Written Assignment 5

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Exercise 1

Details in papers [1] [2] [3] by Crane et al.

In contrast to algorithms that compute shortest paths along a graph (like Dijkstra, A^*), the *heat method* computes the distance to points on a continuous, curved domain. Typically distance computation for some distance function ϕ is formulated using the *Eikonal equation*:

$$|\nabla \phi| = 1 \qquad \text{(distance changes by 1 meter per meter)}$$

However, this results in a system of non-linear hyperbolic equations which is expensive to solve.

The heat method takes inspiration from a result by S. R. Srinivasa Varadhan which says that distance between a source point x and destination y on a curved domain can be recovered via a simple pointwise transformation of the heat kernel $k_{t,x}$:

$$\phi(x, y) = \lim_{t \to 0} \sqrt{-4t \log k_{t,x}(y)}$$

However, reconstructing the heat kernel is prohibitive in practice, and this formula lacks robustness to numerical error. The geodesic distance varies a lot with tiny errors in reconstruction.

The heat method works on arbitrary domains: smooth manifolds, triangle meshes, polygon meshes, point clouds, etc. All we need to define is the gradient, divergence and Laplacian. The algorithm works as follows:

- Integrate the heat flow $\dot{u} = \nabla u$ for some fixed time t.
- Evaluate the vector field $X = -\dfrac{\nabla u_t}{|\nabla u_t|}$.
- Solve the Poisson equation $\Delta \phi = \nabla \cdot X$.

By approximating Step 1 using backward Euler integration $(\mathrm{id} - t\Delta)u_t = \delta_x$ we effectively convert the problem into a linear elliptic equation. For triangle meshes, the authors provide performance and convergence guarantees given some requirements are met – related to mesh resolution (time step equals mean edge length squared) and triangulation quality (Delaunay).

Exercise 2

We make use of the composition theorem from Convex Optimization - Boyd and Vandenberghe.

A nondecreasing convex function of a convex function is convex.

Suppose we have $f(x) = ||x||$ and $g(x) = x^2$. Because f and g are convex, we now have $g \circ f(x) = ||x||^2$ is convex. We can plug differential functions ϕ_1, ϕ_2 into a convex combination:

$$\begin{aligned} \eta||\nabla\phi_1 - X||^2 + (1-\eta)||\nabla\phi_2 - X||^2 &\geq ||\eta(\nabla\phi_1 - X) + (1-\eta)(\nabla\phi_2 - X)||^2 \\ &= ||\eta\nabla\phi_1 + (1-\eta)\nabla\phi_2 - X||^2 \end{aligned}$$

We can make the above claim By integrating on the manifold M:

$$\eta \int_M ||\nabla \phi_1 - X||^2 dA + (1-\eta) \int_M ||\nabla \phi_2 - X||^2 dA \geq \int_M ||\eta \nabla \phi_1 + (1-\eta)\nabla \phi_2 - X||^2 dA$$
$$\eta E(\phi_1) + (1-\eta) E(\phi_2) \geq E(\eta \phi_1 + (1-\eta)\phi_2)$$

$\therefore E$ is convex.

$\boxed{\text{Exercise} \quad 3}$

Restating Green's first identity, with the assumption that $\nabla f \cdot n = 0$:

$$\begin{aligned} &\langle \Delta f, g \rangle = -\langle \nabla f, \nabla g \rangle + \cancelto{0}{\langle \nabla f \cdot n, g \rangle_\partial} \\ \implies &\langle \nabla f, \nabla g \rangle = -\langle \Delta f, g \rangle \end{aligned} \tag{1}$$

The energy functional E can be expressed as:

$$\begin{aligned} E(\phi) &:= \int_M ||\nabla \phi - X||^2 dA \\ &= \int_M \langle\langle \nabla \phi - X, \nabla \phi - X \rangle\rangle dA \\ &= \int_M \underbrace{(\langle\langle \nabla \phi, \nabla \phi \rangle\rangle - 2\langle\langle \nabla \phi, X \rangle\rangle dA + \langle\langle X, X \rangle\rangle)}_{u} \, d\underbrace{A}_{v} \end{aligned}$$

A corollary of the divergence theorem:

$$\langle\langle \nabla \phi, X \rangle\rangle = \langle\langle \phi, -\nabla \cdot X \rangle\rangle \tag{2}$$

Using integration by parts $\int u dv = uv - \int v du$ along with $du = 0$:

$$\begin{aligned} \frac{E(\phi)}{A} &= -\langle\langle \Delta f, g \rangle\rangle - 2\langle\langle \nabla \phi, X \rangle\rangle + ||X||^2 && \text{from (1)} \\ &= -\langle\langle \Delta f, g \rangle\rangle + 2\langle\langle \phi, \nabla \cdot X \rangle\rangle + ||X||^2 && \text{from (2)} \end{aligned}$$

$\boxed{\text{Exercise} \quad 4}$

$$\begin{aligned} E(\phi + \epsilon\psi) &= -\langle\langle \Delta(\phi + \epsilon\psi), \phi + \epsilon\psi \rangle\rangle + 2\langle\langle \phi + \epsilon\psi, \nabla \cdot X \rangle\rangle + ||X||^2 \\ &= -\langle\langle \Delta\phi + \epsilon\Delta\psi, \phi + \epsilon\psi \rangle\rangle + 2\langle\langle \phi + \epsilon\psi, \nabla \cdot X \rangle\rangle + ||X||^2 \\ &= -\langle\langle \Delta\phi, \phi \rangle\rangle - \epsilon\langle\langle \Delta\phi, \psi \rangle\rangle - \epsilon\langle\langle \Delta\psi, \phi \rangle\rangle - \epsilon^2\langle\langle \Delta\psi, \psi \rangle\rangle + 2\langle\langle \phi, \nabla \cdot X \rangle\rangle + 2\epsilon\langle\langle \psi, \nabla \cdot X \rangle\rangle + ||X||^2 \\ &= E(\phi) - \epsilon\langle\langle \Delta\phi, \psi \rangle\rangle - \epsilon\langle\langle \Delta\psi, \phi \rangle\rangle - \epsilon^2\langle\langle \Delta\psi, \psi \rangle\rangle + 2\epsilon\langle\langle \psi, \nabla \cdot X \rangle\rangle \end{aligned}$$

Taking the limit $\epsilon \to 0$ and dropping the term quadratic in ϵ:

$$\begin{aligned} D_\psi E(\phi) &= \lim_{\epsilon \to 0} \frac{E(\phi + \epsilon\psi) - E(\phi)}{\epsilon} \\ &= -\langle\langle \Delta\phi, \psi \rangle\rangle - \langle\langle \Delta\psi, \phi \rangle\rangle + 2\langle\langle \psi, \nabla \cdot X \rangle\rangle \\ &= -2\langle\langle \Delta\phi, \psi \rangle\rangle + 2\langle\langle \psi, \nabla \cdot X \rangle\rangle \\ &= 2\langle\langle \psi, \nabla \cdot X - \Delta\phi \rangle\rangle \end{aligned}$$

$\boxed{\text{Exercise} \quad 5}$

From our previous result, we have:

$$\begin{aligned} &D_\psi E(\phi) = 2\langle\langle \nabla \cdot X - \Delta\phi, \psi \rangle\rangle = \nabla E(\phi), \psi \rangle\rangle \\ \implies &\nabla \cdot X - \Delta\phi = \nabla E(\phi) \end{aligned}$$

$\therefore \nabla E(\phi) = 0$ if and only if $\Delta\phi = \nabla \cdot X$.

$\boxed{\text{Exercise} \quad 6}$

Stokes'/divergence theorem allows us to convert the integral of divergence of X over the domain M into a boundary integral over ∂M.

$$\begin{aligned}
\int_M \nabla \cdot X \; dA &= \int_{\partial M} n \cdot X \; dl \\
&= \sum_j \Big(\frac{e_1}{|e_1|} \cdot X_j \Big) l_1 + \Big(\frac{e_2}{|e_2|} \cdot X_j \Big) l_2 \\
&= \sum_j \frac{l_1}{|e_1|} e_1 \cdot X_j + \frac{l_2}{|e_2|} e_2 \cdot X_j \\
&= \sum_j \frac{1}{2} \cot \theta_1 (e_1 \cdot X_j) + \frac{1}{2} \cot \theta_2 (e_2 \cdot X_j) \\
&= \frac{1}{2} \sum_j \cot \theta_1 (e_1 \cdot X_j) + \cot \theta_2 (e_2 \cdot X_j)
\end{aligned}$$

$\boxed{\text{Exercise} \quad 7, 8, 9}$

TODO