

Written Assignment 5

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Exercise 1

Details in papers [1] [2] [3] by Crane et al.

In contrast to algorithms that compute shortest paths along a graph (like Dijkstra, A^*), the *heat method* computes the distance to points on a continuous, curved domain. Typically distance computation for some distance function ϕ is formulated using the *Eikonal equation*:

$$|\nabla\phi| = 1 \quad (\text{distance changes by 1 meter per meter})$$

However, this results in a system of non-linear hyperbolic equations which is expensive to solve.

The heat method takes inspiration from a result by S. R. Srinivasa Varadhan which says that distance between a source point x and destination y on a curved domain can be recovered via a simple pointwise transformation of the heat kernel $k_{t,x}$:

$$\phi(x, y) = \lim_{t \rightarrow 0} \sqrt{-4t \log k_{t,x}(y)}$$

However, reconstructing the heat kernel is prohibitive in practice, and this formula lacks robustness to numerical error. The geodesic distance varies a lot with tiny errors in reconstruction.

The heat method works on arbitrary domains: smooth manifolds, triangle meshes, polygon meshes, point clouds, etc. All we need to define is the gradient, divergence and Laplacian. The algorithm works as follows:

- Integrate the heat flow $\text{dot}u = \nabla u$ for some fixed time t .
- Evaluate the vector field $X = -\frac{\nabla u_t}{|\nabla u_t|}$.
- Solve the Poisson equation $\Delta\phi = \nabla \cdot X$.

By approximating Step 1 using backward Euler integration $(\text{id} - t\Delta)u_t = \delta_x$ we effectively convert the problem into a linear elliptic equation. For triangle meshes, the authors provide performance and convergence guarantees given some requirements are met – related to mesh resolution and triangulation quality.

Exercise 2

We make use of the composition theorem from Convex Optimization - Boyd and Vandenberghe.

A nondecreasing convex function of a convex function is convex.

Suppose we have $f(x) = \|x\|$ and $g(x) = x^2$. Because f and g are convex, we now have $g \circ f(x) = \|x\|^2$ is convex. We can plug differential functions ϕ_1, ϕ_2 into a convex combination:

$$\begin{aligned} \eta \|\nabla\phi_1 - X\|^2 + (1 - \eta) \|\nabla\phi_2 - X\|^2 &\geq \|\eta(\nabla\phi_1 - X) + (1 - \eta)(\nabla\phi_2 - X)\|^2 \\ &= \|\eta\nabla\phi_1 + (1 - \eta)\nabla\phi_2 - X\|^2 \end{aligned}$$

We can make the above claim By integrating on the manifold M :

$$\begin{aligned} \eta \int_M \|\nabla\phi_1 - X\|^2 dA + (1 - \eta) \int_M \|\nabla\phi_2 - X\|^2 dA &\geq \int_M \|\eta\nabla\phi_1 + (1 - \eta)\nabla\phi_2 - X\|^2 dA \\ \eta E(\phi_1) + (1 - \eta)E(\phi_2) &\geq E(\eta\phi_1 + (1 - \eta)\phi_2) \end{aligned}$$

$\therefore E$ is convex.

Exercise 3

Restating Green's first identity, with the assumption that $\nabla f \cdot n = 0$:

$$\begin{aligned} \langle \Delta f, g \rangle &= -\langle \nabla f, \nabla g \rangle + \underbrace{\langle \nabla f, n, g \rangle}_{\partial} \xrightarrow{0} \\ \implies \langle \nabla f, \nabla g \rangle &= -\langle \Delta f, g \rangle \end{aligned} \quad (1)$$

The energy functional E can be expressed as:

$$\begin{aligned} E(\phi) &:= \int_M \|\nabla \phi - X\|^2 dA \\ &= \int_M \langle \nabla \phi - X, \nabla \phi - X \rangle dA \\ &= \int_M \underbrace{(\langle \nabla \phi, \nabla \phi \rangle - 2\langle \nabla \phi, X \rangle)}_u dA + \underbrace{\langle X, X \rangle}_v dA \end{aligned}$$

A corollary of the divergence theorem:

$$\langle \nabla \phi, X \rangle = \langle \phi, -\nabla \cdot X \rangle \quad (2)$$

Using integration by parts $\int u dv = uv - \int v du$ along with $du = 0$:

$$\begin{aligned} \frac{E(\phi)}{A} &= -\langle \Delta f, g \rangle - 2\langle \nabla \phi, X \rangle + \|X\|^2 && \text{from (1)} \\ &= -\langle \Delta f, g \rangle + 2\langle \phi, \nabla \cdot X \rangle + \|X\|^2 && \text{from (2)} \end{aligned}$$

Exercise 4

$$\begin{aligned} E(\phi + \epsilon \psi) &= -\langle \Delta(\phi + \epsilon \psi), \phi + \epsilon \psi \rangle + 2\langle \phi + \epsilon \psi, \nabla \cdot X \rangle + \|X\|^2 \\ &= -\langle \Delta \phi + \epsilon \Delta \psi, \phi + \epsilon \psi \rangle + 2\langle \phi + \epsilon \psi, \nabla \cdot X \rangle + \|X\|^2 \\ &= -\langle \Delta \phi, \phi \rangle - \epsilon \langle \Delta \phi, \psi \rangle - \epsilon \langle \Delta \psi, \phi \rangle - \epsilon^2 \langle \Delta \psi, \psi \rangle + 2\langle \phi, \nabla \cdot X \rangle + 2\epsilon \langle \psi, \nabla \cdot X \rangle + \|X\|^2 \\ &= E(\phi) - \epsilon \langle \Delta \phi, \psi \rangle - \epsilon \langle \Delta \psi, \phi \rangle - \epsilon^2 \langle \Delta \psi, \psi \rangle + 2\epsilon \langle \psi, \nabla \cdot X \rangle \end{aligned}$$

Taking the limit $\epsilon \rightarrow 0$ and dropping the term quadratic in ϵ :

$$\begin{aligned} D_\psi E(\phi) &= \lim_{\epsilon \rightarrow 0} \frac{E(\phi + \epsilon \psi) - E(\phi)}{\epsilon} \\ &= -\langle \Delta \phi, \psi \rangle - \langle \Delta \psi, \phi \rangle + 2\langle \psi, \nabla \cdot X \rangle \\ &= -2\langle \Delta \phi, \psi \rangle + 2\langle \psi, \nabla \cdot X \rangle \\ &= 2\langle \psi, \nabla \cdot X - \Delta \phi \rangle \end{aligned}$$

Exercise 5

From our previous result, we have:

$$\begin{aligned} D_\psi E(\phi) &= 2\langle \nabla \cdot X - \Delta \phi, \psi \rangle = \nabla E(\phi), \psi \rangle \\ \implies \nabla \cdot X - \Delta \phi &= \nabla E(\phi) \end{aligned}$$

$\therefore \nabla E(\phi) = 0$ if and only if $\Delta \phi = \nabla \cdot X$.

Exercise 6

Stokes'/divergence theorem allows us to convert the integral of integrated divergence of X over the domain M into a boundary integral over ∂M .

$$\begin{aligned}\int_M \nabla \cdot X \, dA &= \int_{\partial M} n \cdot X \, dl \\ &= \sum_j \left(\frac{e_1}{|e_1|} \cdot X_j \right) l_1 + \left(\frac{e_2}{|e_2|} \cdot X_j \right) l_2 \\ &= \sum_j \frac{l_1}{|e_1|} e_1 \cdot X_j + \frac{l_2}{|e_2|} e_2 \cdot X_j \\ &= \sum_j \frac{1}{2} \cot \theta_1 (e_1 \cdot X_j) + \frac{1}{2} \cot \theta_2 (e_2 \cdot X_j) \\ &= \frac{1}{2} \sum_j \cot \theta_1 (e_1 \cdot X_j) + \cot \theta_2 (e_2 \cdot X_j)\end{aligned}$$

Exercise 7, 8, 9

TODO