Week 11

S21 - Filters

S22 - Time Domain Versus Frequency Domain Analysis

Lectures

```
(*In comments we try to follow this notation from A&L. In code anything goes.*)
```

We extended our variable notation to distinguish between total variables, DC operating values, small-signal variables, and complex amplitudes.

- We denote total variables with lowercase letters and uppercase subscripts, for example, v_D.
- DC operating-point variables using all uppercase, for example, V_D.
- Incremental values using all lowercase letters, for example, v_d.
- Complex amplitudes use uppercase letters and lowercase subscripts, for example, V_d.

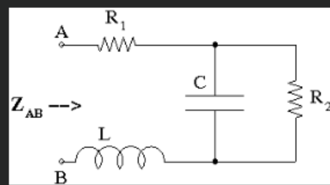
```
(*Recall:
 the behavior of capacitors and inductors to voltage and current impulses*)
```

- Capacitors behave like open circuits when a circuit containing capacitors is driven by a DC voltage source. Conversely, a capacitor behaves like an instantaneous short circuit when inputs make an abrupt transition (for example, a step). (If the capacitor voltage were nonzero, then the capacitor would behave like a voltage source for abrupt transitions.)
- Inductors behave like short circuits when a circuit containing inductors is driven by a DC current source. Conversely, an inductor behaves like an instantaneous open circuit for inputs that make an abrupt transition (for example, a step). (If the inductor current were nonzero, then it would behave like a current source for abrupt transitions.)

S21E1: Second-Order Impedance, Version A

0 points possible (ungraded)

We are presented with a combination of resistors, a capacitor, and an inductor:



The impedance looking into the port with terminals A and B is given:

$$Z_{AB}(\omega) = R_1 + \frac{R_2}{1+j\omega R_2 C} + j\omega L$$

In the space provided below write an algebraic expression that approximates $Z_{AB}(\omega)$ for very small ω. (Remember, write w for ω.)

R1+R2

✔ **Answer:** R1 + R2

$R_1 + R_2$

In the space provided below write an algebraic expression that approximates $Z_{AB}(\omega)$ for very large ω. (Remember, write w for ω.)

j*w*L

✔ **Answer:** j*w*L

$j \cdot \omega \cdot L$

You should notice that at zero frequency the inductor and capacitor are irrelevant, but in different ways: the inductor is a short circuit and the capacitor is an open circuit.

Also notice that as ω increases the impedance is dominated by the inductor.

Detailed Solution:

At low frequencies, capacitors have relatively high impedances and inductors have relatively low impedances.

$$\lim_{\omega \to 0} R_1 + \frac{R_2}{1 + j\omega R_2 C} + j\omega L \approx R_1 + R_2$$

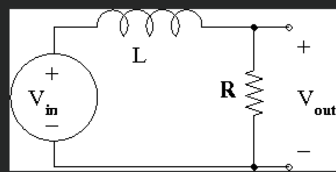
At high frequencies, capacitors have relatively low impedances and inductors have relatively high impedances.

$$\lim_{\omega \to \infty} R_1 + \frac{R_2}{1 + j\omega R_2 C} + j\omega L \approx R_1 + 0 + j\omega L \approx j\omega L$$

S21E2: LR filter

0 points possible (ungraded)

The following circuit is one more combination of an inductor and a resistor driven by a voltage source in a series circuit.



In the space provided below write an algebraic expression for the complex voltage-transfer ratio $\frac{V_{out}}{V_{in}}$. (Remember, write w for ω.)

R/(j*w*L+R) ✔ **Answer:** R/(L*j*w + R)

$$\frac{R}{j \cdot \omega \cdot L + R}$$

What kind of response is this? In the space provided below enter one of the strings: LPF, HPF, or $\mathrm{Neither}$.

LPF ✔ **Answer:** LPF

Solution:

Using the impedance method, we can use the two impedances and treat this system as we would a voltage divider with resistors:

$$\frac{V_{out}}{V_{in}} = \frac{R}{R + j\omega L}$$

When the frequency is low,

$$\lim_{\omega \to 0} \frac{R}{R + j\omega L} = 1$$

and all the voltage is transferred to the resistor because the inductor has relatively low impedance
When the frequency is high,

$$\lim_{\omega \to \inf} \frac{R}{R + j\omega L} = 0$$

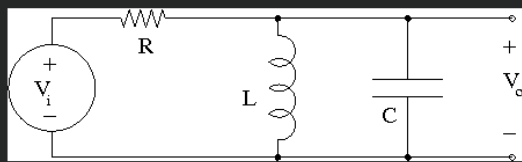
and all the voltage is transferred to the inductor.
This makes this system a **Low Pass Filter**, as the resistor has voltage transfer at low frequencies and none at high frequencies

(*Grader solution for the last part is incorrect, written solution clearly shows a j in numerator*)

S21E3: Thevenin Tank

0 points possible (ungraded)

Consider a parallel combination of an inductor and a capacitor driven by a Thevenin source.



In the space provided below write an algebraic expression for the complex voltage-transfer ratio $\frac{V_o}{V_i}$. (Remember, write w for ω.)

(j*w*L)/(1-w^2*L*C)/(R+(j*w*L)/(1-w^2*L*C)) ✔ **Answer:** (j*L*w)/((R-C*L*R*w^2)+j*L*w)

How does this transfer ratio behave near $\omega = 0$? In the space provided below give an algebraic expression that best approximates this behavior.

(j*w*L)/R ✔ **Answer:** (j*L*w)/R

How does this transfer ratio behave for ω very large? In the space provided below give an algebraic expression that best approximates this behavior.

j/(w*R*C) ✖ **Answer:** 1/(R*C*j*w)

Notice how the inductor and resistor dominates the behavior at low frequencies and how the capacitor and resistor dominates the behavior at high frequencies.

Solution:

Using impedances, this is just a voltage divider between a resistor and a parallel combination of inductor and capacitor:

$$\frac{V_{out}}{V_{in}} = \frac{Z_{CL}}{Z_R + Z_{CL}} = \frac{j\omega L \times 1/j\omega L/\left(j\omega L + 1/j\omega L\right)}{R + j\omega L \times 1/j\omega L/\left(j\omega L + 1/j\omega L\right)}$$

$$\frac{V_{out}}{V_{in}} = \frac{j\omega L}{j\omega L + R - \omega^2 LRC}$$

Now we can find the limits for when the frequency is very large or close to zero:

$$\lim_{\omega \to 0} \frac{V_{out}}{V_{in}} = \frac{0}{0 + R - 0} = 0$$

$$\lim_{\omega \to \infty} \frac{V_{out}}{V_{in}} \approx \frac{j\omega L}{\omega^2 LRC} = 0$$

When the frequency is both large or small, the voltage transfer approaches zero. This matches our intuition, as the capacitor has a low impedance when frequency is large while the inductor has a low impedance when frequency is small. In both limits, their parallel equivalent impedance is low if either one is low.

In[•]:=
```
Zo[w_] := (j * w * L) / (1 - w^2 * L * C);
gain = Zo[w] / (R + Zo[w]);

Limit[gain, w → 0]
Limit[gain, w → ∞]
```

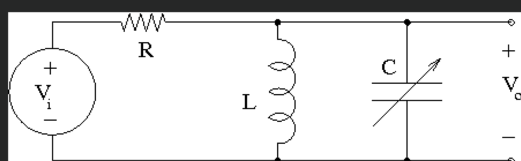
Out[•]= 0

Out[•]= 0

S21E4: AM Radio Tuning

0 points possible (ungraded)

In the United States the frequencies between 535kHz and 1705kHz are allocated to the AM radio broadcast band. Stations are allocated 10kHz channels, so in the worst case two stations may be separated by as little as 10kHz. It is proposed that the channel-selection mechanism for an AM radio could be modeled by the following circuit that uses a variable capacitor to adjust the resonant frequency of a parallel tuned circuit.



In this exercise we will see why this cannot be the whole story.

Assume that the antenna presents an impedance of $R = 10000\Omega$. (This is not very realistic, but the real story is even worse, as we will see!) One traditional tuning capacitor for an AM radio receiver has maximum capacitance of $C_{max} = 365$ pF. This maximum capacitance is used to resonate with the inductor at the lowest frequency.

What is the inductance, in microHenrys, required to resonate with C_{max} at the lower band edge $f = 535$kHz?

0.00024246e6 ✔ **Answer:** 242.45985147315082

Given the inductance you just derived, what is the capacitance, in picoFarads, that the variable capacitor will need to be set to for the circuit to resonate at the higher band edge $f = 1705$kHz?

35.9378 ✔ **Answer:** 35.93781443228043

Now we know all of the part parameters. What is the Q of this tuned circuit at the lower band edge?

12.2695 ✔ **Answer:** 12.269490108594937

What is the Q of this tuned circuit at the upper band edge?

3.84996 ✔ **Answer:** 3.849957306802518

What is the bandwidth Δf, in kHz, of this tuned circuit at the lower band edge?

43604.1e-3 ✔ **Answer:** 43.60409399777954

What is the bandwidth Δf, in kHz, of this tuned circuit at the upper band edge?

442862e-3 ✔ **Answer:** 442.862053817434

This bandwidth is way too large. If we are listening to a station at 1030kHz and there is an interfering station at 1060kHz the carrier of the interfering station will be hardly suppressed. This is a serious problem. In a densely-populated place, New York City for example, WHLI is at 1100kHz, WBBR is at 1130kHz, and WVNJ is at 1160kHz.

City for example, WHLI is at 1100kHz, WBBR is at 1130kHz, and WVNJ is at 1160kHz.

To narrow the bandwidth we would have to raise the Q. But this requires raising the value of R, which is already very high: and antenna radiation resistance is almost never this high, so we would have to use a different way to couple the antenna to the tuned circuit (a loop coupling, making the inductor the secondary of a transformer, would help a lot). But even worse, the bandwidth varies drastically as we tune from one end of the band to the other.

So this tuning scheme cannot be the whole story. It is difficult to make a single tuned circuit that can cover a wide frequency range (a factor of 3 here) with a high enough Q to discriminate stations even 30kHz apart.

The superheterodyne receiver, invented by Edwin Armstrong in 1918, solved this and many other problems.

Solution:

Part 1: C and L

We have the relationship between frequency and angular frequency of a LC circuit, that we will manipulate for each part of this problem:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$L = \frac{1}{4\pi^2 f^2 C_{\max}} = \frac{1}{4 \times \pi^2 \times (535 \times 10^3)^2 \times 365 \times 10^{-12}} = 242.46\mu H$$

We can rework our equation to now solve for capacitance:

$$C_{min} = \frac{1}{4\pi^2 f^2 L} = \frac{1}{4 \times \pi^2 \times (1705 \times 10^3)^2 \times 242.45 \times 10^{-6}} = 35.938pF$$

Part 2: Q

$$Q_L = \frac{R}{2\pi f_L L} = \frac{10000}{2 \times \pi \times 535 \times 10^3 \times 242.45 \times 10^{-6}} = 12.269$$

$$Q_H = R 2\pi f_H C_{min} = 10000 \times 2 \times \pi \times 1705 \times 10^3 \times 35.938 \times 10^{-12} = 3.8499$$

Part 3: Bandwidth

In both cases, we have the equation $\Delta f = \frac{f_0}{Q}$:

$$\Delta f_L = \frac{535 \times 10^3}{12.269} = 43.604\text{kHz}$$

$$\Delta f_H = \frac{1705 \times 10^3}{3.8499} = 442.87\text{kHz}$$

```
(*This is for R||L||C, but it works here*)
```

A useful gauge of the sharpness of the resonance is the ratio of the resonance frequency to the bandwidth:

$$\frac{\text{Resonance frequency}}{\text{Bandwidth}} = \frac{\omega_o}{G/C} = Q = \omega_o RC = \frac{R}{\omega_o L}. \tag{14.47}$$

Again, this very *Quality Factor* was introduced in Chapter 12 (Equations 12.65 and 12.66) from a quite different point of view. There, in the time domain view, Q indicated the length of time for which the circuit would "ring" when excited by an input such as a step. If we know the quality factor Q and the frequency ω_o, we can derive the bandwidth as:

$$\text{Bandwidth} = \frac{\omega_o}{Q}. \tag{14.48}$$

In[•]:=
```
r = 10 000; cmax = 365*^-12;

fl = 535*^3;
ω = 2 π fl;
```
$$\text{Solve}\left[\omega == \frac{1}{\sqrt{l * \text{cmax}}}, l\right] \text{ // N}$$
```
l = 0.0002424598514731508`;

fh = 1705*^3;
ω = 2 π fh;
```
$$\text{Solve}\left[\omega == \frac{1}{\sqrt{l * \text{cmin}}}, \text{cmin}\right] \text{ // N}$$
```
cmin = 3.593781443228042`*^-11;
```
$$\text{Ql} = \frac{r}{2 \pi \text{ fl} * l} \text{ // N}$$

```
Qh = 2 π fh * r cmin // N
```

$$\Delta\text{fl} = \frac{\text{fl}}{\text{Ql}} \text{ // N}$$

$$\Delta\text{fh} = \frac{\text{fh}}{\text{Qh}} \text{ // N}$$

Out[•]= $\{\{l \to 0.00024246\}\}$

Out[•]= $\{\{\text{cmin} \to 3.59378 \times 10^{-11}\}\}$

Out[•]= 12.2695

Out[•]= 3.84996

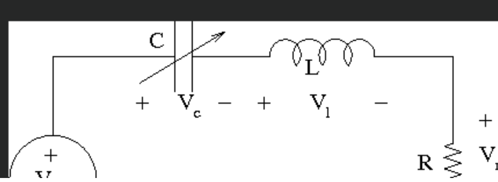
Out[•]= 43 604.1

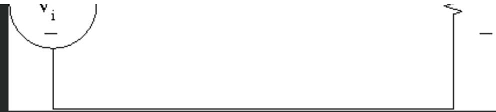
Out[•]= 442 862.

S22E1: Which output?

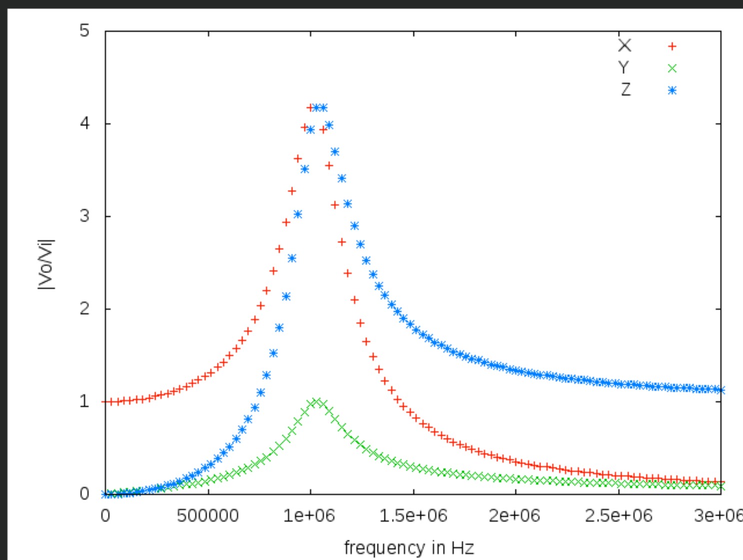
0 points possible (ungraded)

For the following series RLC network we can take as output the voltage across the resistor, the capacitor, or the inductor.





The following graphs show the magnitudes of the voltage-transfer ratios of the amplitudes of these output voltages to the amplitude of the input voltage source V_i.



The graphs are labeled X, Y, Z. Which voltage does each one match?

Which voltage is labeled X? Enter R, L, or C in the space provided.

C ✔ **Answer:** C

Which voltage is labeled Y? Enter R, L, or C in the space provided.

R ✔ **Answer:** R

Which voltage is labeled Z? Enter R, L, or C in the space provided.

L ✔ **Answer:** L

What is the Q of this circuit?

4 ✔ **Answer:** 4.176

Solution:

With Transfer Functions

In the frequency domain, what we have is just three impedances splitting the voltage. We know then that the voltage of the i-th component is then $Z_i = v \times \frac{Z_i}{Z_{total}}$

$$Z = R + j\omega L + \frac{1}{j\omega C}$$

At low frequencies, the capacitor has the highest impedance and should therefore have the most voltage, so X must correspond to the capacitor. At high frequencies, the inductor has the highest impedance and therefore has the most voltage, so Z must correspond to the inductor. This leaves Y for the resistor

With Intuition

By building intuition on frequency domain impedances, we can do relatively fewer computations. We know that capacitors are short circuits at high frequencies and open circuits at low frequencies, which looks like how X behaves. Similarly, inductors are short circuits at low frequencies and open circuits at high frequencies, which looks like how Z behaves. The Q of the circuit is just the peak amplitude of the voltages of L and C.

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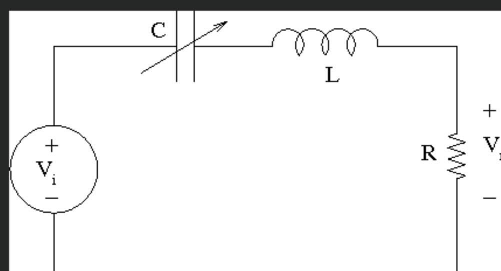
(*The formula for Q given in the solution is incorrect*)
```

S22E2: The filter is ringing

0 points possible (ungraded)

We have analyzed the series RLC circuit several ways



We have determined that the voltage-transfer ratio is

$$\frac{V_r}{V_i} = \frac{j\omega RC}{(1-\omega^2 LC)+j\omega RC}$$

Assume that we are tuned to WCAP-AM at 1.03MHz, and that our inductor has $L = 242\mu$H. What is the capacitance, in picoFarads, needed to resonate at 1.03MHz?

98.6621 ✔ **Answer:** 98.6621210845635

If $R = 375\Omega$ what is the approximate bandwidth (in kiloHertz) of this filter?

246.624 ✔ **Answer:** 246.62439528702794

A nearby lightening bolt produces an impulse in the antenna. For about how many cycles will the filter ring? (Of course, in theory it will ring forever, but a crude estimate is that it takes about Q cycles before the ringing is insignificant. You will be more precise about this in the Homework for this week!)

4.17639 ✔ **Answer:** 4.176391385780223

If $R = 75\Omega$ what is the approximate bandwidth (in kiloHertz) of this filter?

49.3249 ✔ **Answer:** 49.32487905740558

A nearby lightening bolt produces an impulse in the antenna. For about how many cycles will the filter ring?

20.882 ✔ **Answer:** 20.881956928901115

Solution:

Part 1

We resonate when $\omega = 2\pi f = \frac{1}{\sqrt{LC}}$

$$C = \frac{1}{4\pi^2 f^2 L} = 98.66pF$$

Part 2

Note that the answer should be in kHz, so you need to divide by 2π:

$$\Delta f = \frac{R}{2\pi \times L} = 246.62kHz$$

Part 3

$$Q = \frac{\omega}{\Delta f} = 4.176$$

Part 4

$$\Delta f = \frac{R}{2\pi \times L} = 49.32kHz$$

Part 5

$$Q = \frac{\omega}{\Delta f} = 20.88$$

```
f = 1.03*^6; l = 242*^-6;
ω = 2 π f;

Solve[ω == 1/Sqrt[l * c], c] // N

r = 375;
Δf = r/(2 π l) // N (*Convert to kHz after*)
Q = f/Δf // N (*Units must match, beware of solution*)

r = 75;
Δf = r/(2 π l) // N
Q = f/Δf // N
```

Out[∘]= $\{\{c \to 9.86621 \times 10^{-11}\}\}$

Out[∘]= 246 624.

Out[∘]= 4.17639

Out[ ]= 49 324.9

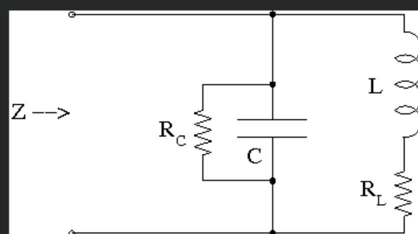
Out[ ]= 20.882

Homework

H3P1: LC Tank

5/5 points (graded)

Parallel resonant "tank" circuits are common in radio equipment. But unfortunately there is always resistance that prevents them from being perfect: in every real inductor, the wire that makes up the inductance has some resistance, and there may be leakage in the capacitor that can be modeled as a resistance. Also, the Norton resistance of the system connected to the tank circut looks like a leakage through the capacitor. So a realistic model for a tank circuit is the following:



In the space provided below write an algebraic expression in terms of the device parameters for the bandwidth $\Delta\omega$ of the impedance Z looking into this tank circuit.

(RC*RL*C+L)/(RC*L*C) ✔ **Answer:** (RL/L + 1/(RC*C))

$$\frac{R_C \cdot R_L \cdot C + L}{R_C \cdot L \cdot C}$$

The antenna tank of a Graymark 536 Radio Kit has an inductance $L \approx 0.65$mH. The resistance of the inductor $R_L \approx 4.0\Omega$. The equivalent resistance across the capacitor is $R_C \approx 490.0\text{k}\Omega$. The capacitor is variable, for tuning.

If we tune to a station at $f = 650.0$kHz what is the capacitance, in picoFarads, of the tuning capacitor?

92.2359 ✔ **Answer:** 92.24

What is the bandwidth, in kHz, of the tank at $f = 650.0$kHz?

4.50088 ✔ **Answer:** 4.5

If next we tune to a station at $f = 1570.0$kHz what is the capacitance, in picoFarads, of the tuning capacitor?

15.8098 ✔ **Answer:** 15.81

What is the bandwidth, in kHz, of the tank at $f = 1570.0$kHz?

21.524 ✔ **Answer:** 21.52

So it is apparent that this is not the only circuit in the radio that selects the desired station from stations on adjacent channels.

Explanation:
The impedance of the capacitor is $Z_C = \frac{1}{j\omega C}$, and the impedance of the inductor is $Z_L = j\omega L$. The terminal impedance of the circuit Z can be found by connecting the components in series and parallel:

$$Z = (Z_C \parallel R_C) \parallel (Z_L + R_L)$$

$$= \frac{R_C}{\quad} \parallel (j\omega L + R_L)$$

$$j\omega R_C C + 1 \parallel (j\omega L + R_L)$$

$$= \frac{j\omega R_C L + R_C R_L}{(j\omega)^2 LCR_C + j\omega\left(R_L R_C C + L\right) + \left(R_L + R_C\right)}$$

$$= \frac{j\omega\left(\frac{R_C L}{LCR_C}\right) + \frac{R_C R_L}{LCR_C}}{(j\omega)^2 + j\omega\left(\frac{R_L R_C C + L}{LCR_C}\right) + \frac{R_L + R_C}{LCR_C}}$$

The denominator of this expression is the characteristic polynomial of the circuit:

$$s^2 + 2\alpha s + {\omega_0}^2 = 0$$

where:

$$s = j\omega$$

$$2\alpha = \frac{R_L R_C C + L}{LCR_C}$$

$${\omega_0}^2 = \frac{R_L + R_C}{LCR_C}$$

Since we know that $\Delta\omega = 2\alpha$ in a second order circuit, we know that the bandwidth of the circuit must be:

$$\Delta\omega = \frac{R_L R_C C + L}{LCR_C} = \frac{R_L}{L} + \frac{1}{R_C C} \tag{1}$$

As previously derived, the angular resonance frequency of the circuit is:

$$\omega_0 = \sqrt{\frac{R_L + R_C}{LCR_C}}$$

Which with some rearranging looks like the following:

$$2\pi f_0 = \sqrt{\frac{1}{LC}\left(\frac{R_L}{R_C} + 1\right)}$$

Remember that in an ideal RLC circuit, $2\pi f_0 = \frac{1}{\sqrt{LC}}$. The $\frac{R_L}{R_C} + 1$ factor in this equation tells us that the two resistors introduce a non-ideal behavior to the circuit, that affects the resonant frequency as this factor exceeds 1. This non-ideal behavior is undesirable in a radio circuit!

Fortunately, the Graymark kit provies a nice inductor with $R_L \approx 4.0\Omega$ and a nice capacitor with $R_C \approx 490k\Omega$. The ratio between their resistances is so small (10^{-5}) that it is approximately zero when added to 1. Because of this, we can simply tune the capacitor using the ideal RLC equation:

$$C = \frac{1}{L(2\pi f_0)^2} \tag{2}$$

The answers are as follows:

(a) $\Delta\omega = \frac{R_L}{L} + \frac{1}{R_C C}$

Plugging in our given parameters into the above equations we find that:

(b) Plugging in $f = 650.0kHz$ and $L = 0.65mH$ into eq(2), we find the tuning capacitance to be $92.24pF$

(c) Plugging in the above capacitance and inductance into eq(1), we find the bandwidth to be $4.5kHz$

(d) Plugging in $f = 1570.0kHz$ and $L = 0.65mH$ into eq(2), we find the tuning capacitance to be $15.81pF$

(e) Plugging in the above capacitance and inductance into eq(1), we find the bandwidth to be $21.52kHz$

You should get an impedance of the form $Z = \frac{A\,(j\omega) + B}{(j\omega)^2 + C\,(j\omega) + D}$ where A, B, C and D are real.

With $s = j\omega$ the characteristic polynomial of this circuit is of the form $s^2 + 2\alpha s + \omega_0^2$ ie $2\alpha = C$ and $\omega_0^2 = D$.

```
In[•]:= l = 0.65*^-3; rl = 4; rc = 490*^3;

Δf[c_] = 1/(2 π) (rc rl c + l)/(rc l c);

f = 650*^3;
ω = 2 π f;
Solve[ω == 1/Sqrt[l*c], c] // N
Δf[9.223594323380773`*^-11] // N

f = 1570*^3;
ω = 2 π f;
Solve[ω == 1/Sqrt[l*c], c] // N
Δf[1.5809844625049196`*^-11] // N
```

Out[•]= $\{\{c \to 9.22359 \times 10^{-11}\}\}$

Out[•]= 4500.88

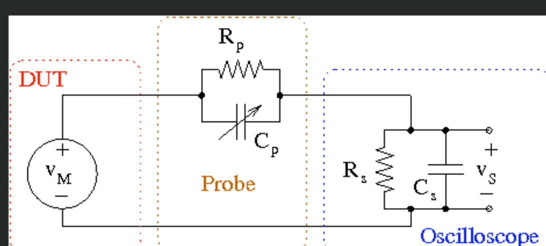
Out[•]= $\{\{c \to 1.58098 \times 10^{-11}\}\}$

Out[•]= 21 524.

H3P2: Scope Probe

4/4 points (graded)

We can model the interaction of an oscilloscope with the device under test (DUT) with the following circuit.



A typical modern oscilloscope has an input resistance of $R_s = 1.0\mathrm{M}\Omega$ and an input capacitance of $12 < C_s < 25$pF. Assume that the input capacitance of our oscilloscope $C_s = 13.0$pF. It is connected to the circuit with a 10X probe with resistance of $R_p = 9.0\mathrm{M}\Omega$. There is a variable capacitor, adjustable with a special tool, that is in parallel with the resistance in the probe. The device under test is modeled as a voltage source with voltage v_M.

If the frequency of the source being measured is very low or a DC value, for example $v_M = 0.9$V, what is the value of the voltage v_S, in Volts, that the oscilloscope sees?

0.09 ✔ **Answer:** 0.09000000000000001

The oscilloscope is used to measure and display waveforms at many frequencies. Suppose that $V_M = 0.08\cos\left(2\pi\, 75.0 \times 10^6\; t\right)$ V. What is the value, in picoFarads, that C_p should be set to so that $V_s = 0.008$V?

1.44

✔ **Answer:** 1.44

Suppose now that $V_M = 0.4\cos\left(2\pi\, 194.0 \times 10^6\ t\right)$ V. What is the value, in picoFarads, that C_p should be set to so that $V_s = 0.04000000000000001$V?

1.44

✔ **Answer:** 1.44

Hmmmmm....

Now suppose that $V_M = 6.0\, u\,(t)$ V and suppose that just before the step, $V_{C_P}\left(0_-\right) = 0$ and $V_{C_S}\left(0_-\right) = 0$. Assume that the value of C_p is what you computed for the last question. What is the value of $v_S\left(0_+\right)$, in Volts?

0.6

✔ **Answer:** 0.6000000000000001

Explanation:
(a) At very low frequencies, the capacitors are approximately open circuits, and can be ignored. Solving the resistor voltage divider circuit, we get:

$$V_{out} = \frac{R_s}{R_s + R_p} \cdot V_{in} = \frac{1.0}{1.0 + 9.0} * 0.9 = 0.09000000000000001V$$

This is what we expect, given that the proe is specified to be 10:1 in the question!
(b) At higher frequencies, the reactance of the capacitors will begin to participate in the attenuation process as well. Let us define the two blocks of impedances as the following:

$$Z_s = R_s \parallel \frac{1}{j\omega C_s}$$

$$Z_p = R_p \parallel \frac{1}{j\omega C_p}$$

We want this ratio of $Z_p : Z_s$ to be $9:1$. Since the ratio of $R_p : R_s$ is already $9:1$, we would like the capacitor reactance ratio to be the same as well:

$$\frac{\frac{1}{j\omega C_p}}{\frac{1}{j\omega C_s}} = 9$$

$$\frac{C_s}{C_p} = 9$$

$$C_p = \frac{C_s}{9}$$

Given that $C_s = 13.0pF$, $C_p = \frac{C_s}{9} = 1.44pF$. Note that the $j\omega$ terms canceled out in this derivation. This suggests that the single C_p we have calculated should be valid for all frequencies.
(c) As mentioned above, $C_p = 1.44pF$
(d) A unit step is simply the superposition of many sine waves of different frequencies. Since the attenuation has been set at $10:1$ for all frequencies, the unit step should be attenuated by a factor of 10 as well. $v_s\left(0^+\right) = 0.6000000000000001V$.

For Part **a** the statement *If the frequency of the source being measured is very low or a DC value . . . .* allows you to ignore some of the circuit elements.

For Part **b** in effect you have two potential divider circuits one consisting of two resistors and the other consisting of two capacitors.
So treating them independently arrange for them both potential dividers to give the same output for a given input.

Look at the circuit as two potential dividers one of which consists of two resistors (the "real" part) and the other which consists of two capacitors (the "imaginary" part).

You have found the relationship between V_M and V_S when dealing with two resistors as a potential divider at very low frequencies.

Now make the relationship between those two voltages the same by consider just the two capacitors as a potential divider.

There is no need to make it too complex. Think about what is unit step func? It is actually the superposition of numerous sinusoidal signals. Thus you can still use the previous conclusions. Here if we decrease the amplitude of the source for 10, how many times we should reduce V_s?

```
Rp[r_List] := 1 / Total[1 / r];
rs = 1*^6; cs = 13*^-12; rp = 9*^6;

(*For a DC voltage, capacitors are open, only consider the resistors.*)
vm = 0.9;
vs = rs/(rp + rs) vm

(*For AC voltage, we consider just the two capacitors as the potential divider.
   The ratio 0.1 = vs/vm = zs/(zs+zp)*)
ω = 2 π * 75*^6;
zs = 1/(ω cs);
zp = 1/(ω cp);
(*The ratio 0.1 is the same for both parts, and ω cancels out*)
Solve[0.1 == zs/(zs + zp)]
```

Out[ ]= 0.09

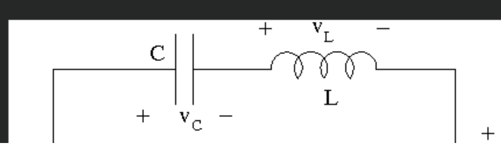
Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

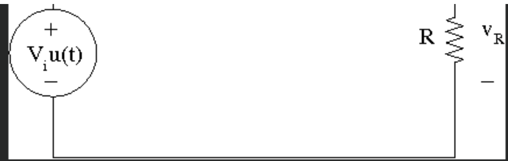
Out[ ]= $\{\{cp \to 1.44444 \times 10^{-12}\}\}$

H3P3: Branch Voltages

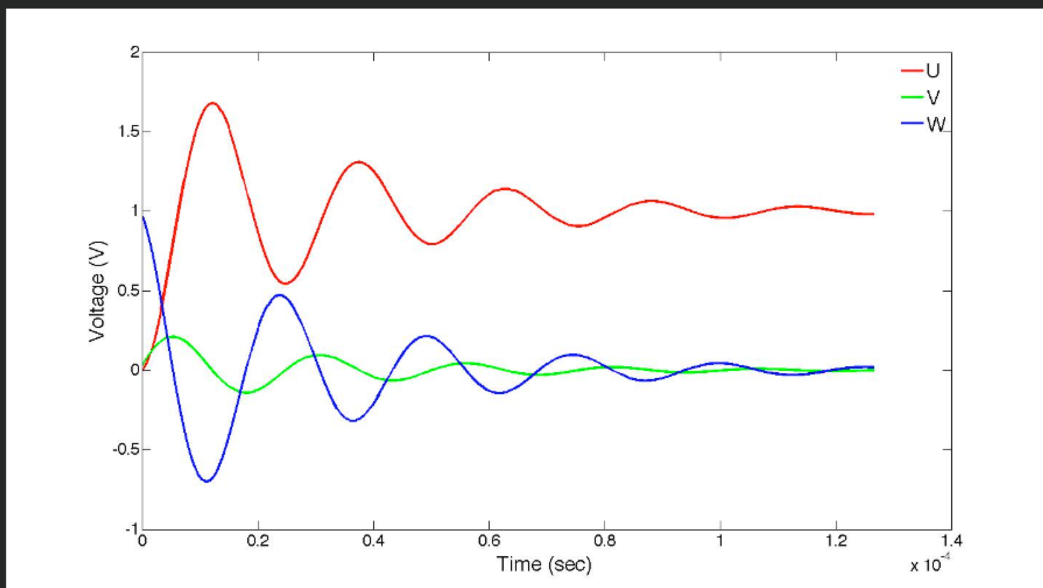
8/8 points (graded)

In the following circuit there are three branch voltages indicated. The voltage source has been off long enough so that all the energy in the circuit has dissipated. The voltage source turns on at $t = 0$ with $V_i = 1V$.





Unfortunately, one of your engineers has played a trick on you and disguised the capacitor, resistor and inductor so that you cannot distinguish them by sight. Fortunately for you, you can measure the voltage across each unknown element versus time. You get the following time-domain voltage traces for each of the three branch voltages



The graphs are labeled U,V, and W. Which voltage corresponds to which element?

Which branch voltage is labeled U? Enter R, L, or C in the space provided.

C

Answer: C

C

Which branch voltage is labeled V? Enter R, L, or C in the space provided.

R

Answer: R

R

Which branch voltage is labeled W? Enter R, L, or C in the space provided.

L

Answer: L

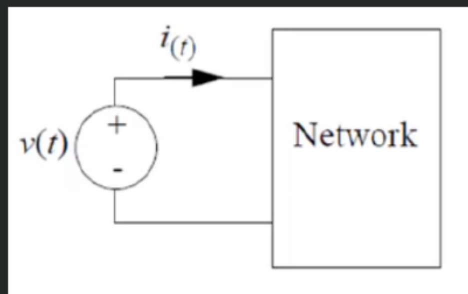
L

From the voltage graphs given above, estimate the Q of the second-order system. Enter your estimation in the space provided below. (Hint: the answer is an integer).

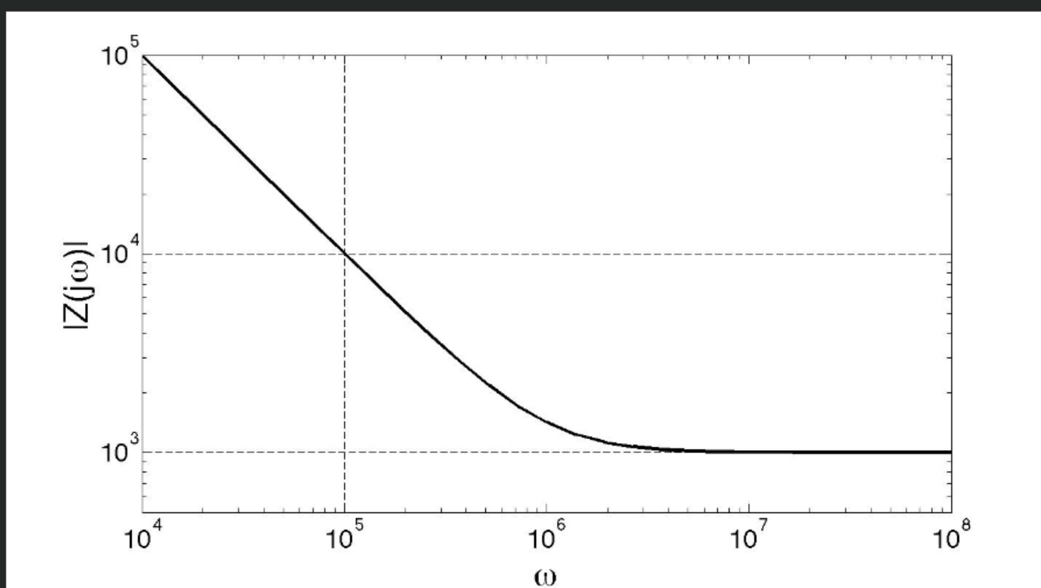
4

Answer: 4

Your engineers are feeling mischevous again and have removed either the inductor or the capacitor from the original circuit, but they won't tell you which. You decide to determine which element they removed by measuring the magnitude of the impedance of the circuit looking from the terminals of the voltage source. You set up a circuit to measure the impedance of the RL or RC network like the one below:



You get the following Frequency-domain plot of the magnitude of the impedance:



The solid black line is the magnitude of the impedance above.

Based off of the graph of the magnitude of the impedance, which element was removed? Enter L or C in the space provided.

L ✔ **Answer:** L

L

What is the value of the resistor in Ohms (Ω)?

1e3 ✔ **Answer:** 1000

What is the value of the remaining element (capacitor or inductor) in the circuit? Enter the capacitance in nF, or the inductance in mH, depending on which element you determined is still in the circuit.

1e9/999999999 ✔ **Answer:** 1

1e9/990000000 ✔ Answer: 1

Based off of the integer value of Q you determined earlier, you could also estimate the value of the element removed from the circuit! Enter the capacitance in nF or the inductance in mH, depending on which element you determined was removed from the circuit.

8e3/495 ✔ **Answer:** 16

Explanation:
(a),(b),(c) Consider exciting a series RLC circuit at rest with a step response. At $t = 0^+$, the common current should remain zero, as the inductor would prevent current flowing through it from changing instantaneously. As $V_R = IR$, this means that $V_R(0^+) = 0V$. Also at $t = 0^+$, the capacitor votage cannot change instantaneously either, such that $V_C(0^+) = 0V$. This means that the only waveform with $V(0^+) \neq 0$ must be the inductor voltage V_L.
After a very long time, the inductor turns into a short circuit, and the capacitor turns into an open circuit. Because of this, we expect V_R to decay back to zero, as the common current flow must decay to zero. Therefore the waveform with $V(0^+) = 0$ and $V(\infty) = 0$ must be the resistor voltage V_R.
This makes the final waveform V_C.
(d) The quality factor Q can be written as the ratio between the damping factor α and the resonant frequency ω_0:

$$Q = \frac{\omega_0}{2\alpha}$$

The period of oscillation is $T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$. Substituting the period of oscillation:

$$Q = \frac{\pi}{T_0\alpha}$$

Noting that the inductor waveform starts off at $1V$, we can interpolate and look for where the envelope decays to $e^{-1} \approx 0.37V$. This occurs in about $3 \times 10^{-5}s$, therefore $\alpha = \frac{1}{3\times10^{-5}s} = \frac{1}{3} \times 10^5 s^{-1}$. The period of oscillation is about $2.5 \times 10^{-5}s$. Substituting, we get the following estimate for Q:

$$Q \approx \frac{\pi}{\left(\frac{2.5}{3}\right)} = 3.7 \approx 4$$

(e) As expected, the impedance is infinite at DC due to the effect of the capacitor. Unexpectedly, the impedance is no longer infinite at $f = \infty$. This suggests that the inductor was removed.
(f) With the inductor gone, the resistor determines the circuit impedance at high frequencies, since the capacitor behaves like a short circuit here. Reading off the impedance chart, $R = 1k\Omega$.
(g) The graph shows that in the capacitor dominated region, $Z = 10^4$ for $\omega = 10^5$. Since $|Z_c|$ = $\frac{1}{\omega C}$, $C = \frac{1}{\omega|Z_c|} = 10^{-9} = 1nF$.
(h) In a series circuit,

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$L = Q^2R^2C$$

With $Q = 4$, $R = 10^3$, $C = 10^{-9}$, $L = 16 \times 10^{-3} = 16mH$.

If you have a series capacitor/resistor or inductor/resistor circuit they will both have the similar characteristics as follows.

As the frequency of the applied voltage increases or decreases the effect of the resistor will dominate and it will be as though there is only a resistor in the circuit.

```
Q = 4;

(*Read the high frequency impedence from the graph,
corresponding to that of a resistor*)
r = 1*^3;

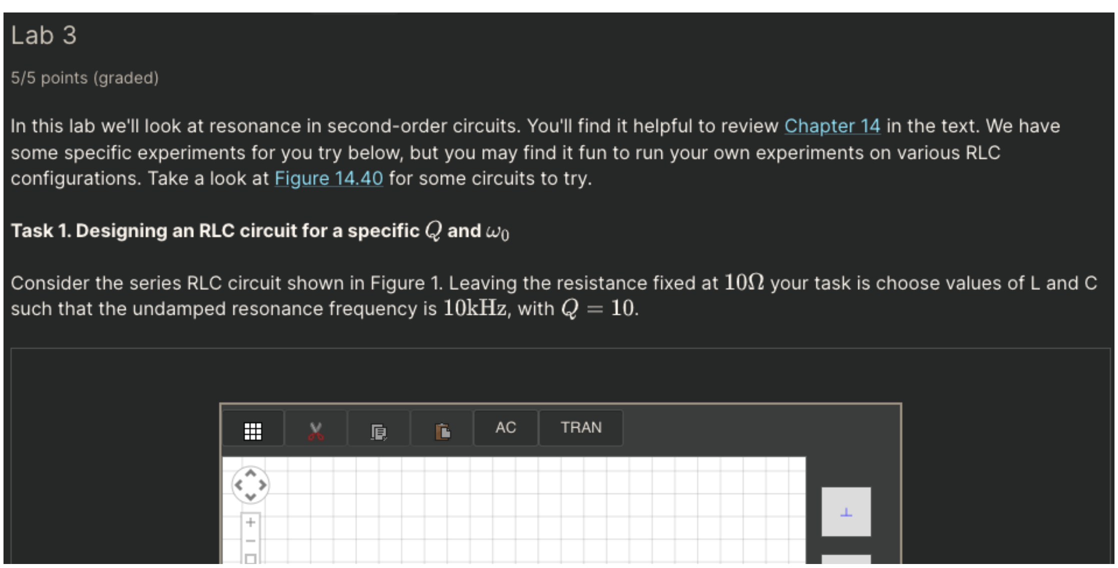
(*Because inductor was removed, this is an RC circuit,
solve for capacitance. Read off the left most value of ω,
intermediate value from the graph does not work.*)
ω = 1*^4; magz = 1*^5;
Solve[magz == r + 1/(ω c), c]

(*Find the resonant frequency first*)
c = 1/990 000 000;
ω0 = 1/Sqrt[l c];
(*Then use the Q relation with R and L, Section 4.3 of A&L*)
Solve[Q == ω0 l/r, l]
```

Out[∘]= $\left\{\left\{c \to \frac{1}{990\,000\,000}\right\}\right\}$

Out[∘]= $\left\{\left\{l \to \frac{8}{495}\right\}\right\}$

Lab



Lab 3

5/5 points (graded)

In this lab we'll look at resonance in second-order circuits. You'll find it helpful to review Chapter 14 in the text. We have some specific experiments for you try below, but you may find it fun to run your own experiments on various RLC configurations. Take a look at Figure 14.40 for some circuits to try.

Task 1. Designing an RLC circuit for a specific Q and ω_0

Consider the series RLC circuit shown in Figure 1. Leaving the resistance fixed at 10Ω your task is choose values of L and C such that the undamped resonance frequency is $10\mathrm{kHz}$, with $Q = 10$.

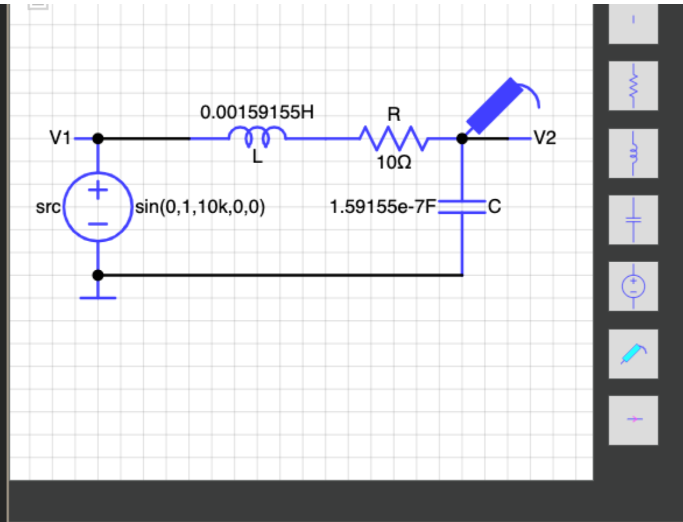


Figure 1. Series RLC circuit

With correctly chosen values for L and C, the results of an AC analysis plotting $|H(s)| = |\frac{V2}{V1}|$ should look like Figure 2. Note that an AC analysis requires that one specify the voltage or current source where the test tone is to be injected -- in this case, it's the voltage source labeled "src".

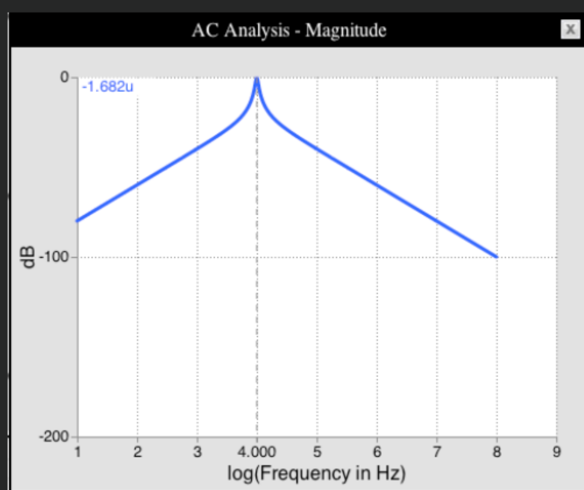


Figure 2. Magnitude of the frequency response

Verify your results by entering your values for L and C in Figure 1 and running an AC analysis. Please enter the values for L and C that give the specified ω_o and Q:

Value for L (in henries)

0.00159155 ✔ **Answer:** 1592e-6

Value for C (in farads)

1.59155e-7 ✔ **Answer:** 159.1e-9

Here's an analysis of the circuit Figure 1 that may help in finding the correct values. Using the impedance model, we can write the system function as

$$H(s) = \frac{V2}{V1} = \frac{R}{R + (sL + 1/sC)}$$

which we can rewrite so that the demonimator is in the standard form $s^2 + 2\alpha s + \omega_0^2$ for second-order systems:

$$H(s) = \frac{s\frac{R}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Matching terms in the standard form to the denominator in the system function, we see that

$$\omega_0 = \sqrt{\frac{1}{LC}} \text{ and } \alpha = \frac{R}{2L}$$

Noting that $Q = \omega_0/2\alpha$, we have the machinery in place to determine L and C given the desired Q, ω_0 and R.

Task 2. Step response of a resonant RLC circuit

Observe that the condition for resonance given by Equation 14.9, $\alpha < \omega_0$, is the same as that for under-damped dynamics described in Section 12.1. So we'd expect the RLC system above to respond to a step input with a ringing response. Using your modified values for L and C, change the voltage source in Figure 1 to a $1V$ step and run a 2 ms TRAN simulation. Your result should like that shown in Figure 3.

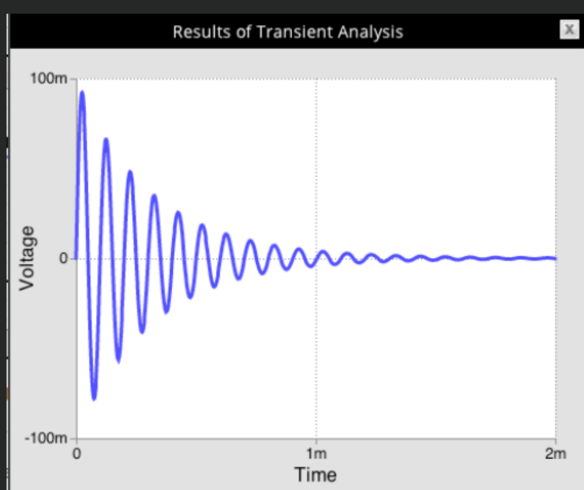


Figure 3. Resonant RLC response to a step input

As expected, the period of the ringing is $\frac{2\pi}{\omega_d} \approx \frac{2\pi}{2\pi \cdot 10000} = 100\mu s$ since the difference between ω_0 and ω_d is small. Measure the amount decay over a span of $Q = 10$ oscillations by taking the ratio of the maximum value in the 11th oscillation at $t \approx 1025\mu s$ to the maximum value of the 1st oscillation at $t \approx 25\mu s$.

Ratio of V2 @ 1025us to V2 @ 25us:

0.0399179 ✔ **Answer:** 0.0356

Looking at Figure 12.16, we'd expect the amplitude of the ringing to decay as $e^{-\alpha t}$, so the computed decay over Q oscillations is

$$e^{-\alpha \cdot Q \cdot \frac{2\pi}{\omega_d}} \approx e^{-\alpha \cdot \frac{\omega_0}{2\alpha} \cdot \frac{2\pi}{\omega_0}} = e^{-\pi} \approx 0.04$$

This leads to a nice rule of thumb: the ringing in the step response for a resonant RLC circuit with a given Q lasts for approximately Q cycles.

Task 3. Response at the resonant frequency

Switch the positions of the R and C in Figure 1, then run an AC analysis. You should see a plot of the magnitude of the system response (now measuring the voltage across the capacitor) like that shown in Figure 4.

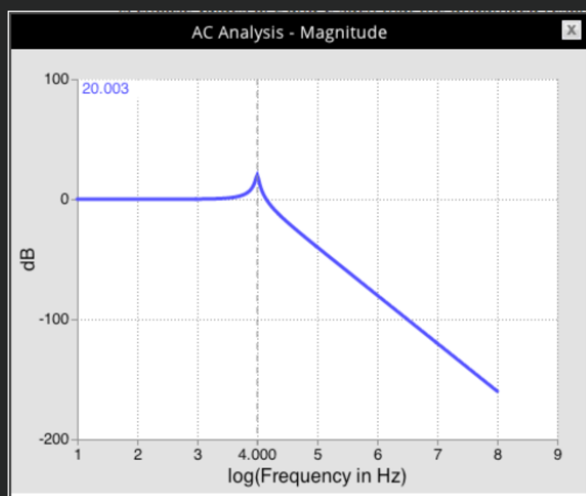


Figure 4. Frequency response with R and C interchanged.

Change the voltage source in Figure 1 back to generating a sinusoid with an amplitude of 1V and a frequency of 10kHz. Run a 2ms TRAN simulation and report the steady-state amplitude of the voltage V2.

Steady-state amplitude of voltage V2 at resonance:

10 ✔ **Answer:** 10

Wow, if we drive the system with a 1V 10kHz sinusoid, we'll see a 10V 10kHz sinusoid across the capacitor! This is consistent with what we learned from the AC analysis shown in Figure 4. The response at $\omega_0 = 10\text{kHz}$ is 20dB, i.e., $20 \log\left(\frac{V2}{V1}\right) = 20$ or $V2 = 10 \cdot V1$.

Where did all this gain come from? Think of pushing a swing. If you time your push to be at the same point of each oscillation, the amplitude of the oscillation will increase slightly with each push, until the energy dissipated during each oscillation due to air resistance and hinge friction exactly equals the energy imparted by the push. In this RLC circuit, the "push" is provided by the sinusoidal voltage source at exactly the frequency of the oscillation (the resonance frequency) and is dissipated by the resistor. As you see in the plot of the TRAN simulation, the amplitude at the output increases until the energy added is balanced by the energy dissipated, at which point we reach a steady state.

Explanation
Design for ω_0 and Q:
To design L and C to achieve a particular ω_0 and Q response, we are given the equations:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{1}{\ }\sqrt{\frac{L}{\ }}$$

These are two simultaneous non-linear equations for the unknowns L and C. With some simple rearranging:

$$LC = \frac{1}{{\omega_0}^2}$$

$$\frac{L}{C} = (QR)^2$$

Now we can substitute to solve the equations:

$$C = \frac{1}{\omega_0 QR}$$

$$L = \frac{QR}{\omega_0}$$

For $f_0 = 10kHz$, $Q = 10$, $R = 10\Omega$, substituting in the equations gives $L = 1.59mH$ and $C = 159nF$.

Step Response of a Resonant RLC Circuit:

The solutions are given in the question. The ration should be $e^{-\pi} \approx 0.04$.

Response at Resonant Frequency:

The solutions are given in the question. At resonance, the circuit gain should be equal to the Q of the system, $V_{SS} = 10V$.

In[◦]:=
```
ω0 = 2 π * 10*^3; r = 10; Q = 10;

Solve[Q == ω0 l / r, l] // N
l = 0.0015915494309189536;
Solve[ω0 == 1/Sqrt[l c], c] // N

(*Reading out the results from transient
 analysis (time on the x-axis instead of frequency),
after changing input from sin → step*)
3.677*^-3 / 92.114*^-3
```

Out[◦]= {{l → 0.00159155}}

Out[◦]= $\{\{c \to 1.59155 \times 10^{-7}\}\}$

Out[◦]= 0.0399179