Week 10

S19 - Sinusoidal Steady State
S20 - The Impedance Model

Lectures

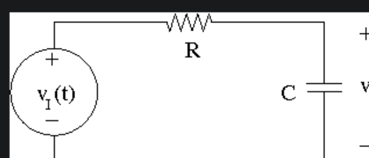
S19E1: Trigonometry Isn't So Bad

0 points possible (ungraded)

We are presented with a familiar first-order circuit: a capacitor driven by a Thevenin source. Now we have a sinusoidal drive that starts at $t = 0$. so the differential equation is

$$RC\frac{dv(t)}{dt} + v(t) = v_I(t) = V_i \cos(\omega t)\, u(t)$$

and we are given the initial state $v(0_-) = 0$.



We know how to solve this problem in the abstract.

1. Find a particular solution, $v_P(t)$.
2. Form the general solution to the homogeneous equation $v_H(t)$.
3. Form the total solution $v(t) = v_P(t) + v_H(t)$.
4. Use the initial conditions to solve for the unknown coefficients in the total solution.

Today we concentrate on finding the particular solution for $t > 0$, where the differential equation is:

$$RC\frac{dv_P(t)}{dt} + v_P(t) = V_i \cos(\omega t)$$

(Notice that we are using an upper-case name with a lower-case subscript for amplitude of the sinusoid.)

The lecturer threw up his hands in disgust after three tries with trigonometric functions. He will go on to exponentials, which is really a better idea. But I think we should try one more possibility... Let's try

$$v_P(t) = A\cos(\omega t) + B\sin(\omega t)$$

If you substitute this into the differential equation you will find that you can solve for the amplitudes A and B fairly easily. (Hint: You will need the fact that $\sin(\omega t)$ and $\cos(\omega t)$ cannot both be zero at the same time.)

In the spaces provided below write algebraic expressions for these amplitudes. (Since we have no easy way for you to enter the Greek letter ω, please use the Roman letter w for the angular frequency.)

$A =$

(1/(1+(R*C*w)^2))*Vi

Answer: (1/(1+(R*C*w)^2))*Vi

$$\left(\frac{1}{1+(R\cdot C\cdot\omega)^2}\right)\cdot V_i$$

$B =$

((R*C*w)/(1+(R*C*w)^2))*Vi

Answer: ((R*C*w)/(1+(R*C*w)^2))*Vi

$$\left(\frac{R\cdot C\cdot\omega}{1+(R\cdot C\cdot\omega)^2}\right)\cdot V_i$$

This wasn't so bad... The big problem here is that

$$A\cos(\omega t)+B\sin(\omega t)=C\cos(\omega t+\phi)$$

and we really want C and ϕ, but the trigonometric computation of these from A and B is pretty nasty.

We will find that exponentials are much nicer and easier to work with.

Solution:

We can plug in our particular solution into our differential equation from KCL to solve for A and B. Using KCL:

$$RC\frac{dv_P(t)}{dt}+v_P(t)=V_i\cos(\omega t)$$

$$RC(-A\omega\sin(\omega t)+B\omega\cos(\omega t))+A\cos(\omega t)+B\sin(\omega t)=V_i\cos(\omega t)$$

$$(-ARC\omega+B)\sin(\omega t)+(BRC\omega+A)\cos(\omega t)=V_i\cos(\omega t)$$

$$\rightarrow(-ARC\omega+B)=0,(BRC\omega+A)=V_i$$

We have two equations, and two unknowns. Skipping some algebra, we get solutions:

$$A=\frac{V_i}{(RC\omega)^2+1}$$

$$B=\frac{V_i\times RC\omega}{(RC\omega)^2+1}$$

S19E2: Exponentials are Nice

0 points possible (ungraded)

We are working on finding the particular solution for the first-order linear constant-coefficient ordinary differential equation:

$$RC\frac{dv_P(t)}{dt} + v_P(t) = V_i \cos(\omega t)$$

The new idea is to find the particular solution for the related equation

$$RC\frac{dv_{PS}(t)}{dt} + v_{PS}(t) = V_i e^{st}$$

because we expect that this solution is easy to get and it will help to find the particular solution for the original equation.

If you try $v_{PS}(t) = V_p e^{st}$ you will find that it will work; you can solve for V_p. In the space provided below write an algebraic expression for V_p in terms of V_i, s, R and C.

Vi/(R*C*s + 1)

✔ **Answer:** (1/(1+R*C*s))*Vi

$$\frac{V_i}{R \cdot C \cdot s + 1}$$

Solution:

We can plug in our particular solution into our differential equation to solve for A and B:

$$RC\frac{dv_P(t)}{dt} + v_P(t) = V_i e^{st}$$

$$RC \times sV_p e^{st} + V_p e^{st} = V_i e^{st}$$

$$V_p = \frac{V_i}{1 + RCs}$$

We got to our answer, but with much less algebra!

S19E3: Complex Numbers

0 points possible (ungraded)

In working with complex exponentials we need some simple rules and short cuts for doing the algebra.

Remember that a complex number z is a vector in the 2-dimensional plane. It has a length (also called "magnitude") $|z|$ and an angle $\angle(z)$ measured counterclockwise from the real axis. You need to know the following facts:

- $\mathrm{Re}(a + jb) = a; \quad \mathrm{Im}(a + jb) = b$
- $|a + jb| = \sqrt{a^2 + b^2}\,; \quad \angle(a + jb) = \mathrm{atan2}(b, a)$

  (atan2(y,x) is the same operation as $\arctan\left(\frac{y}{x}\right)$, except our solution is from $-\pi$ to π to retain phase information. For more information see here).

- $|z_1 z_2| = |z_1||z_2|\,; \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\angle(z_1 z_2) = \angle(z_1) + \angle(z_2); \quad \angle\left(\frac{z_1}{z_2}\right) = \angle(z_1) - \angle(z_2)$

- $|e^{j\theta}| = 1; \quad \angle(e^{j\theta}) = \theta$

Now let's practice on the problem before us. We have a complex particular solution:

$$v_{PS}(t) = V_i \frac{1}{1+j\omega RC} e^{j\omega t}$$

Let's get its magnitude. The complex number before us is a product of factors, so its magnitude is the product of the magnitude of the factors. V_i is a real number, so its magnitude is its absolute value. The second factor is a quotient, so its magnitude is the quotient of the magnitudes of the numerator and the denominator. The third factor is a unit vector; so its magnitude is 1.

What is the magnitude of v_{PS}? Express your result in terms of V_i, ω, R and C. (Please use the Roman letter w for the angular frequency.)

abs(Vi)/sqrt(1+(w*R*C)^2) ✔ **Answer:** abs(Vi)/sqrt(1+(w*R*C)^2)

$$\frac{|V_i|}{\sqrt{1+(\omega \cdot R \cdot C)^2}}$$

Now let's get the angle. The angle of the product is the sum of the angles. The angle of the first factor, the real number V_i is just either 0 if it is positive or π if it is negative. Assume V_i is positive here. The angle of the second factor is just the difference of the angle of the numerator and the angle of the denominator. The angle of the third factor is the imaginary part of the exponent.

What is the angle of v_{PS}? Express your result in terms of V_i, ω, t, R and C. (Please use the Roman letter w for the angular frequency.)

w*t-arctan(w*R*C) ✔ **Answer:** w*t + arctan(-w*R*C)

$$\omega \cdot t - \arctan(\omega \cdot R \cdot C)$$

Solution:

Magnitude

For this part, we can just look at the non-exponential part of this expression because the exponent has an imaginary argument. This is $V_i \times \frac{1}{1+j\omega RC}$. We can get the magnitude of this part:

$$\frac{V_i}{\sqrt{1+(\omega RC)^2}}$$

Angle

We know we are looking for the angle in $v_{PS} = Ae^{j\theta}$, where the angle is the argument of the exponent. We know part of this comes from the ωt in the exponent, and the other part comes from the complex term $\frac{1}{1+j\omega RC}$:

$$\frac{1}{1+j\omega RC} = \frac{1-j\omega RC}{1-(\omega RC)^2}$$

We also know that if we have a complex number $a+bi = ce^{j\theta}$, we can find the angle as $\tan(\theta) = \frac{b}{a}$. We can find the total angle then as:

$$\theta = \omega t + \tan^{-1}(-\omega RC) = \omega t - \tan^{-1}(\omega RC)$$

We got to our answer, but with much less algebra!

S19E4: Magnitudes and Angles

0 points possible (ungraded)

Suppose we have a circuit where the ratio of the complex amplitude of the particular integral to the amplitude of the driving sinusoid is

$$\frac{V_p}{V_i} = \frac{\mathrm{j}\omega RC}{1+\mathrm{j}\omega RC}$$

Given that $R = 0.82\mathrm{k}\Omega$ and $C = 10.0\mathrm{nF}$ we will compute the magnitude and phase of this ratio for various frequencies.

For frequency $f = 170\mathrm{Hz}$ what is the magnitude of $\frac{V_p}{V_i}$?

0.00875842 ✔ **Answer:** 0.008758424369524787

For frequency $f = 170\mathrm{Hz}$ what is the phase of $\frac{V_p}{V_i}$? Enter your answer in radians.

1.56204 ✔ **Answer:** 1.5620377904450544

For frequency $f = 19.409139401450656\mathrm{kHz}$ what is the magnitude of $\frac{V_p}{V_i}$? (Note: we are printing out lots of decimal places here for a reason.)

0.707107 ✔ **Answer:** 0.7071067811865477

For frequency $f = 19.409139401450656\mathrm{kHz}$ what is the phase of $\frac{V_p}{V_i}$? Enter your answer in radians.

0.785398 ✔ **Answer:** 0.7853981633974482

For frequency $f = 17.6\mathrm{MHz}$ what is the magnitude of $\frac{V_p}{V_i}$?

0.999999 ✔ **Answer:** 0.9999993919254444

For frequency $f = 17.6\mathrm{MHz}$ what is the phase of $\frac{V_p}{V_i}$? Enter your answer in radians.

0.00110279 ✔ **Answer:** 0.0011027915643926356

Solution:

Magnitude

If $j\omega RC = Ae^{jb}$ and $1 + j\omega RC = Ce^{jd}$, then $\frac{j\omega RC}{1+j\omega RC} = \frac{A}{C} \times e^{j(b-d)}$. We just need to find $\frac{A}{C}$, which can be found by finding each magnitude separately:

$$|\frac{V_p}{V_i}| = \frac{A}{C} = \frac{\omega RC}{\sqrt{1+(\omega RC)^2}}$$

Now, we can plug in each frequency, rembering the conversion $\omega = 2\pi f$:

$$f = 170 \rightarrow |\frac{V_p}{V_i}| = 0.0087584$$

$$f = 19.4 \times 10^3 \rightarrow |\frac{V_p}{V_i}| = .707$$

$$f = 17.6 \times 10^6 \rightarrow \left| \frac{V_p}{V_i} \right| = 1$$

Angle

Using the same notation as earlier, we just need to find $b - d$. The numerator always has a phase of $\frac{\pi}{2}$ because it is purely imaginary, and the bottom has a magnitude of $\tan^{-1}\left(\frac{\omega RC}{1}\right)$. Now, we can just plug in each frequency, still repemebering that $\omega = 2\pi f$:

$$f = 170 \rightarrow \theta = 1.567$$

$$f = 19.4 \times 10^3 \rightarrow \theta = .785$$

$$f = 17.6 \times 10^6 \rightarrow \theta = 0.001102791$$

In[∘]:=
```
r = 0.82*^3; c = 10*^-9;

f = 170;
ω = 2 π f;
v = (ⅈ ω r c)/(1 + ⅈ ω r c);
Abs[v]
Arg[v]

f = 19.409139401450656*^3;
ω = 2 π f;
v = (ⅈ ω r c)/(1 + ⅈ ω r c);
Abs[v]
Arg[v]

f = 17.6*^6;
ω = 2 π f;
v = (ⅈ ω r c)/(1 + ⅈ ω r c);
Abs[v]
Arg[v]
```

Out[∘]= 0.00875842

Out[∘]= 1.56204

Out[∘]= 0.707107

Out[∘]= 0.785398

Out[∘]= 0.999999

Out[∘]= 0.00110279

S20E1: Inductor Impedance

0 points possible (ungraded)

You just saw a derivation of the impedance of a resistor and the impedance of a capacitor. You are to derive the impedance of an inductor. In the space provided below write an algebraic expression in terms of angular frequency ω and inductance L for the ratio of the complex amplitude of the voltage across an inductor to the complex amplitude of the current through the inductor. (As usual, use w for ω in your expression.)

j*w*L

✔ **Answer:** j*w*L

$j \cdot \omega \cdot L$

Solution:

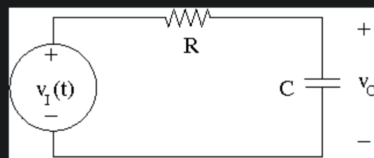
We need to transfer our voltage equation for an inductor from what we already know to the frequency domain

$$V_L\left(t\right) = L\frac{di_L\left(t\right)}{dt} \rightarrow V_L = L\left(j\omega I_L\right)$$

$$Z_L = \frac{V_L}{I_L} = j\omega L$$

S20E2: RC voltage divider

0 points possible (ungraded)



The complex amplitude of $v_I(t)$ is V_i. Derive V_c as a function of R, C, ω, and V_i. In the space below enter your algebraic expression for V_c. (As usual, use w for ω in your expression.)

Vi/(1+R*j*w*C)

Answer: (1/(1+j*w*R*C))*Vi

$$\frac{V_i}{1+R\cdot j\cdot \omega\cdot C}$$

Solution:

We can trasnfer the regular KCL equation into frequency domain:

$$V_i(t) = v_R(t) + v_C(t)$$

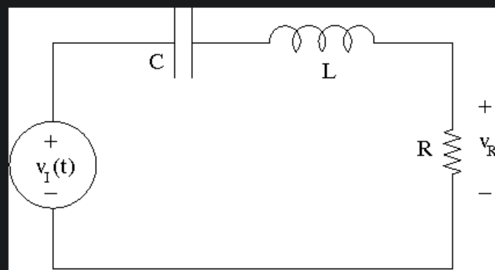
$$v_R(t) = Ri_C(t) = RC\frac{dv_C}{dt}$$

$$V_i = RC(j\omega V_C) + V_C$$

$$V_C = \frac{V_i}{1+j\omega RC}$$

S20E3: LCR voltage divider

0 points possible (ungraded)



The complex amplitude of $v_I(t)$ is V_i. Derive V_r as a function of R, C, L, ω, and V_i. In the space below enter your algebraic expression for V_r. (As usual, use w for ω in your expression.)

(Vi*j*w*R*C)/(1+j*w*R*C-w^2*L*C)

✔ **Answer:** ((j*w*R*C)/(1+j*w*R*C-w^2*L*C))*Vi

$$\frac{V_i \cdot j \cdot \omega \cdot R \cdot C}{1 + j \cdot \omega \cdot R \cdot C - \omega^2 \cdot L \cdot C}$$

Solution:

We can trasnfer the regular KCL equation into frequency domain, just like in the last exercise:

$$V_i(t) = v_R(t) + v_C(t) + v_L(t)$$

$$v_R(t) = Ri_C(t) = RC\frac{dv_C}{dt}$$

$$v_L(t) = L\frac{di_C(t)}{dt} = LC\frac{d^2v_C}{dt^2}$$

$$V_i = RC(j\omega V_C) + V_C + LC(j\omega)^2 v_C$$

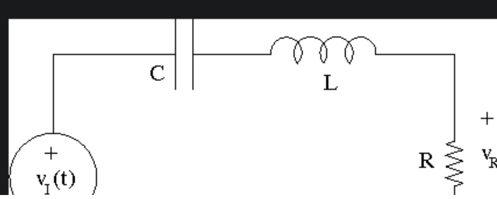
$$V_C = \frac{V_i}{1 - LC\omega^2 + j\omega RC}$$

$$V_R = j\omega RC \times V_C = \frac{j\omega RCV_i}{1 - LC\omega^2 + j\omega RC}$$

S20E4: LCR voltage divider frequency limits

0 points possible (ungraded)

In this circuit we computed the voltage-transfer ratio from the complex amplitude of the source V_i to the complex amplitude of the voltage across the resistor V_r.



This transfer ratio is

$$\frac{V_r}{V_i} = \frac{j\omega RC}{1+j\omega RC-\omega^2 LC}$$

What is the value of the magnitude of this ratio when $\omega\sqrt{LC} = 1$?

1

✔ **Answer:** 1

In the spaces below enter your algebraic expressions in terms of ω, R, C, and L. (As usual, use w for ω in your expressions.)

What is the approximate form of the magnitude of this ratio for small ω?

w*R*C

✔ **Answer:** w*R*C

$\omega \cdot R \cdot C$

What is the approximate form of the magnitude of this ratio for large ω?

R/(L*w)

✔ **Answer:** R/(w*L)

$\frac{R}{L \cdot \omega}$

Solution:

Specific Frequency

We can just plug in our specific frequency into our equations:

$$\frac{V_R}{V_i} = \frac{j\omega RC}{1-1^2+j\omega RC} = 1$$

Limits:

$$\lim_{\omega\to 0}\frac{V_R}{V_i} \approx \frac{0}{1-0+0} = 0$$

$$\lim_{\omega\to\infty}\frac{V_R}{V_i} \approx \frac{jRC}{\omega LC} \approx 0$$

Note that at low frequencies, the capacitor has a high impedance and so little voltage is split to the reisistor. At high frequencies, the inductor has a high impedance and so the resistor yet again has no voltage.

(*Adding some more info as the solutions were sparse here*)

Small ω case:
Here $\omega^2 \to 0$:

$$\lim_{\omega \to 0} \left| \frac{V_R}{V_i} \right| = \frac{|j\omega RC|}{\sqrt{1 + (\omega RC)^2}} = \omega RC$$

Large ω case:
Here the $1 + j\omega RC$ term in the denominator is negligible:

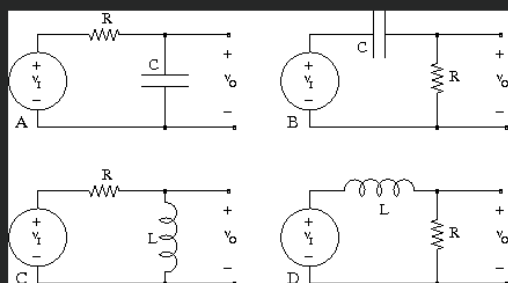
$$\lim_{\omega \to \infty} \left| \frac{V_R}{V_i} \right| = \frac{|j\omega RC|}{|-\omega^2 LC|} = \frac{jRC}{\omega LC} = \frac{R}{L\omega}$$

Homework

H2P1: Magnitude and Angle

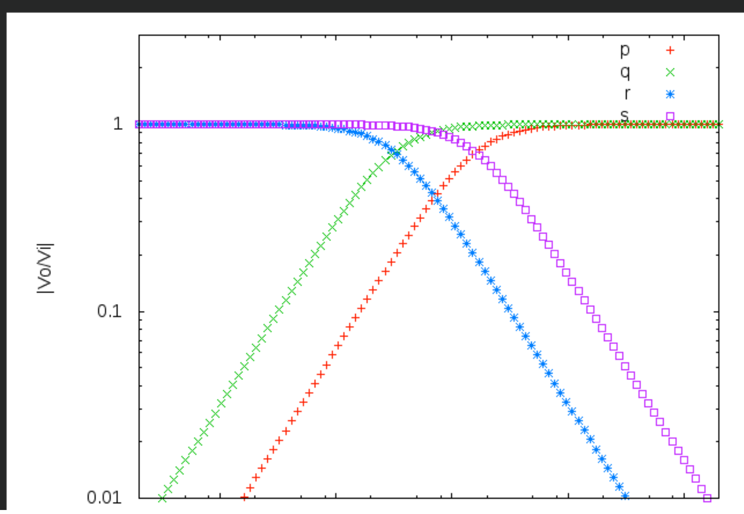
4/4 points (graded)

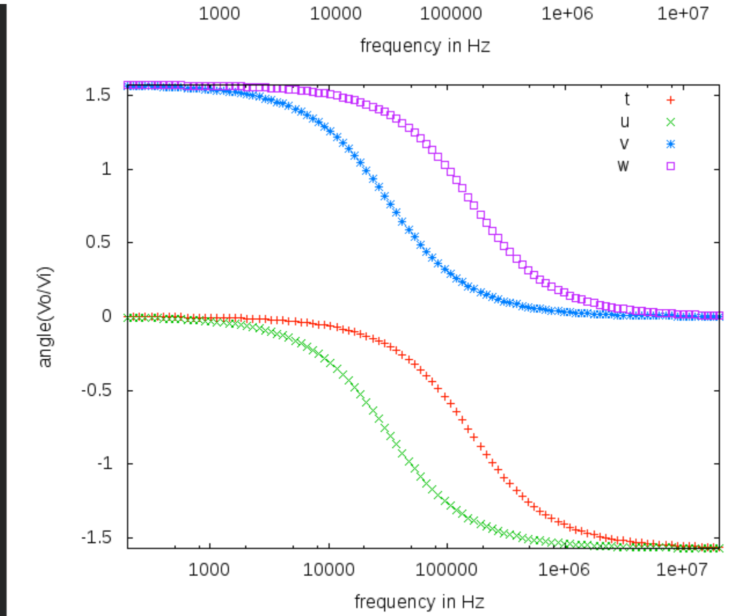
Here are four first-order circuits:



The parameters $R = 1.2\text{k}\Omega$, $C = 4.167$nF, and $L = 1.2$mH.

For each of these circuits there is a magnitude and phase of the voltage-transfer ratio $\frac{V_o}{V_i}$ among the following two graphs. We want you to choose, for each circuit the appropriate magnitude and angle graph.





In the spaces provided please enter your choices. For example, if you chose magnitude p and angle u for circuit A we want you to enter their product $p * u$.

Circuit A:

r*u

Answer: r*u

Circuit B:

q*v

Answer: q*v

Circuit C:

p*w

Answer: p*w

Circuit D:

s*t

Answer: s*t

Explanation:

First, we start by deriving the transfer functions for each of the networks. Replacing each element with its impedance-equivalent ($Z_C = \frac{1}{j\omega C}$, $Z_L = j\omega L$, $Z_R = R$) and solving for the transfer functions (the ratio of v_{out} to v_{in}, $H(j\omega)$:

$$H_A(j\omega) = \frac{1}{j\omega RC + 1}$$

$$H_B(j\omega) = \frac{j\omega RC}{j\omega RC + 1}$$

$$H_C(j\omega) = \frac{j\omega L}{j\omega L + R}$$

$$H_D(j\omega) = \frac{R}{j\omega L + R}$$

To match each transfer function to its magnitude and phase curve, the process of elimination can be used. Let us start by comparing the magnitudes. As our frequency approaches DC ($\omega \to 0$), we have:

$$H_A(0) = 1$$

$$H_B(0) = 0$$

$$H_B(0) = 0$$

$$H_C(0) = 0$$

$$H_D(0) = 1$$

So we can already see that A and D are "low pass filters" and should have a constant magnitude response at low frequencies, (graphs S and R), and B and C are "high pass filters," which roll off at low frequencies but have a constant magnitude response at high frequencies (graphs P and Q). To figure out which partiular filter matches which transfer function, we can also solve for the "3dB break points," the frequency at which the magnitude graphs "break off" and transition from flat to sloped, which also happens to be the frequency at which the real and imaginary parts of the denominators are equal. Recalling that we want frequencies in hertz, and that $\omega = 2\pi f$:

$$f_A = f_B = \frac{1}{2\pi RC}$$

$$f_C = f_D = \frac{R}{2\pi L}$$

So graph S goes with network A and R with network D. Likewise, P matches network B and Q matches network C. The phase graphs can be directly derived from the transfer functions. Again, we can use the process of elimination to make our job slightly easier. Notice a key distinction with two of the network transfer functions, B and C: because their numerator is purely imaginary and positive, they get an extra "bump" in phase shift of $+\frac{\pi}{2}$ everywhere. Thus, V and W will match up with these networks, and the other two graphs (T and$U$) with A and B. Direct calculation can be used to distinguish the specific graphs. The phases of each transfer function are below:

$$\phi_A(j\omega) = -tan^{-1}(\omega RC)$$

$$\phi_B(j\omega) = \frac{\pi}{2} - tan^{-1}(\omega RC)$$

$$\phi_C(j\omega) = \frac{\pi}{2} - tan^{-1}(\omega \frac{L}{R})$$

$$\phi_C(j\omega) = -tan^{-1}(\omega \frac{L}{R})$$

So to summarize, our final answers are:

$$A = S * T$$

$$B = P * W$$

$$C = Q * V$$

$$D = R * U$$

In[∘]:=
```
r = 1.2*^3; c = 4.167*^-9; l = 1.2*^-3;
(*Pg 713*)
zl[ω_] = ⅈ ω l;
zc[ω_] = 1/(ⅈ ω c);
zr = r;

(*We now use voltage divider expressions simplified by impedence model*)
(*Top left
  Vc/VI = Zc/(Zc+Zr) *)
ratio[ω_] := zc[ω]/(zc[ω] + zr);
LogLinearPlot[{Abs[ratio[ω]], Arg[ratio[ω]]},
 {ω, 0, 1*^7}, PlotLabels → {"Magnitude", "Phase"}]
(*Top right
  Vc/VI = Zr/(Zc+Zr) *)
ratio[ω_] := zr/(zc[ω] + zr);
LogLinearPlot[{Abs[ratio[ω]], Arg[ratio[ω]]},
 {ω, 0, 1*^7}, PlotLabels → {"Magnitude", "Phase"}]
(*Bottom left
  Vc/VI = Zl/(Zl+Zr) *)
ratio[ω_] := zl[ω]/(zl[ω] + zr);
LogLinearPlot[{Abs[ratio[ω]], Arg[ratio[ω]]},
 {ω, 0, 1*^7}, PlotLabels → {"Magnitude", "Phase"}]

(*Bottom right
  Vc/VI = Zr/(Zl+Zr) *)
ratio[ω_] := zr/(zl[ω] + zr);
LogLinearPlot[{Abs[ratio[ω]], Arg[ratio[ω]]},
 {ω, 0, 1*^7}, PlotLabels → {"Magnitude", "Phase"}]
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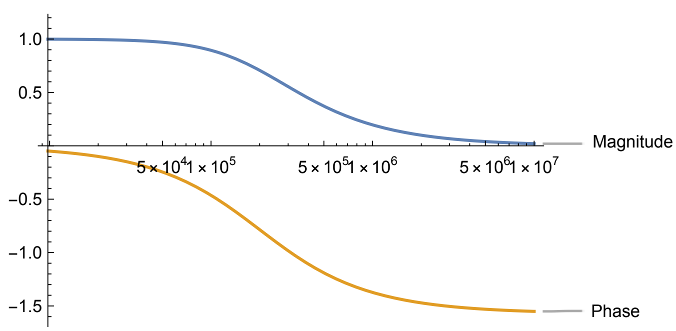
Out[∘]=



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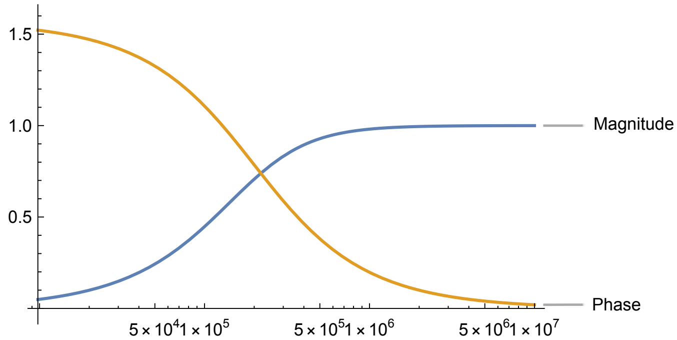

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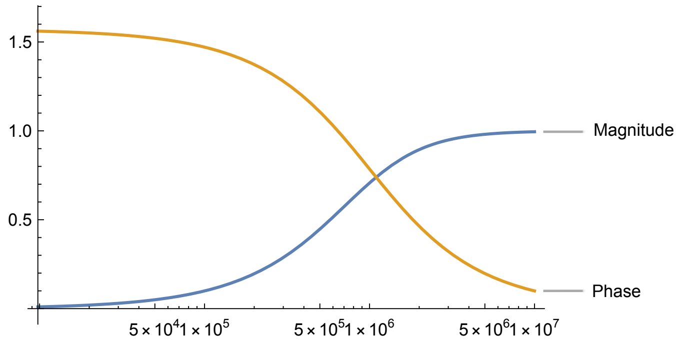

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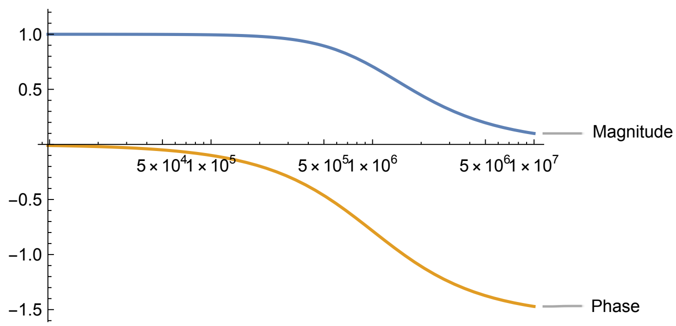
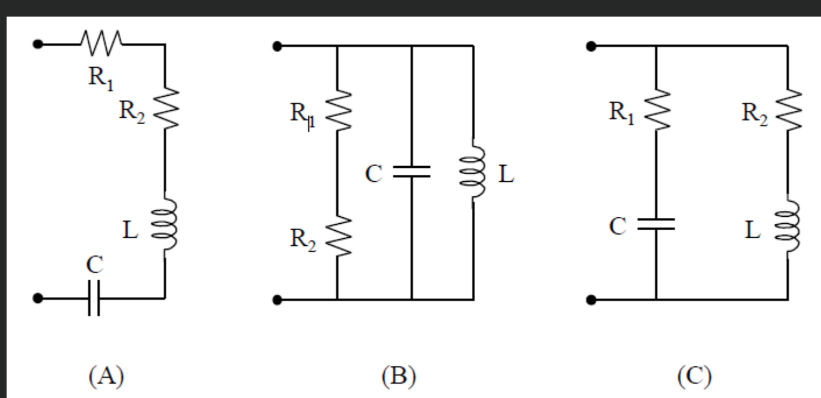


H2P2: Impedence

9/9 points (graded)



Compute the impedance Z looking into the port of each of the above three circuits.

Below write your answer for the impedance as an algebraic expression in terms of R_1, R_2, C, L and ω. (As usual, use "w" for ω in your expressions.) In each case, we also ask how the magnitude of the impedance behaves as $\omega \to 0$ and as $\omega \to \infty$. If the answer is zero, enter a "0," if the answer is a constant enter the algebraic expression for that constant, and if the answer is infinity, enter the symbol "inf"

The impedance of circuit A, $Z_A =$

R1+R2+j*w*L+1/(j*w*C) — ✔ **Answer:** R1+R2+j*w*L+(1/(j*w*C))

$$R_1 + R_2 + j \cdot \omega \cdot L + \frac{1}{j \cdot \omega \cdot C}$$

As $\omega \to 0$, $|Z_A| \to$

inf — ✔ **Answer:** inf

∞

As $\omega \to \infty$, $|Z_A| \to$

inf — ✔ **Answer:** inf

∞

The impedance of circuit B, $Z_B =$

1/(1/(R1+R2)+1/(j*w*L)+j*w*C) — ✔ **Answer:** 1/((1/(R1+R2))+(j*w*C)+(1/(j*w*L)))

$$\frac{1}{\frac{1}{R_1+R_2} + \frac{1}{j \cdot \omega \cdot L} + j \cdot \omega \cdot C}$$

As $\omega \to 0$, $|Z_B| \to$

0 — ✔ **Answer:** 0

0

As $\omega \to \infty$, $|Z_B| \to$

0 — ✔ **Answer:** 0

0

The impedance of circuit C, $Z_C =$

1/(1/(R1+1/(j*w*C))+1/(R2+j*w*L)) — ✔ **Answer:** ((R1*j*w*C +1)*(R2+j*w*L))/(j*w*C*(R2+j*w*L+R1)+1)

$$\frac{1}{\frac{1}{R_1+\frac{1}{j \cdot \omega \cdot C}} + \frac{1}{R_2 + j \cdot \omega \cdot L}}$$

As $\omega \to 0$, $|Z_C| \to$

R2 — ✔ **Answer:** R2

R_2

As $\omega \to \infty$, $|Z_C| \to$

R1 — ✔ **Answer:** R1

R_1

Explanation:
Computing the impedance of networks A, B and C requires us to convert all elements to their impedance equivalent model ($Z_L = j\omega L$, $Z_C = \frac{1}{j\omega C}$ and resistors retain their value, $Z_R = R$).

(a) Circuit A has all four elements in series. We can simply write the answer as the sum of all four element impedances:

$$Z_A = R_1 + R_2 + j\omega L + \frac{1}{j\omega C}$$

(b),(c) At very low frequencies and at DC, capacitors behave as open circuits and inductors behave as short circuits. The open circuit behavior of the capacitor ensures that current will not flow through this network at DC or very low frequencies. Hence the impedance tends to ∞. At very high frequencies, capacitors behave as short circuits and inductors behave as open circuits. As the open circuit caused by the inductor will prevent current from flowing, the impedance of this network at high frequencies is again ∞.

(d) Circuit B has the two series resistors in parallel with the capacitor which is again in parallel with the inductor. Replacing all circuit elements with their impedance equivalents, we get the following expression:

$$Z_B = (R_1 + R_2) \parallel (\frac{1}{j\omega C}) \parallel (j\omega L)$$

Applying the formula for parallel combination of impedances:

$$\frac{1}{Z_B} = \frac{1}{R_1 + R_2} + \frac{1}{\frac{1}{j\omega C}} + \frac{1}{j\omega L}$$

Simplifying, we get:

$$Z_B = \left(\frac{1}{R_1 + R_2} + j\omega C + \frac{1}{j\omega L}\right)^{-1}$$

(e),(f) Again, as the frequency of the input signal approaches DC, we know that capacitors tend to act as open circuits, and inductors tend to act as short circuits. Since network B is a parallel RLC circuit, at DC and low frequencies, the inductor shorts the terminals across which impedance is being measured, resulting in a measured impedance of 0. Similarly, as input frequency tends to ∞, the capacitor will behave as a short circuit, and will effectively short the terminals across which impedance is being measured, resulting once again in an impedance of 0.

(g) Circuit C is the parallel combination of a resistor in series with a capacitor to a resistor in series with an inductor. Converting circuit elements to their impedances, we get:

$$Z_C = (R_1 + \frac{1}{j\omega C}) \parallel (R_2 + j\omega L)$$

Applying the formula for parallel combination of impedances:

$$\frac{1}{Z_C} = \frac{1}{R_1 + \frac{1}{j\omega C}} + \frac{1}{R_2 + j\omega L}$$

Inverting both sides, we get:

$$Z_C = \left(\frac{1}{R_1 + \frac{1}{j\omega C}} + \frac{1}{R_2 + j\omega L}\right)^{-1}$$

(h),(i) As input frequency approaches 0, we can effectively replace the capacitor with an open circuit and the inductor with a short circuit in the network diagram. If we do this, we see that R_1 is only connected to one terminal on one end, and connected to nothing on the other end. This means that we can disregard R_1 in the impedance calculaction of Z_C at low frequencies. This leaves us with the single resistor R_2 as the sole element in between both terminals. The impedance at low frequencies is therefore R_2. At high frequencies, the inductor behaves as an open circuit, which leaves R_2 connected to only a single node and nothing else. Just as before, this allows us to disregard R_2 in the calculation of the network impedance at high frequencies. As the capacitor behaves like a short circuit, the only element in between the terminals of network C is the resistor R_1, and therefore the impedance at high frequencies of network C is simply R_1.

```
Rp[r_List] := 1 / Total[1 / r];

(*Circuit A*)
a = R1 + R2 + j * w * L + 1 / (j * w * C)
Limit[a, w → 0]
Limit[a, w → ∞]

(*Circuit B*)
b = Rp[{r1 + r2, 1 / (j w c), j w l}]
Limit[b, w → 0]
Limit[b, w → ∞]

(*Circuit C - remember there is a conflict where C is the label for the circuit,
capactance, and also speed of light in Mathematica!
 So we call capacitance "cap" when there is a naming conflict*)
(*Had to do the algebra anyway, because we're getting the wrong limit*)
(*TODO: figure out how to make this work and get R2, R1*)
c = Rp[{r1 + 1 / (j w cap), r2 + j w l}]
Limit[b, w → 0]
Limit[b, w → ∞]
```

Out[∘]= $R1 + R2 + \frac{1}{C\, j\, w} + j\, L\, w$

Out[∘]= Indeterminate

Out[∘]= $j\, L\, \infty$

Out[∘]= $\frac{1}{\frac{1}{r1+r2} + \frac{1}{j\, l\, w} + c\, j\, w}$

Out[∘]= 0

Out[∘]= 0

Out[∘]= $\frac{1}{\frac{1}{r1+\frac{1}{cap\, j\, w}} + \frac{1}{r2+j\, l\, w}}$

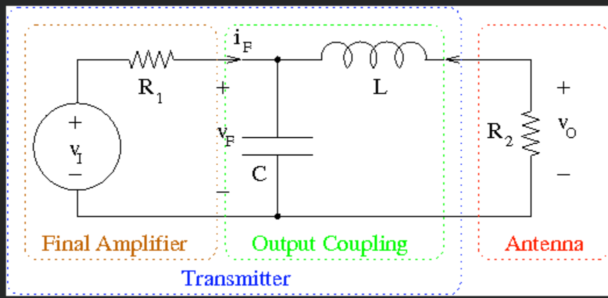
Out[∘]= 0

Out[∘]= 0

H2P3: An L Network

6/6 points (graded)

The inductor and capacitor in the diagram below are part of the output-coupling network of a radio transmitter. The rest of the transmitter (the source of radio-frequency energy) is represented as a Thevenin source, and the antenna load is represented by a resistor.



In this problem we will examine some of the characteristics of this circuit. In the spaces provided below you will write algebraic expressions in terms of the part parameters L, C, R_1, R_2, and the angular frequency ω. (As usual, use w for ω in your expressions.)

One thing we want to know is the voltage-transfer ratio (the ratio of the complex output voltage to the complex input voltage) $\frac{V_o}{V_i}$ of this network, as a function of the operating frequency. Now that we know about impedances this is just like solving a resistive ladder!

In the space provided below write an algebraic expression for this ratio.

R2/((R2+j*w*L)*(j*w*C+1/(R2+j*w*L))*(R1+1/(j*v

✔ **Answer:** R2/((R1+R2-L*C*R1*w^2) + (R1*R2*C + L)*j*w)

$$\frac{R_2}{(R_2 + j\cdot\omega\cdot L)\cdot\left(j\cdot\omega\cdot C + \frac{1}{R_2+j\cdot\omega\cdot L}\right)\cdot\left(R_1 + \frac{1}{j\cdot\omega\cdot C+\frac{1}{R_2+j\cdot\omega\cdot L}}\right)}$$

Look carefully at what you just computed. What is it for $\omega = 0$? What happens as $\omega \to \infty$? You should always examine system functions this way.

Another important value is the driving-point impedance that the final amplifier "sees" looking at the antenna through the coupling network. This is the ratio of the complex voltage across the input port to the complex current into that port. In this circuit it is $\frac{V_f}{I_f}$.

In the space provided below write an algebraic expression for this impedance. (Hint: The algebra is often easier if you invert parallel impedances to make admittances. They then just add.)

1/(j*w*C+1/(R2+j*w*L))

✔ **Answer:** -(R2 + L*j*w)/((L*C*w^2 - 1) - R2*C*j*w)

$$\frac{1}{j\cdot\omega\cdot C + \frac{1}{R_2+j\cdot\omega\cdot L}}$$

Again, look carefully at what you just computed. What is it for $\omega = 0$? What happens as $\omega \to \infty$?

Remember that we found that in resistive circuits the load that absorbs the maximum power from a Thevenin source is the one where the load resistance is the same as the source resistance. Here we have an antenna that we want to transfer power to, but both the amplifier and the load have given resistances R_1 and R_2. Since capacitors and inductors do not dissipate power, they just store the energy temporarily, perhaps if we choose the inductance and capacitance wisely we can couple the amplifier to the antenna very well.

It is possible to find values of L and C that make the driving-point impedance you just computed exactly R_1, if $R_1 > R_2$. This will "match the antenna to the amplifier".

In the space provided below write an algebraic expression for the capacitance C_{match} that allows this match:

sqrt(R1-R2)/(R1*sqrt(R2)*w)

✔ **Answer:** (1/(R1*w))*sqrt((R1-R2)/R2)

$$\frac{\sqrt{R_1 - R_2}}{R_1\cdot\sqrt{R_2}\cdot\omega}$$

In the space provided below write an algebraic expression for the inductance L_{match} that allows this match:

(sqrt(R1 - R2)*sqrt(R2))/w

Answer: (1/w)*sqrt(R1*R2-R2^2)

$$\frac{\sqrt{R_1 - R_2} \cdot \sqrt{R_2}}{\omega}$$

Now let's look at some real numbers. For a big transmitting amplifier the output resistance may be $R_1 = 1000.0\Omega$. A typical antenna has a radiation resistance of $R_2 = 50.0\Omega$. Consider an AM broadcast transmitter at $f = 690.0$kHz. In the spaces provided below, write the numerical values of the capacitance (in picoFarads) and inductance (in microHenrys) for match.

$C_{match} =$

1.00542e3

Answer: 1005

$L_{match} =$

0.000050271e6

Answer: 50.300000000000004

By the way, AM broadcast transmitters can be very large: up to 50kW. The parts used for such power levels are impressive. For example, an inductor may be made of large gauge silver-coated copper tubing. There is a nice picture here.

Explanation:
The voltage transfer ratio is given by the epression:

$$\frac{V_O}{V_I} = \frac{\frac{R2}{j\omega C}}{R_1\left(\frac{1}{j\omega C} + j\omega L + R_2\right) + \frac{1}{j\omega C}(j\omega L + R_2)}$$

And the input impedance to the antenna is given by the parallel combination of C and $L + R_2$:

$$\frac{(j\omega L + R_2)\left(\frac{1}{j\omega C}\right)}{\frac{1}{j\omega C} + R_2 + j\omega L}$$

Finally, to find L and C match, we set the above complex expression equal to R_1. This actually merits two conditions; its magnitude must equal R_1 and its phase must be zero.

$$\frac{(j\omega L + R_2)\left(\frac{1}{j\omega C}\right)}{\frac{1}{j\omega C} + R_2 + j\omega L} = R_1$$

$$\frac{(j\omega L + R_2)}{1 + j\omega R_2 C - \omega^2 LC} = R_1$$

We have one equation and two unknowns, however, we can separate the real and imaginary parts of the equation. this will be easier if we first remove the fraction:

$$(j\omega L + R_2) = R_1\left(1 + j\omega R_2 C - \omega^2 LC\right)$$

Now we can use the fact that the Re{LHS}=Re{RHS} and Im{LHS}}=Im{RHS}:

$$\mathrm{Re}\{\mathrm{LHS}\} = R_2 = R_1\left(1 - \omega^2 LC\right) = \mathrm{Re}\{\mathrm{RHS}\}$$

$$\mathrm{Im}\{\mathrm{LHS}\} = \omega L = \omega R_1 R_2 C = \mathrm{Im}\{\mathrm{RHS}\}$$

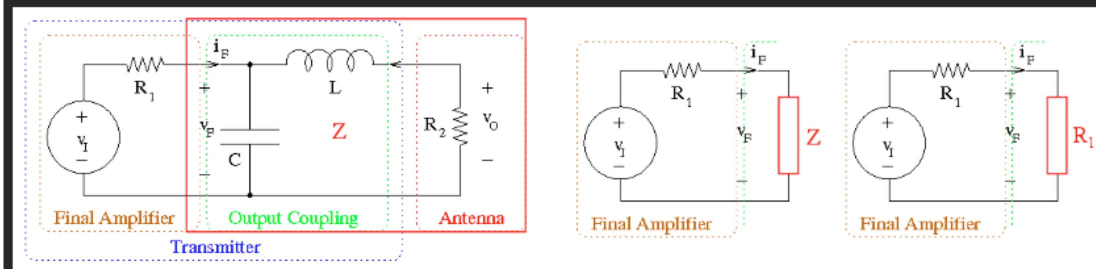
This results in two equations which can be used to solve for L and C match:

$$L_{Match} = \frac{1}{\omega}\sqrt{R_1 R_2 - {R_2}^2}$$

$$C_{Match} = \frac{1}{\omega R_1}\sqrt{\frac{R_1 - R_2}{R_2}}$$

Grove (Community TA)
2 years ago

Another important value is the driving-point impedance that the final amplifier "sees" looking at the antenna through the coupling network.
In part 2 you have written *an algebraic expression for this impedance* and I have called this impedance Z.



Z is equal to inductor L in series with resistor R_2 and this combination in parallel with capacitor C.

Z will be equal to something like $\frac{a+jb}{c+jd}$.
This is an illustrative fraction and the actual form of the fraction will be slightly different.

In general Z will be the sum of a real and imaginary components but by choosing suitable values of C and L, Z can be made to equal R_1.

ie $Z = \frac{a+jb}{c+jd} = R_1 \Rightarrow a + jb = cR_1 + jdR_1$.

This gives you two equations when you make the real part on each side equal, ie $a = cR_1$ and the imaginary parts on each side of the equation equal, ie $b = dR_1$

```
Rp[r_List] := 1 / Total[1 / r];

(*We use the voltage divider relationships in Section 13.3.3.
  Different impedences but same idea.
  We consider the overall parallel resistance in the 1st voltage divider.
  The capacitor is like a "voltage source" in the 2nd voltage divider.*)
zl = j * w * l;
zc = 1 / (j * w * c);
ratio = Rp[{zl + R2, zc}] / (Rp[{zl + R2, zc}] + R1) * R2 / (zl + R2)

zf = Rp[{zl + R2, zc}]

(*Use the hint above to solve system of equations.
   We know Re[z1/z2] ≠ Re[z1]/Re[z2].
    This causes problems in Mathematica, it doesn't know how to get Re[z1/z2] anyway.
  Tried solving for Cmatch, then Refine[Re[cmatch]],
assuming R1,R2,l,w is real, does not work.*)
a = R2;
b = w * l;
c = -w^2 * l * cap + 1;
d = w * cap * R2;
Solve[a == c R1 && b == d R1, {cap, l}]

R1 = 1000; R2 = 50; f = 690*^3;
w = 2 π f;
(*Choose positive solutions*)
Sqrt[R1 - R2] / (R1 Sqrt[R2] w) // N
Sqrt[R1 - R2] Sqrt[R2] / w // N
```

Out[ ]= $$\frac{R2}{(R2 + j\ l\ w)\left(c\ j\ w + \frac{1}{R2 + j\ l\ w}\right)\left(R1 + \frac{1}{c\ j\ w + \frac{1}{R2 + j\ l\ w}}\right)}$$

Out[ ]= $$\frac{1}{c\ j\ w + \frac{1}{R2 + j\ l\ w}}$$

Out[ ]= $$\left\{\left\{cap \to -\frac{\sqrt{R1 - R2}}{R1\sqrt{R2}\ w},\ l \to -\frac{\sqrt{R1 - R2}\ \sqrt{R2}}{w}\right\},\ \left\{cap \to \frac{\sqrt{R1 - R2}}{R1\sqrt{R2}\ w},\ l \to \frac{\sqrt{R1 - R2}\ \sqrt{R2}}{w}\right\}\right\}$$

Out[ ]= 1.00542×10^{-9}

Out[ ]= 0.000050271

Lab

Lab 2

7/7 points (graded)

In this lab we'll look at the response of first-order circuits as a function of frequency, the topic covered in Chapter 13 in the text.

Consider the series RC and RL circuits shown in Figure 1. The resistors are each $1k\Omega$; the C and L values are unknown.

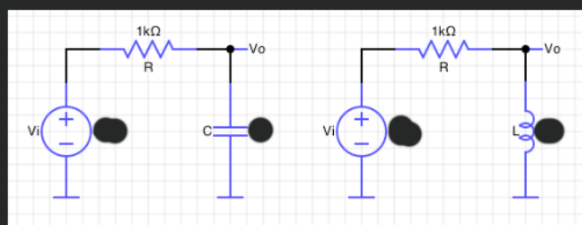


Figure 1. Series RC and RL circuits

Since we'll be asking you to complete a filter design below, let's analyze the frequency response of each of the circuits.

Recalling the component transfer functions on page 733 of the text

- $$H_R(j\omega) = R$$
- $$H_L(j\omega) = j\omega L$$
- $$H_C(j\omega) = 1/j\omega C$$

we can derive $H(j\omega) = V_o/V_i$ for the series RL and RC circuits using these impedances and the voltage divider technique shown in Example 13.4:

- $$H_{RC}(j\omega) = \frac{1/RC}{j\omega + 1/RC}$$
- $$H_{RL}(j\omega) = \frac{j\omega}{j\omega + R/L}$$

Think about the magnitude of the transfer function $|H(j\omega)|$ for both low frequencies ($\omega \to 0$) and high frequencies ($\omega \to \infty$). Are these filter circuits low-pass or high-pass? Hint: there's one of each!

The phase of the transfer function $\angle H(j\omega)$ is given by

- $$\angle H(j\omega) = \tan^{-1}\frac{y}{x}$$

where x is the real part of $H(j\omega)$ and y is the imaginary part. With a little algebra we can conclude

- $$\angle H_{RC}(j\omega) = \tan^{-1}\frac{-\omega}{1/RC}$$
- $$\angle H_{RL}(j\omega) = \tan^{-1}\frac{R/L}{\omega}$$

The *break frequency* is the frequency at which the filter response changes from essentially unity gain to a steadily decreasing response. At the break frequency the phase $\angle H(j\omega)$ is $\pm\pi/4$ radians or $\pm 45°$. In other words, the break frequency is the value of ω that makes argument to $\tan^{-1}$ equal to ± 1. The break frequency is also where where the magnitude of the response reaches $0.707 = 1/\sqrt{2}$ of its peak value. Please give the formula for the break frequencies in radians/sec for the series RC and RL circuits:

Break frequency in radians/sec for RC circuit (formula using R and C):

1/(R*C) ✔ **Answer:** 1/(R*C)

$\frac{1}{R \cdot C}$

Break frequency in radians/sec for RL circuit (formula using R and L):

R/L ✔ **Answer:** R/L

$\frac{R}{L}$

With all this analysis under our belts, let's look at plots of the frequency response for the circuits in Figure 1. The on-line lab tool will calculate the frequency response of a circuit if you click on the AC analysis tool. You'll be asked to specify the beginning and ending frequencies (the default values are fine) and where in the circuit to inject the tone burst, a sinusoidal input of the form $u(t)\,e^{j\omega t}$ (it has to be one of the named voltage sources). For each voltage probe, ratio of the voltage at the probe to the voltage at the specified source is computed for frequencies in the specified range. The results appear in two plots, both of which use a log scale for the frequency (x) axis. One plot shows the magnitude of the response $|H(j\omega)|$ and the other the phase $\angle H(j\omega)$. The magnitude is plotted in decibels = $20 \log(V_o/V_i)$ -- for more on decibels, see footnote 8 on page 738 of the text.

The circuits in Figure 1 were entered using the schematic tool with voltage probes added to the V_o nodes. An AC analysis was performed; the resulting plots are shown in Figure 2. The red plots come from one circuit, the blue plots from the other, but it'll be up to you to figure which plots came from which circuit.

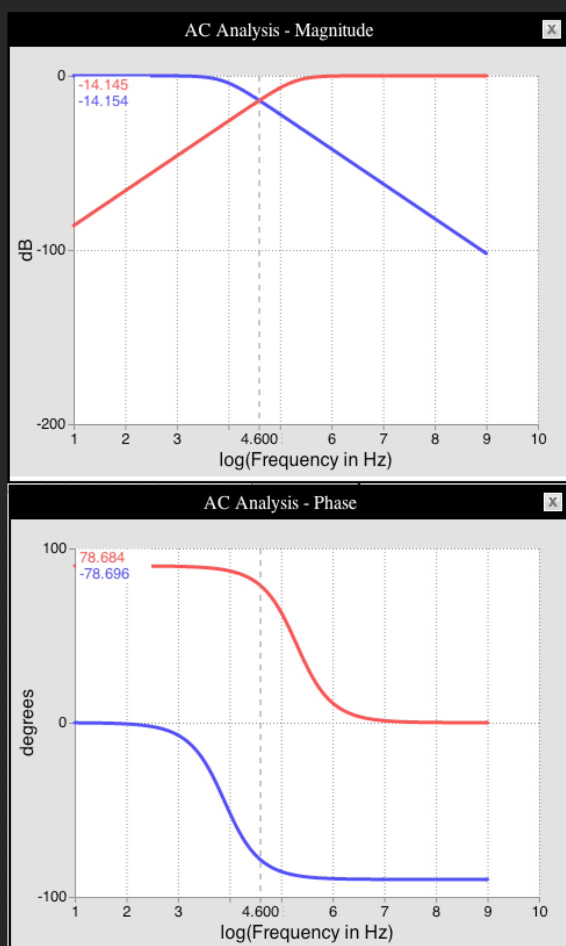


Figure 2. Results of AC analysis

Your job is to figure out the unknown C and L values based on the information in the plot. Remember that the R value for both circuits is $1k\Omega$. You can perform your own experiments using the on-line lab tool or you can use the analysis above to determine the unknown component types and values. Note that the formulas for the phase use the frequency ω in radians/sec but that the plots show frequencies in Hz. Recall that $1\ \mathrm{Hz} = 2\pi\ \mathrm{radians/sec}$.

Type of unknown component in circuit that produced the red plot (C or L):

L ✔ **Answer:** L

Value of unknown component in circuit that produced the red plot (in farads or henries):

0.000799999 ✔ **Answer:** 800e-6

Type of unknown component in circuit that produced the blue plot (C or L):

C ✔ **Answer:** C

Value of unknown component in circuit that produced the blue plot (in farads or henries):

1.9978e-8 ✔ **Answer:** 20e-9

Now that you're an expert :) use the on-line lab tool below to build a *band-pass filter* with break frequencies at $1kHz$ and $1MHz$. A band-pass filter passes frequencies between the break frequencies but attenuates frequencies above and below the break frequencies, as shown in Figure 3, which shows the magnitude of the frequency response V_o/V_1 using the logarithmic decibel scale. The expected response is approximately $-3\ \mathrm{dB}$ (i.e., a ratio of 0.707) at the two break frequencies and close to $0\ \mathrm{dB}$ (a ratio of 1) inbetween.

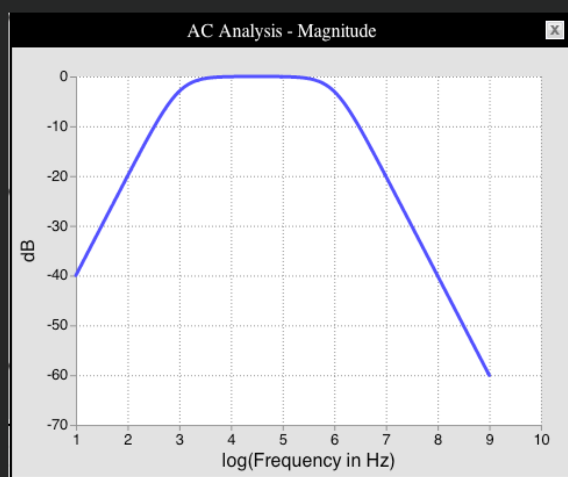
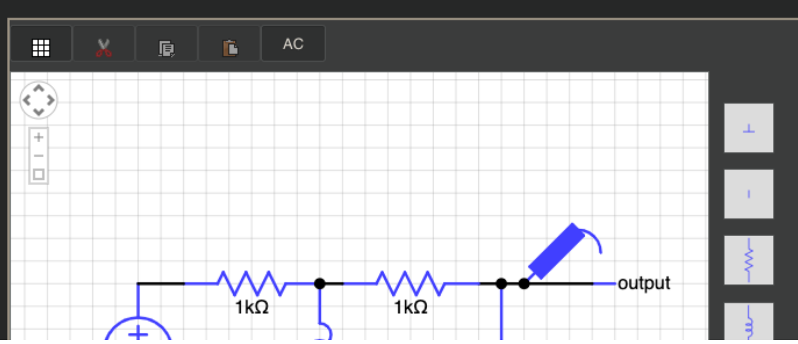
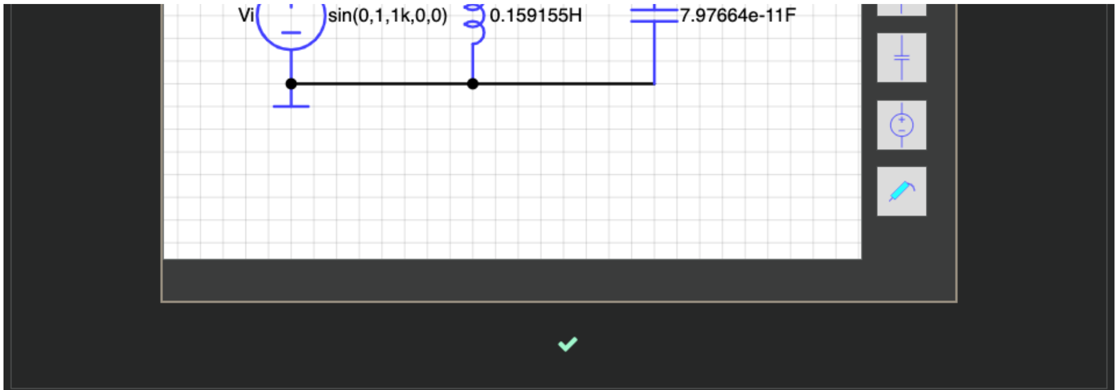


Figure 3. AC analysis of a band-pass filter (magnitude plot)

Hint: Cascade a low-pass and high-pass filter -- their effects are cumulative. We've just finished analyzing one of each above! Complete the circuit below, adding the appropriate circuitry between the voltage source and V_o, the node labeled "output" in the schematic.





After you've completed your design and run the AC analysis, click CHECK to have the system verify correct operation.

Explanation:
(a),(b) We are told that the break frequency is the frequency at which the phase of the transfer function equals ± 45 degrees ($\frac{\pi}{4}$ radians), which is equivalent to setting the argument of the inverse tangent function equal to 1. So we have for the RC and RL circuit respectively:

$$-\omega RC = \pm 1$$

$$\frac{R}{L\omega} = \pm 1$$

We take the value which results in a positive frequency, so for the RC and RL respectively:

$$\omega = \frac{1}{RC}$$

$$\omega = \frac{R}{L}$$

(c),(d),(e),(f) To figure out which plot (red or blue) matches with which circuit, we only have to look at the magnitude plots. Inspect the transfer functions for the RC and RL circuits, shown below:

$$H_{RC}(j\omega) = \frac{1}{j\omega RC + 1}$$

$$H_{RL}(j\omega) = \frac{j\omega}{j\omega + \frac{R}{L}}$$

As we can see, the RC circuit is a low pass filter (unity gain at $\omega = 0$, 0 at $\omega \gg \frac{1}{RC}$), and the RL circuit is a high pass filter (approaches unity gain at $\omega \gg \frac{R}{L}$, 0 at $\omega = 0$).
So the component L matches the red graph, and C the blue graph.
We can also determine the values of both components, purely from the measurements shown on the magnitude graph. For both transfer functions, at a value of $4.6Hz$ on the log frequency scale, we have a value of $-14.15dB$. Keeping in mind that:

$$Gain = 10^{\frac{dB}{20}}$$

This corresponds to the transfer function having a magnitude gain of about 0.196 at a frequency of about $40KHz$($80\pi K\frac{radians}{sec}$). The last thing we have to do is find an expression for the magnitude gain, which amounts to taking the magnitude of each transfer function (which can be accomplished by taking the magnitudes of the numerators and denominators seperately):

$$|H_{RC}| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$$|H_{RL}| = \frac{\omega}{\sqrt{\left(\frac{R}{L}\right)^2 + \omega^2}}$$

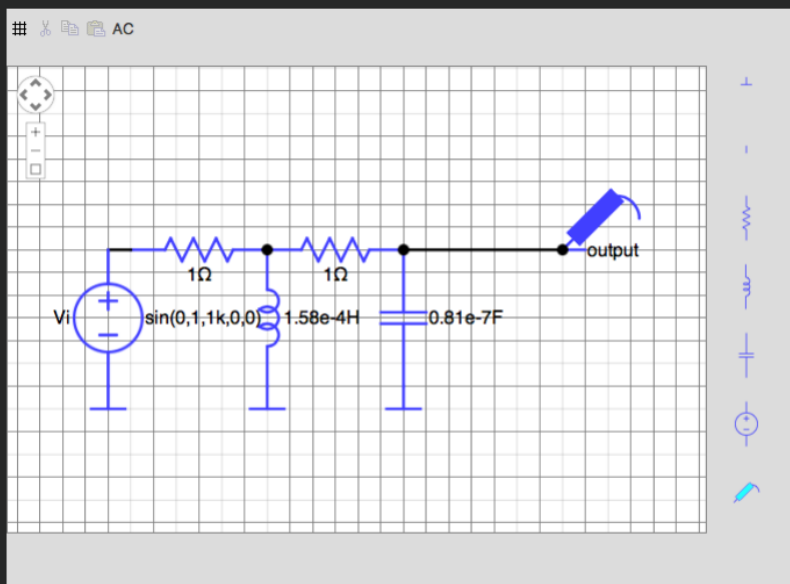
We plug in $|H| = 0.196, \omega = 80,000\pi$ and $R = 1k\Omega$ into both of the above expressions, and solve for C and L respectively:

respectively.

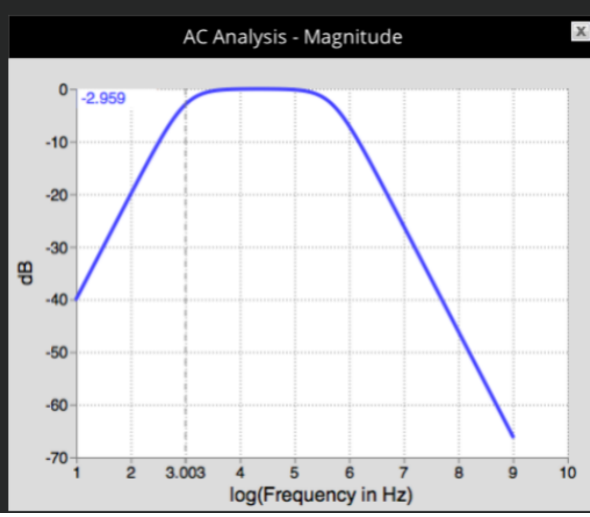
$$L = 8 \times 10^{-4} H$$

$$C = 2 \times 10^{-4} F$$

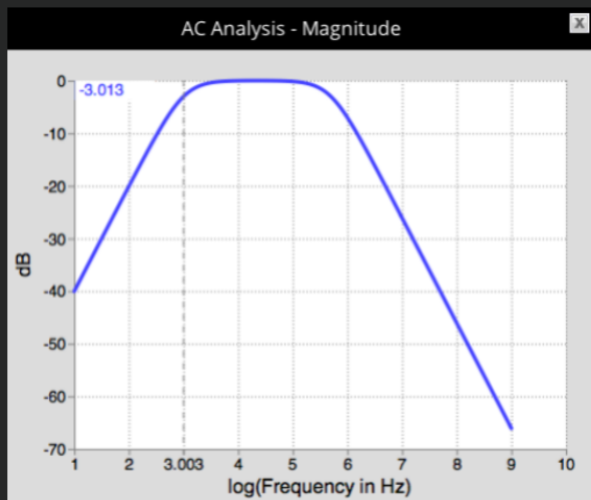
(g) Take the hint - cascade a high pass and low pass filter! The circuit shown below should work:



Where did the component values come from? First, because we want close to unity gain at midband, we want to make R as small as possible, which will allow us to drop a minimal amount of the input voltage across the resistors when travelling to the output (making $v_{out} \approx v_{in}$). So we set $R = 1\Omega$.
This sets the initial values to try for L and C. The low-frequency break point is set by the high pass filter (L), and the high-frequency break point is set by the low pass filter (C). Plugging in to the previous break frequency formulas we end up with $L = 1.6 \times 10^{-4} H$ and $C = 1.6 \times 10^{-7}$.
However, if we use these exact values, our circuit won't check out. Why's that? It's because the low pass and high pass filters we cascaded aren't completely separate; the extra impedance of the capacitor and second resistor affects the transfer function of the high pass block, and vice-versa. There are second-order effects at work which make our calculations close, but not exact.
How can we reconcile this? We could go back and recalculate the full transfer function, but it's an incredibly messy and tedious endeavor. Instead, it is much faster to look at the plot we get and use some intuition to change the values and make it right.



Shown here is the low-frequency break point with a value of $L = 1.6 \times 10^{-4} H$. As we can see, at $1000 Hz$, we are down $-2.959 dB$- almost $-3 dB$, but not quite within the tolerance of the lab simulation. We need to lower the value of L slightly, which makes the break frequency larger, and effectively "shifts" the curve to the right. Shown again is the same point on the same plot, but with $L = 1.58 \times 10^{-4} H$:



We are now almost exactly at $-3 dB$, at least within the tolerance of the lab simulation.
A similar trial-and-error process applies for the capacitor- at $1 MHz$, $(log[f] = 6)$, we again want to be at $-3 dB$. I ended up with a value of about $0.81 \times 10^{-7} F$. Simulating the circuit with the aforementioned values shold pass the check.

In[∘]:=
```
Solve[ArcTan[-ω/(1/(r c))] == π/4, ω]

Solve[ArcTan[(r/l)/ω] == π/4, ω]

r = 1*^3;
(*Converting decibels → Hz, then Hz → rad/s*)
ω = 2 π * 10^(4.6);
(*The y-axis of the phase plot is in deg,
so must be converted from deg → rad*)
(*Red phase plot - Inductor*)
Solve[78.684 * (2 π/360) == ArcTan[(r/l)/ω], l]
(*Blue phase plot - Capacitor*)
Solve[-78.684 * (2 π/360) == ArcTan[-ω/(1/(r c))], c]

r = 1000;(*Just an arbit choice for resistance*)
(*Combining the two plots to make a band-pass filter*)
(*Break 1: Inductor which breaks at 1k Hz*)
f1 = 1*^3;
ω = 2 π f1;
Solve[ω == r/l, l] // N
(*Break 2: Capacitor which breaks at little over 1M Hz.
   Using exactly 1M Hz as per the problem statement does not work.
   I had to shift this second break
  frequency by 0.3 decibels to get the grader to accept.*)
f2 = 10^6.3;
ω = 2 π * f2;
Solve[ω == 1/(c r), c] // N
```

Out[∘]= $\left\{\left\{\omega \to -\frac{1}{c\, r}\right\}\right\}$

Out[∘]= $\left\{\left\{\omega \to \frac{r}{l}\right\}\right\}$

Out[∘]= $\{\{l \to 0.000799999\}\}$

Out[∘]= $\{\{c \to 1.9978 \times 10^{-8}\}\}$

Out[∘]= $\{\{l \to 0.159155\}\}$

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[ ]= $\{\{c \to 7.97664 \times 10^{-11}\}\}$