Written Assignment 2

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$\boxed{\text{Exercise} \quad 1}$

(a)

$$
\begin{aligned}
\frac{d}{ds}\gamma &= \left(1, 2s, 3s^2\right) \\
T(s) &= \frac{1}{\sqrt{9s^4+4s^2+1}}\left(1, 2s, 3s^2\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\kappa(s) &= \frac{2}{9s^4+4s^2+1} \\
N(s) &= \frac{1}{\sqrt{(9s^4+4s^2+1)(9s^4+9s^2+1)}}\left(-9s^3-2s, -9s^4+1, 6s^3+3s\right)
\end{aligned}
$$

(c)

$$
\begin{aligned}
B(s) &= \frac{1}{\sqrt{9s^4+9s^2+1}}\left(3s^2, -3s, 1\right) \\
\tau(s) &= \frac{3}{9s^4+9s^2+1}
\end{aligned}
$$

$\boxed{\text{Exercise} \quad 2}$

(a) The 2-norm of the vector $f(u,v)$ is 1:

$$
\begin{aligned}
f(u,v) &= \sqrt{\frac{4u^2}{\left(u^2+v^2+1\right)^2} + \frac{4v^2}{\left(u^2+v^2+1\right)^2} + \frac{\left(u^2+v^2-1\right)^2}{\left(u^2+v^2+1\right)^2}} \\
&= 1
\end{aligned}
$$

So $f(u,v)$ describes a 2-sphere of radius 1 centered at the origin.

(b) Differential df is:

$$
\begin{aligned}
df = &\left(\frac{-2u^2+2v^2+2}{\left(u^2+v^2+1\right)^2}, \quad -\frac{4uv}{\left(u^2+v^2+1\right)^2}, \quad \frac{4u}{\left(u^2+v^2+1\right)^2} \right) du \\
&+ \left(-\frac{4uv}{\left(u^2+v^2+1\right)^2}, \quad \frac{2\left(u^2-v^2+1\right)}{\left(u^2+v^2+1\right)^2}, \quad \frac{4v}{\left(u^2+v^2+1\right)^2} \right) dv
\end{aligned}
$$

(c) Metric g induced by map f is the first fundamental form $\mathbf{I} = J_f^{\intercal} J_f$:

$$
\mathbf{I} = \frac{4}{\left(u^2+v^2+1\right)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Because our induced metric $\mathbf{I}$ is a positive rescaling of the 2D Euclidean metric we can say that our parameterization f is conformal, i.e., f is a conformal map.

(d) Gauss map N is $\frac{df(\frac{\partial}{\partial u})\times df(\frac{\partial}{\partial v})}{df(\frac{\partial}{\partial u})\times df(\frac{\partial}{\partial v})}$:

$$\begin{aligned} N &= \frac{1}{(u^2+v^2+1)}\left(-2u,-2v,1-u^2-v^2\right) \\ &= -f \end{aligned}$$

In our case Gauss map is a constant multiple of the sphere itself.

(e) Shape operator dN is:

$$\begin{aligned} dN &= -df \\ &= -\left(\frac{-2u^2+2v^2+2}{(u^2+v^2+1)^2}, \quad -\frac{4uv}{(u^2+v^2+1)^2}, \quad \frac{4u}{(u^2+v^2+1)^2} \right) du \\ &\quad -\left(-\frac{4uv}{(u^2+v^2+1)^2}, \quad \frac{2\left(u^2-v^2+1\right)}{(u^2+v^2+1)^2}, \quad \frac{4v}{(u^2+v^2+1)^2} \right) dv \end{aligned}$$

$\boxed{\text{Exercise} \quad 3}$

(a) Differential df is:

$$\begin{aligned} df &= \big(\; -\sin\theta(r\cos\varphi+R), \quad \cos\theta(r\cos\varphi+R), \quad 0 \; \big) \; d\theta \\ &+ \big(\; -r\cos\theta\sin\varphi, \quad -r\sin\theta\sin\varphi, \quad r\cos\varphi \; \big) \; d\varphi \end{aligned}$$

(b) Metric g induced by map f is the first fundamental form $\mathbf{I} = J_f^\intercal J_f$:

$$\mathbf{I} = \begin{pmatrix} \cos^2\theta(r\cos\varphi+R)^2+\sin^2\theta(r\cos\varphi+R)^2 & 0 \\ 0 & r^2\sin^2\theta\sin^2\varphi+r^2\cos^2\theta\sin^2\varphi+r^2\cos^2\varphi \end{pmatrix}$$

(c) Gauss map N is $\frac{df(\frac{\partial}{\partial u})\times df(\frac{\partial}{\partial v})}{df(\frac{\partial}{\partial u})\times df(\frac{\partial}{\partial v})}$:

$$N = \frac{1}{\sqrt{\cos\theta\cos^2\varphi+\cos\varphi\sin^2\theta+\sin^2\varphi}}\left(\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi\right)$$

Normals of the torus have no dependence on the radii r, R.

(d) Shape operator dN is:

$$\begin{aligned} dN = &\frac{1}{\left(\cos(\phi)\left(\cos(\theta)\cos(\phi)+\sin^2(\theta)\right)+\sin^2(\phi)\right)^{3/2}} \\ &\Big(\frac{-2\sin(2\theta)\cos^3(\phi)-2\sin(\theta)(\cos(\phi)+2\cos(2\phi)-\cos(3\phi)+2)}{8}, \\ &\frac{(\cos(2\theta)+3)\cos^3(\phi)+4\cos(\theta)\sin^2(\phi)\cos(\phi)}{4}, \\ &\frac{\sin(\theta)\sin(\phi)\cos(\phi)(\cos(\phi)-2\cos(\theta))}{2}\Big) \, du \\ &+\Big(-\frac{\cos(\theta)\left(\sin^2(\theta)\sin(2\phi)+4\sin(\phi)\right)}{4}, \\ &-\frac{\sin(\theta)\left(\sin^2(\theta)\sin(2\phi)+4\sin(\phi)\right)}{4}, \\ &\frac{4\cos(\theta)\cos(\phi)+\sin^2(\theta)(\cos(2\phi)+3)}{4}\Big) \, dv \end{aligned}$$

Exercise 4

Solving the system of equations:

$$\kappa_1 = H \pm \sqrt{H^2 - K},$$
$$\kappa_2 = H \mp \sqrt{H^2 - K}$$

We see that one of these conditions are possible: $\kappa_1 > \kappa_2$, $\kappa_1 < \kappa_2$ or $\kappa_1 = \kappa_2$.
The principal curvatures are eigenvalues of the shape operator, and are equal at so called *umbilic points.*

Exercise 5

We know that the length of u remains fixed after rotation: $|\mathcal{J}u| = |u|$.
Direction of the gradient is the direction of counter-clockwise rotation to increase ψ: $\dfrac{\mathcal{J}u}{|\mathcal{J}u|} = \dfrac{\mathcal{J}u}{|u|}$.
Lets say we move u in this direction by one unit. To get the magnitude of the gradient, we use the small angle approximation: $d\psi = \sin(d\psi) = \dfrac{1}{|u|}$.
The gradient of ψ with respect to u is: $\nabla_u \psi = \dfrac{|\mathcal{J}u|}{|u|^2}$

Exercise 6

We can understand this proof by thinking of the point q as being the origin. So shifting the origin around equates to rigid motion of the polygon.
This would mean different values for the cross product of pairs of vectors (p_i, p_{i+1}). The sign of $p_i \times p_{i+1}$ depends on the position of q. But excess positive area would cancel out with negative areas.
Furthermore Stokes' theorem shows us that the position of q does not matter at all, and we only need the boundary to describe the area.

Exercise 7

Let X, Y be two vectors before immersion. Using the cross product in $\mathbb{R}^3$ we get the area of an infinitesimal patch of the immersion f:

$$\begin{aligned} df \wedge df(X, Y) &= df(X) \times df(Y) - df(Y) \times df(X) \\ &= 2df(X) \times df(Y) \\ &= 2NdA(X, Y) \end{aligned}$$

where dA (the area form) is a surface patch and N the normal of that patch (or the Gauss map).
If we integrate this over the entire surface M, we recover the vector area:

$$\begin{aligned} N_{\mathcal{V}} = \int_M NdA &= \frac{1}{2} \int_M df \wedge df \\ &= \frac{1}{2} \int_M df \wedge df + (-1) f \wedge d(df) \\ &= \frac{1}{2} \int_M d(f \wedge df) && \text{because } d(df) = 0 \\ &= \frac{1}{2} \int_{\partial M} f \wedge df && \text{(using Stokes' thorem)} \end{aligned}$$

Exercise 8

Let X_1, X_2 be the principal directions, the eigenvectors of the shape operator.

$$\begin{aligned} df \wedge dN(X_1, X_2) &= df(X_1) \times dN(X_2) - df(X_2) \times dN(X_1) \\ &= df(X_1) \times \kappa_2 df(X_2) - df(X_2) \times \kappa_1 df(X_1) \\ &= (\kappa_1 + \kappa_2)\ df(X_1) \times df(X_2) \\ &= (\kappa_1 + \kappa_2) N dA(X_1, X_2) \\ &= 2HNdA(X_1, X_2) \end{aligned}$$

$$\begin{aligned} dN \wedge dN(X_1, X_2) &= dN(X_1) \times dN(X_2) - dN(X_2) \times dN(X_1) \\ &= \kappa_1 df(X_1) \times \kappa_2 df(X_2) - \kappa_2 df(X_2) \times \kappa_1 df(X_1) \\ &= (2\kappa_1\kappa_2)\ df(X_1) \times df(X_2) \\ &= (2\kappa_1\kappa_2) N dA(X_1, X_2) \\ &= 2KNdA(X_1, X_2) \end{aligned}$$

We can express any tangent vector Y as a linear combination of the principal directions X_1 and X_2, because these directions form an orthonormal basis for the tangent space T_pM. Moreover, the differential is a linear operator. So we conclude that:

$$\begin{aligned} df \wedge dN &= 2HNdA \\ dN \wedge dN &= 2KNdA \end{aligned}$$

Exercise 9

Direction of the gradient is the direction perpendicular to u : $\dfrac{\mathbf{u}^\perp}{|\mathbf{u}^\perp|} = \dfrac{\mathbf{u}^\perp}{|u|}$.
Lets say we move p in this direction by one unit. Change in area:

$$dA_\sigma = \frac{1}{2}\, |\mathbf{u}^\perp + 1 - \mathbf{u}^\perp|\, |u| = \frac{1}{2}|u|$$

The gradient of A_σ with respect to p is: $\nabla_p A_\sigma = \dfrac{1}{2}\mathbf{u}^\perp$

Exercise 10

Volume of a single tet: $\mathcal{V} = \dfrac{1}{3}Ah$.
Following the same reasoning as the previous exercise, volume gradient for a single tet: $\nabla_p \mathcal{V} = \dfrac{1}{3}AN$ where N is the unit normal to the base.
For a tet mesh M, to express the gradient of the enclosed volume with respect to a given vertex p, we simply sum up the gradients for the tetrahedra containing p:

$$\nabla_p \mathcal{V} = \sum_i \mathcal{V}_i = \frac{1}{3}\sum_i A_i N_i = \frac{1}{3}\int_M N dA = \frac{1}{3}N_{\mathcal{V}}$$

Exercise 11

Here I reproduce the proof found in the book The Shape of Space by Jeffrey Weeks, and combine it with the concept of *diangles* in the course text. The book proof uses *lunes*, which have half the area of the diangles. The continuation of a lune is a diangle.

We have that a diangle with maximum angle π would cover the whole sphere of area 4π. So a diangle of angle α has area 4α.

Each side of the spherical triangle can be continued into a great circle. Doing this we can construct an *antipodal triangle*, which is a mirror image (flipped) version of the triangle on the opposite half of the sphere.

For a spherical triangle with area A and interior angles $\alpha_1, \alpha_2, \alpha_3$, we can construct their respective diangles, with areas A_1, A_2, A_3. These diangles overlap both the main and antipodal triangles.

$$\begin{array}{ccccccccccc} \Big(\text{Diangle for } \alpha_1\Big) & + & \Big(\text{Diangle for } \alpha_2\Big) & + & \Big(\text{Diangle for } \alpha_3\Big) & = & 2\Big(\text{Main triangle}\Big) & + & 2\Big(\text{Antipodal triangle}\Big) & + & \Big(\text{Entire sphere}\Big) \\ 4\alpha_1 & + & 4\alpha_2 & + & 4\alpha_3 & = & 2A & + & 2A & + & 4\pi \end{array}$$

$$\implies A = \alpha_1 + \alpha_2 + \alpha_3 - \pi$$

Exercise 12

Paraphrasing Weeks:

> The area of any n-gon is the sum of the areas of $n-2$ triangles. The area of each spherical triangle is the sum of it's angles minus π. Therefore the area of the spherical n-gon is the sum of all of the angles minus $(n-2)\pi$.

$$A = (2-n)\pi + \sum_{i=1}^{n} \beta_i$$

In general though, for a surface with constant Gaussian curvature K we have:

$$KA = (2-n)\pi + \sum_{i=1}^{n} \beta_i$$

Exercise 13

As the dihedral angle on the surface approaches π by flattening out the star of the vertex v, the area of the spherical polygon approaches zero, because the normals all meet at a sinlge point on the Gauss map.
By doing the opposite, the dihedral angles can approach zero or 2π, and the area of the spherical polygon approaches the hemisphere area 2π. Dihedral angles on the surface become interior angles on the sphere and vice versa.
The relationship between the dihedral angle θ and interior angle β can be summarized as $\theta = \pi - \beta$.

Alternatively, two adjacent normals N_i and N_j would have a dihedral angle: $\theta = \cos^{-1} \dfrac{N_i \cdot N_i}{|N_i||N_j|}$ which further illustrates this complementary relationship.

Using our earlier proof:

$$\begin{aligned}
A &= (2-n)\pi + \sum_{i=1}^{n} \beta_i \\
&= (2-n)\pi + \sum_{i=1}^{n} (\pi - \theta_i) \\
&= 2\pi - \sum_{i=1}^{n} \theta_i \\
&= d(v)
\end{aligned}$$

$\boxed{\text{Exercise} \quad 14}$

A simplicial surface is a triangle mesh, with $n = 3$ sides. Each edge touches two faces, so:

$$\frac{nF}{2} = E \implies E = \frac{3F}{2}$$

Using our previous result:

$$\begin{aligned}
\sum_{v \in V} d(v) &= \sum_{v \in V} \Big[2\pi - \sum_{f \in F_v} \angle_f(v) \Big] \\
&= \sum_{v \in V} 2\pi - \sum_{v \in V} \sum_{f \in F_v} \angle_f(v) \\
&= 2\pi |V| - \sum_{f \in F} \pi && \text{(sum of interior angles} = \pi) \\
&= 2\pi |V| - \pi |F| \\
&= 2\pi (|V| - \frac{|F|}{2}) \\
&= 2\pi (|V| + |F| - \frac{3|F|}{2}) \\
&= 2\pi (|V| - \frac{3|F|}{2} + |F|) \\
&= 2\pi (|V| - |E| + |F|) \\
&= 2\pi \chi
\end{aligned}$$

This is just a discrete version of the smooth Gauss-Bonnet theorem which says that the total Gaussian curvature is always equal to 2π times the Euler characteristic.

$$\int_M K \, dA = 2\pi \chi$$

For a surface with boundary, an extra boundary term must be included.
For smooth surfaces we integrate geodesic curvature along the boundary, for discrete surfaces we add boundary angle defects.

$\boxed{\text{Exercise} \quad 15}$

Faces add slabs of thickness r, hence volume contributed is: $r \sum_{ijk \in F} A_{ijk}$

$\boxed{\text{Exercise} \quad 16}$

Cylinders have volume $\pi r^2 L$.
Edges add cylindrical wedges of volume $\dfrac{1}{2}\ell_{ij}\varphi_{ij} r^2$, hence volume contributed is: $r^2 \sum_{ij \in E} H_{ij}$ where H_{ij} is the total mean curvature for edge ij.

$\boxed{\text{Exercise} \quad 17}$

Spheres have volume $\frac{4}{3}\pi r^3$.
Vertices add spherical cones of volume $\frac{1}{3}\Omega_i r^3$, hence volume contributed is: $\frac{r^3}{3} \sum_{i \in V} K_i$ because sum of angle defects is equal to sum of Gauss curvature.

$\boxed{\text{Exercise} \quad 18}$

Volume of mollified polyhedron is a polynomial in radius r:

$$\mathcal{V}(r) = \mathcal{V} + r \sum_{ijk \in F} A_{ijk} + r^2 \sum_{ij \in E} H_{ij} + \frac{r^3}{3} \sum_{i \in V} K_i$$

Derivatives w.r.t r give:

$$\text{volume}_r \xrightarrow{\frac{d}{dr}} \text{area}_r \xrightarrow{\frac{d}{dr}} \text{Mean curvature}_r \xrightarrow{\frac{d}{dr}} \text{Gauss curvature}_r \xrightarrow{\frac{d}{dr}} 0$$

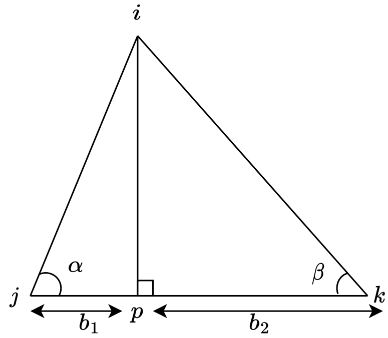
Exercise 19

This uses a proof found in DDG course by Etienne Vouga.
We have proved that gradient of triangle area is $\nabla_{f_i} A_{ijk} = \dfrac{1}{2}\mathbf{u}_i^{\perp}$ where vertex i is located at position f_i and

$$\mathbf{u}_i^{\perp} = \frac{f_i - p}{|f_i - p|}(b_1 + b_2)$$

p can be written as a linear combination:

$$p = \frac{b_1 f_k + b_2 f_j}{b_1 + b_2}$$



This lets us rewrite the gradient as:

$$\begin{aligned}
\nabla_{f_i} A_{ijk} &= \frac{1}{2}\mathbf{u}_i^{\perp} \\
&= \frac{1}{2}\frac{f_i - p}{|f_i - p|}(b_1 + b_2) \\
&= \frac{1}{2}\frac{f_i - \left(\dfrac{b_1 f_k + b_2 f_j}{b_1 + b_2}\right)}{|f_i - p|}(b_1 + b_2) && \text{(substituting } p\text{)} \\
&= \frac{1}{2}\frac{b_1 f_i + b_2 f_i - b_1 f_k - b_2 f_j}{|f_i - p|} \\
&= \frac{1}{2}\left[\frac{b_1}{|f_i - p|}(f_i - f_k) + \frac{b_2}{|f_i - p|}(f_i - f_j)\right] \\
&= \frac{1}{2}\Big[\cot(\angle ijp)(f_i - f_k) + \cot(\angle ikp)(f_i - f_j)\Big] \\
&= \frac{1}{2}\Big[\cot(\alpha_{ip})(f_i - f_k) + \cot(\beta_{ip})(f_i - f_j)\Big]
\end{aligned}$$

Now we express the gradient of total surface area expressed in terms of edges ij oriented from $i \to j$.
The label p in our diagram we now call j when referring to edges:

$$\begin{aligned}
\nabla_{f_i} \sum_{ijk \in F} A_{ijk} &= \frac{1}{2}\sum_{ij \in E} \mathbf{u}_i^{\perp} - \mathbf{u}_j^{\perp} \\
&= \frac{1}{2}\sum_{ij \in E} \cot(\alpha_{ij}) + \cot(\beta_{ij})(f_i - f_j)
\end{aligned}$$

Exercise 20

Gradient of the total discrete scalar mean curvature gives:

$$\begin{aligned}
\nabla_{f_i} \frac{1}{2}\sum_{ij \in E} \theta_{ij} l_{ij} &= \frac{1}{2}\sum_{ij \in E} \cancelto{0}{(\nabla_{f_i}\theta_{ij}) l_{ij}} + \theta_{ij}(\nabla_{f_i} l_{ij}) && \text{(product rule)} \\
&= \frac{1}{2}\sum_{ij \in E} \theta_{ij}(\nabla_{f_i} l_{ij}) && \text{using Schläfli formula: } (\nabla_{f_i}\theta_{ij}) l_{ij} = 0 \\
&= \frac{1}{2}\sum_{ij \in E} \theta_{ij}\frac{f_j - f_i}{l_{ij}} \\
&= \frac{1}{2}\sum_{ij \in E} \frac{\theta_{ij}}{l_{ij}}(f_j - f_i) \\
&= KN_i
\end{aligned}$$

Equates to sum of the Gauss curvature normals over the dual cell of vertex i.

$\boxed{\text{Exercise} \quad 21}$

The gradient of the total discrete scalar Gauss curvature is zero, because the sum of Gauss curvatures K_i is the sum of angle defects $d(v)$, and by Gauss-Bonnet theorem results in a constant multiple of the Euler characteristic χ.

$\chi = (2 - 2g)$ is a topological invariant that does not change unless we change the genus of the surface, or change the number of mesh elements in a way that changes χ: $|V|, |E|, |F|$. Another way of stating this result is that the gradient of $\sum_i K_i$ w.r.t motion of any vertex i is zero.

$$\nabla_{f_i} \sum_{\ell \in V} \left[2\pi - \sum_{jk\ell \in F} \varphi^{jk} \right] = 2\pi \nabla_{f_i} \chi = 0$$