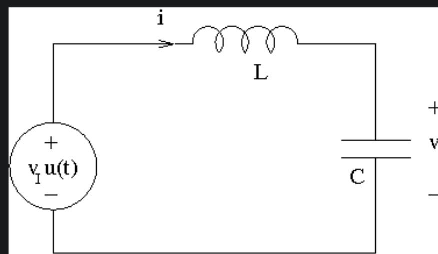
Week 8

Lectures

S17E1: Particular Solution

0 points possible (ungraded)

We have the circuit



with the input $V = V_I u(t)$ and with initial conditions $v(0) = 0$ and $i(0) = 0$.

The capacitor voltage in this system is described by a second-order linear constant-coefficient ordinary differential equation:

$$LC\frac{d^2 v(t)}{dt^2} + v(t) = V_I u(t)$$

After $t = 0$ the step becomes a constant.

So first we need a particular solution $v_P(t)$ to the equation:

$$LC\frac{d^2 v_P(t)}{dt^2} + v_P(t) = V_I$$

(Any particular solution will do, because we will use the associated homogeneous equation to match up the initial conditions.)

We can use $v_P(t)$ as the value that satisfies the equation and makes all the derivative terms in the equation zero.

What is this particular solution $v_P(t)$?
Remember, algebraic expressions are case sensitive, and use the format "vXYZ" to specify v_{XYZ} when you have subscripts in the answer boxes.

VI

✔ **Answer:** VI

V_I

This particular solution is not yet the solution to our problem, because it does not match the initial conditions.

Solution:

One easy solution is a steady-state where all the voltage is on the capacitor and none is on the inductor:

$$v_P = V_I$$

$$LC\frac{d^2 v_P(t)}{dt^2} = 0$$

S17E2: Characteristic Equation

0 points possible (ungraded)

We now need the general solution to the second-order homogeneous linear constant-coefficient ordinary differential equation associated with our problem, to give us the freedom to match the initial conditions. The homogeneous equation is:

$$LC\frac{d^2 v_H(t)}{dt^2} + v_H(t) = 0$$

All homogeneous linear constant-coefficient ordinary differential equations have solutions of the form $v_H(t) = Ae^{st}$ because $\frac{de^{st}}{dt} = se^{st}$. If we substitute in this trial solution we get the characteristic equation: a polynomial equation in s.

Since this is second order the polynomial is of the form:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

For our differential equation what are the values of α and ω_0. Write algebraic expressions for them in the spaces below.

$\alpha =$

0 ✔ **Answer:** 0

0

$\omega_0 =$

1/sqrt(L*C) ✔ **Answer:** 1/sqrt(L*C)

$\frac{1}{\sqrt{L \cdot C}}$

Solution:

We start off by plugging in $v_H + Ae^{st}$:

$$LC\left(s^2 Ae^{st}\right) + Ae^{st} = 0$$

$$LCs^2 + 1 = 0 \rightarrow s^2 + \frac{1}{LC} = 0$$

$$\rightarrow \alpha = 0, \omega_0 = \frac{1}{\sqrt{LC}}$$

S17E3: Matching Initial Conditions

0 points possible (ungraded)

Our total solution is formed from the sum of the particular solution we found and the general solution of the associated homogeneous equation. The undetermined coefficients in the homogeneous part are determined by the initial conditions.

$$v(t) = V_I + A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t}$$

We were given that $v(0) = 0$ and $i(0) = C\frac{dv(t)}{dt}|_{t=0} = 0$.

This is enough information to solve for A_1 and A_2.

In the spaces provided write algebraic expressions for A_1 and A_2.

$A_1 =$

-VI/2

✔ **Answer:** -VI/2

$-\frac{V_I}{2}$

$A_2 =$

-VI/2

✔ **Answer:** -VI/2

$-\frac{V_I}{2}$

Solution:

We can use our initial conditions to build equations to solve for our variables:

$$v(0) = 0, i(0) = C\frac{dv(0)}{dt} = 0$$

$$v(0) = V_1 + A_1 + A_2 = 0$$

$$i(0) = 0 + j\omega A_1 - j\omega A_2 = 0 \rightarrow A_1 = A_2$$

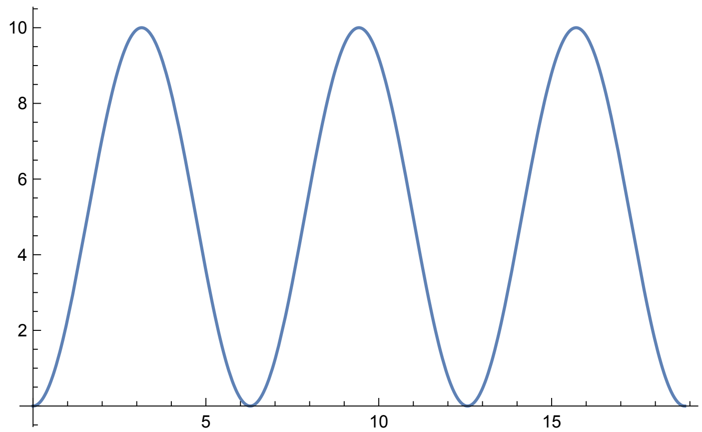
$$v(0) = V_1 + A_1 + A_1 = 0$$

$$A_1 = A_2 = -\frac{V_1}{2}$$

```
(*Summary of driven LC oscillator*)
DSolve[{l c v''[t] + v[t] == V, v'[0] == 0, v[0] == 0}, v[t], t]
(*Plotting this solution for initial conditions: voltage V_I=
 5 V and oscillator frequency ω_0= 1/√LC = 1*)
Plot[5 - 5 Cos[t], {t, 0, 6 π}]
```

Out[]= $\left\{\left\{v[t] \to V - V \operatorname{Cos}\left[\frac{t}{\sqrt{c}\ \sqrt{l}}\right]\right\}\right\}$

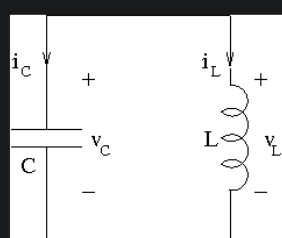
Out[]=



S17E4: An LC circuit

0 points possible (ungraded)

Here is another way to attack a simple L-C circuit:



There are two differential equations among the circuit variables i_L and v_C:

$$C\frac{dv_C(t)}{dt} = -i_L\left(t\right)$$

$$L\frac{di_L(t)}{dt} = v_C\left(t\right)$$

These are called the "state equations" for the circuit.

Suppose that we are given the initial state: $v_C\left(0\right) = V$ V and $i_L\left(0\right) = 0$A. We want to find $v_C\left(t\right)$ and $i_L\left(t\right)$ for all time t.

Let's try $v_C\left(t\right) = A\cos\left(\omega t\right)$ and $i_L\left(t\right) = B\sin\left(\omega t\right)$: (I know this works!)

For each of the following questions write an algebraic expression in terms of L, C, and V in the space provided.

What is the value of ω?

1/sqrt(L*C)

✔ **Answer:** 1/sqrt(L*C)

$\frac{1}{\sqrt{L \cdot C}}$

What is the value of A?

V

✔ **Answer:** V

V

What is the value of B?

V*sqrt(C/L)

✔ **Answer:** V*sqrt(C/L)

$V \cdot \sqrt{\frac{C}{L}}$

Solution:

Start off with knowing that the angular frequency of our system is still $\omega = \frac{1}{\sqrt{LC}}$

$$v_C(0) = V = A + 0 \rightarrow A = V$$

$$v_c(t) = -v_L(t) = L\frac{di_L(t)}{dt}$$

$$A\cos(\omega t) = B\omega\cos(\omega t)$$

$$B = \frac{A}{\omega} = A\sqrt{LC}$$

```
(*Part 4 solution is repeated (typo)*)
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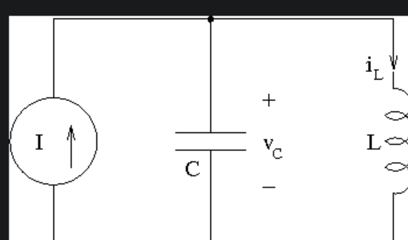
S17E5: An ILC circuit

0 points possible (ungraded)

Consider the circuit:



In the space provided below write an algebraic expression for the second derivative of the inductor current in terms of the source current I, the inductor current i_L, the inductance L, and the capacitance C.

$\frac{d^2 i_L}{dt^2} =$

(I - iL)/(L*C)

✔ **Answer:** (I - iL)/(L*C)

Let $I(t) = I_0 u(t)$, a step of height $I_0 = 3.0$A starting at $t = 0$.

Suppose also that just after the step $i_L(0^+) = 0.0A$ and $v_C(0^+) = 0.0V$.

For $t > 0$ what would $i_L(t)$ have to be, in Amperes, to make $\frac{d^2 i_L(t)}{dt^2} = 0$?

3

✔ **Answer:** 3.0

Your answer should have been $I_0 = 3.0$. So $I_P(t) = 3.0$ is a particular solution of the differential equation for $t > 0$.

Unfortunately, our particular solution cannot be the solution to our problem because it does not match the initial conditions. So we must add in an appropriate amount of solution of the homogeneous equation to match the initial conditions.

In the space provided below write the characteristic polynomial for this differential equation in terms of the indeterminate s and the parameters L and C.

L*C*s^2 + 1

✔ **Answer:** L*C*s^2 + 1

If we want the natural frequency of oscillation of this circuit to be $f = 13.0$MHz and we have a capacitor of $C = 100.0$pF what inductance L, in microHenrys, should we use? Note: $\omega_0 = 2\pi f$.

1.4988

✔ **Answer:** 1.4988340775493751

A general form of the solution to the homogeneous equation associated with this circuit is $i_H(t) = A\sin(\omega_0 t + \phi)$. So the solution to our problem for $t > 0$ can be written $i_L(t) = i_P(t) + i_H(t)$, or $i_L(t) = I_0 + A\sin(\omega_0 t + \phi)$.

We now know ω_0 but we need the values of A and ϕ to complete the solution. But these are determined by the initial conditions. We have two equations:

$i_L(0) = I_0 + A\sin(\phi)$

$v_C(0) = L\frac{di_L(t)}{dt}|_{t=0} = AL\omega_0\cos(\phi)$

Given the numerical values we now know for ω and for the initial conditions, solve the last two equations for A and ϕ. Note: There are mathematically equivalent solutions with A positive or A negative. Please give the solution with positive A.
$A =$

3

✔ **Answer:** 3.0

$\phi =$ (in radians $-\pi < \phi < \pi$)

-1.57

✔ **Answer:** -1.5707963267948966

Solution:

Part 1

$$i_C = C\frac{dv_C}{dt}$$

$$v_L = L\frac{di_L}{dt}$$

$$i_C = v_L \to i_C = LC\frac{di_L}{dt^2}$$

$$i_C + i_L = I = i_L + LC\frac{d^2 i_L}{dt^2}$$

$$\frac{d^2 i_L}{dt^2} = \frac{I - i_L}{LC}$$

Part 2

$$\frac{d^2 i_L}{dt^2} = 0$$

$$i_L = I = 3A$$

Part 3

We have seen this situation before:

$$LCs^2 + 1 = 0$$

Part 4

Remember the relationship between frequency and angular frequency $\omega = 2\pi f$:

$$\omega = 2\pi f = \frac{1}{\sqrt{LC}}$$

$$L = \frac{1}{4\pi^2 f^2 C} = 1.4988\mu H$$

Part 4

Remember the relationship between frequency and angular frequency $\omega = 2\pi f$:

$$\omega = 2\pi f = \frac{1}{\sqrt{LC}}$$

$$L = \frac{1}{4\pi^2 f^2 C} = 1.4988\mu H$$

Part 5

$$i_{L+}(0) = 0 = I_0 + A\sin(\phi) \rightarrow A = -\frac{I_0}{\sin(\phi)}$$

$$v_C(0) = L\frac{di_L}{dt} = AL\omega_0\cos(\phi) = 0$$

$$\cos(\phi) = 0$$

Thus, our phase is $-\frac{\pi}{2} = -1.57\text{rad}$ and $A = -\frac{I_0}{-1} = I_0$. We do not choose$\phi = \frac{\pi}{2} \rightarrow A = -I_0$ because we want a value with positive A.

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(*Now we analyze RLC circuits (damped oscillations)*)
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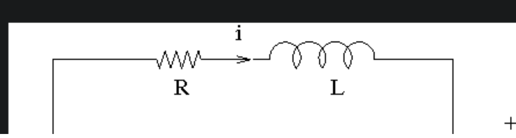
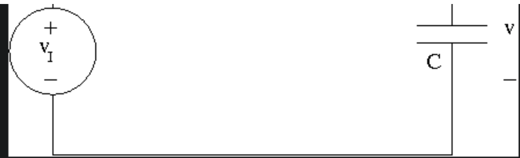
S18E1: Particular Solution

0 points possible (ungraded)

We have the circuit





with the input $v_I(t) = V_I u(t)$ and with initial conditions $v(0) = 0$ and $i(0) = 0$.

The capacitor voltage in this system is described by a second-order linear constant-coefficient ordinary differential equation:

$$LC\frac{d^2v(t)}{dt^2} + RC\frac{dv(t)}{dt} + v(t) = V_I u(t)$$

After $t = 0$ the step becomes a constant.

So first we need a particular solution $v_P(t)$ to the equation:

$$LC\frac{d^2v_P(t)}{dt^2} + RC\frac{dv_P(t)}{dt} + v_P(t) = V_I$$

(Any particular solution will do, because we will use the associated homogeneous equation to match up the initial conditions.)

We can use $v_P(t)$ as the value that satisfies the equation and makes all the derivative terms in the equation zero.

What is this particular solution $v_P(t)$?

VI

✔ **Answer:** VI

V_I

This particular solution is not yet the solution to our problem, because it does not match the initial conditions.

Solution:

Like in the first exercise of this week, we can choose a nice, steady-state solution where the derivative and second derivatives are both zero. This models the situation where the capacitor is at the same voltage as the battery and no current whatsoever flows though our system:

$$v_P(t) = V_I$$

$$LC\frac{d^2v_P(t)}{dt^2} = RC\frac{d^2v_P(t)}{dt^2} = 0$$

Solutions for more complicated systems will be discussed in the next exercises.

```
(*Summary of driven RLC dampled oscillator*)
DSolve[{l * c * v''[t] + r * c * v'[t] + v[t] == V, v'[0] == 0, v[0] == 0}, v[t], t]
(*Looks different from the total solution derived in lecture,
TODO make it look similar, not use hyperbolic fns*)
```

$$-\frac{1}{2\sqrt{-4\,l+c\,r^2}}\left(-\sqrt{c}\,e^{\frac{1}{2}\left(-\frac{r}{l}-\frac{\sqrt{-4\,l+c\,r^2}}{\sqrt{c}\,l}\right)t}\,r+\sqrt{c}\,e^{\frac{1}{2}\left(-\frac{r}{l}+\frac{\sqrt{-4\,l+c\,r^2}}{\sqrt{c}\,l}\right)t}\,r-2\sqrt{-4\,l+c\,r^2}+e^{\frac{1}{2}\left(-\frac{r}{l}-\frac{\sqrt{-4\,l+c\,r^2}}{\sqrt{c}\,l}\right)t}\sqrt{-4\,l+c\,r^2}+e^{\frac{1}{2}\left(-\frac{r}{l}+\frac{\sqrt{-4\,l+c\,r^2}}{\sqrt{c}\,l}\right)t}\sqrt{-4\,l+c\,r^2}\right)V\ \text{// ExpToTrig}$$

Out[ ]= $$\left\{\left\{v[t]\to-\frac{1}{2\sqrt{-4\,l+c\,r^2}}\left(-\sqrt{c}\,e^{\frac{1}{2}\left(-\frac{r}{l}-\frac{\sqrt{-4\,l+c\,r^2}}{\sqrt{c}\,l}\right)t}\,r+\sqrt{c}\,e^{\frac{1}{2}\left(-\frac{r}{l}+\frac{\sqrt{-4\,l+c\,r^2}}{\sqrt{c}\,l}\right)t}\,r-2\sqrt{-4\,l+c\,r^2}+e^{\frac{1}{2}\left(-\frac{r}{l}-\frac{\sqrt{-4\,l+c\,r^2}}{\sqrt{c}\,l}\right)t}\sqrt{-4\,l+c\,r^2}+e^{\frac{1}{2}\left(-\frac{r}{l}+\frac{\sqrt{-4\,l+c\,r^2}}{\sqrt{c}\,l}\right)t}\sqrt{-4\,l+c\,r^2}\right)V\right\}\right\}$$

Out[ ]= $$V-\frac{1}{2}V\,\text{Cosh}\left[\frac{r\,t}{2\,l}-\frac{\sqrt{-4\,l+c\,r^2}\,t}{2\sqrt{c}\,l}\right]-\frac{\sqrt{c}\,r\,V\,\text{Cosh}\left[\frac{r\,t}{2\,l}-\frac{\sqrt{-4\,l+c\,r^2}\,t}{2\sqrt{c}\,l}\right]}{2\sqrt{-4\,l+c\,r^2}}-$$
$$\frac{1}{2}V\,\text{Cosh}\left[\frac{r\,t}{2\,l}+\frac{\sqrt{-4\,l+c\,r^2}\,t}{2\sqrt{c}\,l}\right]+\frac{\sqrt{c}\,r\,V\,\text{Cosh}\left[\frac{r\,t}{2\,l}+\frac{\sqrt{-4\,l+c\,r^2}\,t}{2\sqrt{c}\,l}\right]}{2\sqrt{-4\,l+c\,r^2}}+$$
$$\frac{1}{2}V\,\text{Sinh}\left[\frac{r\,t}{2\,l}-\frac{\sqrt{-4\,l+c\,r^2}\,t}{2\sqrt{c}\,l}\right]+\frac{\sqrt{c}\,r\,V\,\text{Sinh}\left[\frac{r\,t}{2\,l}-\frac{\sqrt{-4\,l+c\,r^2}\,t}{2\sqrt{c}\,l}\right]}{2\sqrt{-4\,l+c\,r^2}}+$$
$$\frac{1}{2}V\,\text{Sinh}\left[\frac{r\,t}{2\,l}+\frac{\sqrt{-4\,l+c\,r^2}\,t}{2\sqrt{c}\,l}\right]-\frac{\sqrt{c}\,r\,V\,\text{Sinh}\left[\frac{r\,t}{2\,l}+\frac{\sqrt{-4\,l+c\,r^2}\,t}{2\sqrt{c}\,l}\right]}{2\sqrt{-4\,l+c\,r^2}}$$

S18E2: Homogeneous Equation Solution

0 points possible (ungraded)

We now need the general solution to the second-order homogeneous linear constant-coefficient ordinary differential equation associated with our problem, to give us the freedom to match the initial conditions. The homogeneous equation is:

$$LC\frac{d^2 v_H(t)}{dt^2} + RC\frac{dv_H(t)}{dt} + v_H(t) = 0$$

All homogeneous linear constant-coefficient ordinary differential equations have solutions of the form $v_H(t) = Ae^{st}$ because $\frac{de^{st}}{dt} = se^{st}$. If we substitute in this trial solution we get the characteristic equation: a polynomial equation in s.

Since this is second order the polynomial is of the form:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

For our differential equation what are the values of α and ω_0? Write algebraic expressions for them in the spaces below.

$\alpha =$

R/(2*L) ✔ **Answer:** R/(2*L)

$\frac{R}{2 \cdot L}$

$\omega_0 =$

1/sqrt(L*C) ✔ **Answer:** 1/sqrt(L*C)

$\frac{1}{\sqrt{L \cdot C}}$

Solution:

Plug in the suggested expression into our differential equation:

$$LC\left(s^2 Ae^{st}\right) + RC\left(sAe^{st}\right) + As^{st} = 0$$

$$LCs^2 + RCs + 1 = 0$$

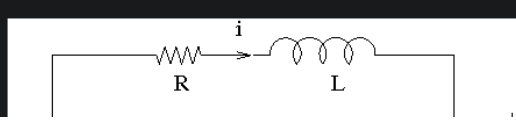
$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 = s^2 + 2\alpha s + \omega^2$$

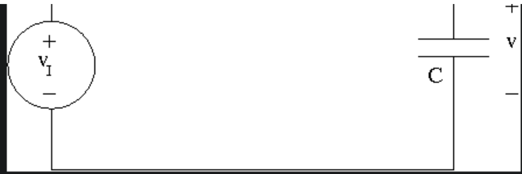
$$\rightarrow \omega = \frac{1}{\sqrt{LC}}, \alpha = \frac{R}{2L}$$

S18E3: Total Solution

0 points possible (ungraded)

Back to our circuit





The input is $v_I(t) = V_I u(t)$. The initial conditions are $v(0) = 0$ and $i(0) = 0$.

The capacitor voltage in this system is described by a second-order linear constant-coefficient ordinary differential equation:

$$LC\frac{d^2v(t)}{dt^2} + RC\frac{dv(t)}{dt} + v(t) = V_I u(t)$$

For $t > 0$ we now know a particular solution $v_P(t) = V_I$ and the general solution of the associated homogeneous equation.

Suppose $L = 10.0\mu$H, $C = 10.0$pF, and $R = 1.0$kΩ. This gives an underdamped system: The general solution of the homogeneous equation is an exponentially-decaying sinusoid of the form:

$$v_H(t) = Ae^{-\alpha t}\cos(\omega_d t + \phi)$$

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ and ω_0 and α are derived from the device parameters.

What is the numerical value of α in this system?

50000000 ✔ **Answer:** 50000000.00000001

What is the numerical value of ω_0?

100000000 ✔ **Answer:** 100000000.00000001

What is the numerical value of ω_d, the natural oscillation frequency of this damped system?

8.66025e7 ✔ **Answer:** 86602540.37844388

The remaining parameters, A and ϕ cannot be determined yet. They are determined by matching the total solution (=particular+homogeneous) to the initial conditions. The total solution is of the form:

$$v(t) = V_I + Ae^{-\alpha t}\cos(\omega_d t + \phi)$$

We know two initial conditions: $v(0) = 0$ and $i(0) = 0$.

First, from $i(0) = 0$ we know that $\frac{dv(t)}{dt}|_{t=0} = 0$. From this we can deduce the value of ϕ. In the space provided below give an algebraic expression for ϕ in terms of α and ω_d. (Use wd for ω_d and a for α in your expression.)

arctan(-a/wd) ✔ **Answer:** arctan(-a/wd)

$$\arctan\left(-\frac{\alpha}{\omega_d}\right)$$

Now, from our knowledge of ϕ and using the other condition, $v(0) = 0$, we can deduce the value of A. In the space provided below write an algebraic expression for A in terms of V_I, ω_0, and ω_d. (Use wd for ω_d and $w0$ for ω_0 in your expression.)

-VI*(w0/wd)

Answer: -VI*(w0/wd)

$-V_I \cdot \left(\frac{\omega_0}{\omega_d}\right)$

Assume that $V_I = 3.0$V. Now, we can bring it all together numerically. We have the form of the solution:

$$v(t) = V_I + Ae^{-\alpha t}\cos(\omega_d t + \phi)$$

and the only two coefficients that we don't have numerically are A and ϕ.

What is the numerical value of A?

-3.4641

Answer: -3.4641016151377544

What is the numerical value of ϕ?

-0.523599

Answer: -0.5235987755982988

At this point you should check that $v(0) = 0$ and $i(0) = 0$, as required.

Solution:

Calculating Values

From exercise 2, we've already derived equations for alpha and omega in this situation:

$$\alpha = \frac{R}{2L} = 5 \times 10^7 \mathrm{rad/s}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10^8 \mathrm{rad/s}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 8.66 \times 10^7 \mathrm{rad/s}$$

Deriving Values

$$i(0) = 0 \rightarrow \frac{dv(0)}{dt} = -\omega_d Ae^{-\alpha t}\sin(\omega_d t + \phi) - \alpha Ae^{-\alpha t}\cos(\omega_d t + \phi) = 0$$

$$\omega_d \sin(\phi) = \alpha \cos(\phi)$$

$$\phi = \tan^{-1}(\frac{\alpha}{\omega_d})$$

$$v(0) = 0 = V_I + A\cos(\phi)$$

$$A = -\frac{V_I}{\cos(\phi)}$$

$$\tan(\phi) = \frac{\alpha}{\omega_d} \rightarrow \frac{\omega_d}{\sqrt{\alpha^2 + \omega_d^2}} = \frac{\omega_d}{\sqrt{\alpha^2 + (\omega_0^2 - \alpha^2)}} = \frac{\omega_d}{\omega_0}$$

$$A = V_I \times \frac{\omega_0}{\omega_d}$$

For the last two problems, plug in our values to find that $A = -3.464, \phi = 0.524 \mathrm{rad}$

```
(*For some reason the official solution removes the
 sign in the atan but grader has the correct solution*)
l = 10*^-6; c = 10*^-12; r = 1*^3;
```

$$\alpha = \frac{r}{2\,l}$$

$$\omega 0 = \frac{1}{\sqrt{l\,c}}$$

$$\omega d = \sqrt{\omega 0^2 - \alpha^2} \;//\; N$$

```
VI = 3;
```

$$\mathrm{ArcTan}\left[\frac{-\alpha}{\omega d}\right]$$

$$-VI * \left(\frac{\omega 0}{\omega d}\right)$$

Out[∘]= 50 000 000

Out[∘]= 100 000 000

Out[∘]= 8.66025×10^7

Out[∘]= -0.523599

Out[∘]= -3.4641

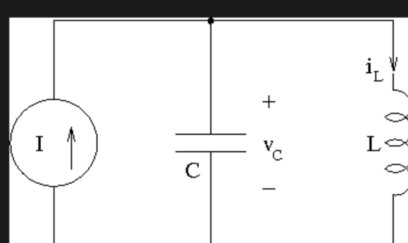
Homework

H1P1: Response to a Delayed Impulse

8/8 points (graded)

Note: In this problem we have chosen numbers for the part parameters to make it easier to compute an answer :-). By the way, it is also hard to arrange zero resistance, except with superconducting materials at very low temperatures.

In the circuit shown below $L = 35.0H$ and $C = 11.58mF$.



The current source puts out an impulse of area $A = 2/\pi = 0.637$ Coulomb at time $t = 9.0s$.

At $t = 0$ the state is: $v_C\,(0) = 0.0V$ and $i_L\,(0) = 1.0A$.

The equation governing the evolution of the inductor current in this circuit is

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{LC} i_L(t) = \frac{A}{LC}\delta(t-9.0)$$

What is the natural frequency, in Hertz, of this circuit?

0.249995 ✔ **Answer:** 0.25

For the remaining parts of the question, you should round the above value off to two decimal places to make calculations easier.

At the initial time what is the total energy, in Joules, stored in the circuit?

35/2 ✔ **Answer:** 17.5

At the time just before the impulse happens $t = 9.0s_-$ what is the total energy, in Joules, stored in the circuit?

35/2 ✔ **Answer:** 17.5

At the time just before the impulse happens what is the current $i_L(9.0s_-)$, in Amperes, through the inductor?

0 ✔ **Answer:** 0

At the time just before the impulse happens what is the voltage $v_C(9.0s_-)$, in Volts, across the capacitor?

-54.9768 ✔ **Answer:** -54.98

At the time just after the impulse happens what is the current $i_L(9.0s_+)$, in Amperes, through the inductor?

0 ✔ **Answer:** 0

At the time just after the impulse happens what is the voltage $v_C(9.0s_+)$, in Volts, across the capacitor?

-0.00103535 ✔ **Answer:** 0

At the time just after the impulse happens what is the total energy, in Joules, stored in the circuit?

5.8549e-6 ✔ **Answer:** 0

Explanation:
(a) If we look at the homogenous equation:

$$\frac{d^2 i_L(t)}{dt^2} = -\frac{i_L(t)}{LC}$$

We recognize this equation is of the familiar form of a simple harmonic oscillator: $x'' = -\omega^2 x$. The natural frequency of oscillation is therefore given by the square root of the constant out in front of the right hand side term:

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(35.0H)(11.58mF)}} = 1.5707667448701708 rad/s$$

$$f_0 = \frac{1.5707667448701708 rad/s}{2\pi} = 0.24999529189045372 Hz$$

(b) At any time the energy in the circuit is given by:

$$E_i = \frac{1}{2}CV_c(t)^2 + \frac{1}{2}Li_L(t)^2$$

Because $V_c(0) = 0$, the initial energy is purely stored in the inductor. Plugging in to the above expression, we get:

$$E_i = 17.5\ Joules$$

(c) The energy stored in the circuit will remain constant in the absence of damping (having a resistor) or outside influences. Because we have none of these in the interval $t = 0$ to $t = 9.0$, the energy remains the same, at $17.5\ Joules$.

(d) At time $t = 0$, we have all the current through the inductor and none through the capacitor. To find out what happens at $t = 9.0$, we should first find the period of oscillation of the current (and correspondingly, the voltage) in the circuit. This is related to the natural frequency:

$$T = \frac{2\pi}{\omega_0} = 4$$

And its units are seconds. So now we know that at $t = 4$, we're in the same situation we were in when we started. Note that this same analysis holds for any number of integral multiples of the period (so we can start at $t = 8$ and proceed identically, (or $t = 12$, or $t = 0$), etc. The only difference comes from where in time we eventually decide to place our impulse input that we haven't considered yet).
When dealing with pure oscillations in an LC circuit, it is often useful to break down what's happening in the circuit into quarter-periods. As time starts, all of our energy is in the inductor. After a quarter period, our situation will be reversed; all of our energy will now be in the capacitor. Then the energy goes back into the inductor (but the current flowing in the opposite direction) after half a period. At $\frac{3}{4}$ period all the energy is back in the capacitor again, and finally after a full period we're back to the beginning. $t = 9.0$ (or 13.0 or 17.0) is a quarter period after this initial state, so we know all of our energy will be in the capacitor. This realization makes the rest of this question easy: since the energy in the inductor is zero the current must be zero.
(e) By our previous discussion, we know all the energy will be in the capacitor, so once again by substituting into the above energy equation:

$$V_c\left(t = 9.0^-\right) = -54.98V$$

Note that the sign is negative. If you look carefully at the circuit, we're actually just starting to charge up the bottom plate of the capcitor at $t = 4$ and the polarity we've assigned to the capacitor plates in the question is opposite of this.
(f) What is the initial transient behavior of an inductor when subjected to a DC input? It looks like a high-impedance wall. Inductors do not respond well to sharp changes (when $\omega \to \infty$, so does Z_L, the inductor's impedance), so because of the excessive impedance, zero current flows instantaneously at $t = 9.0^+$.
(g) Because no current flows through the inductor path, all the impulse does is inject a finite amount of charge onto the capacitor plates at the instant $t = 9.0$, equal to the impulse's "area," $\frac{2}{\pi} coulombs$. Because $V = \frac{Q}{C}$, the impulse injects a voltage onto the capacitor equal to:

$$V_{impulse} = \frac{2}{\pi C} = 54.978V$$

Note that this is equal and opposite to the voltage already on the capacitor at $t = 9.0^-$, so therefore the net voltage across the capacitor ends up being zero!
(h) What the impulse effectively did was rob all of our energy in the circuit, by cancelling out the voltage on the capacitor. If you think of the flowing current in the circuit as a moving car, the impulse was a force that stopped it right in its tracks. Because there are no other sources for all time, the energy in the circuit remains zero forever.

In[ ]:=
```
(*This is more of a theory
 question. Mathematica can be useless/misleading here*)
l = 35; c = 11.58*^-3;
ω0 = 1/Sqrt[l c];
f = ω0/(2 π)

vc[0] = 0; il[0] = 1;
(*Initially all energy is stored in the inductor*)
1/2 l*il[0]^2
(*Because time period T=4s, and this is an LC oscillator without dissipation,
t=9⁻s is the exact moment when all energy has sloshed into the capacitor*)
Solve[35/2 == 1/2 c*v^2, v]
(*At t=9⁺s we get a current impulse,
remember that capacitor acts like an instantaneous short
 circuit and inductor acts like an instantaneous open circuit*)
q = 2/π;(*Using the rounded value 0.637 gives wrong answer*)
-54.976836070455974` + q/c
1/2 c*(-0.0010353478669031801`)^2
```

Out[ ]= 0.249995

Out[ ]= $\frac{35}{2}$

Out[ ]= {{v → -54.9768}, {v → 54.9768}}

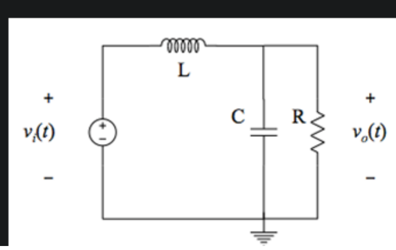
Out[ ]= -0.00103535

Out[ ]= 6.20656×10^{-9}

H1P2: Second Order Filter

7/7 points (graded)

Consider the second order low pass filter circuit shown below.



The differential equation associated with the input voltage $v_i\,(t)$ and the output voltage $v_o\,(t)$ of this system can be written in the following form:

$$A \cdot \frac{d^2 v_o}{dt^2} + B \cdot \frac{dv_o}{dt} + C \cdot v_o = v_i$$

Where the coefficients A, B, and C can all be written in terms of the component values L, C and R

(a) What is the algebraic expression for coefficient A associated with the $\frac{d^2 v_o}{dt^2}$ term?

L*C ✔ **Answer:** L*C

$L \cdot C$

(b) What is the algebraic expression for coefficient B associated with the $\frac{dv_o}{dt}$ term?

L/R ✔ **Answer:** L/R

$\frac{L}{R}$

(c) What is the algebraic expression for coefficient C associated with the v_o term?

1 ✔ **Answer:** 1

1

(d)The circuit would ring when excited with a step response $v_i = u\,(t)$. Assuming that there is negligible damping in the circuit, what is the undamped natural frequency (in Hertz) in terms of the component values?

1/(2*pi*sqrt(L*C)) ✔ **Answer:** 1/(2*pi*sqrt(L*C))

$\frac{1}{2 \cdot \pi \cdot \sqrt{L \cdot C}}$

(e) The ringing will be damped by the factor $e^{-\alpha t}$. What is the expression for α in terms of the component values?

1/(2*R*C) ✔ **Answer:** 1/(2*R*C)

$\frac{1}{2 \cdot R \cdot C}$

(f) What is the expression for the "Quality Factor" Q of this circuit, in terms of the component values?

(R*C)/sqrt(L*C) ✔ **Answer:** R*sqrt(C/L)

$\frac{R \cdot C}{\sqrt{L \cdot C}}$

(g) Suppose we need to suppress the ringing. We could change the value of R to make this circuit critically damped (Hint: make the Q = 0.5). What is the expression for this critical R, in terms of L and/or C?

sqrt(L*C)/(2*C) ✔ **Answer:** sqrt(L/(4*C))

$$\frac{\sqrt{L \cdot C}}{2 \cdot C}$$

Explanation:
(a),(b),(c) To derive the differential equation for this circuit, we start at the v_{out} node. Applying KCL, we get:

$$C\frac{dv_o}{dt} + \frac{v_o}{R} = i_L$$

And because we know:

$$V_L = L\frac{di_L}{dt} = v_i - v_o$$

We substitute the derivatie of our first expression for i_L into the second one and get:

$$LC\frac{d^2v_o}{dt^2} + \frac{L}{R}\frac{dv_o}{dt} + v_o = v_i$$

Which immediately shows:

$$A = LC$$

$$B = \frac{L}{R}$$

$$C = 1$$

(d) While the addition of the resistor will cause damping in the response, the natural frequency of the circuit is still the same as the purely LC one we analyzed in the previous question:

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

(e) The characteristic equation of a typical second order RLC is of the form:

$$s^2 + 2\alpha s + {\omega_0}^2 = 0$$

To get our own characteristic equation in such a form, we divide through by LC to get:

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

$$\alpha = \frac{1}{2RC}$$

(f) The quality factor is given by:

$$Q = \frac{\omega_0}{2\alpha} = R \cdot \sqrt{\frac{C}{L}}$$

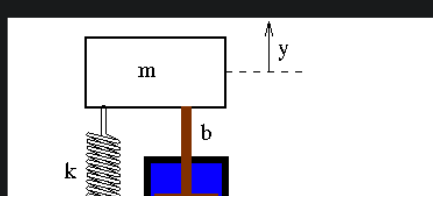
(g) We simply plug in $Q = 0.5$ and solve for R:

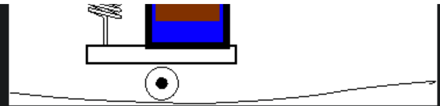
$$R = \frac{1}{2}\sqrt{\frac{L}{C}}$$

H1P3: Designing a Shock Absorber

4/4 points (graded)

One way to model a vehicle is as a massive object that is connected by springs and shock absorbers to the wheels. Here is a simple model:



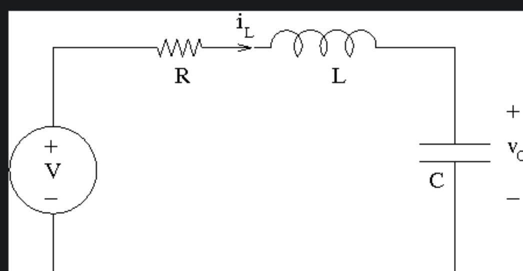


There is a mass m suspended by a spring with spring constant k and a oil-filled shock-absorber with viscous friction constant b over a wheel base. The wheels ride over a rough road that moves up and down. We will only model the vertical motion, with vertical displacement y. If you remember your basic physics you will be able to see that the balance-of-forces equation governing the motion in the y direction is just a second-order linear constant-coefficient ordinary differential equation:

$$m\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + k\left(y(t) - Y_{road}(t)\right) = 0$$

(Don't worry if you don't remember this: we are giving you the equation.)

Now, this equation looks very much like the equation governing a driven RLC circuit:



$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = V(t)$$

Indeed, if we think of the charge on the capacitor q as analogous to the displacement y, and if we note that $q = Cv$ we can rewrite this equation as:

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) - V(t) = 0$$

So the inductance is analogous to the mass (inertia), the resistance is analogous to the viscous friction, the reciprocal of the capacitance is analogous to the spring constant, and the variations of height of the road is analogous to the voltage drive (scaled by C.

When we hit a bump, a vehicle may oscillate, just like an underdamped circuit.

Since this is second order the characteristic polynomial is of the form:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

For our differential equation what are the values of α and ω_0? Write algebraic expressions for them in terms of the mechanical parameters m, b, and k, in the spaces below.

$\alpha =$

b/(2*m)

✔ **Answer:** b/(2*m)

b

$2 \cdot m$

$\omega_0 =$

sqrt(k/m) ✔ **Answer:** sqrt(k/m)

$\sqrt{\frac{k}{m}}$

Suppose that the shock absorber is dead, so $b = 0$. Write an algebraic expression for the frequency, in Hertz, of the bouncing of our vehicle. Your expression should be in terms of m and k.

sqrt(k/m)/(2*pi) ✔ **Answer:** sqrt(k/m)/(2*pi)

$\frac{\sqrt{\frac{k}{m}}}{2 \cdot \pi}$

We can make the frequency of the bouncing go to zero by picking an appropriate value of the viscous-friction coefficient b. (This is called critical damping.) Write an algebraic expression for the value of b that will make this vehicle critically damped.

2*sqrt(k*m) ✔ **Answer:** 2*sqrt(m*k)

$2 \cdot \sqrt{k \cdot m}$

Explanation:
(a),(b) Similar to the previous problem, all we have to do is take the characteristic equation for the mechanical differential equation and divide out by the constant on the first term (this time it's m) to get the characteristic in its standard form. This time:

$$\alpha = \frac{b}{2m}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

(c) It will simply bounce at the natural frequency, ω_0. We only have to convert this to hertz.

$$f_0 = \frac{\omega_0}{2\pi}$$

(d) Again similar to the previous problem, $Q = \frac{\omega_0}{2\alpha}$. We plug in $Q = 0.5$ and solve for b:

$$b = 2\sqrt{km}$$

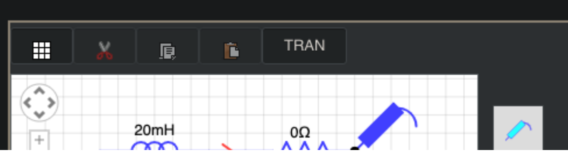
Lab

Lab 1

3/3 points (graded)

In this lab we'll be exploring the properities of *second-order circuits*, i.e., circuits with two energy storage elements. You may find it useful to review Chapter 12 in the text.

Figure 1 below shows the circuit we'll be using to explore the step response of an RLC circuit. The voltage source produces a $1V$ step at $t = 0$. Initially the resistor has been set to 0Ω. We'll be probing the voltage across the capacitor, which will indicate the amount of charge on the plates of the capacitor ($q = Cv$), and the current through the inductor, which will indicate the flux linkage of the magnetic field of the inductor ($\lambda = Li$).



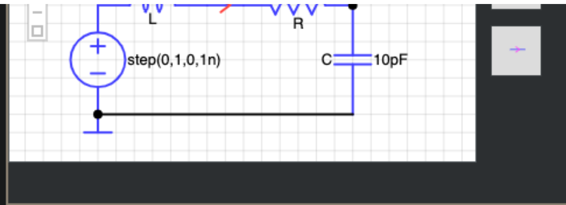
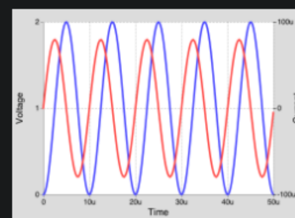


Figure 1. RLC Circuit driven by a voltage step

Run a $50\mu s$ TRAN simulation on the circuit. You'll see that the energy imparted by the voltage step oscillates back and forth between the flux of the inductor and the charge of the capacitor (as indicated by the oscillating voltage across the capacitor and the oscillating current through the inductor).

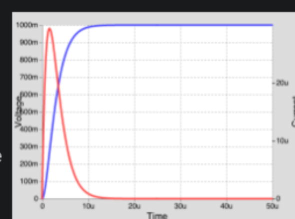


Leaving the inductance L as-is, adjust the capacitance C so that the frequency of oscillation is $100\ \mathrm{kHz}$ with a period of $10\mu s$. You can do this by experimentation, but a more effective approach would be to use Equation 12.45 to compute the correct value for C given L and the desired ω_0. Note that the units of ω_0 are in radians/sec, so you'll have to convert Hz to radians/sec using $1\mathrm{Hz} = 2\pi\ \mathrm{rad/s}$. Your plots should like those shown to the right.

Please enter the adjusted capacitance C below:

Adjusted capacitance *C* (in farads):

1.26651e-10 ✔ **Answer:** 126.5e-12



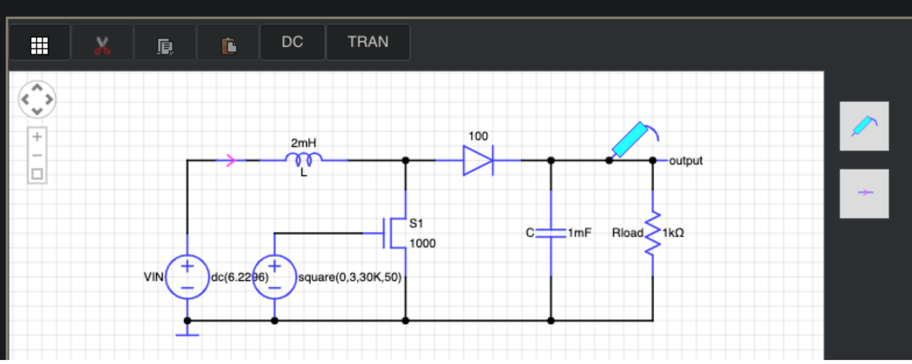
Keeping the adjusted C, now adjust the resistance R so that the system is just operating in the over-damped region, i.e., so that the voltage across the capacitor makes a single $0 \rightarrow 1$ transition, never exceeding 1V. Again you can use the analysis in Section 12.2.2 to compute the R necessary to produce a plot like that to the right. Hint: if there's any voltage sample that is greater than 1V, the voltage scale will have a maximum of 2V, so one quick way to tell if the voltage is staying less than or equal to 1V is when the maximum value on the voltage scale is 1V.

Please enter the adjusted resistance R below:

Adjusted resistance *R* (in ohms):

25e3 ✔ **Answer:** 25000

Now let's use the properties of second-order systems to build a *boost converter*, a DC-to-DC power supply useful where high voltages are required but not directly available. Powering the flash bulb in a camera is one such example. A boost converter circuit is shown in Figure 2. In this case the supply voltage is $3V$ and the goal is produce a relatively stable supply of $6V$ to drive a load, here represented as a $1k\Omega$ resistor. Your task to is adjust L, C and the duty cycle D of the square wave controlling the MOSFET switch so that the output voltage falls between $5.9V$ and $6.1V$ (6V with a maximum of 0.1V ripple).



✔

Figure 2. Boost converter

The operation of the boost converter is analyzed in detail in Section 12.10 of the text. The circuit above differs from the example in how the switches are implemented. S_1 in the Example is implemented by a MOSFET switch, where a square-wave voltage is used to turn the switch on and off. S_2 in the Example is implemented using a diode, which conducts when the voltage on the inductor side sufficiently exceeds the voltage on the capacitor side. In the steady state this happens when $S1$ is off, just what we wanted for the correct operation of the circuit.

The boost converter operates in a two-state cyclic manner. During the first state, the MOSFET is turned on. This connects the inductor across the power supply, so the inductor current i_L increases. During this time, the diode is reverse biased and so behaves as an open circuit, disconnecting the capacitor and load resistor from the remainder of the circuit. In this state, the capacitor powers the load resistor, discharging in the process.

During the second state, the MOSFET is turned off but the inductor current cannot go to zero instaneously. So the voltage on the diode increases making the diode forward biased and the decreasing current from the inductor flows through the diode to power the load and simultaneously replenish the charge in the capacitor. The inductor flux diminishes in the process.

This cycle repeats itself indefinitely. Let the duration of the first state be DT, and the duration of the second state be $(1 - D)\,T$, where D is the fraction of a cycle for which the square wave controlling the MOSFET is high (sometimes called the *duty cycle*) and T is the switching period (set to $1/30\text{kHz}$ in the circuit above).

Try a $2ms$ TRAN simulation to see how the circuit works, paying attention to the current through the inductor and the output voltage. Start by adjusting the capacitance C so that the ripple on the output voltage is $0.1V$ or less using the equation $i\Delta t = C\Delta v$ to help choose the appropriate value. Here i is the desired load current $6V/1k\Omega$, Δt is the discharge time in one cycle DT and Δv is the desired ripple $0.1V$.

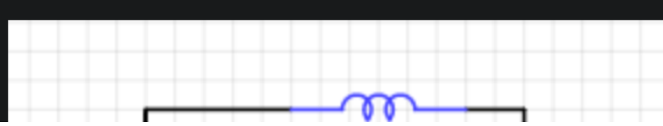
Now choose values for the inductance L and duty cycle D to produce the desired output voltage. As explained in the text, the average output voltage is roughly $\text{VIN}/\,(1 - D)$, so increasing the duty cycle will raise the output voltage. One wants to choose D and the cycle time T so that the inductor current reaches zero just as the next cycle begins.

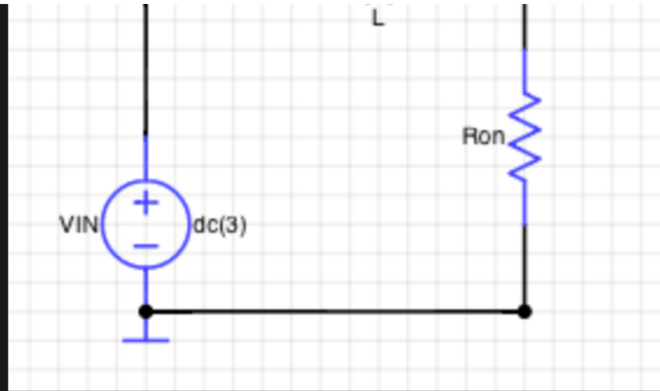
Experiment with the various parameters until the output voltage is $6V$ with a maximum $0.1V$ ripple. As a final check, perform a $10ms$ TRAN simulation and then click CHECK. The on-line system will be veryifying that the output voltage meets the specification for $9.5ms < t < 10ms$.

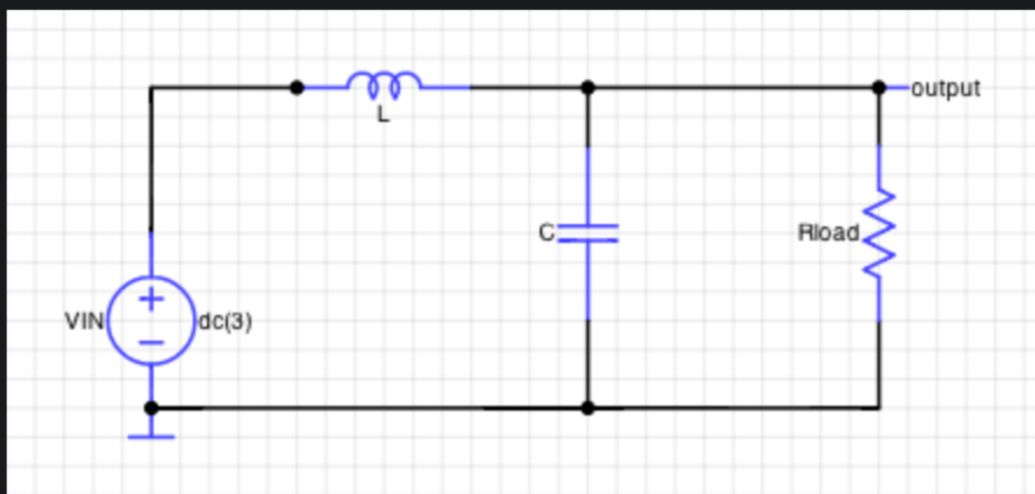
Explanation: TEST
The goal of this lab is to explore the properties of second-order circuits. We first wish to change the ω_0 of a series LC circuit. The inductance L is $20mH$, and we want to change the capacitance C so that the frequency of oscillation is $100kHz$.
The ω_0 of this circuit is $\sqrt{\frac{1}{LC}}$. Therefore, we want $\omega_0 = 2\pi \cdot 100kHz$. This gives us a capacitance of about $1.27 \times 10^{-10}F$.
Now we wish to increase R while leaving L and C the same so that the system is just operating in the over-damped region. The differential equation governing this circuit is:

$$\frac{1}{LC}V_i = \frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c$$

Where v_c is the capacitor voltage and V_i is the voltage step input. For this circuit, $\omega_0 = \sqrt{\frac{1}{LC}}$ and $\alpha = \frac{R}{2L}$. The circuit enters the over-damped region of operation once $\alpha \geq \omega_0$. So for over-damped operation, $R \geq 2\sqrt{\frac{L}{C}} \approx 25098\Omega$.
Now we want to use our understanding of second-order circuits to build a boost converter. Intuitively, the boost converter switches between the following two circuits:







In the first circuit, the inductor current rises, and the inductor charges up while the capacitor's energy is dissipated by the load resistor. In the second circuit, the inductor's current flows through the capacitor, and the capacitor charges. If the energy dissipated by the resistor can dissipate in one cycle, then the capacitor voltage can keep rising until the energy deposited and the energy dissipated over one cycle become approximately the same.
We want to adjust C so that the output voltage ripple is less than $0.1V$. Using the equation given:

$$i\Delta t = C\Delta v$$

with $i = \frac{6V}{1k\Omega}$, $\Delta t = DT = 0.5 \times \frac{1}{30kHz}$, and $\Delta v = 0.1V$, we get a capacitance of about $1\mu F$.
Now we wish to adjust the duty cycle D and the inductance L so that the output voltage reaches $6V$. If the average output voltage is roughly $\frac{V_{IN}}{(1-D)}$ and $V_{IN} = 3V$, then $D \approx 0.5$. With the given $2mH$ inductor, a duty cycle of 0.5 is too low to reach $6V$. You could find the right duty cycle via experimentation, as analytic analysis may be too difficult for this problem. A set of values that yields roughly a $6V$ output voltage with less than $0.1V$ ripple is $L = 2mH$, $C = 1\mu F$, and $D = 0.57$. Intuitively, raising C reduces the voltage ripple as the capacitor takes longer to discharge and raising D increases the average voltage at the output.

In[ ]:=
```
(*Part 1*)
l = 20*^-3;
Solve[2 π * 100*^3 == 1/Sqrt[l c], c] // N
(*Part 2 - Solved with experimentation*)
(*Part 3 - Solved with experimentation*)
```

Out[ ]= $\{\{c \to 1.26651 \times 10^{-10}\}\}$