Week 8

S15 - Ramps, Steps, and Impulses

S16 - State and Memory

Lectures

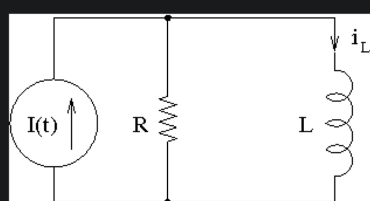
S15E1: Review: A Step Up

0 points possible (ungraded)

We now have a name u for the "unit step" function that turns on at zero:

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Now, for a circuit we have seen before,



we can say that $I(t) = I_S \; u(t)$ to specify that the current source rises at time $t = 0$ from 0 to I_S.

Suppose that $I(t)$ rises from 0 to I_S at a time $t_0 \neq 0$. In the space provided below write an algebraic expression for $I(t)$. Please be very careful to format your answer without spaces and using standard ordering conventions.

IS*u(t-t0) ✔ **Answer:** IS*u(t-t0)

Now, back to our circuit. Suppose that $I(t) = I_S \; u(t - t_0)$; and suppose that the inductor current $i_L(t) = 0$ for $t < t_0$. In the box provided below write an algebraic expression for the value of $i_L(t)$ for $t \geq t_0$.

IS*(1-e^(-(R/L)*(t-t0))) ✔ **Answer:** IS*(1-e^(-(R/L)*(t-t0)))

$$I_S \cdot \left(1 - e^{-\left(\frac{R}{L}\right)\cdot(t-t_0)}\right)$$

S15E2: Review: A Step Down

0 points possible (ungraded)

We can make a unit falling step from a unit rising step by $1 - u(t)$.

Continuing with our parallel L-R-I circuit from before, suppose now that the independent current source falls at time $t = t_0$ from I_S to 0.

We can denote this using our unit step by $I(t) = I_S \ (1 - u(t - t_0))$.

Suppose that the inductor current $i_L(t) = I_0$ for $t < t_0$. In the box provided below write an algebraic expression for the value of $i_L(t)$ for $t \geq t_0$.

I0*e^(-(R/L)*(t-t0))

✔ **Answer:** I0*e^(-(R/L)*(t-t0))

$I_0 \cdot e^{-\left(\frac{R}{L}\right)\cdot(t-t_0)}$

S15E3: Review: A Pulse is Step Up then Step Down

0 points possible (ungraded)

We can use unit steps to make a unit pulse $u(t - t_0) - u(t - t_1)$. This pulse is one unit high, starting at t_0 and ending at t_1.

Continuing with our parallel L-R-I circuit, suppose now that our independent current source puts out a pulse of height I_S that starts at time $t = 0$ and ends at time $t = T$.

We can denote this using our unit step by $I(t) = I_S \ (u(t) - u(t - T))$.

Suppose that the inductor current $i_L(t) = 0$ for $t < 0$. In the box provided below write an algebraic expression for the value of $i_L(T)$ in terms of I_S, R, L, and T.

IS*(1-e^(-T*(R/L)))

✔ **Answer:** IS*(1-e^(-(R/L)*T))

$I_S \cdot \left(1 - e^{-T\cdot\left(\frac{R}{L}\right)}\right)$

Let's define $I_T = i_L(T)$ that you just computed.

In the box provided below write an algebraic expression for the value of $i_L(t)$ for $T \leq t$ in terms of I_T, R, L, T, and t.

IT*e^(-(t-T)*(R/L))

✔ **Answer:** IT*e^(-(R/L)*(t-T))

$I_T \cdot e^{-(t-T)\cdot\left(\frac{R}{L}\right)}$

S15E4: Area

0 points possible (ungraded)

Now, let's find out what happens when the pulse becomes very short but very high. If we choose $Is = A/T$ then the area of the pulse is kept constant A, independent of the length of the pulse.

So now, our independent current source puts out a pulse of height $I_S = A/T$ that starts at time $t = 0$ and ends at time $t = T$.

Thus $I(t) = \frac{A}{T}(u(t) - u(t - T))$.

Continuing with the inductor current $i_L(t) = 0$ for $t < 0$.

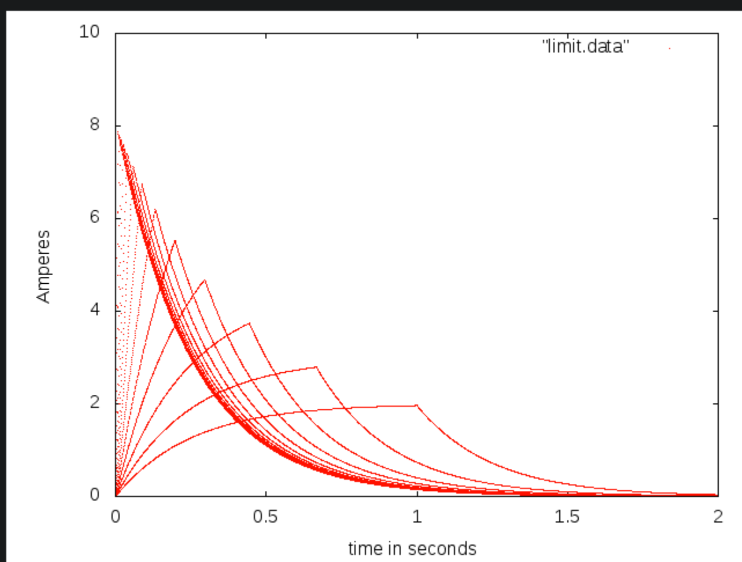
In the box provided below write an algebraic expression for the value of $\lim_{T \to 0} i_L(t)$.

A*R/L*e^(-t*R/L)

✔ **Answer:** (A*(R/L))*e^(-(R/L)*t)

$A \cdot \frac{R}{L} \cdot e^{-t \cdot \frac{R}{L}}$

To see this in operation, here is a set of response graphs for a circuit with time-constant 1/4 second, to a sequence of narrowing pulses of 1/4 second and area 2.



Because this limit exists we can invent a new function symbol representing an infinitely tall and infinitely narrow pulse. We define $\delta(t)$ as an infinitesimal pulse of area 1 at $t = 0$ and zero everywhere else. This is called an "impulse".

Mathematically, introducing impulses is a big step, because δ does not satisfy the usual requirements for a function. It is a a distribution, which generalizes the notion of a function. But we do not expect you to learn that very advanced material for use in this class.

Notice what the impulse did: it "instantaneously" changed the current through the inductor, thus setting the initial conditions for the exponential decay that follows. In fact, it incremented the current in the inductor by the area of the impulse divided by the time constant of the circuit, as seen from the inductor.

Notice also that in this problem the area is the product of current and time, hence charge: we put out an impulse whose area can be expressed in Coulombs (C).

I don't understand why there is a (A*(R/L)) multiplier before the exponent.

This post is visible to everyone.

Grove (Community TA)
2 years ago - marked as answer 2 years ago by **MIT_Lover_UA** (Staff)

During the "charging" phase the current through the inductor after a time T is given by $i_{\mathrm{L}}(T) = I_{\mathrm{S}}\left(1 - e^{-\frac{RT}{L}}\right)$.

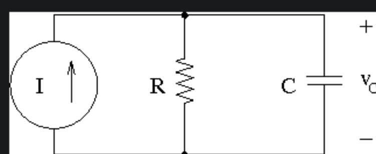
Now expand $e^{-\frac{RT}{L}}$ to the first two terms $e^{-\frac{RT}{L}} \approx 1 - \frac{RT}{L} + \ldots.$ and substitute it into the equation for $i_{\mathrm{L}}(T)$.

Note that the approximation gets better and better as time T tends to zero.

S15E5: Initial Conditions

0 points possible (ungraded)

Consider the following circuit:



Let $R = 100.0\Omega$ and $C = 10.0\mu F$. Suppose time is measured in milliseconds and we know that $I(t) = I_0 + Q \cdot \delta(t)$, where $I_0 = 10mA$, $Q = -20\mu C$, and $\delta(t)$ has the units of s^{-1} and is defined as follows:

$\delta(t) = 0$ for $t \neq 0$

$\int_{-\infty}^{t} \delta(t)\, \mathrm{d}x = u(t) \rightarrow \delta(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t}$

$\int_{-\infty}^{\infty} \delta(t)\, \mathrm{d}x = 1$

Also, assume that we measured the voltage across the capacitor at $t = -1.0$milliseconds, before the impulse is delivered: $v_C(-1.0) = 3.0$V.

What is the voltage, in Volts, across the capacitor at $t = 0_-$, a time infinitesimally before the impulse is delivered?

1.73576 ✔ **Answer:** 1.7357588823428847

What is the voltage, in Volts, across the capacitor at $t = 0_+$, a time infinitesimally after the impulse is delivered?

-0.264241 ✔ **Answer:** -0.26424111765711533

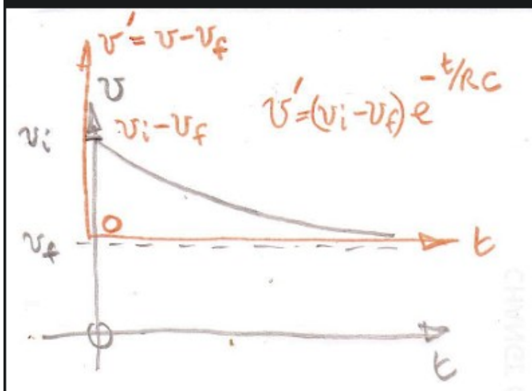
What is the voltage, in Volts, across the capacitor at $t = 1.0$ms, after the impulse is delivered?

0.534912 ✔ **Answer:** 0.5349116841303407

Grove (Community TA)
3 years ago - endorsed 3 years ago by **MIT_Lover_UA** (Staff)

The voltage acros the capacitor is $3.0\,\mathrm{V}$ so the current through the resistor is $\frac{3.0}{100} = 30\,\mathrm{mA}$.
The current source can only deliver $10\,\mathrm{mA}$ so where does the other $30 - 10 = 20\,\mathrm{mA}$ come from?
The capacitor discharging.

The simple type of decay to deal with is one from an initial value v_i to a final value of zero, $v = v_\mathrm{i}\, e^{-\frac{t}{RC}}$.
So what happens if the final value, v_f, is not zero as is the case here.



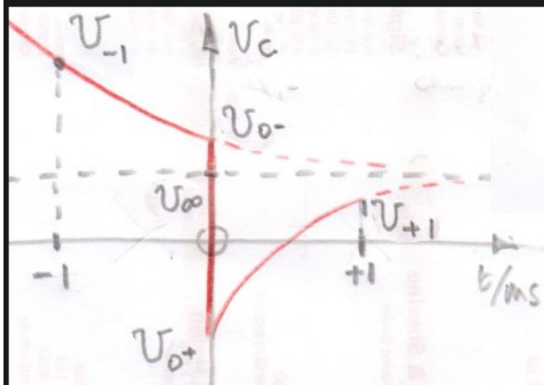
One way of dealing with such a situation is to offset the voltage axis by v_f as shown in the graph above and define a voltage $v' = v - v_\mathrm{f}$.
So the voltage v' now falls from $v_\mathrm{i} - v_\mathrm{f}$ to zero.
So what you have to figure out is the value of v_f in this example.

An equivalent procedure can also be used when the capacitor is being charged.

I am sorry that I have given the impression that it is a linear function.
The line is slightly curved and was supposed to represent an exponential function!
I have annotated my sketch graph to show this.

Grove (Community TA)
2 years ago - marked as answer 2 years ago by **MIT_Lover_UA** (Staff)

Let me summarise in a graph the whole cycle of changes to the voltage across the capacitor, v_C.



At the start with time $t < 0$ a capacitor is discharging towards a steady state voltage v_∞ which is determined by the current source current, I, and the resistance of the resistor, R.

At time $t = -1\,\mathrm{ms}$ you are given the voltage across the capacitor v_{-1}.
For Part **a** you are to find the voltage across the capacitor v_{0^-} at time $t = 0^-$ as the capacitor discharges asymptotically towards voltage v_∞.

At time $t = 0$ a quantity of charge is dumped on the capacitor and so the voltage across the capacitor changes to v_{0^+}.
You have to find that voltage and that is the answer to Part **b**.

Now the capacitor is "charging" towards a steady state voltage v_∞ which is determined by the current source current, I, and the resistance of the resistor, R. For Part **c** you are to find the voltage across the capacitor v_{+1} at time $t = +1\,\mathrm{ms}$ as the capacitor "charges" asymptotically towards voltage v_∞.

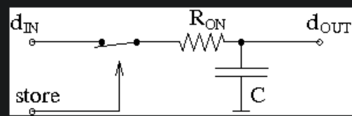
```
r = 100; c = 10*^-6; io = 10*^-3; q = -20*^-6;
(*Before the current impulse, the current through capacitor is a combination
 of current source and capacitor discharge (see 1st hint above)*)
(*vf below is the same as v∞ in the 2nd hint*)
t = 1*^-3; vi = 3; vf = io * r;
Solve[v == vf + (vi - vf) Exp[-t / (r c)], v] // N
(*Change in capacitor voltage should be
 due to the charge dumped by the current impulse*)
v0minus = 1.7357588823428844`;
v0plus = v0minus + q/c
(*After impulse the capacitor goes back to steady state*)
t = 1*^-3; vi = v0plus; vf = io * r;
Solve[v == vf + (vi - vf) Exp[-t / (r c)], v] // N
```

```
Out[]= {{v → 1.73576}}

Out[]= -0.264241

Out[]= {{v → 0.534912}}
```

S16E1: Charging and Discharging

0 points possible (ungraded)

Consider a proposal for using a capacitor as a digital memory:



Suppose that STORE is high so the switch is on. Also assume that the switch has $R_{ON} = 2000.0\Omega$ and the capacitor has a capacitance of 5.0fF.

How much time, in picoseconds, does it take to charge the capacitor from $V_{OL} = 1.0$V to $V_{OH} = 4.0$V, with $d_{IN} = 5.0$V?

13.8629 ✔ **Answer:** 13.862943611198904

Now, suppose that $d_{IN} = 1.0$V. How much time, in picoseconds, does it take to discharge the capacitor from $V_{OH} = 4.0$V to $V_{IL} = 2.0$V?

10.9861 ✔ **Answer:** 10.986122886681096

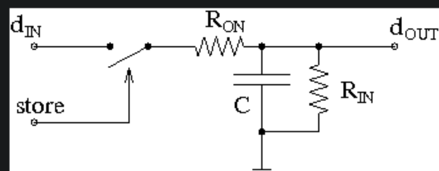
```
ron = 2000; c = 5*^-15; vol = 1; voh = 4; din = 5;
(*I just took c₁ to be zero, and also the imaginary part.
  Also remember to multiply by 10 for conversion to picosec*)
Solve[voh == vol + (din - vol) (1 - Exp[-t / (ron * c)]), t] // N
din = 1; vil = 2;
Solve[vil == din + (voh - din) Exp[-t / (ron * c)], t] // N
```

Out[∘]= $\{\{t \to 1. \times 10^{-11} \times (1.38629 + (0. + 6.28319\, i)\, c_1) \text{ if } c_1 \in \mathbb{Z}\}\}$

Out[∘]= $\{\{t \to 1. \times 10^{-11} \times (1.09861 + (0. + 6.28319\, i)\, c_1) \text{ if } c_1 \in \mathbb{Z}\}\}$

S16E2: Time to Decay

0 points possible (ungraded)



Suppose that STORE is low, so the switch is off, but now we have a leakage resistance to worry about.

The capacitor has a capacitance of 5.0fF and the leakage resistance $R_{IN} = 10.0\text{G}\Omega$.

How much time, in microseconds, does it take to discharge the capacitor such that the voltage across it goes from $V_S = 5.0$V to $V_{OH} = 4.0$V?

11.1572 ✔

```
c = 5*^-15; rin = 10*^9; vs = 5; voh = 4;
Solve[voh == vs Exp[-t / (rin * c)], t] // N
(*Parsing the nonsense below, convert to microsec*)
t = 0.00005` * (0.22314355131420976`) * 1*^6
```

Out[∘]= $\{\{t \to 0.00005 \times (0.223144 + (0. + 6.28319\, i)\, c_1) \text{ if } c_1 \in \mathbb{Z}\}\}$

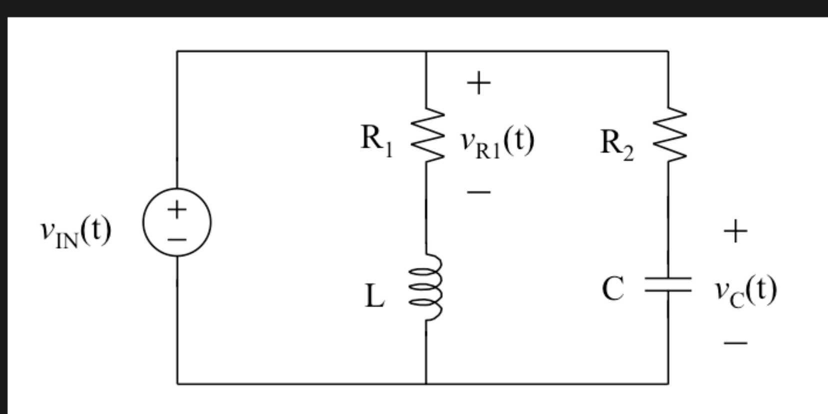
Out[∘]= 11.1572

Homework

H4P1: Impulse

5/5 points (graded)

The impulse response of a circuit is its response to a unit impulse, $\delta(t)$. Knowing the impulse response of a linear circuit is extremely valuable as we can figure out the circuit's response to an arbitrary input from it. In this problem you will find the response of a circuit when it is driven by a unit impulse.



Consider the circuit shown above in which $v_{IN}(t) = 1\delta(t)$ volt-seconds. That is, v_{IN} is a unit impulse at time t = 0. In the circuit, $L = 5mH$, $R_1 = 10\Omega$, $C = 33nF$ and $R_2 = 10k\Omega$. Note that since this circuit is driven by an impulse and there is no other source of energy, the capacitor voltage and inductor current at t = 0 will be zero.

(a) What is the value of $v_{R_1}(t)$ at $t = 0^-$ in Volts (V)?

0 ✔ **Answer:** 0

(b) What is the value of $v_{R_1}(t)$ at $t = 0^+$ in volts (V)? Hint: Recall that that $\int_{0^-}^{0^+} \delta(t)\, dt = 1$

2000 ✔ **Answer:** 2000

(c) What is the value for $v_{R_1}(t)$ at $t = 1ms$ in volts(V)? (Hint: The L-R_1 and C-R_2 branches of the circuit are decoupled.)

270.671 ✔ **Answer:** 270.6

(d) What is the value of $v_C(t)$ at $t = 0^+$ in volts (V)?

100000/33 ✔ **Answer:** 3030.3

(e) What is the value for $v_C(t)$ at $t = 1ms$ in volts (V)?

146.367 ✔ **Answer:** 146.36

Explanation:

1. The Votlage across resistor R_1 at $t = 0^-$ is calculated as $v_{r1} = R_1 \cdot i_L$, where i_L is the current through the inductor. From the problem description, we know that there are no other sources in the circuit apart from the voltage impulse source and therefore the current through the inductor before the impulse is zero. From that, we can calculate $v_{r1} = 0$ at $t = 0^-$

2. After the impulse, some energy has been transferred into the circuit. Following the analysis done during lecture sequece 15, we know that the flux linkage of the voltage source $V \cdot T = \Lambda$ is immediatly transferred to the inductor. Since $\Lambda = LI$, we can calculate the current through the inductor immediatly after the impulse to be $I = \frac{\Lambda}{L}$. Using the fact that $\int_{-\inf}^{inf} v_{IN}\, dt = 1$, then $\Lambda = 1$ and $i_L = \frac{1}{5 \cdot 10^{-3}} = 200A$. Then:

$$v_{r1} = R_1 \cdot i_L = (10\Omega) \cdot (200A) = 2000V$$

3. After the impulse, the v_{IN} is zero (a short circuit), and the inductor will dissipate its energy throught R_1. The equation describing this energy dissipation was studied before and is:

$$i_L = \frac{\Lambda}{L} \cdot e^{-t \cdot \frac{R}{L}} = (200A) \cdot e^{-1 \cdot 10^{-3} s \cdot \frac{10\Omega}{5 \cdot 10^{-3} H}} = 27.06A$$

   Voltage across the resistor is then:

$$v_{r1} = R_1 \cdot i_L = (10\Omega) \cdot (27.06A) = 270.6V$$

4. To analyze the capacitor branch it is easier to transform the voltage source in series with R_2 (remember that both branches are decoupled), to its Norton equivalent using the equations: $R_{No} = R_{Th}$ and $I_{No} = \frac{V_{th}}{R_{No}}$. The new circuit is a current impulse source with $i_{IN} = \frac{1}{R_2} v_{IN}$.

   As studied in lecture sequence 15, some charge will be transferred immediatly from the current impulse source to the capacitor. This charge is simply the area under the curve of the current source or, mathematically, $Q = \int_{-\inf}^{inf} i_{IN}\, dt = \frac{1}{R_2} = 1 \cdot 10^{-4} C$. The voltage in the capacitor after the impulse is then calculated as:

$$v_C = \frac{Q}{C} = \frac{1 \cdot 10^{-4} C}{33 \cdot 10^{-9} F} = 3030.3V$$

5. Finally, after the impulse, the current i_{IN} is zero (open circuit) and the capacitor discharges through R_2. The expression for this energy dissipation is:

$$v_C = \frac{Q}{C} \cdot e^{\frac{-t}{RC}} = (3030.3V) \cdot e^{-\frac{1 \cdot 10^{-3} s}{(10k\Omega)(33 \cdot 10^{-9})}} = 146.36V$$

In[∘]:=
```
l = 5*^-3; r1 = 10; c = 33*^-9; r2 = 10*^3;
(*We use the properties that:
   integral from 0⁻ to 0⁺ of 1δ(t)=1,
flux linkage Φ=Li and Φ=∫ v dt*)
ir1 = 1/l;
vr1 = ir1 r1
(*The capacitor and inductor branches are decoupled ODEs*)
t = 1*^-3;
Solve[v == vr1 Exp[-t * (r1 / l)], v] // N
(*In the second branch we convert the source to
  a current impulse in units of Amperes*sec. We can than use:
   integral from 0⁻ to 0⁺ of 1δ(t)=1 and
   capacitive charge Q=Cv and Q=∫ i dt*)
iIN = 1/r2;
vc = iIN/c
Solve[v == vc Exp[-t / (r2 c)], v] // N
```

Out[∘]= 2000

Out[∘]= {{v → 270.671}}

Out[∘]= $\frac{100000}{33}$

Out[∘]= {{v → 146.367}}

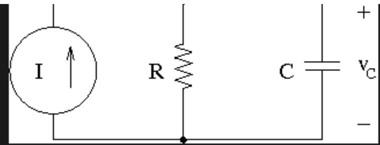
H4P2: Physiological Model

4/4 points (graded)

The electrical circuit language is a powerful language for modeling other kinds of systems, such as mechanical, chemical, thermal, and fluidic systems. Here is an example from physiology: drug delivery.

Suppose we want to understand what happens when a medical doctor injects a drug into an person. For example, the doctor may need to know how the concentration of an antibiotic will vary at a target site with time. To be effective, the concentration must be high enough for long enough to kill the invading bacteria, but not so high as to damage healthy tissues.

We can model the amount of the drug by charge and the concentration of the drug by voltage. Thus, we get a very simple model:



The interpretation is as follows:

- I models the rate of injection of the drug into the body.
- v_C models the concentration of the drug in the body.
- $1/R$ models the rate of elimination of the drug.
- C models the size of the body.

Of course, this model is very crude, but it can be elaborated. With more circuitry we could model the concentrations in the blood, the elimination in the kidneys, the concentrations in various organs, the degradation of the drug in the liver, the elimination of the degraded products by the kidneys, etc. So this can be very complicated, for another time!

If the time constants for these processes are on the order of hours, and the injection takes tens of seconds, it is apparent that the exact profile of the rate of injection does not matter. If the doctor takes 5 seconds or 10 seconds to inject the same amount of antibiotic, the profile of concentration in the blood should be about the same for times on the order of hours. Thus, an injection can be modeled as an impulse. (This is just a mental experiment: most antibiotics cannot be injected this way, they can be given intravenously, over a period of an hour, thus making a pulse rather than an impulse.)

So, let's do some numbers.

Consider the consequences of injecting $Q = 400.0$mg of cipro into a $C = 70$kg person. Assume the person is entirely made of water, which weighs 1kg/L. By the end of the injection the antibiotic is uniformly distributed in all of the body tissues.

What is the initial concentration after the injection $v_C\left(0_+\right)$, of the antibiotic in the body, in mg/L?

40/7 ✔ **Answer:** 5.7142857142857135

The half-life of the antibiotic in the person is about $t_h = 4.0$Hours. Thus, the concentration will drop to half of its initial value in that time. What is the resistance, in seconds/liter, in our model circuit, that will produce this rate of loss of concentration?

296.783 ✔ **Answer:** 296.78297984001534

If the doctor wants to reach 1/4 of the initial concentration, what is longest time, in hours, that may be allowed to elapse before a new injection is required?

8 ✔ **Answer:** 8.0

What dose, in milligrams, will be required at that time to get the concentration back to 2/3 of the initial concentration?

500/3 ✔ **Answer:** 166.66666666666666

Note that now the doctor can repeat this calculation to determine how often he must give a this new dose to maintain the concentrations in the range he believes will be effective.

Explanation:

1. If we model the injection of drug into the body as an impulse, then our circuit is a parallel RC circuit with a impulse current source. We studied these circuits in S15V14 to S15V16, where it is explained how all the charge (drug) is transferred into the capacitor at $t = 0_+$. Therefore, the initial concentration of drug in the body (equal to the voltage in the capacitor) is:

$$v_C\left(0_+\right) = \frac{Q}{C} = \frac{400mg}{70L} = 5.714\frac{mg}{L}$$

2. In the RC circuit we will see an exponential decay of drug concentration (i.e. capacitor voltage) with a time constant of $\tau = R \cdot C$. The expression for drug concentration V_c will be:

$$v_C(t) = \frac{Q}{C} \cdot e^{-\frac{t}{R \cdot C}}$$

We are given that the exponential term is one half when $t = 4.0h = 14,400sec$. Therefore we can solve for R as:

$$-\frac{14,400s}{R \cdot C} = ln\left(\frac{1}{2}\right)$$

$$R = -\frac{1}{C} \cdot \frac{14,400s}{ln\left(\frac{1}{2}\right)} = -\frac{1}{70L} \cdot \frac{14,400s}{ln\left(\frac{1}{2}\right)} = 296.78\frac{sec}{L}$$

3. The new injection should be given whenever the drug concentration v_C reaches 1/4 of the original. An easy way to see the solution to this problem is to remember the definition of half-life: the time it takes concentration to reduce to half starting at any point in time. Therefore, if we know that it took 4 hours for concentration to go from 1 to 1/2, then it will take 4 more hours (one more half-life) to go from 1/2 to 1/4 in a total of $t = 8.0$ hours.

   This can also be found by solving for t in the expression given above for v_C and setting the concentration value to one fourth of the original. That means that the exponential term is 1/4 and we get:

$$-\frac{t}{R \cdot C} = ln\left(\frac{1}{4}\right)$$

   Then:

$$t = -R \cdot C \cdot ln\left(\frac{1}{4}\right) = -(70L) \cdot (296.78sec/L) \cdot ln\left(\frac{1}{4}\right) = 28,800sec = 8.0h$$

4. The concentration at that time will be $v_C\left(t_{\frac{1}{4}}\right) = 1.4285$ mg/L. We need the drug concentration to increase to 3.8095 mg/L, therefore, the new injection should contain enough drug (charge) to increase the voltage (concentration) to that value. Since charge will be transferred immediatly to the capacitor, we can compute

$$\frac{Q_{new}}{C} + v_C\left(t_{\frac{1}{4}}\right) = 3.8095mg/L$$

   Where $t_{\frac{1}{4}}$ is the time it took the concentration to reach 1/4 of the original (i.e. the solution to part (c)). Solving for Q_{new} we get:

$$Q_{new} = (70L) \cdot (3.8095mg/L) - (70L) \cdot (1.4285mg/L) = 166.67mg$$

In[•]:=
```
q = 400; c = 70;
vc = q / c
th = 4;
Solve[vc/2 == vc Exp[-th / (r c)], r]
(*Parsing the output, taking constant to be 0,
and also converting hours to seconds*)
r = 2/(35 Log[2]);
r * 3600 // N
Solve[vc/4 == vc Exp[-t / (r c)], t]
(*We need to find concentration, modeled by charge,
so we model this dose as a voltage impulse that takes us from vold to vnew*)
vnew = 2/3 vc; vold = vc/4;
(vnew - vold) * c
```

Out[•]= $\frac{40}{7}$

Out[•]= $\left\{\left\{r \to \boxed{\frac{2}{35 \times (2\, i\, \pi\, c_1 + \text{Log}[2])} \quad \text{if} \quad c_1 \in \mathbb{Z}}\right\}\right\}$

Out[•]= 296.783

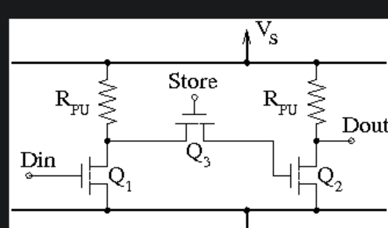
Out[•]= $\left\{\left\{t \to \boxed{4 \times \left(2 + \frac{2\, i\, \pi\, c_1}{\text{Log}[2]}\right) \quad \text{if} \quad c_1 \in \mathbb{Z}}\right\}\right\}$

Out[•]= $\frac{500}{3}$

H4P3: Memory

5/5 points (graded)

One possible implementation of one bit of a digital memory might be something like the circuit below. The memory element here is the gate-to-source capacitance of transistor Q_2. Since the capacitance is leaky, this can hold a value only for a time determined by the time constant of the capacitance and the leakage resistance. Thus, such a memory has to be "refreshed" periodically, by reading out its value and rewriting that value.



We assume an "ancient" 1μm technology satisfying the static discipline:

$$V_S = 5.0\text{V}, V_{OH} = 3.5\text{V}, V_{IH} = 3.0\text{V}, V_{IL} = 0.9\text{V}, V_{OL} = 0.5\text{V}$$

The gate-source capacitance $C_{GS} = 3.5\text{fF}$. Furthermore, assume the following component parameters $V_T = 1.0\text{V}$, $R_{ON} = 2100.0\Omega$, $R_{OFF} = 110.0\text{M}\Omega$ and $R_{PU} = 10.0\text{k}\Omega$.

The specifications for this logic family say that we have to refresh this memory every 64ms. So, if we charge the gate capacitance to V_{OH} it will take more than 64ms to decay to V_{IH}.

If it takes 0.1s to decay that far, what must be the parasitic resistance, in TerraOhms from the gate to the source of Q_2? Note: (1 T$\Omega = 10^{12}\Omega$).

185.347 ✔ **Answer:** 185.34740556088227

On a chip, a MOSFET is physically symmetrical: the source and the drain are interchangeable. For the purposes of the formulae that describe an n-channel MOSFET the source is just the electrode with the lowest potential an Calculator one with the highest potential. (That is, the voltage from the drain to the source v_{DS} is positive.) For Q_1 and Q_2 this is not an issue, but for Q_3 sometimes we are charging the gate of Q_2, to store a "0", and sometimes we are discharging it, to store a "1". (Note that there is an inversion on input and output.)

When D_{IN} is low Q_1 turns off and its drain goes high. What is the maximum value of voltage on the drain of Q_1, in Volts?

5 ✔ **Answer:** 4.999545495864012

Now, suppose the drain of Q_1 is high, as above, and the store line is held at the same voltage as the drain of Q_1. What is the maximum voltage, in Volts, that the gate of Q_2 can be charged to? Note, this value must be larger than V_{OH} to satisfy the static discipline.

3.8 ✔ **Answer:** 3.9995454958640124

The drain of Q_1 is still high and the gate of Q_2 is at V_{OL}. A STORE pulse comes in and turns on Q_3. How long must the STORE pulse be on, in picoseconds, to charge the gate capacitance of Q_2 to V_{OH}?

46.5313 ✔ **Answer:** 46.52

Note: In fact, the SR model is not very accurate for Q_3 in this circuit because Q_3 is in saturation, but use the SR model anyway, for simplicity.

Now, suppose D_{IN} is high, the drain of Q_1 is low, and the gate of Q_2 is at V_{OH}. A STORE pulse comes in and turns on Q_3. How long must the STORE pulse be on, in picoseconds, to discharge the gate capacitance of Q_2 to V_{IL}?

59.1028 ✔ **Answer:** 59.1

Explanation:

1. The capacitor will discharge through the parasitic resistance that we are asked to find. It is an RC circuit with the voltage described by the following equation:

$$v_C(t) = V_{OH} \cdot e^{-\frac{t}{R \cdot C}}$$

We can solve for R by setting $v_C = V_{IH} = 3V$ and using the known values for V_{OH} and C.

$$R = -\frac{1}{C} \cdot \frac{t}{ln\left(\frac{V_{IH}}{V_{OH}}\right)} = -\frac{1}{(3.5 \cdot 10^{-15}F)} \cdot \frac{0.1s}{ln\left(\frac{3.0V}{3.5V}\right)} = 185.34 \cdot 10^{12}\Omega = 185.34T\Omega$$

2. When D_{IN} is zero, then the MOSFET behaves as a resistor with $R_{off} = 110.0M\Omega$. Then, we have a voltage divider for the supply voltage Vs. The drain voltage of the MOSFET Q_1 is:

$$V_D = \frac{R_{off}}{R_{off} + R_{PU}} \cdot V_s = \frac{110.0 \cdot 10^6\Omega}{110.0 \cdot 10^6\Omega + 16 \cdot 10^3\Omega} \cdot 5V = 4.9995V$$

3. To allow the MOSFET Q_3 to be turned on, we need its gate-source voltage to be greater than the threshold, i.e. $V_{gs_3} > V_T$. In this case, the drain of Q3 is the same as the drain of Q_1, and therefore the gate-source voltage can be

calculated as $V_{gs_3} = V_{store} - V_{gs_2}$. The subscript denote the MOSFET to which the variable referes to. We are given that $V_{store} = 4.9995V$ so that solving for V_{gs_2}, we get $V_{gs_2} = V_{store} - V_{gs_3}$. The maximum value of V_{gs_2} occurs at the minimum V_{gs_3} given by V_T, therefore:

$$V_{gs_2} \leq V_{store} - V_T = 3.9995V$$

4. The gate capacitance will be charged by the supply voltage. However, the resistors R_{PU} and R_{on} (due to the MOSFET Q_3 are in series between the supply and the capacitor. This is a RC circuit again, with the capacitor voltage given by the expression:

$$v_C\,(t) = V_s + (V_{OL} - V_s) \cdot e^{-\frac{t}{R \cdot C}}$$

We are asked to find the time when $v_C\,(t) = V_{OH}$, so that we need to solve for t from the above equation. This gives:

$$t = -R \cdot C \cdot ln\left(\frac{V_{OH} - V_s}{V_{OL} - V_s}\right)$$

Using the known values for $R = R_{PU} + R_{on}$, C, V_{OH} and V_{OL}, we get:

$$t = -\,(2100.0 + 10000.0\Omega) \cdot (3.5 \cdot 10^{-15}) \cdot ln\left(\frac{-1.5V}{-4.5V}\right) = 46.52ps$$

5. To solve this problem, we can create a Thevenin equivalent at the capacitors terminals to simplify the analysis. The Thevenin voltage is simply a voltage divider:

$$V_{th} = \frac{R_{on}}{R_{PU} + R_{on}} \cdot V_s = \frac{2100.0\Omega}{2100.0\Omega + 10000.0\Omega} \cdot 5.0V = 0.87V$$

The Thevenin resistace is:

$$R_{th} = \frac{R_{on} \cdot R_{PU}}{R_{PU} + R_{on}} + R_{on} = 3835.54\Omega$$

Then, we have an RC circuit with R = R_{th} and V = V_{th}. The equation for the capacitor voltage is then:

$$v_C\,(t) = V_{th} + (V_{OH} - V_{th}) \cdot e^{-\frac{t}{R_{th}C}}$$

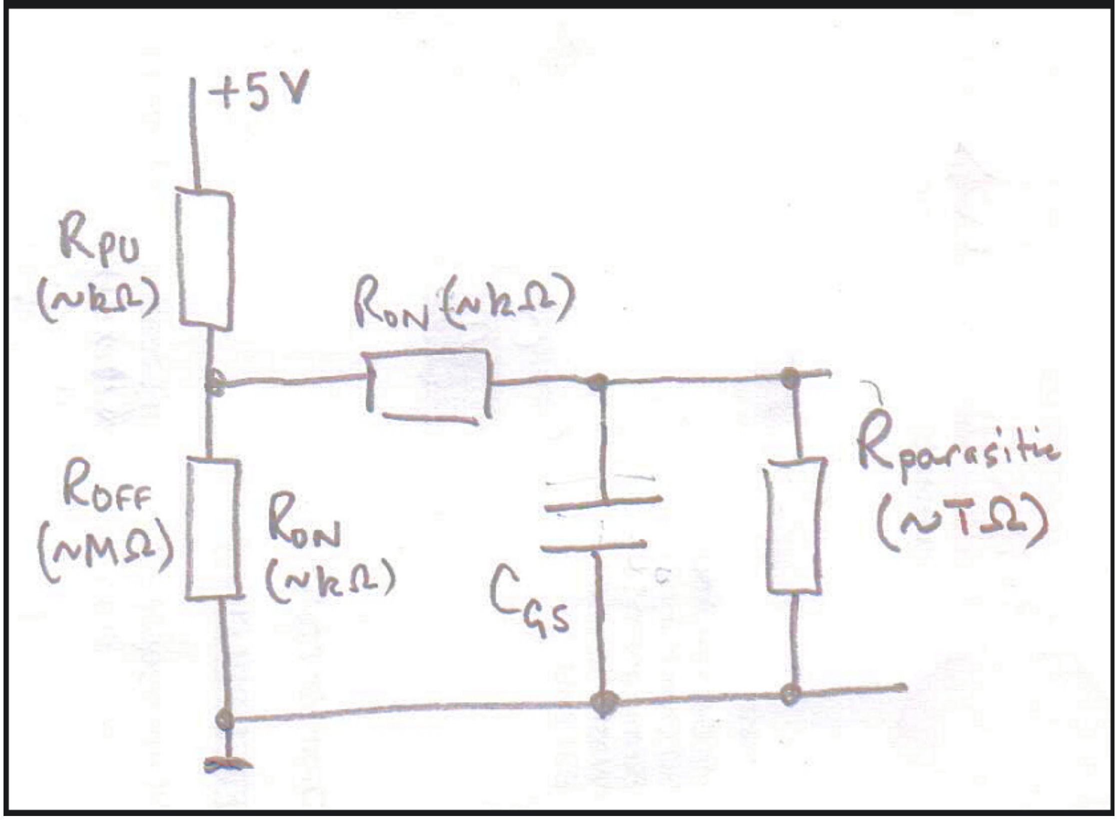
Solving for the time when $v_C\,(t) = V_{IL}$ we get:

$$t = -R_{th} \cdot C \cdot ln\left(\frac{V_{IL} - V_{th}}{V_{OH} - V_{th}}\right) = -\,(3835.54\Omega) \cdot (3.5fF) \cdot ln\left(\frac{0.9V - 0.87V}{3.5V - 0.87V}\right) = 59.1ps$$

Grove (Community TA)
3 years ago

For part **4** there is a voltage source and two resistors (R_{OFF} is very large compared with the other two resistors) to deal with.
This combination, reduced to V_{TH} and R_{TH}, **charges** the capacitor from V_{OL} to V_{OH} with the maximum possible voltage being V_{TH}.
This is where I would convert this to a charging from $V_{\mathrm{OL}} - V_{\mathrm{OL}} = 0$ to $V_{\mathrm{OH}} - V_{\mathrm{OL}}$ with the maximum voltage being $V_{\mathrm{TH}} - V_{\mathrm{OL}}$, ie offset the voltage scale by V_{OL}.

For part **5** there is a voltage source and three resistors (R_{ON} is comparable with the other two resistors) to deal with.
This combination, reduced to V_{TH} and R_{TH}, **discharges** the capacitor from V_{OH} to V_{IL} with the minimum possible voltage being V_{TH}.
This is where I would convert this to a discharging from $V_{\mathrm{OH}} - V_{\mathrm{TH}}$ to $V_{\mathrm{IL}} - V_{\mathrm{TH}}$ with the minimum voltage being $V_{\mathrm{TH}} - V_{\mathrm{TH}} = 0$, ie offset the voltage scale by V_{TH}.



In[ ]:=
```
vs = 5; voh = 3.5; vih = 3; vil = 0.9; vol = 0.5;
cgs = 3.5*^-15; vt = 1; ron = 2100; roff = 110*^6; rpu = 10*^3;

t = 0.1;
(*Remember to convert Ohms to TerraOhms*)
Solve[vih == voh Exp[-t / (r cgs)], r]
Clear[t]
```

$$rth = ron + \frac{1}{1/rpu + 1/roff};$$

$$vth = \frac{roff}{rpu + roff} * vs;$$

```
(*Charge from V_OL → V_OH with max voltage V_TH.
   Remember to convert to picosec*)
Solve[voh == vol + (vth - vol) (1 - Exp[-t / (rth cgs)]), t]
```

$$rth = ron + \frac{1}{1/rpu + 1/ron};$$

$$vth = \frac{ron}{rpu + ron} * vs;$$

```
(*Now discharge from V_OH → V_IL with min voltage V_TH.
   Remember to convert to picosec*)
Solve[vil == vth + (voh - vth) Exp[-t / (rth cgs)], t]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[ ]= $\{\{r \to 1.85347 \times 10^{14}\}\}$

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[ ]= $\{\{t \to 4.65313 \times 10^{-11}\}\}$

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[ ]= $\{\{t \to 5.91028 \times 10^{-11}\}\}$

Lab

Lab 4

20/20 points (graded)

In this lab we'll be exploring how linear circuits respond to changes in their inputs. You may find it useful to review the first three sections of Chapter 10 in the text.

Figure 1 below shows the circuit we'll be using to explore the response of two linear devices hooked in series with the input driven by a voltage source that has a step from $0V$ to $1V$ at $t = 0$.

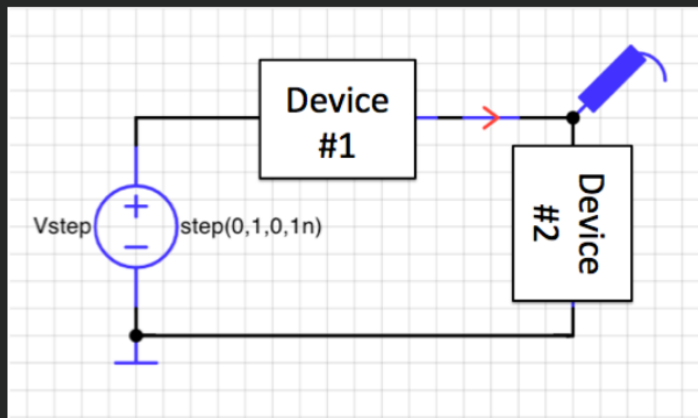


Figure 1. Circuit for testing step response

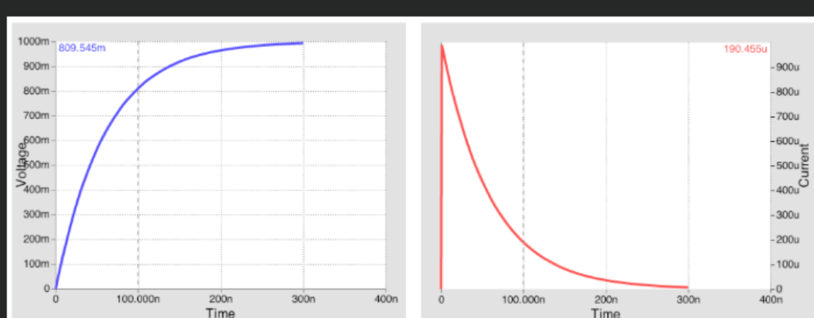
The circuit includes probes for measuring the voltage across Device #2 and the current entering the "+" terminal of Device #2. We'll be asking you determine the type and value for each of the two devices based on the voltage and current plots shown below.

For each of the voltage and current plot pairs shown below, give the component type (R, C, L) and value in the appropriate units (ohm, farads, henries) for each device. One of the two devices will always be a $1k\Omega$ resistor; it could be in either position. The other device is a resistor (R), capacitor (C) or inductor (L) of unknown value.

You may find it helpful to run experiments on your own copy of the test circuit in the circuit sandbox, which can be found in the Overview section of the website. Remember to click CHECK at the bottom of that page in order to save your schematic between visits. If you include both the current and voltage probes, both curves will appear in the same plot and may overlay each other. To get separate plots like below, you'll need to insert the probes one at a time.

The step response for series connections of RC and RL circuits is discussed in Chapter 10. For series CR and LR circuits, you'll need to complete (or find!) a similar derivation in order to have the formulas you'll need to compute the unknown component values. Looking at the shape of the voltage and current curves is enough to determine the configuration of the series circuit given that one component is known to be a resistor. To determine the value of the other component, use the measurements of (t, v) or (t, i) shown on the plots to solve the formula for the component value, remembering that the other component is a $1k\Omega$ resistor.

Voltage and current plots for Circuit 1



Type of Device #1 (R, C, L):

R ✔ **Answer:** R

Component value for Device #1:

1e3 ✔ **Answer:** 1000

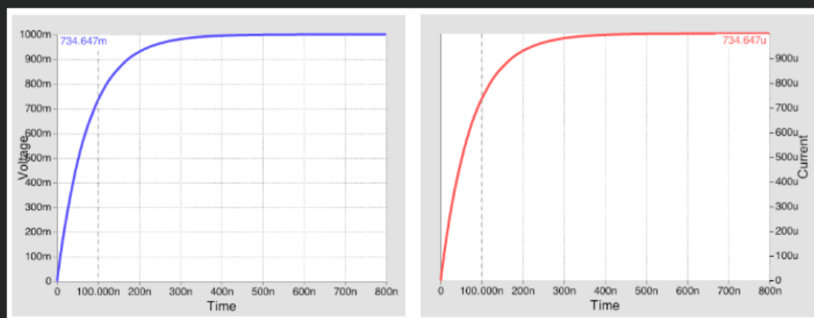
Type of Device #2 (R, C, L):

C ✔ **Answer:** C

Component value for Device #2:

6.03013e-11 ✔ **Answer:** 60e-12

Voltage and current plots for Circuit 2



Type of Device #1 (R, C, L):

L ✔ **Answer:** L

Component value for Device #1:

0.0000753753 ✔ **Answer:** 75e-6

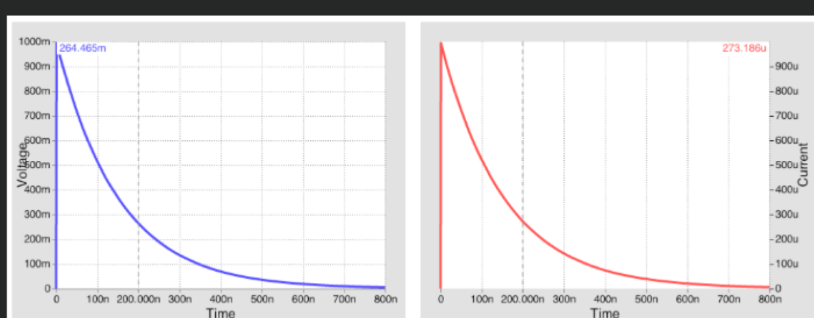
Type of Device #2 (R, C, L):

R ✔ **Answer:** R

Component value for Device #2:

1e3 ✔ **Answer:** 1000

Voltage and current plots for Circuit 3



Type of Device #1 (R, C, L):

C ✔ **Answer:** C

Component value for Device #1:

1.50371e-10 ✔ **Answer:** 150e-12

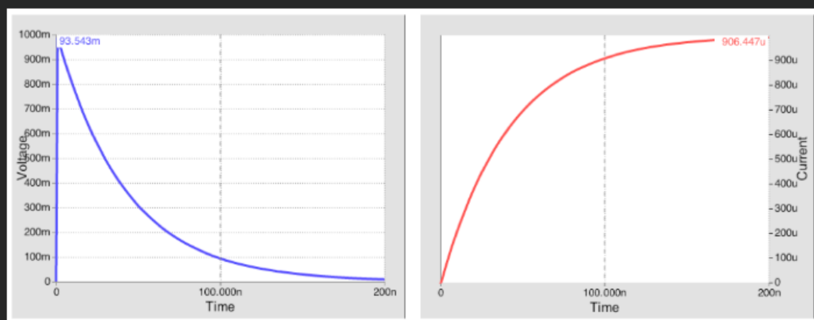
Type of Device #2 (R, C, L):

R ✔ **Answer:** R

Component value for Device #2:

1e3 ✔ **Answer:** 1000

Voltage and current plots for Circuit 4



Type of Device #1 (R, C, L):

R ✔ **Answer:** R

Component value for Device #1:

1e3 ✔ **Answer:** 1000

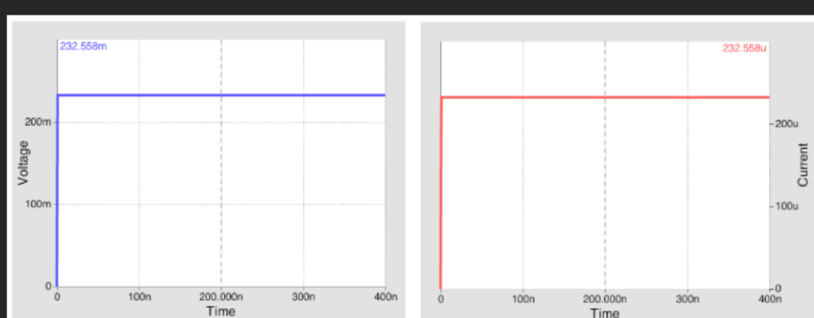
Type of Device #2 (R, C, L):

L ✔ **Answer:** L

Component value for Device #2:

0.000042206 ✔ **Answer:** 42e-6

Voltage and current plots for Circuit 5



Type of Device #1 (R, C, L):

R ✔ **Answer:** R

Component value for Device #1:

3300 ✔ **Answer:** 3300

Type of Device #2 (R, C, L):

R ✔ **Answer:** R

Component value for Device #2:

1e3 ✔ **Answer:** 1000

Explanation:
1. Circuit 1 voltage increases slowly from zero towards a constant value of $1V$, while current increases rapidly to $1mA$ and then decays to zero. We know that the circuit is composed of two elements in series, one of them being a $1k\Omega$ resistor. Notice that for the plots to be true, we need the second element to behave as a short circuit (zero voltage) at $t = 0$ and an open circuit (zero current) as $t \to \infty$. This is exactly the behavior of a capacitor. Also, notice that the plots show the measured voltage of Device 2, and since the plot starts with zero voltage, we know that Device 1 is a $1k\Omega$ resistor and Device 2 is a capacitor.
The capacitor value can be determined with the information given in the plot. We see that a certain voltage is obtained when time reaches $100ns$. We can solve for the time constant as:

$$v_C = V_s \cdot (1 - e^{-\frac{t}{\tau}})$$

Then:

$$\tau = \frac{-t}{ln\,(1 - \frac{v_C}{V_s})}$$

Plugging in the values from the plot, we get:

$$\tau = \frac{-100 \times 10^{-9}s}{ln\,(1 - \frac{0.809V}{1V})} = 6.03 \times 10^{-8}s$$

Since $\tau = RC$, we can solve for C, which equals: $60 \times 10^{-12} = 60pF$
2. Circuit 2 shows a plot of voltage slowly increasing from zero to a constant value of $1V$ and current also increasing from zero to a value of $1mA$. It might seem that since voltage is increasing slowly, the circuit contains a capacitor, but this would be inconsistant with the current plot. In fact, since the elements are in series, the current is independent of which element is Device 1 and which element is Device 2, but that is not true for the voltage. The current plot requires the second element to behave as an open circuit (zero current) at $t = 0$ and that can only be achieved with an inductor. Since the inductor behaves as an open circuit at $t = 0$, it will have $1V$ across it and will slowly decay towards zero as $t \to \infty$. Notice, however, that the resistor voltage is $v_R = 1V - v_L$ and that is precisely what the voltage plot is showing. We can conclude then that Device 1 is an inductor and Device 2 is a $1k\Omega$ resistor.
The inductor value can be determined with the information given in the plot. We see that a certain value v_L is obtained when time reaches $100ns$. We can solve for the time constant as:

$$v_L = V_s \cdot (1 - e^{-\frac{t}{\tau}})$$

Then:

$$\tau = \frac{-t}{ln\,(1 - \frac{V_L}{V_s})}$$

Plugging in the values from the plot, we get:

$$\tau = \frac{-100 \times 10^{-9}s}{ln\,(1 - \frac{0.734V}{1V})} = 7.55 \times 10^{-8}s$$

Since $\tau = \frac{L}{R}$, we find that: $L = 75.5 \times 10^{-6} = 75.5\mu H$
3. Circuit 3 shows two decaying plots for voltage and current. Again, since the elements are in series, it is a good idea to establish the requirements for the second device looking at the current (because it is independent of device location). In this case, we see that the second element should behave as a short circuit (zero resistance) at $t = 0$ and as an open circuit (zero current) as $t \to \infty$. This is the behavior of a capacitor. Since voltage is decaying, we know we are not measuring it in the capacitor (since its voltage must increase as described while analyzing Circuit 1). However, the voltage across the resistor $v_R = 1V - v_C$, is consistent with the voltage plot. Therefore, Device 1 is a capacitor and Device 2 is a $1k\Omega$ resistor.
The value of the capacitor can be determined with the information given in the plot. We see that a certain voltage is obtained when time reaches $200ns$. We can solve for the time constant as follows:

$$v_C = V_s \cdot e^{-\frac{t}{\tau}}$$

Then:

$$\tau = \frac{-t}{ln\left(\frac{v_C}{V_s}\right)}$$

Plugging in the values from the plot, we get:

$$\tau = \frac{-200 \times 10^{-9} s}{ln\left(\frac{0.264V}{1V}\right)} = 1.5 \times 10^{-7} s$$

Since $\tau = RC$, we find that: $C = 150 \times 10^{-12} = 150pF$.

4. Circuit 4 shows a decreasing plot for voltage and an increasing plot for current. Starting from the current plot, we know that there is an inductor in the circuit since initially the second element should behave as an open circuit. From the voltage plot, we see that Device 2 should be an inductor for the measured voltage to start at the supply value $(1V)$ at $t = 0$ implying the element behaves as an open circuit, and then decaying to zero implying the element behaves as a short circuit as $t \to \infty$. Therefore Device 1 is a $1k\Omega$ resistor and Device 2 is an inductor.

The inductor value can be determined with the information given in the plot. We see that a certain voltage is obtained when time reaches $100ns$. We can solve for the time constant as follows:

$$v_L = V_s \cdot e^{-\frac{t}{\tau}}$$

Then:

$$\tau = \frac{-t}{ln\left(\frac{v_L}{V_s}\right)}$$

Plugging in the values from the plot, we get:

$$\tau = \frac{-100 \times 10^{-9} s}{ln\left(\frac{0.09354V}{1V}\right)} = 4.24 \times 10^{-8} s$$

Since $\tau = \frac{L}{R}$ we find that: $L = 42 \times 10^{-6} = 42\mu H$

5. Circuit 5 shows a step transition for voltage and current when the source is turned on. Therefore, there is no capacitor nor inductor in the circuit, and we simply have two resistors. The total resistance should be $\frac{V}{I} = \frac{1V}{0.232mA} = 4.3k\Omega$. So that one resistor has a resistance of $1k\Omega$ and the other one of $3.3k\Omega$. To find their position in the circuit, we apply a voltage divider equation for both. The voltage across the $1k\Omega$ resistor is:

$$V_{1k\Omega} = \frac{1k\Omega}{4.3k\Omega}(1V) = 232.55mV$$

The voltage across the $3.3k\Omega$ resistor is:

$$V_{3.3k\Omega} = \frac{3.3k\Omega}{4.3k\Omega}(1V) = 767.44mV$$

Since the measured voltage is $232.55mV$, Device 1 is a $3.3k\Omega$ resistor and Device 2 (across which we measure the voltage) is a $1k\Omega$ resistor.

In[ ]:=
```
(*Circuit 1: RC*)
r = 1*^3; v = 809.545*^-3; t = 100*^-9; vf = 1000*^-3;
Solve[v == vf (1 - Exp[-t / (r c)]), c]

(*Circuit 2: LR*)
r = 1*^3; v = 734.647*^-3; t = 100*^-9; vf = 1000*^-3;
Solve[v == vf (1 - Exp[-t * (r / l)]), l]

(*Circuit 3: CR*)
r = 1*^3; v = 264.465*^-3; t = 200*^-9; vf = 1000*^-3;
Solve[v == vf Exp[-t / (r c)], c]

(*Circuit 4: RL*)
r = 1*^3; v = 93.543*^-3; t = 100*^-9; vf = 1000*^-3;
Solve[v == vf Exp[-t * (r / l)], l]

(*Circuit 5: Voltage divider*)
r2 = 1*^3; v = 1; v2 = 232.558*^-3; i = 232.558*^-6;
Solve[v == i (r1 + r2), r1]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[ ]= $\{\{c \to 6.03013 \times 10^{-11}\}\}$

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[ ]= $\{\{l \to 0.0000753753\}\}$

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[ ]= $\{\{c \to 1.50371 \times 10^{-10}\}\}$

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[ ]= $\{\{l \to 0.000042206\}\}$

Out[ ]= $\{\{r1 \to 3300.\}\}$