Written Assignment 0

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Exercise 2.1

Proving $\chi = V - E + F = 1$ for a polygonal disk:

If we start with a polygonal disk which has non-triangular faces, we can triangulate it by adding diagonals. Each diagonal increases the number of edges and faces by 1. This process of triangulation leaves χ invariant. Now that all the faces are triangular, the remainder of the proof must show that $\chi = 1$. We start by removing triangles from the boundary which involves either:

- Removing 1 face and 1 edge
- Removing 1 vertex, 2 edges and 1 face

Both would leave χ the same. We are left with the base case which is just a triangle which has $\chi = 3 - 3 + 1 = 1$.

Proving $\chi = V - E + F = 2$ for a polygonal sphere:

We project the polyhedron to the 2D plane to re-use the previous result. To do this imagine shining a light from the top and casting a shadow on a surface placed on the bottom. This would yield a projection which has the same number of edges and vertices as before. If this is not the case, then we are allowed to reposition the vertices so that it casts a proper shadow. We are allowed to do this since χ is a topological property.
The shadow we cast will have one less face than the original because that face is now the boundary of our shadow. So $\chi = 1 + 1 = 2$ for the polygonal sphere.

Exercise 2.2

Angles argument:

- **Triangles**. The interior angle of an equilateral triangle is 60 degrees. Thus on a regular polyhedron, only 3, 4, or 5 triangles can meet a vertex. If there were more than 6 their angles would add up to at least 360 degrees which they can't. Consider the possibilities:
  - 3 triangles meet at each vertex, giving rise to a Tetrahedron
  - 4 triangles meet at each vertex, giving rise to an Octahedron
  - 5 triangles meet at each vertex, giving rise to an Icosahedron
- **Squares**. Since the interior angle of a square is 90 degrees, at most three squares can meet at a vertex. This is indeed possible and it gives rise to a hexahedron or cube.
- **Pentagons**. As in the case of cubes, the only possibility is that three pentagons meet at a vertex. This gives rise to a Dodecahedron.
- **Hexagons** or regular polygons with more than six sides cannot form the faces of a regular polyhedron since their interior angles are at least 120 degrees.

We end up with 5 platonic solids.

Connectivity argument:

Because this is a regular polytope/mesh, the valence of each vertex is equal, so we argue each face is an identical n-gon, for some positive n.
Being regular implies $n \geq 3$.
With the same argument, each vertex is identical, so let d be the degree of vertices.

Being regular implies $d \geq 3$.

As usual V is number of vertices, E is number of edges and F is number of faces.
Each edge touches two faces, so $\frac{nF}{2} = E \implies F = \frac{2E}{n}$.
Each edge touches two vertices, so $\frac{dV}{2} = E \implies V = \frac{2E}{d}$.

Using Euler's formula:

$$\begin{aligned} \chi = 2 &= V - E + F \\ &= \frac{2E}{d} - E + \frac{2E}{n} \\ &= E\left(\frac{2}{d} - 1 + \frac{2}{n}\right) \end{aligned}$$

From earlier $n \geq 3$ and $d \geq 3$, so we get $\frac{1}{n} \leq \frac{1}{3}$ and $\frac{1}{d} \leq \frac{1}{3}$.
E must be positive so:

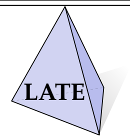
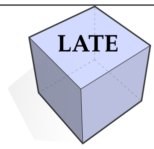
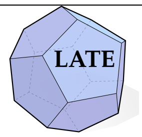
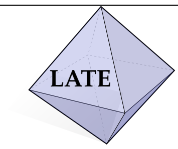
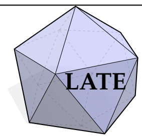
$$\begin{aligned} \frac{2}{d} - 1 + \frac{2}{n} &> 0 \\ \frac{1}{d} > \frac{1}{2} - \frac{1}{n} &> \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \\ 3 \leq d &< 6 \end{aligned}$$

When $d = 3$, $\frac{1}{n} > \frac{1}{6}$, so $n = 3,\ 4$ or 5.

When $d = 4$, $\frac{1}{n} > \frac{1}{4}$, so $n = 3$.

When $d = 5$, $\frac{1}{n} > \frac{3}{10}$, so $n = 3$.

Overall, this gives us the following table of platonic solids:

d	n	V	E	F	Solid	Mesh
3	3	4	6	4	Tetrahedron	
3	4	8	12	6	Cube	
3	5	20	30	12	Dodecahedron	
4	3	6	12	8	Octahedron	
5	3	12	30	20	Icosahedron	

The whole 'LATE' thing is a joke that can be found here.

Exercise 2.3

Previous formulas apply here, with $n = 3$ and $d = 6$. We apply the Euler-Poincaré formula:

$$\begin{aligned}\chi = 2 - 2g &= V - E + F \\ &= \frac{2E}{d} - E + \frac{2E}{n} \\ &= \frac{2E}{6} - E + \frac{2E}{3} \\ &= 0\end{aligned}$$

So we get $2(1-g) = 0 \implies g = 1$, which is a torus.

Exercise 2.4

Let the number of vertices with irregular valence be n.
The valences of these n vertices are $v_1, v_2, \ldots, v_n$, and we assume $v_i \geq 3$. Using the previous formula for regular triangle mesh with degree d: $dV = 2E = 3F$. This degree for irregular mesh is not uniformly d so we now have:

$$\begin{aligned}6(V-n) + \sum_i^n v_i &= 2E = 3F \\ \implies F &= \frac{6(V-n) + \sum_i^n v_i}{3}\end{aligned}$$

We apply the Euler-Poincaré formula, and express in terms of V:

$$\begin{aligned}\chi = 2 - 2g &= V - E + F \\ &= V - \frac{3}{2}F + F \\ &= V - \frac{1}{2}F \\ &= V - \frac{6(V-n) + \sum_i^n v_i}{6} \\ &= n - \frac{\sum_i^n v_i}{6}\end{aligned}$$

So $n = 2 - 2g + \frac{\sum_i^n v_i}{6}$. Because $v_i \geq 3$:

$$\sum_i^n v_i \geq 3n \implies \frac{\sum_i^n v_i}{6} \geq \frac{n}{2} \implies n - 2 + 2g \geq \frac{n}{2} \implies n \geq 4 - 4g$$

<u>When $g = 0$:</u> we have $n \geq 4$.

<u>When $g = 1$:</u> we have $n = 0$ from Exercise 2.3

<u>When $g \geq 2$:</u> we have $n \leq -4$.
n is non-negative, and if $n = 0 = 2\chi$, implies $\chi = 0$ for genus $g \geq 2$ which is invalid. So the valid values start from $n \geq 1$.

Which gives us our result:

$$m(K) = \begin{cases} 4, & g = 0 \\ 0, & g = 1 \\ 1, & g \geq 2 \end{cases}$$

Exercise 2.5

Triangle mesh:
Each edge has 2 faces on either side, each face is bounded by 3 edges. So $3F = 2E$ or $E : F = 3 : 2$.

We apply the Euler-Poincaré formula:

$$\begin{aligned}\chi = 2 - 2g &= V - E + F \\ &= V - E + \frac{2}{3}E \\ \implies E &= 3(V - 2 + 2g)\end{aligned}$$

An edge connects two vertices, but we can say that the edge belongs to only one of the vertices. So, mean valence for a triangle mesh of large V:

$$\lim_{V\to\infty} \frac{2E}{V} = \lim_{V\to\infty} \frac{6(V-2+2g)}{V} = 6$$

Exercise 2.6

Quad mesh:
Very similar to the previous calculation, except each face is bounded by 4 edges, so $E : F = 4 : 2 = 2 : 1$.
We apply the Euler-Poincaré formula and get: $E = 2(V - 2 + 2g)$
Mean valence for a quad mesh of large V:

$$\lim_{V\to\infty} \frac{2E}{V} = 4$$

Exercise 2.7

Tet mesh:
TODO: find a proper explanation:
I found an explanation from Stack Exchange which gives us the following result:

$$V : E : F : T = 1 : 4 : 6 : 3$$

whereas the data included in the problem gives us something like:

$$V : E : F : T = 2 : 14 : 3 : 1$$

So, I don't know what to make of this proof.

Let d be a positive integer and M a d-dimensional triagularizable geometric object. Let F_j denote the number of j-dimensional faces of a triangularization T of M. (For example, F_0 is the number of vertices and F_1 is the number of edges.) Each time a new vertex is added into the interior of an n-simplex in T, we see that the new triangularization T' satisfies

$$F'_j = \begin{cases} F_j + \binom{d+1}{j}\,, & \text{if } j = 0, 1, 2, \ldots, d-1\,, \\ F_d + d\,, & \text{if } j = d\,, \end{cases}$$

where F'_j is the number of d-dimensional faces in T' for each $j = 0, 1, 2, \ldots, d$. Hence, if you refine the mesh on M nicely (i.e., avoid fiddling with boundaries of all n-simplices), then the ratio

$$F_0 : F_1 : \ldots : F_{d-1} : F_d$$

should tend to

$$\binom{d+1}{0} : \binom{d+1}{1} : \ldots : \binom{d+1}{d-1} : d\,,$$

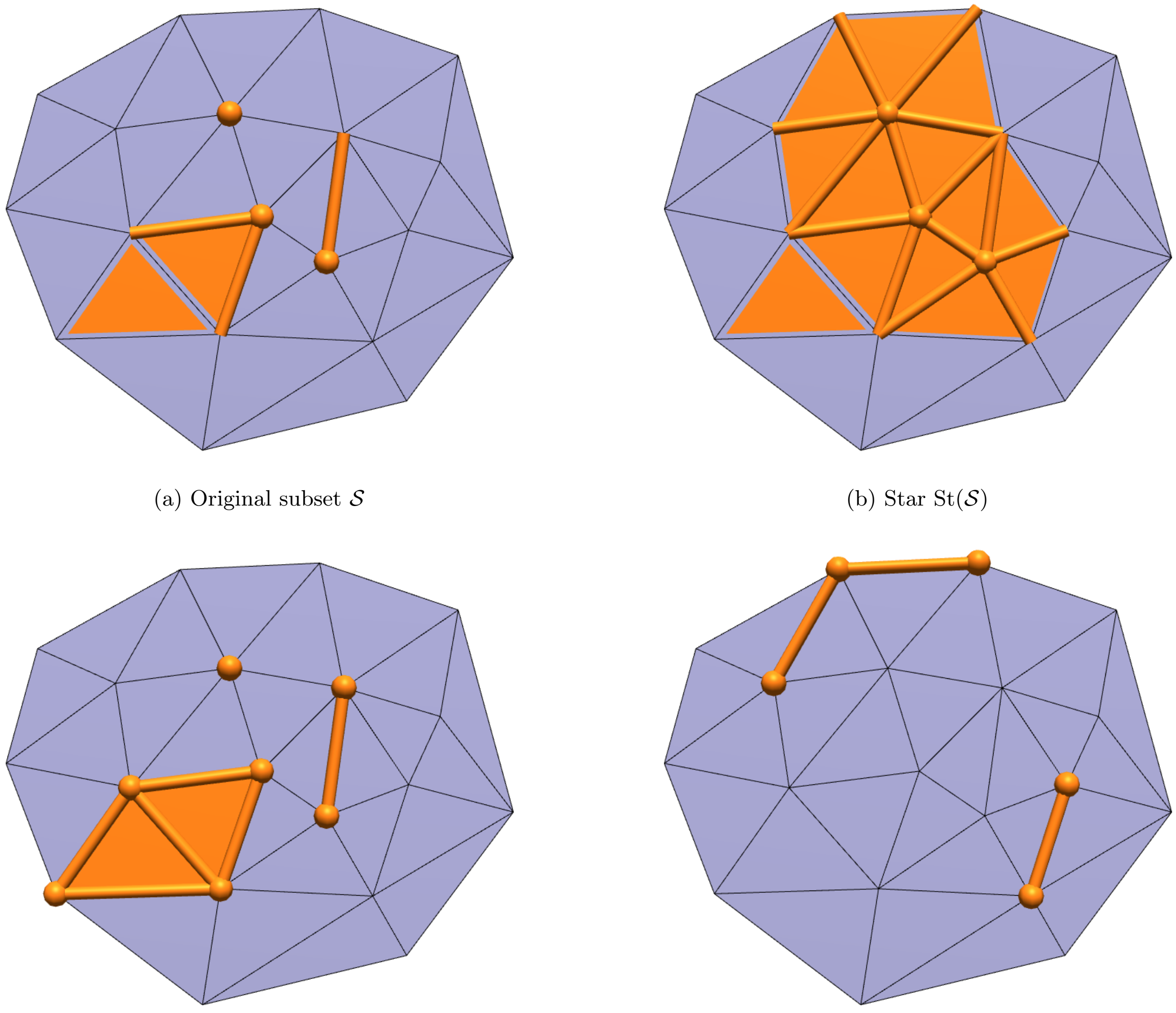
as the number of vertices increases. In particular, for $d = 3$, one would expect

$$\frac{F_1}{F_0} \approx \frac{\binom{3+1}{1}}{\binom{3+1}{0}} = 4\,.$$

If we are allowed to play with the boundaries of n-simplices, then the ratios $\frac{F_j}{F_{j-1}}$ for $j = 1, 2, \ldots, d$ may not have limits, or can tend to arbitrarily large values, provided that $d \geq 3$.

$\boxed{\text{Exercise} \quad 2.8}$

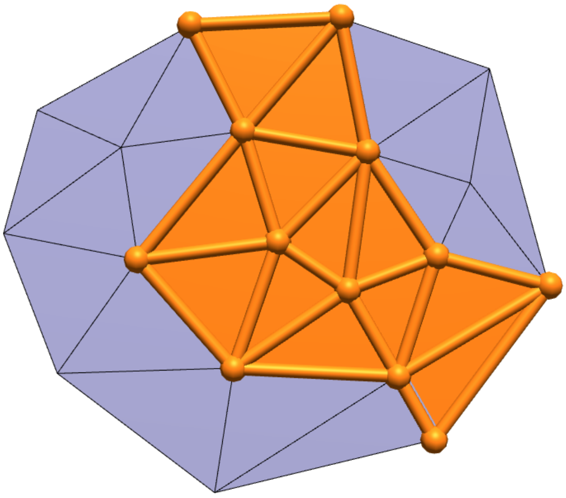
Luckily the mesh in this problem matches the one in the coding exercise!



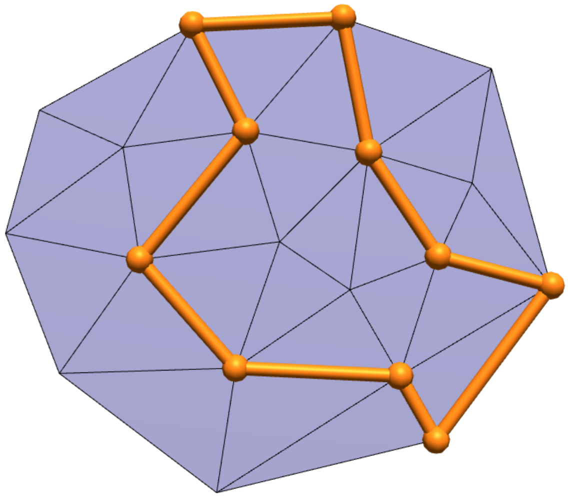
(a) Original subset $\mathcal{S}$ (b) Star $\mathrm{St}(\mathcal{S})$

(c) Closure $\mathrm{Cl}(\mathcal{S})$ (d) Link $\mathrm{Lk}(\mathcal{S})$

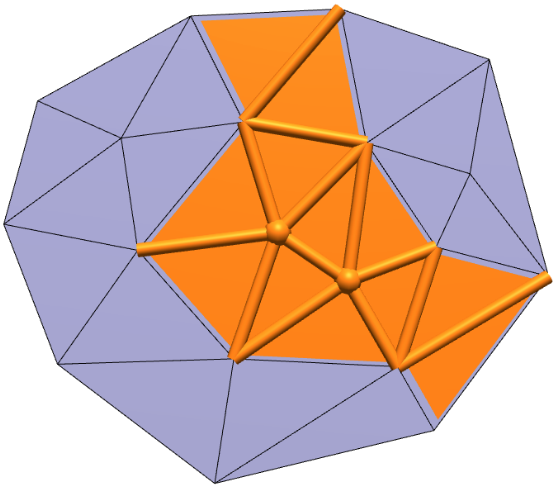
Exercise 2.9



(a) Original subset $\mathcal{K}'$



(b) Boundary $\mathrm{bd}(\mathcal{K}')$



(c) Interior $\mathrm{int}(\mathcal{K}')$

Exercise 2.10

Twin

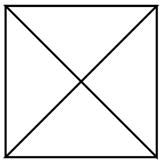
h	0	1	2	3	4	5	6	7	8	9
$\eta(h)$	4	2	1	5	0	3	7	6	9	8

Next

h	0	1	2	3	4	5	6	7	8	9
$\rho(h)$	1	2	0	4	5	6	3	9	7	8

$\boxed{\text{Exercise} \quad 2.11}$

Looks like this:



$\boxed{\text{Exercise} \quad 2.12}$

$$A_0 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\boxed{\text{Exercise} \quad 2.13}$

For a simplicial $k-$manifold, the link of every vertex (0-simplex) looks like a $(k-1)-$dimensional sphere.
A simplicial $1-$complex is just a graph, but a simplicial $1-$manifold is not an arbitrary graph.
The degree of each vertex is no greater than 2, so the link of every vertex should be a pair of vertices.
So it cannot contain anything other than isolated paths of edges and closed loops of edges.

$\boxed{\text{Exercise} \quad 2.14}$

The boundary of a simplicial surface will have zero or more closed loops.
Each connected set of vertices with a boundary will generate a closed loop for its boundary.

$\boxed{\text{Exercise} \quad 2.15}$

Here are a few ways to show that $\mathrm{bd}(\mathrm{bd}(\mathcal{K})) = \emptyset$:

- Taking the boundary $\mathrm{bd}(\mathcal{K})$ means removing the interior: $\mathrm{bd}(\mathcal{K}) = \mathrm{Cl}(\mathcal{K}) \setminus \mathrm{int}(\mathcal{K})$. We could say that $\mathrm{int}(\mathrm{bd}(\mathcal{K})) = \mathrm{Cl}(\mathrm{bd}(\mathcal{K}))$ which gives us our result.
- Given a k-dimensional point set $\mathcal{K}$, for all points p in the boundary of $\mathcal{K}$, the intersection of some open ball around p is homeomorphic to an open $(k-1)$-ball.
- In DDG the boundary is defined as the closure of the set of all simplices σ that are proper faces of exactly one simplex of $\mathcal{K}$. The result of taking the boundary is a $(k-1)-$submanifold without any proper faces of exactly one simplex. This means σ for $\mathrm{bd}(\mathcal{K})$ becomes $\emptyset$.

$$\mathrm{bd}(\mathrm{bd}(\mathcal{K})) = \mathrm{Cl}(\sigma) = \mathrm{Cl}(\emptyset) = \emptyset.$$