Written Assignment 4

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Exercise 1

Important to note that we can only prove this for a 1-form α in $\mathbb{R}^2$. In $\mathbb{R}^n$ the Hodge star of a k-form is an $(n-k)$-form. We now need some way of relating the operators Hodge star $\star$ and complex structure $\mathcal{J}$.

Adopting the convention $\star\alpha(X) = \alpha(\mathcal{J}X)$:

$$\begin{aligned}
\star\alpha(X) \wedge \alpha(X, \mathcal{J}X) &= \star\alpha(X)\alpha(\mathcal{J}X) - \star\alpha(\mathcal{J}X)\alpha(X) \\
&= \alpha(\mathcal{J}X)\alpha(\mathcal{J}X) - \alpha(\mathcal{J}\mathcal{J}X)\alpha(X) && \mathcal{J}^2 = \text{-id} \\
&= \alpha(\mathcal{J}X)\alpha(\mathcal{J}X) + \alpha(X)\alpha(X) \\
&\geq 0
\end{aligned}$$

X and $\mathcal{J}X$ form an orthogonal basis for the tangent space T_pM. This means for any two real valued 1-forms $\alpha, \beta \geq 0$ we claim that $\langle\langle \alpha, \beta \rangle\rangle = \int_M \star\alpha \wedge \beta$ is positive definite.

With the equal and opposite convention $\star\alpha(X) = -\alpha(\mathcal{J}X)$:

$$\begin{aligned}
\alpha(X) \wedge \star\alpha(X, \mathcal{J}X) &= \alpha(X) \star \alpha(\mathcal{J}X) - \alpha(\mathcal{J}X) \star \alpha(X) \\
&= -\alpha(X)\alpha(\mathcal{J}\mathcal{J}X) + \alpha(\mathcal{J}X)\alpha(\mathcal{J}X) && \mathcal{J}^2 = \text{-id} \\
&= \alpha(X)\alpha(X) + \alpha(\mathcal{J}X)\alpha(\mathcal{J}X) \\
&\geq 0
\end{aligned}$$

In this case we claim that $\langle\langle \alpha, \beta \rangle\rangle = \int_M \alpha \wedge \star\beta$ is positive definite.

Exercise 2

$$\begin{aligned}
(\star \star \alpha) \wedge (\star\alpha)(X, \mathcal{J}X) &= \star \star \alpha(X) \star \alpha(\mathcal{J}X) - \star \star \alpha(\mathcal{J}X) \star \alpha(X) \\
&= \star\alpha(\mathcal{J}X) \star \alpha(\mathcal{J}X) + \star\alpha(X) \star \alpha(X) \\
&= \alpha(\mathcal{J}X)\alpha(\mathcal{J}X) + \alpha(X)\alpha(X) \\
&= \star\alpha \wedge \alpha(X, \mathcal{J}X)
\end{aligned}$$

Therefore,

$$\begin{aligned}
|| \star \alpha|| &= \sqrt{\langle\langle \star\alpha, \star\alpha \rangle\rangle} \\
&= \sqrt{\int_M (\star \star \alpha) \wedge (\star\alpha)} \\
&= \sqrt{\int_M \star\alpha \wedge \alpha} \\
&= \sqrt{\langle\langle \alpha, \alpha \rangle\rangle} = ||\alpha||
\end{aligned}$$

The geometric intuition here is that in $\mathbb{R}^2$, the Hodge star of a 1-form is just a 90° rotation, and rotations preserve length.

$\boxed{\text{Exercise} \quad 3}$

Let $u = a + ib$ and $v = c + id$, then

$$\begin{aligned}\bar{u}v &= (a - ib)(c + id) \\ &= ac + iad - ibc + bd \\ &= (ac + bd) + i(ad - bc) \\ &= u \cdot v + i(u \times v)\end{aligned}$$

$\boxed{\text{Exercise} \quad 4}$

We show that inner product $\langle ., . \rangle$ in $\mathbb{C}$ is Hermitian:

$$\begin{aligned}\langle u, v \rangle &= u \cdot v + i(u \times v) \\ &= v \cdot u - i(v \times u) \\ &= \overline{v \cdot u - i(v \times u)} \\ &= \overline{\langle v, u \rangle}\end{aligned}$$

Next we show that inner product is positive definite for $u \neq 0$:

$$\begin{aligned}\langle u, u \rangle &= u \cdot u + i(u \times u) \\ &= u \cdot u + 0 \\ &> 0\end{aligned}$$

$\boxed{\text{Exercise} \quad 5}$

$$\begin{aligned}\star\bar{\alpha} \wedge \alpha(X, \mathcal{J}X) &= \star\bar{\alpha}(X)\alpha(\mathcal{J}X) - \star\bar{\alpha}(\mathcal{J}X)\alpha(X) \\ &= \bar{\alpha}(\mathcal{J}X)\alpha(\mathcal{J}X) + \bar{\alpha}(X)\alpha(X) \\ &= \langle \alpha(\mathcal{J}X), \alpha(\mathcal{J}X) \rangle + \langle \alpha(X), \alpha(X) \rangle \\ &\geq 0\end{aligned}$$

$$\begin{aligned}\star\bar{\alpha} \wedge \beta(X, \mathcal{J}X) &= \star\bar{\alpha}(X)\beta(\mathcal{J}X) - \star\bar{\alpha}(\mathcal{J}X)\beta(X) \\ &= \bar{\alpha}(\mathcal{J}X)\beta(\mathcal{J}X) + \bar{\alpha}(X)\beta(X) \\ &= \langle \alpha(\mathcal{J}X), \beta(\mathcal{J}X) \rangle + \langle \alpha(X), \beta(X) \rangle\end{aligned}$$

$$\begin{aligned}\star\bar{\beta} \wedge \alpha(X, \mathcal{J}X) &= \langle \beta(\mathcal{J}X), \alpha(\mathcal{J}X) \rangle + \langle \beta(X), \alpha(X) \rangle \\ &= \overline{\langle \alpha(\mathcal{J}X), \beta(\mathcal{J}X) \rangle} + \overline{\langle \alpha(X), \beta(X) \rangle} \\ &= \overline{\star\bar{\alpha} \wedge \beta(X, \mathcal{J}X)}\end{aligned}$$

Using the identity $z + \bar{z} = \operatorname{Re} z$, $\langle\langle \alpha, \beta \rangle\rangle = \operatorname{Re} \int_M \star\bar{\alpha} \wedge \beta$ is Hermitian and positive definite.

$\boxed{\text{Exercise} \quad 6}$

Because u, v are 0-forms, du, dv are 1-forms.
We will use this in the product rule for exterior derivatives when considering Stokes' theorem on the manifold M. Assuming that du, dv are zero at the boundary ∂M:

$$\mathrm{Re} \int_{\partial M} v \star \overline{du} = 0$$

$$\begin{aligned} 0 &= \mathrm{Re} \int_{\partial M} v \star \overline{du} \\ &= \mathrm{Re} \int_M d(v \star \overline{du}) \\ &= \mathrm{Re} \int_M dv \wedge \star\overline{du} + \mathrm{Re} \int_M (-1)^0 v \wedge d \star \overline{du} \\ &= -\,\mathrm{Re} \int_M \star\overline{du} \wedge dv + \mathrm{Re} \int_M v \wedge d \star \overline{du} \\ \implies \mathrm{Re} \int_M \star\overline{du} \wedge dv &= \mathrm{Re} \int_M v \wedge d \star \overline{du} \end{aligned}$$

where we used Stokes' theorem to convert $\int_{\partial M} \cdots \to \int_M d(\cdots)$

From our previous result for the inner product of complex 1-forms, we have:

$$\begin{aligned} \langle\langle du, dv \rangle\rangle &= \mathrm{Re} \int_M \star\overline{du} \wedge dv \\ &= \mathrm{Re} \int_M v \wedge d \star \overline{du} \\ &= -\,\mathrm{Re} \int_M d \star \overline{du} \wedge v \\ &= \mathrm{Re} \int_M (\star\star) d \star \overline{du} \wedge v && \star^2 = \text{-id when } n = 2, k = 1 \\ &= \mathrm{Re} \int_M \star(\star d \star d)\bar{u} \wedge v && \overline{du} = d\bar{u} \\ &= \mathrm{Re} \int_M \star\Delta\bar{u} \wedge v && \Delta = \star d \star d \\ &= \langle\langle \Delta u, v \rangle\rangle \end{aligned}$$

$\boxed{\text{Exercise} \quad 7}$

Area on a Riemann surface (topological disk M) can be expressed in terms of exterior calculus as $X \times \mathcal{J}X$. A conformal parameterization $z : M \to \mathbb{C}$ must satisfy the Cauchy-Riemann equation:

$$dz(\mathcal{J}X) = i dz(X) \implies \star dz = i dz$$

Total signed area of the region $z(M)$:

$$\begin{aligned} \mathcal{A}(z) &= \int_M dz \times d\bar{z} \\ &= \frac{1}{2} \int_M \star(dz \wedge d\bar{z}) && \text{Signed vector area formula} \\ &= \frac{i}{2} \int_M dz \wedge d\bar{z} && \text{using Cauchy-Riemann} \\ &= -\frac{i}{2} \int_M d\bar{z} \wedge dz \end{aligned}$$

Exercise 8

Using this paper by John E. Hutchinson as reference, whos argument works for the smooth case.

The conformal energy $E_C(z)$ is defined as the failure of the map z to be conformal, and as such, satisfying the discrete Cauchy-Riemann equations.

We prove that $E_C(z)$ is the difference of Dirichlet energy $E_D(z) = \frac{1}{2}\langle\langle \Delta z, z\rangle\rangle$ and total signed area $\mathcal{A}$:

$$\begin{aligned}
E_C(z) &= \frac{1}{4}|| \star dz - idz||^2 \\
&= \frac{1}{4}\langle\langle \star dz - idz, \star dz - idz\rangle\rangle && \text{Inner product is distributive} \\
&= \frac{1}{4}\Big(\langle\langle \star dz, \star dz\rangle\rangle - \langle\langle \star dz, idz\rangle\rangle - \langle\langle idz, \star dz\rangle\rangle + \langle\langle idz, idz\rangle\rangle\Big) \\
&= \frac{1}{4}\Big(2\langle\langle \star dz, \star dz\rangle\rangle - 2i\langle\langle \star dz, dz\rangle\rangle\Big) && \text{Inner product is homogeneous} \\
&= \frac{1}{2}\langle\langle \star dz, \star dz\rangle\rangle - \frac{i}{2}\langle\langle \star dz, dz\rangle\rangle \\
&= \frac{1}{2}\langle\langle \Delta z, z\rangle\rangle - \frac{i}{2}\int_M (\star\star d\bar{z}) \wedge dz \\
&= \frac{1}{2}\langle\langle \Delta z, z\rangle\rangle + \frac{i}{2}\int_M d\bar{z} \wedge dz \\
&= E_D(z) - \mathcal{A}(z)
\end{aligned}$$

Exercise 9

Using this paper by Keenan Crane as reference, which reminded me of our previous result for the complex inner product which we use below.

We had proved in a past assignment that signed area of a polygon is the sum of cross products of neighbouring oriented boundary edges, which sums up n terms for an n-gon.

Using this result for n-gons to find the area of piecewise linear region $z(M)$:

$$\begin{aligned}
\mathcal{A}(z) &= \frac{1}{2}\sum_{ij\in\partial E_\partial} z_i \times z_j \\
&= \frac{1}{2}\sum_{ij\in\partial E_\partial} \mathrm{Im}(\bar{z}_i z_j) && z_i \times z_j := \mathrm{Im}(\bar{z}_i z_j) \\
&= \frac{1}{2}\sum_{ij\in\partial E_\partial} \frac{\bar{z}_i z_j - \bar{z}_j z_i}{2i} \\
&= \frac{1}{2}\sum_{ij\in\partial E_\partial} \frac{\bar{z}_i z_j - \bar{z}_j z_i}{2i} \\
&= -\frac{i}{4}\sum_{ij\in\partial E_\partial} \bar{z}_i z_j - \bar{z}_j z_i
\end{aligned}$$

Exercise 10

From the Boundary First Flattening paper by Sawhney et. al.

Conformal maps can also be expressed as pairs of conjugate harmonic functions. A real function $a : M \to \mathbb{R}$ is harmonic if it sits in the kernel of the Laplace-Beltrami operator Δ, i.e., it solves the Laplace equation $\Delta a = 0$.

Suppose we express a holomorphic map as $z = a + ib$ for a pair of coordinate functions $a, b : M \to \mathbb{R}$. Then (by Cauchy-Riemann)

$$\mathcal{J}\nabla a = \nabla b$$

i.e., the gradients ∇ of the two coordinates are orthogonal and have equal magnitude.

Since a quarter-rotation of a gradient field is divergence-free, we have

$$\Delta a = \nabla \cdot \nabla a = -\nabla \cdot (\mathcal{J}\nabla b) = 0$$

and similarly, $\Delta b = 0$. In other words, the two real components of a holomorphic function are both harmonic — we say that a and b form a *conjugate harmonic* pair.

Exercise 11

If a harmonic function $\varphi : M \to \mathbb{C}$ is real valued then we do not satisfy Cauchy-Riemann conditions any more, so φ will not be holomorphic/conformal.
Geometrically we can interpret φ as mapping vectors in M to the real line in $\mathbb{C}$, so the angles in M aren't being preserved.

A more mathematical treatment comes from Eells & Wood, which says that:

> If $f : M_1 \to M_2$ is a harmonic map and
>
> $$\chi(M_1) + |\deg(f)\chi(M_2)| > 0,$$
>
> then f is either holomorphic or anti-holomorphic (where $\chi(M)$ is the Euler characteristic of M, and $\deg(f)$ is the topological degree of f).

We know that $\chi(M) = 1$ because M is a topological disk but both topological degree of φ and Euler characteristic of the image of φ are undefined, so φ cannot be holomorphic or anti-holomorphic.

Exercise 12

This proof is adapted from Linear Algebra Done Right by Sheldon Axler:

Suppose $A = A^*$ is a self-adjoint operator on a complex inner-product space V, with Hermitian inner product $\langle\langle .\,, . \rangle\rangle$. Let λ be an eigenvalue of A, and let v be a nonzero vector in V such that $Av = \lambda v$. Then

$$\lambda ||v||^2 = \langle\langle \lambda v, v\rangle\rangle = \langle\langle Av, v\rangle\rangle = \langle\langle v, A^* v\rangle\rangle = \langle\langle v, Av\rangle\rangle = \langle\langle v, \lambda v\rangle\rangle = \bar{\lambda}||v||^2$$

So $\lambda = \bar{\lambda}$, which means all the eigenvalues of a self-adjoint operator are real.

Exercise 13

From Axler:

An operator on an inner product space is called normal if it commutes with its adjoint, i.e., $AA^* = A^*A$. If A is normal so is $A - \lambda I$ which gives us:

$$0 = ||(A - \lambda I)v|| = ||(A - \lambda I)^* v|| = ||(A^* - \bar{\lambda} I)v||$$

Suppose λ_i, λ_j are distinct eigenvalues of A with corresponding eigenfunctions e_i, e_j. We have $Ae_i = \lambda_i e_i$ and $Ae_j = \lambda_j e_j$. Furthermore, $A^* e_j = \bar{\lambda_j} e_j$. Thus:

$$\begin{aligned}(\lambda_i - \lambda_j)\langle\langle e_i, e_j\rangle\rangle &= \langle\langle \lambda_i e_i, e_j\rangle\rangle - \langle\langle e_i, \bar{\lambda_j} e_j\rangle\rangle \\ &= \langle\langle Ae_i, e_j\rangle\rangle - \langle\langle e_i, A^* e_j\rangle\rangle \\ &= 0\end{aligned}$$

Because $\lambda_i \neq \lambda_j$ this implies $\langle\langle Ae_i, e_j\rangle\rangle = 0$, i.e., e_i and e_j are orthogonal.
Every self-adjoint operator should be normal. If A is self-adjoint: $A^* = A$ and $\bar{\lambda} = \lambda$.
This proves that eigenvectors of a self-adjoint operator with unique eigenvalues must be orthogonal.

Exercise 14

Below is a restatement of Example 1.27 from Numerical Algorithms by Justin Solomon:

Our goal is to minimize $x^T Ax$ for a PSD symmetric matrix A subject to our contraint $||x||^2 = 1$, which is equivalent to $\langle\langle x, x\rangle\rangle = ||x||^2 = 1$.

Without the constraint the function is minimized at $x = 0$. We define the Lagrange multiplier function:

$$\Lambda(x, \lambda) = x^T Ax - \lambda(||x||^2 - 1) = x^T Ax - \lambda(x^T x - 1)$$

Differentiating w.r.t x, we find $0 = \nabla_x \Lambda = 2Ax - 2\lambda x$. In other words, critical points of x are exactly the eigenvectors of the matrix A:

$$Ax = \lambda x, \quad \text{with} \quad ||x||^2 = 1$$

At these critical points, we can evaluate the objective function as $x^T Ax = x^T \lambda x = \lambda ||x||^2 = \lambda$.
Hence, the minimizer of $x^T Ax$ subject to $||x||^2 = 1$ is the eigenvector x with minimum eigenvalue λ.

Exercise 15

Section 6.3.1 from Solomon:

Assume that $A \in \mathbb{R}^{n\times n}$ is non-defective and nonzero with all real eigenvalues, e.g., A is symmetric. By definition, A has a full set of eigenvectors $x_1, \ldots, x_n \in \mathbb{R}^n$; we sort them such that their corresponding eigenvalues satisfy $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

Take an arbitrary vector $v \in \mathbb{R}^n$. Since the eigenvectors of A span $\mathbb{R}^n$, we can write v in the x_i basis as $v = c_1 x_1 + \cdots + c_n x_n$. Applying A to both sides:

$$\begin{aligned}
Av &= c_1 A x_1 + \cdots + c_n A x_n \\
&= c_1 \lambda_1 x_1 + \cdots + c_n \lambda_n x_n && \text{since } Ax_i = \lambda x_i \\
&= \lambda_1 \Big(c_1 x_1 + c_2 \frac{\lambda_2}{\lambda_1} x_2 + \cdots + c_n \frac{\lambda_n}{\lambda_1} x_n \Big) \\
A^2 v &= \lambda_1^2 \Big(c_1 x_1 + c_2 \Big(\frac{\lambda_2}{\lambda_1}\Big)^2 x_2 + \cdots + c_n \Big(\frac{\lambda_n}{\lambda_1}\Big)^2 x_n \Big) \\
&\vdots \\
A^k v &= \lambda_1^k \Big(c_1 x_1 + c_2 \Big(\frac{\lambda_2}{\lambda_1}\Big)^k x_2 + \cdots + c_n \Big(\frac{\lambda_n}{\lambda_1}\Big)^k x_n \Big)
\end{aligned}$$

As $k \to \infty$, the ratio $\left(\frac{\lambda_i}{\lambda_1}\right)^k \to 0$ unless $\lambda_i = \pm\lambda_1$, since λ_1 has the largest magnitude of any eigenvalue by construction. If x is the projection of v onto the space of eigenvectors with eigenvalues λ_1, then – at least when the absolute values $|\lambda_i|$ are unique – as $k \to \infty$ the following approximation begins to dominate: $A^k v \approx \lambda_1 x$.

By composing this algorithm, called *power iteration*, we produce vectors v_k more and more parallel to the desired x_1 as $k \to \infty$.

$\boxed{\text{Exercise} \quad 16}$

Because A has an inverse:

$$Ax = \lambda x \implies x = \lambda A^{-1} x \implies A^{-1} x = \frac{1}{\lambda} x$$

So we retain the same eigenvectors, but now $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. We can visualize this as the opposite of stretching/squishing that happened to the eigenvectors of A, because we now scale them by $\frac{1}{\lambda}$ instead.