interior  $\text{int}(\mathcal{K}')$

Twin

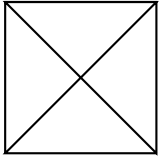
$h$	0	1	2	3	4	5	6	7	8	9
$\eta(h)$	4	2	1	5	0	3	7	6	9	8

Next

$h$	0	1	2	3	4	5	6	7	8	9
$\rho(h)$	1	2	0	4	5	6	3	9	7	8

Exercise 2.11

Looks like this:



Exercise 2.12

$$A_0 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 2.13

For a simplicial  $k$ -manifold, the link of every vertex (0-simplex) looks like a  $(k - 1)$ -dimensional sphere. A simplicial 1-complex is just a graph, but a simplicial 1-manifold is not an arbitrary graph. The degree of each vertex is no greater than 2, so the link of every vertex should be a pair of vertices. So it cannot contain anything other than isolated paths of edges and closed loops of edges.

Exercise 2.14

The boundary of a simplicial surface will have zero or more closed loops. Each connected set of vertices with a boundary will generate a closed loop for its boundary.

Exercise 2.15

Taking the boundary  $\text{bd}(\mathcal{K})$  means removing the interior:  $\text{bd}(\mathcal{K}) = \text{Cl}(\mathcal{K}) \setminus \text{int}(\mathcal{K})$ . The boundary is defined to be the closure of the set of all simplices  $\sigma$  that are proper faces of exactly one simplex of  $\mathcal{K}$ . The result of taking the boundary is a  $(k - 1)$ -submanifold without any proper faces of exactly one simplex. This means  $\sigma$  for  $\text{bd}(\mathcal{K})$  becomes  $\emptyset$ .

$$\text{bd}(\text{bd}(\mathcal{K})) = \text{Cl}(\sigma) = \text{Cl}(\emptyset) = \emptyset.$$