Written Assignment 3

Niraj Venkat

Exercise 1

An *analytic* function has infinitely many derivatives in its domain (C^∞). For our proof, we only need a function to be twice-differentiable.
We state Liouville's theorem:

> Suppose f is analytic in the open set that contains a ball $B_r(z_0)$ of radius r centered at z_0, and that $|f(z)| \leq c$ holds on $\partial B_r(z_0)$ for some constant c. Then for all $k \geq 0$ we have:
>
> $$|f^{(k)}(z_0)| \leq \frac{k!\, c}{r^k}$$

Simply put: if f is bounded, f must be constant.
This holds for *harmonic* functions as well, in particular our function $\phi : M \to \mathbb{R}$ where $k = 2$ and $\Delta\phi = 0$.

Exercise 2

This admits different proofs:

- Suppose we had a function f such that $\Delta f = \Delta\phi = c$. Then by linearity of the Laplacian we have:

  $$\Delta(\phi - f) = \Delta(f - \phi) = 0$$

  Liouville's Theorem says that both functions $\phi - f$ and $f - \phi$ must be constant.
  $c = 0$ is the only value that satisfies this condition.

- Stokes' theorem (divergence theorem) says the following:

  > For any vector field, we can relate the divergence of the vector field within the volume V to the outward flux of the vector field through the surface $S = \partial V$.

  In our case, we apply this theorem to the vector field which is the gradient $\nabla\phi$.
  The Laplacian ($\Delta = \nabla \cdot \nabla$) is the divergence of gradient, so for some compact domain M without boundary ($\partial M = \emptyset$) we have:

  $$\int_M \nabla \cdot \nabla\phi \, dV = \int_{\partial M} \nabla\phi \cdot \hat{n} \, dS = 0$$
  $$\implies 0 = \int_M \Delta\phi \, dV = \int_M c \, dV = c \, \mathrm{vol}(M)$$

  Because M has some non-negative volume, c cannot be an arbitrary constant. c has to be zero.

Exercise 3

According to Stokes' theorem:

$$\begin{aligned}
\int_\partial g \star df &= \int d(g \star df) \\
&= \int dg \wedge \star df + \int g \wedge d \star df \\
&= \int dg \wedge \star df + \int g \wedge \star \Delta f \\
&= \langle \nabla g, \nabla f \rangle + \langle g, \Delta f \rangle
\end{aligned}$$

On the other hand:

$$\int_\partial g \star df = \langle g, N \cdot \nabla f \rangle_\partial$$

This proves Green's first identity:

$$\langle \Delta f, g \rangle = -\langle \nabla f, \nabla g \rangle + \langle N \cdot \nabla f, g \rangle_\partial$$

Exercise 4

If there is no boundary:

$$\langle g, N \cdot \nabla f \rangle_\partial = 0$$

If $f = g$, then Green's first identity becomes:

$$\begin{aligned}
\langle \Delta f, f \rangle &= -\langle \nabla f, \nabla f \rangle + 0 \\
&= -||\nabla f||^2 \leq 0
\end{aligned}$$

This means the operator Δ is negative semidefinite, and $-\Delta$ is positive semidefinite (PSD). The set of PSD matrices is convex, as such many tools in numerical linear algebra (eg: semidefinite programming) can be used to find the minimum of the quadratic functions they describe (eg: quadratic forms $\vec{x}^T A \vec{x}$).

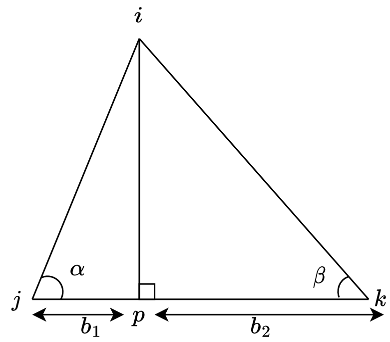
Exercise 5

Using figure on the next page as a reference:

$$\begin{aligned}
w &= b_1 + b_2 \\
h &= |\overline{ip}| \\
\cot \alpha &= \frac{b_1}{h} \\
\cot \beta &= \frac{b_2}{h} \\
\frac{w}{h} &= \cot \alpha + \cot \beta
\end{aligned}$$

Exercise 6

Triangle ijk has base $b = |e| = b_1 + b_2$ and height h.
Let e be the edge vector for the base edge jk, with i as the opposite vertex.



The interpolating hat functions ϕ are similar to their smooth counterpart, the Dirac delta δ. It is equal to one at their associated vertex and zero at all other vertices. We associate ϕ with vertex i.

We can expand the interpolating hat function ϕ in a Taylor series expansion about the point x_0. Since the hat function is linear, all higher order terms go to zero.

$$\phi(x) = \phi(x_0) + \nabla\phi \cdot (x - x_0)$$

Plugging our vertices into this expression:

$$\phi(i) = 1 \tag{1}$$
$$\phi(j) = \phi(i) + \nabla\phi \cdot (j - i) = 0 \tag{2}$$
$$\phi(k) = \phi(i) + \nabla\phi \cdot (k - i) = 0 \tag{3}$$

Subtracting (3) - (2):

$$\begin{aligned} \phi(k) - \phi(j) &= \nabla\phi \cdot (k - i) - \nabla\phi \cdot (j - i) \\ &= \nabla\phi \cdot (k - j) = 0 \end{aligned}$$

We get that direction of $\nabla\phi$ is perpendicular to e. Next we calculate the magnitude of $\nabla\phi$ by substituting (1) into (2):

$$\begin{aligned} 0 &= \phi(i) + \nabla\phi \cdot (j - i) = 0 \\ &= 1 + \nabla\phi \cdot (j - i) \\ 1 &= \nabla\phi \cdot (i - j) \\ &= |\nabla\phi||i - j| \cos(\pi - \alpha) \\ &= |\nabla\phi|\, h \\ |\nabla\phi| &= \frac{1}{h} = \frac{b}{bh} = \frac{|e|}{2A} \end{aligned}$$

We call $e^\perp$ the vector perpendicular to e with magnitude $|e|$. This gives our desired gradient:

$$\nabla\phi = \frac{e^\perp}{2A}$$

Exercise 7

Using our previous result $\nabla\phi = \dfrac{e^\perp}{2A}$:

$$\begin{aligned} \langle \nabla\phi, \nabla\phi \rangle &= \frac{e^\perp \cdot e^\perp}{4A} \\ &= \frac{b^2}{4bh} \\ &= \frac{1}{2}\frac{b}{2h} \\ &= \frac{1}{2}(\cot\alpha + \cot\beta) \end{aligned}$$

Exercise 8

We consider vertices i and j with opposite edge vectors making angle θ (labelled β in our figure). The opposite edge vectors as labelled in our figure are $e_i = \overline{jk}$ and $e_j = \overline{ki}$

$$\begin{aligned}
\langle \nabla\phi_i, \nabla\phi_j \rangle &= \frac{e_i^\perp \cdot e_j^\perp}{4A} \\
e_i^\perp \cdot e_j^\perp &= |e_i||e_j|\cos(\pi-\theta) = -|e_i||e_j|\cos\theta \\
A &= \frac{1}{2}(e_j \times -e_i) = \frac{1}{2}(e_i \times e_j) = \frac{1}{2}|e_i||e_j|\sin(\pi-\theta) = \frac{1}{2}|e_i||e_j|\sin\theta
\end{aligned}$$

Altogether:

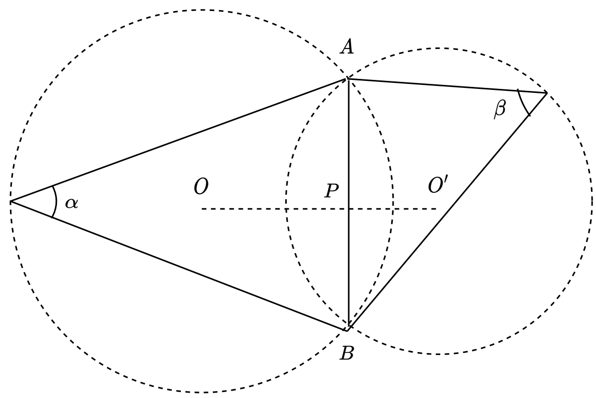
$$\langle \nabla\phi_i, \nabla\phi_j \rangle = -\frac{1}{2}\cot\theta$$

Exercise 9

In our diagram, primal mesh elements are solid, dual are dashed.

Since each dual edge connects the centers of two circles with the corresponding primal edge as a common chord, each dual edge is a perpendicular bisector of the primal edge.
This can easily be proven:

- Let the two circumcircles have circumcentres O and O' where AB is the common chord, with intersection point P.
- $OA = OB$ and $O'A = O'B$ (radii of same circle) So, $\triangle OAO'$ and $\triangle OBO'$ are congruent.
- With this we can prove $AP = BP$ and $\angle APO = \angle BPO = \frac{\pi}{2}$.
- $\therefore OO'$ is the perpendicular bisector of AB.



Using inscribed angle theorem, we get:

$\alpha = \frac{1}{2}\angle AOB$ and $\beta = \frac{1}{2}\angle AO'B$.

Using angle bisector theorem, we get:

$\angle AOO' = \frac{1}{2}\angle AOB = \alpha$ and $\angle BO'O = \frac{1}{2}\angle BO'A = \beta$.

From the previous result relating the ratio w/h to sum of cotangents we have:

$$\begin{aligned}
\frac{OO'}{AP} &= \frac{OO'}{AB/2} \\
\frac{OO'}{AB} &= \frac{1}{2}(\cot\alpha + \cot\beta) \\
\implies \frac{|e_{ij}^*|}{|e_{ij}|} &= \frac{1}{2}(\cot\alpha + \cot\beta)
\end{aligned}$$