Journal of Computational and Applied Mathematics 320 (2017) 202–220

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A review on implied volatility calculation

Giuseppe Orlando [a,b,*], Giovanni Taglialatela [a]

[a] Università degli Studi di Bari "Aldo Moro" - Department of Economics and Mathematical Methods, Via C. Rosalba 53, Bari, I-70124, Italy

[b] Università degli Studi di Camerino - School of Science and Technologies, Via M. delle Carceri 9, Camerino, I-62032, Italy

ARTICLE INFO

Article history:
Received 18 September 2016

JEL classification:
G10
C02
C88

MSC:
65-02
91G20
91G60

Keywords:
Implied volatility
Quantitative methods
Numerical calculus

ABSTRACT

This paper aims to summarizing the different approaches in determining the implied volatility for the options. This value is of particular importance since it is the main component of the option's price and because, among traders, options are quoted in terms of volatility rather than price. After a discussion on the approximation methods, a numerical approach is explained. It is shown that, in order to ensure a fast and reliable convergence, the selection of an appropriate starting point is key. The authors' suggestion for choosing the first order approximation or the inflexion as initial point is also illustrated.

Contents

* Corresponding author at: Università degli Studi di Bari "Aldo Moro" - Department of Economics and Mathematical Methods, Via C. Rosalba 53, Bari, I-70124, Italy.

E-mail addresses: giuseppe.orlando@uniba.it (G. Orlando), giovanni.taglialatela@uniba.it (G. Taglialatela).

http://dx.doi.org/10.1016/j.cam.2017.02.002
0377-0427/

0. Preface

For all goods as well as for the options, the main problem when engaging in trading is to assign them a value. The classical Black & Scholes (1973) [1] formula, later extended by Merton, provides, within the Capital Asset Pricing Model (CAPM) framework, an elegant answer to the above problem by identifying a relation between the value of the stock and its option. This was not an easy task which required some "strong" assumptions (e.g. log-normal distribution, risk neutral valuation) and a formal link between the option and several factors such as time, underlying and volatility [2]. In particular, as the latter parameter was not directly observable, it raised some concerns about the practical suitability of the formula. In reality the Black & Scholes formula has been proven over the years to stand quite strong, and from that moment on, the problem to price an option has become to identify correctly the market volatility i.e. (once again) to find a value. As noted by Latané & Rendleman (1976) [3], the solution to this problem is given by extrapolation of the market volatility from the prices of the options, or otherwise from the 'observation' of the implied volatility. Due to the structure of the Black & Scholes formula, the implied volatility cannot be found in closed form but only through numerical approximation methods. Many articles have addressed the issue of trying to better approximate the "true" value of the volatility with more or less accurate results.

This paper is organized as follows: Section 1 gives an account of the literature on implied volatility, Section 2 is about the most common closed form approximations (and the related limits), Section 3 illustrates numerical methods (such as the Newton–Raphson algorithm) which can return precise results in very few steps provided a "good" starting point is assigned. An explanation about the mathematical conditions needed for the algorithm to converge and on its efficiency (i.e. convergence in a very few steps) is given. Finally some examples are shown comparing numeric and approximation methods.

1. Background

The interest on identifying the market implied volatility has not waned over time especially for traders and for those that claim some predictive power over future volatility. According to Bandi and Perron (2006) [4], for example, "although little can be said about short-term unbiasedness, our results largely support a notion of long-run unbiasedness of implied volatility as a predictor of realized volatility". A clear indicator of the growing interest on the topic, not only from scholars but also from the market, was the VIX (i.e. a trademarked ticker of tradable stock market volatility) that the CBOE commissioned to Whaley (1993) [5].

Implied volatility is so important that options are often quoted in terms of volatility rather than price. This is because when an option is held as part of a delta neutral strategy (i.e. a portfolio hedged against small changes of the underlying's price) then the most important factor in determining the value of the option is the implied volatility. In addition to the VIX there are also a number of volatility indices such as the VXN (Nasdaq 100 index futures volatility), QQV (QQQ volatility), etc. and they span over a wide range of asset types: stocks, ETFs, commodity, interest rates, etc. (for example see CBOE Volatility Indexes [6]).

Generally speaking the investigations have been conducted in two directions: one pragmatic, which aims to provide approximations that can be easily calculated in a spreadsheet, and the other theoretical, which explores the mathematical properties of the implied volatility (e.g. Dupire (1992) [7], (1994) [8], Koekebakker and Lien (2004) [9]).

Within the Black and Scholes framework, the implied volatility represents the volatility of the underlying; therefore options with a different strike price, written on the same underlying and with same time to expiration, should have the same implied volatility. In fact the reality shows that the volatility depends on the maturity and the strike; i.e. an option does not varies with a constant volatility $\hat{\sigma}_{t_1-t_0}$ and then with a constant volatility $\tilde{\sigma}_{t_2-t_0}$ with $t_2 \geq t_1 \geq t_0$. Merton (1973) [10] was one of the first to solve the time-dependency problem by describing the spot price process S as follows:

$$\frac{dS}{S} = r(t)dt + \sigma(t)dW$$

where $r(t)$ is the instantaneous forward rate of maturity t implied from the yield curve.

The relationship between implied volatility and exercise price is not constant (see Fig. 1) and may look like a smile, a skew, a smirk, etc. (for simplicity are all called "smiles"). From the theoretical point of view this problem is more tricky to solve and its explanations are related, mostly, to the unrealistic model's assumptions on perfect market and the stochastic process followed by the underlying. Some claim, also, that computational errors may play a role [12].

In order to compute the theoretical smile, researchers have introduced a "non-traded source of risk such as jumps, stochastic volatility or transaction costs, thus losing the completeness (ability to hedge options with the underlying asset) of the model" [8]. Therefore one the most fruitful approaches [8] was to discretize the risk-neutral diffusion spot process

$$\frac{dS}{S} = r(t)dt + \sigma(S,t)dW$$

by the means of binomial/trinomial tree (Nelson and Ramaswamy (1990) [13], Hull and White (1990) [14]) and to ensure the compatibility with the observed smiles recurring to the forward induction (Jamshidian (1991) [15], Hull and White (1993) [16]).

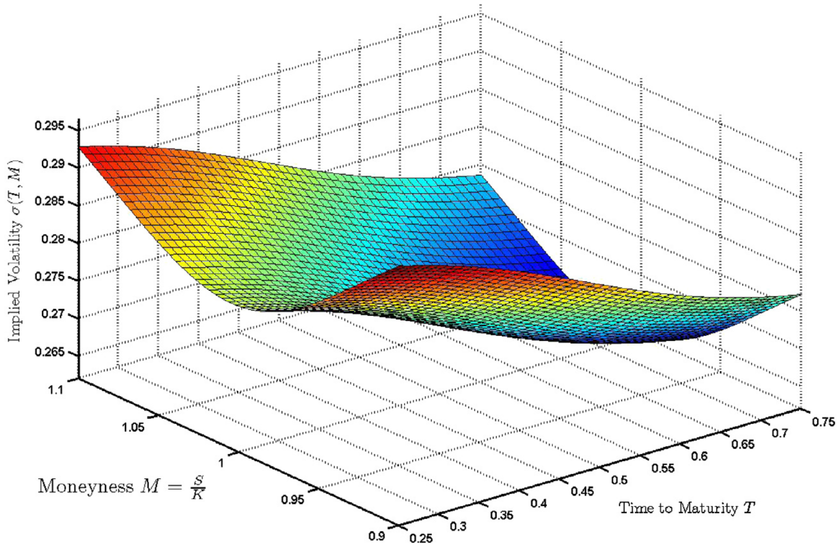

Fig. 1. Volatility smile obtained with [11].

2. Closed form approximations

The Black & Scholes formula for pricing a European call is described by the following equation:

$$C = S\,N(d_1) - X\,N(d_2) \tag{2.1}$$

where

- C is the price of a call option;
- S is the value of the underlying;
- K is the strike price;
- $X = K\,e^{-rT}$ is the present value of the strike price;
- r is the interest rate;
- T is the time to maturity in terms of a year;
- $N(x)$ is the cumulative distribution function of the standard normal i.e.

$$N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt$$

- $d_1 := \frac{\log(S/K)}{\sigma\sqrt{T}} + \frac{1}{2}\,\sigma\sqrt{T}$ is the first parameter of probability i.e. "factor by which the present value of contingent receipt of the stock, contingent on exercise, exceeds the current value of the stock" [17];
- $d_2 := \frac{\log(S/K)}{\sigma\sqrt{T}} - \frac{1}{2}\,\sigma\sqrt{T}$ is the second parameter of probability which represents the risk-adjusted probability of exercise;
- σ is the volatility.

Given the so called "error function", or erf(x), defined as the probability of a random variable with normal distribution of mean 0 and variance 1/2 falling in the range $[-x, x]$

$$\operatorname{erf}(x) := \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2}\,dt$$

it can be shown that

$$N(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$$

and for a generic normal distribution with mean μ and deviation σ the cumulative distribution function is

$$F(x) = N\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right].$$

Integrating by parts the cumulative distribution function of the standard normal distribution we get

$$F(x) = N(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}\left[x + \frac{x^3}{3} + \frac{x^5}{3\cdot 5} + \cdots + \frac{x^{2n+1}}{(2n+1)!!} + \cdots\right] \tag{2.2}$$

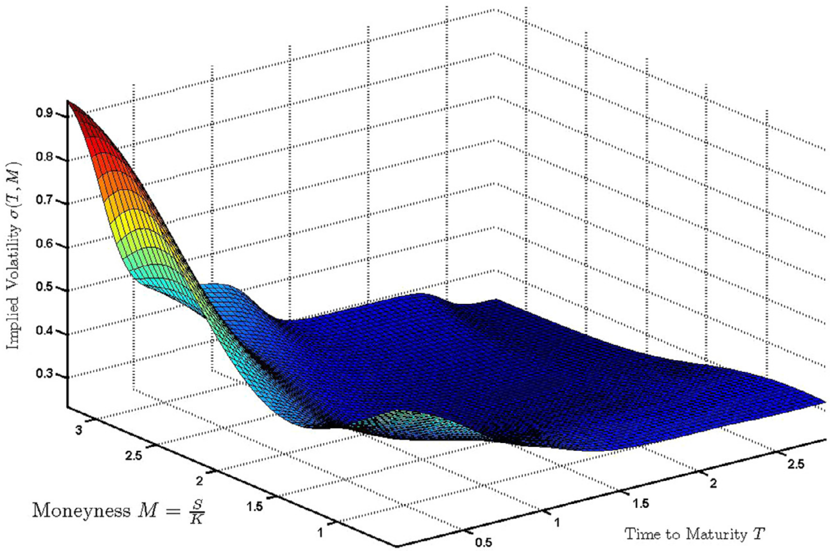

Fig. 2. Implied volatility surface obtained with [11].

where !! is the double factorial defined as follows

$$n!! := \prod_{k=0}^{\lceil n/2 \rceil - 1} (n - 2k) = n(n-2)(n-4)\ldots.$$

Finally the second-order Taylor expansion of $N(x)$ in a is

$$\begin{aligned} N(x) &= N(a) + N'(a)(x-a) + \frac{N''(a)}{2!}(x-a)^2 + O((x-a)^3) \\ &= N(a) - \frac{e^{-a^2/2}}{\sqrt{2\pi}}(x-a) + \frac{e^{-a^2/2}a}{\sqrt{2\pi}}(x-a)^2 + O((x-a)^3). \end{aligned} \tag{2.3}$$

Even though the Black and Scholes framework is not consistent with the non constant volatility observed on the market (see Fig. 2), it is an easy and a handy tool for professionals. For this reason it is common practice of the financial industry to take the observed implied volatility as an input and, from it, to derive back the options' value in a process defined "the wrong number in the wrong formula to get the right price" [18].

2.1. Brenner & Subrahmanyam formula

Focusing for simplicity on European options, Brenner & Subrahmanyam (1988) [19] provide a formula that approximates fairly well the volatility when the stock price is very close to the strike price, but it very quickly loses accuracy as it departs from it.

In fact for $S = X$ is $d_1 = \frac{1}{2}\sigma\sqrt{T} = -d_2$ and

$$C = SN\left(\frac{1}{2}\sigma\sqrt{T}\right) - S\,N\left(-\frac{1}{2}\sigma\sqrt{T}\right).$$

Moreover the first-order Maclaurin expansion of $N(x)$ is

$$N(x) := N(0) + N'(0)x + \frac{N''(0)}{2!}x^2 + O(x^2) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}x + O(x^2)$$

and

$$C \approx S\left(\frac{1}{2} + \frac{1}{\sqrt{2\pi}}\,\frac{1}{2}\sigma\sqrt{T} - \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\,\frac{1}{2}\sigma\sqrt{T}\right) = \frac{1}{\sqrt{2\pi}}\,S\,\sigma\sqrt{T}$$

which yields the same result when (2.2) is calculated in d_1 and d_2 respectively:

$$\begin{aligned} N(d_1) &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\,\frac{1}{2}\sigma\sqrt{T} \\ N(d_2) &= 1 - N(d_1) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\,\frac{1}{2}\sigma\sqrt{T}. \end{aligned}$$

Table 1
Approximation errors: Bharadia, Christofides and Salkin model.

Strike price ($)	Time to expiration				Mean of absolute values
	0.25	0.50	0.75	1.00	
70	−0.4092	−0.2358	−0.1717	−0.1383	0.2388
75	−0.2685	−0.1534	−0.1124	−0.0913	0.1564
80	−0.1554	−0.0894	−0.0665	−0.0549	0.0915
85	−0.0727	−0.0433	−0.0333	−0.0282	0.0444
90	−0.0214	−0.014	−0.0116	−0.0103	0.0143
95	−0.0002	−0.0002	−0.0003	−0.0003	0.0002
100	−0.0061	−0.0001	0.0019	0.0029	0.0027
105	−0.0349	−0.0116	−0.0039	0	0.0126
110	−0.0819	−0.033	−0.0163	−0.0079	0.0348
115	−0.1425	−0.0622	−0.0343	−0.0202	0.0648
120	−0.2126	−0.0977	−0.0569	−0.0361	0.1008
125	−0.2888	−0.1379	−0.0831	−0.0549	0.1412
130	−0.3684	−0.1815	−0.1122	−0.0761	0.1846
Mean of absolute values	0.1587	0.0816	0.0542	0.0401	0.0836

Estimation error between the true volatility (0.41) and the model estimate of volatility obtained for a call option with the following input $T = 1$ year, $X = \$95.21$, $\sigma = 0.35$, $K = \$100$, $r = 5\%$ and $S = \$13.21$. Values taken from Table 4 in Chambers and Nawalkha (2001) [21].

Therefore the Brenner & Subrahmanyam approximation in his first formulation is:

$$\sigma_{BS} = \sqrt{\frac{2\pi}{T}}\,\frac{C}{S}$$

and, more generally, for $S \neq X$ is:

$$\sigma_{BS} = \sqrt{\frac{2\pi}{T}}\,\frac{(C+P)/2}{S}.$$

Hence using put–call parity

$$P = C + X - S$$

for European options (where P is the put), we obtain:

$$\sigma_{BS} = \sqrt{\frac{2\pi}{T}}\,\frac{C-\delta}{S} \tag{2.4}$$

where $\delta = (S - X)/2$.

2.2. Bharadia, Christofides and Salkin formula

Bharadia, Christofides, and Salkin (1995) [20] suggest an improvement given by the following simplified formula:

$$\sigma_{BCS} = \sqrt{\frac{2\pi}{T}}\,\frac{C-\delta}{S-\delta}. \tag{2.5}$$

Also in this case, as well as for the Brenner and Subrahmanyam formula (2.4), the accuracy of the approximation worsens as soon as the option departs from the at-the-money position (see Table 1).

2.3. Chance model

Chance (1996) [22] starts with the implied volatility $\sigma^* = \sigma_{BS}$ of an at-the-money-call C^* option obtained with the Brenner–Subrahmanyam approximation formula (2.4) (see Table 2).

By expanding to second order of Taylor series, Chance finds out that the solution has a quadratic form:

$$\Delta\sigma^* = \frac{-b + \sqrt{b^2 - 4aq}}{2a} \tag{2.6}$$

Table 2
Approximation errors: Chance model.

Strike price ($)	Time to expiration				Mean of absolute values
	0.25	0.50	0.75	1.00	
70	0.1115	0.0312	0.0143	0.0081	0.0413
75	0.0636	0.0197	0.01	0.0063	0.0249
80	0.0342	0.0121	0.0068	0.0046	0.0144
85	0.017	0.0068	0.004	0.0027	0.0076
90	0.007	0.003	0.0017	0.001	0.0032
95	0.0015	0.0004	0	−0.0003	0.0005
100	−0.0004	−0.0006	−0.0007	−0.0007	0.0006
105	0.0021	0.0007	0.0002	−0.0001	0.0008
110	0.0108	0.005	0.0031	0.0021	0.0052
115	0.0282	0.0132	0.0085	0.0061	0.014
120	0.0566	0.026	0.0168	0.0124	0.028
125	0.0977	0.0443	0.0285	0.021	0.0479
130	0.1529	0.0685	0.0438	0.0322	0.0744
Mean of absolute values	0.0449	0.0178	0.0106	0.0075	0.0202

Estimation error between the true volatility (0.41) and the model estimate of volatility. The implied volatility from Brenner–Subrahmanyam formula is 0.348 obtained for a call option with the following input $T = 1$ year, $X = \$95.21$, $\sigma = 0.35$, $K = \$100$, $r = 5\%$ and $S = \$13.21$. Values taken from Table 1 in Chance (1996) [22] and Table 1 in Chambers and Nawalkha (2001) [21].

Table 3
Brenner & Subrahmanyam vs. improved quadratic formula.

Stock name	Stock price ($)	Call price ($)	Strike price ($)	Riskfree	Maturity	σ_{BS}	σ_{CM}	σ
A	83.250	4.625	80	4.750%	32	28.8165%	25.0461%	25.20%
	83.250	1.750	85	4.750%	32	24.8975%	24.0335%	24.04%
B	52.875	3.500	50	4.750%	32	31.3587%	23.5762%	24.31%
	52.875	0.875	55	4.750%	32	29.1910%	25.9481%	26.01%

where:

$$a = \frac{1}{2} \frac{\partial^2 C^*}{\partial \sigma^* \partial \sigma^*}$$
$$b = \frac{\partial C^*}{\partial \sigma^*} + \frac{\partial^2 C^*}{\partial \sigma^* \partial K^*} \Delta K^*$$
$$q = C^* - C + \frac{\partial C^*}{\partial K^*} + \frac{1}{2} \frac{\partial^2 C^*}{\partial K^* \partial K^*} \Delta K^{*2}.$$

As expected more the distance from the at-the money more is the approximation error which may lead to a mispricing.

2.4. Corrado & Miller approach: The "improved quadratic formula"

To increase the accuracy of the volatility provided by the "Brenner & Subrahmanyam formula" Corrado & Miller (1996) [23,24] have proposed the following "improved quadratic formula":

$$\sigma_{CM} = \sqrt{\frac{2\pi}{T}} \frac{1}{S - X} \left[C - \delta + \sqrt{(C - \delta)^2 - \frac{4\,\delta^2}{\pi}} \right]. \tag{2.7}$$

It is clear that the above formula is not defined for

$$C \in \left(\delta - \frac{2\delta}{\sqrt{\pi}}, \delta + \frac{2\delta}{\sqrt{\pi}} \right).$$

From Fig. 3 it can be seen that in the case exemplified by them, that the true implied volatility, equal to 20%, is much better approximated by the improved formula compared to that obtained with the Brenner & Subrahmanyam formula.

In addition, reporting a simulation on real data (see Table 3), we can observe that the improved formula provides us convincingly with better values.

However, although alleviating the imprecision of the solution by Brenner & Subrahmanyam, the solution by Corrado & Miller has the same problem of ever-increasing error when the value of the stock price diverges from the strike price, i.e. when there is the highest indeterminacy of the market. As often happens it is in this context that the accuracy of the information is essential; for this reason it would be highly valuable to have a formula that provides a better accuracy.

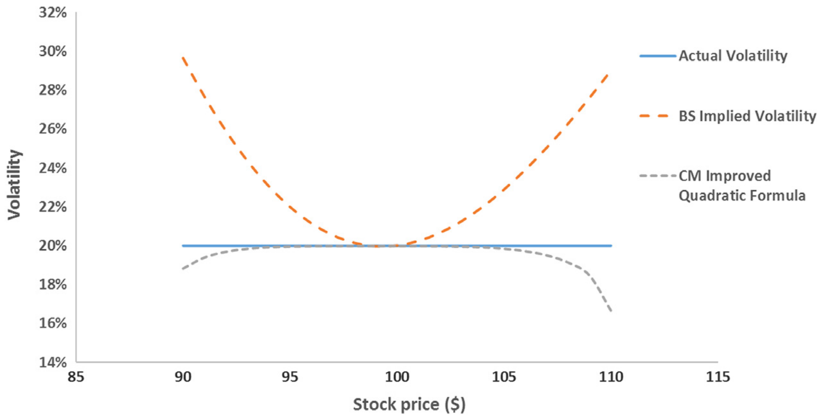

Fig. 3. True volatility vs. implied volatility.

Table 4
Brenner & Subrahmanyam vs. improved quadratic formula for 10% in- and out-of-the-money options.

Stock price ($)	Call price ($)	Strike price ($)	Riskfree	Maturity	σ_{BS}	$\sigma_{BS}-\sigma$	σ_{CM}	$\sigma_{CM}-\sigma$	σ
90	0.8682315	100	4.75%	90	29.65%	9.65%	18.83%	−1.17%	20%
91	1.0694896	100	4.75%	90	27.67%	7.67%	19.40%	−0.60%	20%
92	1.3032421	100	4.75%	90	25.90%	5.90%	19.69%	−0.31%	20%
93	1.5718081	100	4.75%	90	24.37%	4.37%	19.85%	−0.15%	20%
94	1.8772032	100	4.75%	90	23.06%	3.06%	19.93%	−0.07%	20%
95	2.2210861	100	4.75%	90	21.99%	1.99%	19.97%	−0.03%	20%
96	2.6047172	100	4.75%	90	21.15%	1.15%	19.98%	−0.02%	20%
97	3.0289328	100	4.75%	90	20.54%	0.54%	19.99%	−0.01%	20%
98	3.4941361	100	4.75%	90	20.15%	0.15%	19.99%	−0.01%	20%
99	4.0002779	100	4.75%	90	19.98%	−0.02%	19.99%	−0.01%	20%
100	4.5468389	100	4.75%	90	20.01%	0.01%	19.99%	−0.01%	20%
101	5.1329993	100	4.75%	90	20.25%	0.25%	19.99%	−0.01%	20%
102	5.7575111	100	4.75%	90	20.66%	0.66%	19.98%	−0.02%	20%
103	6.4188171	100	4.75%	90	21.25%	1.25%	19.96%	−0.04%	20%
104	7.1151023	100	4.75%	90	22.00%	2.00%	19.92%	−0.08%	20%
105	7.8443455	100	4.75%	90	22.89%	2.89%	19.85%	−0.15%	20%
106	8.6043722	100	4.75%	90	23.92%	3.92%	19.72%	−0.28%	20%
107	9.3929083	100	4.75%	90	25.05%	5.05%	19.51%	−0.49%	20%
108	10.20763	100	4.75%	90	26.29%	6.29%	19.14%	−0.86%	20%
109	11.046211	100	4.75%	90	27.62%	7.62%	18.46%	−1.54%	20%
110	11.906363	100	4.75%	90	29.02%	9.02%	16.65%	−3.35%	20%

To get an idea of the amplitude of the 6 deviation, we report the Table 4, which we have reproduced the values calculated with the two formulas, and with which we have drawn Fig. 3.

The table was constructed by calculating option value given the strike price, the time interval, the rate, the stock and the volatility constantly equal to 20%. Instead the values identified with the Brenner & Subrahmanyam and the improved quadratic formula were calculated by proceeding in the inverse manner.

From Table 4 we can see that a deviation of 10% from the at-the-money, results in an error in the calculated values with the improved quadratic formula that goes overall from 1.17% (in-the-money) to −3.35% (out-of-the-money). A further problem that emerges from the Improved Quadratic Formula is that, unlike the Brenner & Subrahmanyam formula, in some cases it cannot be calculated as it contains a square root.

2.5. Li formula

Li (2005) [25] considers first the at-the-money case $S = X$; taking the third order Taylor expansion of the standard normal cumulative distribution function

$$N(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\,x - \frac{1}{\sqrt{6\pi}}\,x^2 + O(x^5)$$

and substituting it in (2.1) where $N(d_1) = N(1/2\,\sigma\sqrt{T}\,)$ and $N(d_2) = N(-1/2\,\sigma\sqrt{T}\,)$ yields

$$\frac{\sqrt{2\pi}}{S}\,C \approx \sigma\sqrt{T} + \frac{1}{3}(1/2\,\sigma\sqrt{T}\,)^3. \tag{2.8}$$

By solving (2.8) for σ, Li obtains

$$\sigma_L^{ATM} = \frac{2\sqrt{2}}{\sqrt{T}} z - \frac{1}{\sqrt{T}} \sqrt{8z^2 - \frac{6\alpha}{z\sqrt{2}}} \tag{2.9}$$

with:

$$\alpha = \frac{\sqrt{2\pi}}{S} C$$
$$z = \cos\left(\frac{1}{3} \cos^{-1}\left(\frac{3\alpha}{32}\right)\right).$$

More generally, denoting

$$\eta = \frac{X}{S}$$
$$\rho = \frac{|\eta - 1|}{(C/S)^2} = \frac{|K - S|S}{C^2}$$
$$\tilde{\alpha} = \frac{\sqrt{2\pi}}{1+\eta}\left[\frac{2C}{S} + \eta - 1\right]$$
$$\tilde{z} = \cos\left(\frac{1}{3} \cos^{-1}\left(\frac{3\tilde{\alpha}}{32}\right)\right)$$

the Li formula for the implied volatility is

$$\sigma_L = \begin{cases} \dfrac{2\sqrt{2}}{\sqrt{T}}\tilde{z} - \dfrac{1}{\sqrt{T}}\sqrt{8\tilde{z}^2 - \dfrac{6\tilde{\alpha}}{\tilde{z}\sqrt{2}}}, & \text{if } \rho \le 1.4 \\ \dfrac{1}{2\sqrt{T}}\left[\tilde{\alpha} + \sqrt{\tilde{\alpha}^2 - \dfrac{4(\eta-1)^2}{1+\eta}}\right], & \text{if } \rho > 1.4. \end{cases}$$

As shown in Table 5, the Li formula approximates better the true volatility when this is high.

3. Numerical methods

Many have suggested numerical methods which involve the use of a root-finding procedure for obtaining the given value of σ such that the function

$$f(\sigma) = C - g(\sigma) \tag{3.1}$$

vanishes i.e. that the price of the call C observed on the market is the same as that expressed by the value

$$g(\sigma) = S\,N(d_1) - X\,N(d_2)$$

proposed by the algorithm.

Alternatives vary from the robust but inefficient/inaccurate so called "shotgun method" by Kritzman (1991) [26] to the bisection method (see Brown (1990) [27] and Chriss (1996) [28]), which requires a starting interval where the initial guess lays in the middle. However, as pointed out by Chance et al. (2013) [12], "computational issues that arise from the numerical estimation of implied volatility can generate implied volatilities that produce smiles, skews, and smirks" therefore "even if investors price options in accordance with the Black–Scholes–Merton model using a single volatility, the implied volatilities across strikes almost always vary".

To overcome the kind of problem described above and obtain greater accuracy in the calculation of volatility, it is possible to think about a combination of an algorithm with a high degree of convergence as the Newton–Raphson (or the Secant) method (which avoids prolonged iterations) and a data (used as initial point) which have some bearing on the true volatility. The Newton–Raphson (resp. Secant) method consists of a numerical solution of the equation $f(x) = 0$. The solution x^* is found after a number of iterations that are progressively closer to the true value x^* (in our case the value of the volatility implied in the market price of the option). In the following we summarize some properties of two most common algorithms used for computing the implied volatility (for more details see for example [29–31]).

Table 5
Comparison of Brenner & Subrahmanyam, Corrado & Miller and Li estimated implied volatilities for at-the-money calls.

Time to expiration	True volatility								
	15%			35%			55%		
	σ_{BS}	σ_{CM}	σ_L	σ_{BS}	σ_{CM}	σ_L	σ_{BS}	σ_{CM}	σ_L
0.1	14.96%	14.91%	12.16%	34.89%	34.95%	28.52%	54.79%	54.91%	55.00%
0.2	14.92%	14.83%	14.04%	34.79%	34.89%	33.05%	54.59%	54.82%	55.00%
0.3	14.88%	14.74%	15.46%	34.69%	34.84%	36.51%	54.38%	54.72%	55.00%
0.4	14.85%	14.65%	16.64%	34.58%	34.78%	39.41%	54.18%	54.63%	55.00%
0.5	14.81%	14.56%	17.67%	34.48%	34.73%	41.96%	53.98%	54.54%	55.00%
0.6	14.77%	14.47%	18.60%	34.38%	34.68%	44.26%	53.78%	54.45%	55.00%
0.7	14.73%	14.38%	19.44%	34.28%	34.62%	46.37%	53.58%	54.36%	55.00%
0.8	14.69%	14.29%	20.22%	34.17%	34.57%	48.32%	53.38%	54.27%	55.01%
0.9	14.66%	14.20%	20.94%	34.07%	34.51%	50.15%	53.19%	54.17%	55.01%
1	14.62%	14.10%	21.62%	33.97%	34.46%	51.88%	52.99%	54.08%	55.01%
1.1	14.58%	14.00%	22.25%	33.87%	34.41%	53.52%	52.80%	53.99%	55.01%
1.2	14.55%	13.91%	22.86%	33.77%	34.35%	55.08%	52.60%	53.90%	55.01%
1.3	14.51%	13.81%	23.44%	33.68%	34.30%	56.57%	52.41%	53.81%	55.01%
1.4	14.47%	13.70%	23.99%	33.58%	34.24%	58.00%	52.22%	53.72%	55.02%
1.5	14.44%	13.60%	24.51%	33.48%	34.19%	59.38%	52.03%	53.63%	55.02%

Time to expiration	True volatility								
	75%			95%			135%		
	σ_{BS}	σ_{CM}	σ_L	σ_{BS}	σ_{CM}	σ_L	σ_{BS}	σ_{CM}	σ_L
0.1	74.64%	74.81%	75.00%	94.41%	94.63%	95.00%	133.65%	133.97%	135.01%
0.2	74.28%	74.62%	75.00%	93.82%	94.26%	95.00%	132.32%	132.96%	135.03%
0.3	73.92%	74.43%	75.00%	93.24%	93.90%	95.01%	131.00%	131.96%	135.07%
0.4	73.57%	74.24%	75.01%	92.66%	93.54%	95.02%	129.71%	130.97%	135.12%
0.5	73.22%	74.05%	75.01%	92.09%	93.18%	95.03%	128.44%	130.00%	135.19%
0.6	72.87%	73.86%	75.01%	91.53%	92.82%	95.05%	127.19%	129.04%	135.28%
0.7	72.52%	73.67%	75.02%	90.97%	92.46%	95.06%	125.95%	128.09%	135.39%
0.8	72.17%	73.48%	75.02%	90.41%	92.11%	95.08%	124.73%	127.15%	135.53%
0.9	71.83%	73.29%	75.03%	89.86%	91.76%	95.11%	123.54%	126.23%	135.68%
1	71.49%	73.11%	75.04%	89.31%	91.41%	95.13%	122.35%	125.31%	135.86%
1.1	71.15%	72.92%	75.05%	88.77%	91.06%	95.16%	121.19%	124.41%	136.07%
1.2	70.82%	72.74%	75.06%	88.24%	90.72%	95.20%	120.04%	123.52%	136.31%
1.3	70.49%	72.55%	75.07%	87.70%	90.37%	95.23%	118.91%	122.65%	136.58%
1.4	70.16%	72.37%	75.08%	87.18%	90.03%	95.27%	117.80%	121.78%	136.88%
1.5	69.83%	72.19%	75.09%	86.65%	89.69%	95.31%	116.70%	120.92%	137.23%

For a comparison with Tables 1 and 2 [25] each entry represents the estimated implied volatilities for at-the-money calls with an exercise price of \$100, and risk free-rate of 5% p.a. Implied volatilities have been calculated with Brenner & Subrahmanyam (σ_{BS}), Corrado & Miller (σ_{CM}) and Li formula (σ_L), respectively. The prices of all calls used in this table are generated with the Black & Scholes model for a given volatility ranging from 15% to 135% and maturity ranging from 0.1 to 1.5 years.

3.1. The Newton–Raphson method

The Newton–Raphson method is a special case of the method of fixed point iterations. Given an equation $f(x) = 0$, a sequence $x_0, \; x_1, \ldots$ can be defined using an initial approximation x_0, and the formula

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}.$$

When the initial point is in a neighbourhood of the zero and $f'(x) \neq 0$ the method will usually converge (and the convergence is at least quadratic if the zero has multiplicity 1).

If $f(x)$ is regular enough, it can be proved that the Newton–Raphson method provides quadratic convergence:

$$e_{n+1} \leq c\, e_n^2 \quad \text{with } c \approx \frac{1}{2}\left|\frac{f''(x^*)}{f'(x^*)}\right|,$$

where $e_n := |x_n - x^*|$.

In other terms a good convergence is ensured when the constant k is small and $e_n \ll 1$.

3.2. The secant method

Given an equation $f(x) = 0$, it can be defined a sequence $x_0, \; x_1, \ldots$ using two initial approximations x_0 and x_1, and the formula

$$x_{j+1} = x_j - \frac{f(x_j)\,(x_j - x_{j-1})}{f(x_i) - f(x_{i-1})}.$$

The convergence of the Secant method is *super-linear*, i.e.:

$$e_{n+1} \leq c_f e_n^{\alpha} \quad \text{with } \alpha = \frac{1+\sqrt{5}}{2} \simeq 1.61803,$$

where c_f is a constant dependent on the derivative of f.

For a comparison between the two methods, first of all it can be noted that the Secant method requires only one function evaluation per iteration whilst the Newton–Raphson method requires two evaluations per iteration: the function and the derivative. Therefore in terms of speed it is:

$$e_{n+2} \leq c_f e_{n+1}^{\alpha} \leq c_f (c_f\, e_n^{\alpha})^{\alpha} = c_f^{\alpha+1} e_n^{\alpha^2}.$$

Since $\alpha^2 > 2$ the Secant method performs better than the Newton–Raphson method but for the remainder of the discussion we will only refer to the Newton–Raphson method as the results of the two algorithms are equivalent.

3.3. The zero for the Black and Scholes function

The Newton–Raphson algorithm is applied to (3.1) in order to obtain the given value of σ. The reason previously given for the use of an initial sigma (e.g. resulting from Brenner & Subrahmanyam formula) close to the "true" implied volatility, not only lies in an attempt to provide a satisfactory answer to the problem of finding an "educated guess" but, also, lies in the fact that the function $f(\sigma)$ could present a starting point with a high "allocation error", or would lead the algorithm to diverge too far.

To better understand this point we proceed with the study of the function $g(\sigma)$.

In the following we note $k_1 := \frac{\log(S/X)}{\sqrt{T}}$, $k_2 := \frac{1}{2}\sqrt{T}$, hence we have

$$d_1 := \frac{k_1}{\sigma} + \sigma\, k_2 \qquad d_2 := \frac{k_1}{\sigma} - \sigma\, k_2.$$

Note that:

$$d_1' = -\frac{k_1}{\sigma^2} + k_2 = -\frac{d_2}{\sigma} \qquad d_2' = -\frac{k_1}{\sigma^2} - k_2 = -\frac{d_1}{\sigma}, \tag{3.2}$$

and

$$d_1'' = -\frac{d_2'\sigma - d_2}{\sigma^2} = \frac{d_1 + d_2}{\sigma^2} \qquad d_2'' = -\frac{d_1'\sigma - d_1}{\sigma^2} = \frac{d_1 + d_2}{\sigma^2},$$

that is

$$d_1'' = d_2'' = \frac{2\,k_1}{\sigma^3}.$$

Proposition 1. *The function $g(\sigma)$ has the following properties:*

1.i $g'(\sigma) = S\sqrt{\frac{T}{2\pi}}\, e^{-d_1^2/2}$;
1.ii *is strictly increasing for* $\sigma > 0$;
1.iii $g''(\sigma) = S\sqrt{\frac{T}{2\pi}}\, e^{-d_1^2/2}\, \frac{d_1\, d_2}{\sigma}$;
1.iv *if* $k_1 = 0$, *then* $g(\sigma)$ *is concave in* $]0, +\infty[$, *whereas if* $k_1 \neq 0$, *then* $g(\sigma)$ *is convex in* $]0, \sigma_F[$ *and concave in* $]\sigma_F, +\infty[$, *where*

$$\sigma_F := \sqrt{\frac{|k_1|}{k_2}} = \sqrt{\frac{2\,|\log(S/K)|}{T}}$$

is the inflection point of $g(\sigma)$;
1.v *as* $\sigma \to 0$ *then* $g(\sigma)$ *decreases to the pay-off, that is:*

$$\lim_{\sigma\to 0^+} g(\sigma) = \max(0, S - X);$$

1.vi $\lim_{\sigma\to+\infty} g(\sigma) = S$.

To prove the first point we need the following lemma:

Lemma 3.1. *We have*

$$S\, N'\big(d_1(\sigma)\big) - X\, N'\big(d_2(\sigma)\big) = 0. \tag{3.3}$$

Proof of the Lemma. We have:

$$S\, N'\big(d_1(\sigma)\big) - X\, N'\big(d_2(\sigma)\big) = \frac{S}{\sqrt{2\pi}}\left[e^{-d_1^2/2} - e^{-d_2^2/2}\,\frac{X}{S}\right].$$

Noting that $e^{-2\,k_1\,k_2} = \frac{X}{S}$, we have:

$$e^{-d_2^2/2}\,\frac{X}{S} = \exp\left[-\frac{1}{2}\left(\frac{k_1}{\sigma} - \sigma\, k_2\right)^2 - 2\, k_1\, k_2\right] = \exp\left[-\frac{1}{2}\left(\frac{k_1}{\sigma} + \sigma\, k_2\right)^2\right] = e^{-d_1^2/2}. \quad \square$$

Proof of 1.i and 1.ii. Using (3.2) and (3.3):

$$\begin{aligned} g'(\sigma) &= S\, N'\big(d_1(\sigma)\big)\, d_1' - X\, N'\big(d_2(\sigma)\big)\, d_2' = S\, N'\big(d_1(\sigma)\big)\,(d_1' - d_2') \\ &= \frac{1}{\sqrt{2\pi}}\,\frac{S}{\sigma}\, e^{-d_1^2/2}\,(d_1 - d_2) = S\sqrt{\frac{T}{2\pi}}\, e^{-d_1^2/2} \end{aligned}$$

thus $g'(\sigma) > 0$ for $\sigma > 0$, thus g is strictly increasing. $\square$

Proof of 1.iii and 1.iv. Now, using the previous result:

$$g''(\sigma) = S\sqrt{\frac{T}{2\pi}}\, e^{-d_1^2/2}\,(-d_1)d_1' = S\sqrt{\frac{T}{2\pi}}\, e^{-d_1^2/2}\,\frac{d_1\, d_2}{\sigma}$$

and

$$g''(\sigma) = 0 \iff d_1\, d_2 = 0 \iff \frac{k_1^2}{\sigma^2} - \sigma^2\, k_2^2 = 0 \iff \sigma = \sigma_F,$$

which shows that $g(\sigma)$ is convex in $]0, \sigma_F[$ and that $g(\sigma)$ is concave in $]\sigma_F, +\infty[$. $\square$

Proof of 1.v. We need to distinguish three cases according to the sign of k_1.

If $k_1 > 0$ then

$$\lim_{\sigma\to 0^+} d_1(\sigma) = \lim_{\sigma\to 0^+} d_2(\sigma) = +\infty,$$

then

$$\lim_{\sigma\to 0^+} N\big(d_1(\sigma)\big) = \lim_{\sigma\to 0^+} N\big(d_2(\sigma)\big) = 1,$$

hence

$$\lim_{\sigma\to 0^+} g(\sigma) = S - X.$$

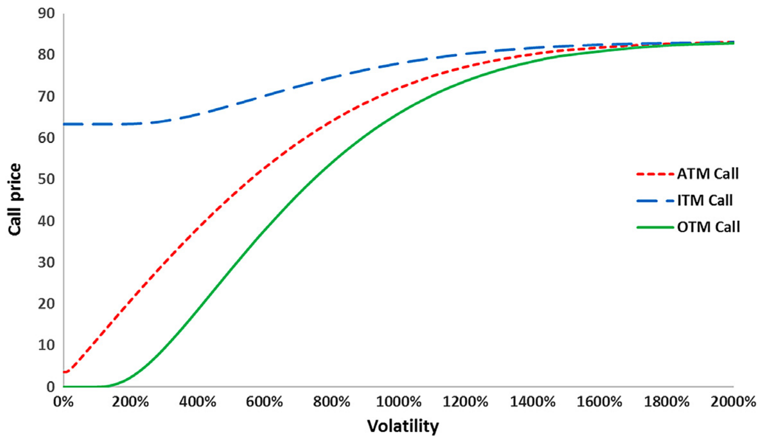

Fig. 4. Three examples of options vs. volatility. ATM, ITM and OTM calls versus the volatility.

If $k_1 = 0$, that is $S = X$, then

$$\lim_{\sigma\to 0^+} d_1(\sigma) = \lim_{\sigma\to 0^+} d_2(\sigma) = 0,$$

hence

$$\lim_{\sigma\to 0^+} N\big(d_1(\sigma)\big) = \lim_{\sigma\to 0^+} N\big(d_2(\sigma)\big) = \frac{1}{2},$$

therefore

$$\lim_{\sigma\to 0^+} g(\sigma) = \frac{1}{2}(S - X) = 0.$$

If $k_1 < 0$ then

$$\lim_{\sigma\to 0^+} d_1(\sigma) = \lim_{\sigma\to 0^+} d_2(\sigma) = -\infty,$$

then

$$\lim_{\sigma\to 0^+} N\big(d_1(\sigma)\big) = \lim_{\sigma\to 0^+} N\big(d_2(\sigma)\big) = 0,$$

hence

$$\lim_{\sigma\to 0^+} g(\sigma) = 0. \quad \square$$

Proof of 1.vi. We have

$$\lim_{\sigma\to +\infty} d_1(\sigma) = +\infty \quad \text{and} \quad \lim_{\sigma\to +\infty} d_2(\sigma) = -\infty,$$

then

$$\lim_{\sigma\to +\infty} N\big(d_1(\sigma)\big) = 1 \quad \text{and} \quad \lim_{\sigma\to +\infty} N\big(d_2(\sigma)\big) = 0,$$

hence

$$\lim_{\sigma\to +\infty} g(\sigma) = S. \quad \square$$

$g(\sigma)$ is the option value with respect to the volatility of the underlying asset (see Fig. 4).

$g'(\sigma)$ is the measure of the sensitivity to volatility, i.e. the derivative of the option value with respect to the volatility of the underlying asset. $g'(\sigma)$ is called *Vega* and usually denoted by ν (see Fig. 5).

Remark 1. If $k_1 = 0$, that is $S = X$, then $d_1 = \sigma\, k_2$ and $d_2 = -\sigma\, k_2 = -d_1$, hence the function g reduces to

$$\begin{aligned} g(\sigma) &= S\,N\big(d_1(\sigma)\big) - S\,N\big(-d_1(\sigma)\big) = \frac{S}{\sqrt{2\pi}} \int_{-\sigma\, k_2}^{\sigma\, k_2} e^{-t^2/2}\, dt \\ &= \frac{2\,S}{\sqrt{2\pi}} \int_0^{\sigma\, k_2} e^{-t^2/2} = S\ \mathrm{erf}\Big(\frac{\sigma\, k_2}{\sqrt{2}}\Big), \end{aligned}$$

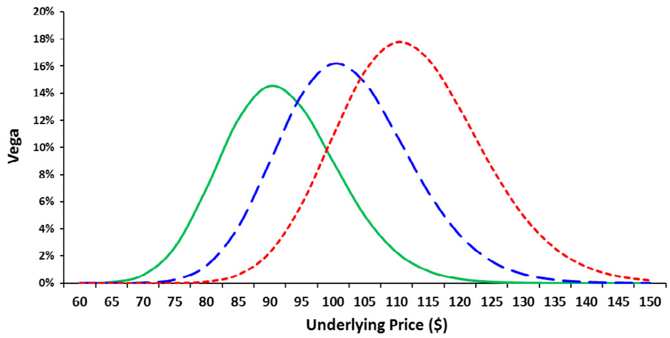

Fig. 5. Three examples of Vega vs. underlying price. Three cases of bell shaped Vega. Vega is maximum when the option is at-the-money.

where erf is the error function:

$$\mathrm{erf}(x) := \int_0^x \exp(-t^2)\,dt.$$

Thus the equation $f(\sigma) = 0$ has a unique solution for any choice of $C \in]0, S[$, and can be expressed by the *inverse error function*:

$$\sigma = \frac{\sqrt{2}}{k_2}\,\mathrm{inverf}\left(\frac{C}{S}\right).$$

Philip (1960) [32], Strecok (1968) [33] and Fettis (1974) [34] studied the inverse of the complementary error function ($\mathrm{inverfc}(x)$) and the asymptotic behaviour for small x.

Remark 2. If $S > X$, we can replace the $g(\sigma)$ for the call options, with the corresponding $\widetilde{g}(\sigma)$ for the put options, that is:

$$\widetilde{g}(\sigma) = g(\sigma) - S + X = S\,N(d_1) - X\,N(d_2) - S + X.$$

Now, since $N(x) = 1 - N(-x)$, we can write:

$$\widetilde{g}(\sigma) = X\,N(-d_2) - S\,N(-d_1).$$

The function $\widetilde{g}(\sigma)$ enjoys the same properties 1.i–1.iv of the function $g(\sigma)$, whereas properties 1.v and 1.vi are replaced by

2.v. as $\sigma \to 0$ then $\widetilde{g}(\sigma)$ decreases to 0, that is:

$$\lim_{\sigma\to 0^+} \widetilde{g}(\sigma) = 0;$$

2.vi. $\lim_{\sigma\to+\infty} \widetilde{g}(\sigma) = X$.

The Newton's algorithm is locally convergent. In order to have global convergence, a convexity hypothesis is sufficient. More precisely the following well-known Theorem holds true:

Theorem 1. *Let $f : I \to \mathbb{R}$ be a convex (or concave) differentiable function on an interval $I \subseteq \mathbb{R}$, with at least one root. Then the following sequence $\{x_n\}$ obtained from Newton's method,*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3.4}$$

will converge to a root of f, provided that $f'(x_0) \neq 0$ and $x_1 \in I$ for the given starting point $x_0 \in I$.

Let σ_* be the unique solution of the equation

$$g(\sigma) = C.$$

If $x_0 = \sigma_F$ then (3.4) converges to σ_*.

A way to discriminate whether the initial guessing point σ_0 is feasible is that it must satisfy the following conditions (see Theorem 1 in Chance et al. (2013) [12]):

$$\sigma_0 < \sigma_F \quad \text{and} \quad g(\sigma_0) > C \text{ if } C < g(\sigma_F)$$
$$\sigma_0 > \sigma_F \quad \text{and} \quad g(\sigma_0) < C \text{ if } C > g(\sigma_F)$$

Table 6
Brenner & Subrahmanyam, improved quadratic formula and Newton–Raphson.

Stock name	Stock price ($)	Call price ($)	Strike price ($)	Riskfree	Maturity	σ_{BS}	σ_{CM}	σ	σ_{NR}
A	83.250	4.625	80	4.750%	32	28.8165%	25.04608%	25.20%	25.2044393%
	83.250	1.75	85	4.750%	32	24.8975%	24.03348%	24.04%	24.0421981%
B	52.875	3.5	50	4.750%	32	31.3587%	23.5762%	24.31%	24.3057959%
	52.875	0.875	55	4.750%	32	29.1910%	25.9481%	26.01%	26.0092441%

Table 7
Brenner & Subrahmanyam, improved quadratic formula and Newton–Raphson for 10% In- and Out-of-the-money options.

Stock price ($)	Call price ($)	Strike price ($)	Riskfree	Maturity	σ_{BS}	σ_{CM}	σ	σ_{NR}
90	0.8682315	100	4.75%	90	29.65%	18.83%	20.00%	20.000000000%
91	1.0694896	100	4.75%	90	27.67%	19.40%	20.00%	20.000000000%
92	1.3032421	100	4.75%	90	25.90%	19.69%	20.00%	20.000000000%
93	1.5718081	100	4.75%	90	24.37%	19.85%	20.00%	20.000000000%
94	1.8772032	100	4.75%	90	23.06%	19.93%	20.00%	20.000000000%
95	2.2210861	100	4.75%	90	21.99%	19.97%	20.00%	20.000000000%
96	2.6047172	100	4.75%	90	21.15%	19.98%	20.00%	20.000000000%
97	3.0289328	100	4.75%	90	20.54%	19.99%	20.00%	20.000000000%
98	3.4941361	100	4.75%	90	20.15%	19.99%	20.00%	20.000000000%
99	4.0002779	100	4.75%	90	19.98%	19.99%	20.00%	20.000000000%
100	4.5468389	100	4.75%	90	20.01%	19.99%	20.00%	20.000000000%
101	5.1329993	100	4.75%	90	20.25%	19.99%	20.00%	20.000000000%
102	5.7575111	100	4.75%	90	20.66%	19.98%	20.00%	20.000000000%
103	6.4188171	100	4.75%	90	21.25%	19.96%	20.00%	20.000000000%
104	7.1151023	100	4.75%	90	22.00%	19.92%	20.00%	20.000000000%
105	7.8443455	100	4.75%	90	22.89%	19.85%	20.00%	20.000000000%
106	8.6043722	100	4.75%	90	23.92%	19.72%	20.00%	20.000000000%
107	9.3929083	100	4.75%	90	25.05%	19.51%	20.00%	20.000000000%
108	10.20763	100	4.75%	90	26.29%	19.14%	20.00%	20.000000000%
109	11.046211	100	4.75%	90	27.62%	18.46%	20.00%	20.000000000%
110	11.906363	100	4.75%	90	29.02%	16.65%	20.00%	20.000000000%

or, equivalently:

$$\text{if } \sigma_* < \sigma_F \text{ and } \sigma_* < \sigma_0 < \sigma_F$$

or

$$\text{if } \sigma_* > \sigma_F \text{ and } \sigma_F < \sigma_0 < \sigma_*.$$

3.4. Newton–Raphson with "Brenner & Subrahmanyam formula" as initial point

As mentioned above the idea of using the Newton–Raphson algorithm is not new but key is the selection of a "good" starting point x_0 (possibly "near" to the zero of $g(x)$ where $g'(\sigma) \not\cong 0$) necessary to ensure first of all the convergence and then the efficiency (i.e. convergence in few steps). As the Brenner & Subrahmanyam is the first order approximation of the solution, it can be considered as a good candidate as initial point for the algorithm.

In Tables 6 and 7 it is possible to compare the values obtained from the solution just proposed with those seen previously.

3.5. Newton–Raphson with the inflection as initial point

In Tables 9–11 it is possible to compare the values obtained for ATM, OTM and ITM options with the Newton–Raphson method for both cases: Brenner & Subrahmanyam and inflection as starting point. In Table 8 it is shown the performance of Newton–Raphson Method on a test run over 10,000 options. The algorithm, written on VBA, is quite fast in finding the 10,000 volatilities for the corresponding ATM, ITM and OTM options and, as expected, it is slightly faster when σ_F is given as initial point for ATM and ITM, whilst is sensibly faster for OTM options.

The reason why σ_{BS} as starting point stands well with σ_F, is that for ATM and ITM options may happen that the solution (i.e. the true volatility) lies closer to σ_{BS} than to σ_F as depicted in Fig. 6.

Table 8
Performance of Newton–Raphson method (seconds).

	Newton–Raphson with as starting point		Δ performance $\left(\frac{a}{b}-1\right)$
	σ_{BS} (a)	σ_F (b)	
ATM	1.095703125	1.092773	0.27%
ITM	1.2021484375	1.19921875	0.24%
OTM	1.3505859375	1.2275390625	10.02%

Performance of a VBA macro for finding 10,000 volatilities for each option's type (i.e. ATM, ITM, OTM) with the Newton–Raphson method and initial point the Brenner & Subrahmanyam (σ_{BS}) or the inflection volatility (σ_F). As shown in Fig. 4 the inflection is clearly distinguishable (and far from the minimum and the maximum of $g(\sigma)$) for OTM options. For this reason the Newton–Raphson algorithm performs better with the inflection as initial point.

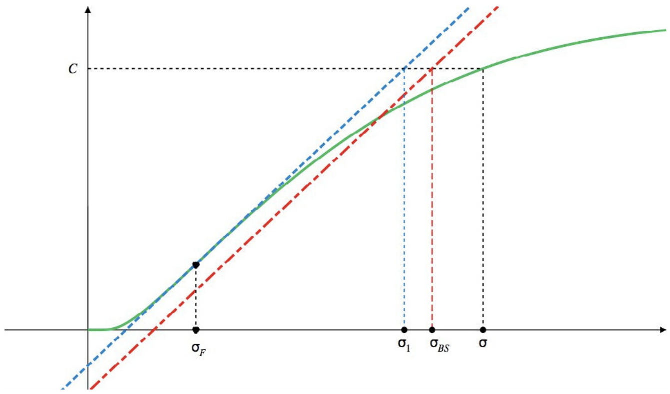

Fig. 6. σ_{BS} vs. σ_F as starting point. In blue the first step of the Newton–Raphson algorithm with σ_F as starting point, and in red the approximation given by the Brenner & Subrahmanyam formula. In the figure the true volatility is closer to σ_{BS} therefore the latter is a better initial point.

So far we have shown that the Newton–Raphson method gives accurate solutions until the volatility does not determine any longer the option's value (e.g. a change in volatility has an impact on the option's price lower than 10^{-5}). However, it must be noted that in those cases the solution lies where the derivative is almost zero. Therefore the Newton–Raphson method, where a suitable initial point is provided and under some given condition, is by far more precise and reliable than any other closed form approximations and it should be preferred when an exact solution is needed.

4. Conclusions

In this review we have illustrated the most common approaches to calculating the implied volatility with both closed-form solutions (which provide approximated results) and numerical-methods (which, within the specified conditions, provide more precise results). We have verified that the combination of the Newton–Raphson algorithm with initial point the Brenner & Subrahmanyam or the inflection volatility allows to build a function capable of extracting the implied volatility on the market with great precision and in real time.

In addition, it has been explained that the Brenner & Subrahmanyam volatility is preferred for those cases in which the inflection point is near the zero. Last but not least, it has been shown that such numerical methods do not work properly in those cases in which $g(\sigma)$ is flat. It can be argued however that in those cases the volatility matters little to none in determining the option's price. In literature hybrid methods are available, such as bisection with the Newton–Raphson algorithm, which can provide stability and ensure convergence to a desired root. Future research will explain in more detail how to identify the areas in which initial points should lie in order to ensure the convergence and will provide suggestions on how to find the zero where the Newton–Raphson method fails.

Appendix

Tables from 9 to 11 display the results of Newton–Raphson Method for ATM, OTM and ITM options.

Table 9
Newton–Raphson method for ATM options.

Stock price ($)	Call price ($)	Strike price ($)	σ_{BS}	NR Vol. (σ_{BS})	σ_F	NR Vol. (σ_F)	True Vol.
83.25	3.58246	80	18.21%	3.19682%	0.950%	3.24090906%	0.50%
83.25	3.58246	80	18.21%	3.19682%	0.950%	3.24090906%	1.00%
83.25	3.58246	80	18.21%	3.19682%	0.950%	3.24090906%	1.50%
83.25	3.58246	80	18.21%	3.19682%	0.950%	3.24090852%	2.00%
83.25	3.58246	80	18.21%	3.19418%	0.950%	3.23893306%	2.50%
83.25	3.58246	80	18.21%	2.86539%	0.950%	3.02681275%	3.00%
83.25	3.58245	80	18.21%	2.83073%	0.950%	1.81514464%	3.50%
83.25	3.58242	80	18.21%	3.12453%	0.950%	2.25809072%	4.00%
83.25	3.58241	80	18.21%	2.75438%	0.950%	0.94535317%	4.50%
83.25	3.58257	80	18.22%	4.42488%	0.950%	4.42487711%	5.00%
83.25	3.65187	80	18.92%	9.88530%	0.950%	9.88529798%	10.00%
83.25	3.88521	80	21.29%	14.92983%	0.950%	14.92983081%	15.00%
83.25	4.21908	80	24.69%	19.94882%	0.950%	19.94881924%	20.00%
83.25	4.60524	80	28.62%	24.95957%	0.950%	24.95956613%	25.00%
83.25	5.02055	80	32.84%	29.96653%	0.950%	29.96653115%	30.00%
83.25	5.45342	80	37.24%	34.97143%	0.950%	34.97142779%	35.00%
83.25	5.89756	80	41.76%	39.97506%	0.950%	39.97506433%	40.00%
83.25	6.34928	80	46.35%	44.97787%	0.950%	44.97787438%	45.00%
83.25	6.80632	80	51.00%	49.98011%	0.950%	49.98011223%	50.00%
83.25	7.26716	80	55.68%	54.98194%	0.950%	54.98193717%	55.00%
83.25	7.73078	80	60.40%	59.98345%	0.950%	59.98345421%	60.00%
83.25	8.19648	80	65.13%	64.98474%	0.950%	64.98473543%	65.00%
83.25	8.66371	80	69.89%	69.98583%	0.950%	69.98583199%	70.00%
83.25	9.13208	80	74.65%	74.98678%	0.950%	74.98678121%	75.00%
83.25	9.60129	80	79.42%	79.98761%	0.950%	79.98761098%	80.00%
83.25	10.07109	80	84.20%	84.98834%	0.950%	84.98834255%	85.00%
83.25	10.54130	80	88.98%	89.98899%	0.950%	89.98899240%	90.00%
83.25	11.01175	80	93.76%	94.98957%	0.950%	94.98957352%	95.00%
83.25	11.48231	80	98.55%	99.99010%	0.950%	99.99009629%	100.00%
83.25	12.42334	80	108.12%	109.99100%	0.950%	109.99099873%	110.00%
83.25	13.36366	80	117.68%	119.99175%	0.950%	119.99175030%	120.00%
83.25	14.30271	80	127.23%	129.99239%	0.950%	129.99238594%	130.00%
83.25	15.24002	80	136.76%	139.99293%	0.950%	139.99293058%	140.00%
83.25	16.17521	80	146.27%	149.99340%	0.950%	149.99340246%	150.00%
83.25	18.96482	80	174.64%	179.99450%	0.950%	179.99450305%	180.00%
83.25	21.72480	80	202.70%	209.99529%	0.950%	209.99528885%	210.00%
83.25	24.44891	80	230.41%	239.99588%	0.950%	239.99587804%	240.00%
83.25	27.13168	80	257.69%	269.99634%	0.950%	269.99633621%	270.00%
83.25	29.76813	80	284.50%	299.99670%	0.950%	299.99670271%	300.00%
83.25	34.04713	80	328.01%	349.99717%	0.950%	349.99717386%	350.00%
83.25	38.16626	80	369.90%	399.99753%	0.950%	399.99752719%	400.00%
83.25	42.10948	80	410.00%	449.99780%	0.950%	449.99780199%	450.00%
83.25	45.86351	80	448.17%	499.99802%	0.950%	499.99802181%	500.00%
83.25	49.41777	80	484.31%	549.99820%	0.950%	549.99820166%	550.00%
83.25	52.76445	80	518.35%	599.99835%	0.950%	599.99835154%	600.00%
83.25	58.81709	80	579.89%	699.99859%	0.950%	699.99858705%	700.00%
83.25	64.01063	80	632.71%	799.99876%	0.950%	799.99876367%	800.00%
83.25	68.37048	80	677.04%	899.99890%	0.950%	899.99890105%	900.00%
83.25	71.95121	80	713.46%	999.99901%	0.950%	999.99901095%	1000.00%
83.25	74.82839	80	742.71%	1099.99910%	0.950%	1099.99910086%	1100.00%
83.25	77.09021	80	765.71%	1199.99918%	0.950%	1199.99917579%	1200.00%
83.25	78.82980	80	783.40%	1299.99924%	0.950%	1299.99923919%	1300.00%
83.25	80.13878	80	796.71%	1399.99929%	0.950%	1399.99929354%	1400.00%
83.25	81.10242	80	806.51%	1499.99934%	0.950%	1499.99934064%	1500.00%
83.25	82.47025	80	820.42%	1749.99943%	0.950%	1749.99943483%	1750.00%
83.25	83.00022	80	825.81%	1999.99951%	0.950%	1999.99950548%	2000.00%

Table 10
Newton–Raphson method for OTM options.

Stock price ($)	Call price ($)	Strike price ($)	σ_{BS}	NR Vol. (σ_{BS})	σ_F	NR Vol. (σ_F)	True Vol.
83.25	–	200	589.39%	Vol. N/A	4.470%	Vol. N/A	0.50%
83.25	–	200	589.39%	Vol. N/A	4.470%	Vol. N/A	1.00%
83.25	–	200	589.39%	Vol. N/A	4.470%	Vol. N/A	1.50%
83.25	–	200	589.39%	Vol. N/A	4.470%	Vol. N/A	2.00%
83.25	–	200	589.39%	Vol. N/A	4.470%	Vol. N/A	2.50%
83.25	–	200	589.39%	Vol. N/A	4.470%	Vol. N/A	3.00%
83.25	–	200	589.39%	Vol. N/A	4.470%	Vol. N/A	3.50%
83.25	–	200	589.39%	Vol. N/A	4.470%	Vol. N/A	4.00%
83.25	–	200	589.39%	Vol. N/A	4.470%	Vol. N/A	4.50%
83.25	–	200	589.39%	Vol. N/A	4.470%	Vol. N/A	5.00%
83.25	0.00000	200	589.39%	60.47162%	4.470%	61.94556424%	10.00%
83.25	0.00000	200	589.39%	60.47162%	4.470%	61.94556424%	15.00%
83.25	0.00000	200	589.39%	60.47162%	4.470%	61.94556424%	20.00%
83.25	0.00000	200	589.39%	60.47162%	4.470%	61.94556424%	25.00%
83.25	0.00000	200	589.39%	60.47162%	4.470%	61.94556424%	30.00%
83.25	0.00000	200	589.39%	60.47162%	4.470%	61.94556424%	35.00%
83.25	0.00000	200	589.39%	60.47162%	4.470%	61.94556429%	40.00%
83.25	0.00000	200	589.39%	60.47166%	4.470%	61.94558733%	45.00%
83.25	0.00000	200	589.39%	60.47483%	4.470%	61.94740740%	50.00%
83.25	0.00000	200	589.39%	60.55699%	4.470%	61.99467300%	55.00%
83.25	0.00000	200	589.39%	61.45478%	4.470%	62.53595698%	60.00%
83.25	0.00001	200	589.39%	65.02619%	4.470%	65.21004357%	65.00%
83.25	0.00007	200	589.39%	69.98663%	4.470%	69.98711451%	70.00%
83.25	0.00028	200	589.39%	74.98743%	4.470%	74.98742769%	75.00%
83.25	0.00084	200	589.40%	79.98815%	4.470%	79.98814686%	80.00%
83.25	0.00215	200	589.41%	84.98879%	4.470%	84.98879156%	85.00%
83.25	0.00479	200	589.44%	89.98937%	4.470%	89.98937226%	90.00%
83.25	0.00955	200	589.48%	94.98990%	4.470%	94.98989766%	95.00%
83.25	0.01742	200	589.57%	99.99038%	4.470%	99.99037505%	100.00%
83.25	0.04716	200	589.87%	109.99121%	4.470%	109.99120921%	110.00%
83.25	0.10388	200	590.44%	119.99191%	4.470%	119.99191304%	120.00%
83.25	0.19721	200	591.39%	129.99251%	4.470%	129.99251433%	130.00%
83.25	0.33526	200	592.80%	139.99303%	4.470%	139.99303361%	140.00%
83.25	0.52408	200	594.72%	149.99349%	4.470%	149.99348639%	150.00%
83.25	1.42387	200	603.87%	179.99455%	4.470%	179.99455181%	180.00%
83.25	2.81903	200	618.05%	209.99532%	4.470%	209.99531963%	210.00%
83.25	4.64722	200	636.65%	239.99590%	4.470%	239.99589869%	240.00%
83.25	6.82390	200	658.78%	269.99635%	4.470%	269.99635073%	270.00%
83.25	9.26764	200	683.63%	299.99671%	4.470%	299.99671330%	300.00%
83.25	13.74894	200	729.20%	349.99718%	4.470%	349.99718054%	350.00%
83.25	18.52972	200	777.82%	399.99753%	4.470%	399.99753167%	400.00%
83.25	23.43142	200	827.66%	449.99781%	4.470%	449.99780513%	450.00%
83.25	28.32774	200	877.45%	499.99802%	4.470%	499.99802411%	500.00%
83.25	33.12855	200	926.27%	549.99820%	4.470%	549.99820339%	550.00%
83.25	37.76916	200	973.46%	599.99835%	4.470%	599.99835287%	600.00%
83.25	46.39916	200	1061.22%	699.99859%	4.470%	699.99858788%	700.00%
83.25	53.99804	200	1138.49%	799.99876%	4.470%	799.99876423%	800.00%
83.25	60.48797	200	1204.49%	899.99890%	4.470%	899.99890144%	900.00%
83.25	65.88282	200	1259.35%	999.99901%	4.470%	999.99901123%	1000.00%
83.25	70.25589	200	1303.82%	1099.99910%	4.470%	1099.99910108%	1100.00%
83.25	73.71636	200	1339.01%	1199.99918%	4.470%	1199.99917596%	1200.00%
83.25	76.39135	200	1366.21%	1299.99924%	4.470%	1299.99923932%	1300.00%
83.25	78.41221	200	1386.76%	1399.99929%	4.470%	1399.99929364%	1400.00%
83.25	79.90467	200	1401.94%	1499.99934%	4.470%	1499.99934072%	1500.00%
83.25	82.03124	200	1423.56%	1749.99943%	4.470%	1749.99943489%	1750.00%
83.25	82.85867	200	1431.98%	1999.99951%	4.470%	1999.99950551%	2000.00%

Table 11
Newton–Raphson method for ITM options.

Stock price ($)	Call price ($)	Strike price ($)	σ_{BS}	NR Vol. (σ_{BS})	σ_F	NR Vol. (σ_F)	True Vol.
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	0.50%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	1.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	1.50%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	2.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	2.50%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	3.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	3.50%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	4.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	4.50%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	5.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	10.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	15.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	20.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	25.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	30.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	35.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	40.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	45.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	50.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914144%	55.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914146%	60.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914153%	65.00%
83.25	63.33311	20	322.02%	97.81027%	5.700%	99.90914591%	70.00%
83.25	63.33311	20	322.02%	97.81046%	5.700%	99.90925852%	75.00%
83.25	63.33311	20	322.02%	97.81315%	5.700%	99.91087972%	80.00%
83.25	63.33311	20	322.02%	97.83771%	5.700%	99.92570388%	85.00%
83.25	63.33311	20	322.02%	97.99318%	5.700%	100.02000137%	90.00%
83.25	63.33311	20	322.02%	98.70029%	5.700%	100.45937823%	95.00%
83.25	63.33312	20	322.02%	100.90095%	5.700%	101.94538434%	100.00%
83.25	63.33313	20	322.02%	109.99104%	5.700%	110.00010671%	110.00%
83.25	63.33321	20	322.02%	119.99149%	5.700%	119.99148952%	120.00%
83.25	63.33349	20	322.02%	129.99218%	5.700%	129.99218198%	130.00%
83.25	63.33429	20	322.03%	139.99277%	5.700%	139.99276799%	140.00%
83.25	63.33615	20	322.05%	149.99327%	5.700%	149.99327074%	150.00%
83.25	63.35666	20	322.26%	179.99443%	5.700%	179.99442738%	180.00%
83.25	63.42216	20	322.92%	209.99524%	5.700%	209.99524140%	210.00%
83.25	63.55903	20	324.31%	239.99585%	5.700%	239.99584634%	240.00%
83.25	63.78256	20	326.59%	269.99631%	5.700%	269.99631399%	270.00%
83.25	64.09623	20	329.78%	299.99669%	5.700%	299.99668653%	300.00%
83.25	64.80398	20	336.97%	349.99716%	5.700%	349.99716369%	350.00%
83.25	65.70091	20	346.09%	399.99752%	5.700%	399.99752039%	400.00%
83.25	66.73310	20	356.59%	449.99780%	5.700%	449.99779721%	450.00%
83.25	67.85123	20	367.96%	499.99802%	5.700%	499.99801833%	500.00%
83.25	69.01427	20	379.79%	549.99820%	5.700%	549.99819905%	550.00%
83.25	70.18948	20	391.74%	599.99835%	5.700%	599.99834952%	600.00%
83.25	72.48078	20	415.04%	699.99859%	5.700%	699.99858578%	700.00%
83.25	74.58827	20	436.47%	799.99876%	5.700%	799.99876282%	800.00%
83.25	76.44139	20	455.31%	899.99890%	5.700%	899.99890045%	900.00%
83.25	78.01357	20	471.30%	999.99901%	5.700%	999.99901051%	1000.00%
83.25	79.30705	20	484.45%	1099.99910%	5.700%	1099.99910054%	1100.00%
83.25	80.34206	20	494.98%	1199.99918%	5.700%	1199.99917554%	1200.00%
83.25	81.14902	20	503.19%	1299.99924%	5.700%	1299.99923900%	1300.00%
83.25	81.76276	20	509.43%	1399.99929%	5.700%	1399.99929338%	1400.00%
83.25	82.21848	20	514.06%	1499.99934%	5.700%	1499.99934051%	1500.00%
83.25	82.87204	20	520.71%	1749.99943%	5.700%	1749.99943475%	1750.00%
83.25	83.12815	20	523.31%	1999.99951%	5.700%	1999.99950542%	2000.00%

References

[1] F. Black, M. Scholes, The pricing of options and corporate liabilities, J. Polit. Econ. (1973) 637–654.
[2] J.C. Hull, Options, Futures, and Other Derivatives, Pearson Education India, 2006.
[3] H. Latané, R. Rendleman, Standard deviation of stock price raions implied by option premia, J. Finance 31 (1976) 29–58.
[4] F.M. Bandi, B. Perron, Long memory and the relation between implied and realized volatility, J. Financ. Econom. 4 (4) (2006) 636–670.
[5] R.E. Whaley, Derivatives on market volatility: Hedging tools long overdue, J. Deriv. 1 (1) (1993) 71–84.
[6] Chicago Board Options Exchange, CBOE Volatility Indexes, Mar. 2016. http://www.cboe.com/products/vix-index-volatility/volatility-indexes.
[7] B. Dupire, Arbitrage pricing with stochastic volatility. Technical Report, Société Générale, 1992.

[8] B. Dupire, Pricing with a smile, Risk 7 (1) (1994) 18–20.
[9] S. Koekebakker, G. Lien, Volatility and price jumps in agricultural futures prices evidence from wheat options, Am. J. Agric. Econ. 86 (4) (2004) 1018–1031.
[10] R.C. Merton, Theory of rational option pricing, Bell J. Econ. Manag. Sci. (1973) 141–183.
[11] R. Sitter, Volatility Surface, Mar. 2009. http://www.mathworks.com/matlabcentral/fileexchange/23316-volatility-surface.
[12] D.M. Chance, T.A. Hanson, W. Li, J. Muthuswamy, The impact of computational error on the volatility smile. Technical Report, The University of Adelaide, 2013, Available at SSRN 2243589.
[13] D.B. Nelson, K. Ramaswamy, Simple binomial processes as diffusion approximations in financial models, Rev. Financ. Stud. 3 (3) (1990) 393–430.
[14] J.C. Hull, A. White, Valuing derivative securities using the explicit finite difference method, J. Finan. Quant. Anal. 25 (1) (1990) 87–100.
[15] F. Jamshidian, Forward induction and construction of yield curve diffusion models, J. Fixed Income 1 (1) (1991) 62–74.
[16] J.C. Hull, A. White, One-factor interest-rate models and the valuation of interest-rate derivative securities, J. Financ. Quant. Anal. 28 (2) (1993) 235–254.
[17] L.T. Nielsen, Understanding $n(d_1)$ and $n(d_2)$: Risk adjusted probabilities in the Black-Scholes model, J. French Financ. Assoc. 14 (1993) 95–106.
[18] R. Rebonato, Volatility and Correlation in the Pricing of Equity, FX and Interest-Rate Options, John Wiley and Sons, New York, 2000.
[19] M. Brenner, M. Subrahmanyam, A simple formula to compute implied volatility, Financ. Anal. J. 44 (1998) 80–83.
[20] M. Bharadia, N. Christofides, G. Salkin, Computing the Black-Scholes implied volatility: generalization of a simple formula, Adv. Futures Options Res. 8 (1995) 15–30.
[21] D.R. Chambers, S.K. Nawalkha, An improved approach to computing implied volatility, Financ. Rev. 36 (3) (2001) 89–100.
[22] D.M. Chance, A generalized simple formula to compute the implied volatility, Financ. Rev. 31 (4) (1996) 859–867.
[23] C.J. Corrado, T.W. Miller, A note on a simple, accurate formula to compute implied standard deviations, J. Bank. Finance 20 (3) (1996) 595–603.
[24] C.J. Corrado, T.W. Miller, Volatility without tears: A simple formula for estimating implied volatility from options prices over a wide range of strike prices, Risk 9 (7) (1996) 49–52.
[25] S. Li, A new formula for computing implied volatility, Appl. Math. Comput. 170 (1) (2005) 611–625.
[26] M. Kritzman, What practitioners need to know about estimating volatility. Part 1, Financ. Anal. J. 47 (4) (1991) 22–25.
[27] S.J. Brown, Estimating volatility, Financ. Options: Theory Pract. 516 (1990) 537.
[28] N. Chriss, Black Scholes and Beyond: Option Pricing Models, McGraw-Hill, 1996.
[29] P. Wriggers, Nonlinear Finite Element Methods, Springer-Verlag, Berlin, 2008.
[30] M.J. Miranda, P.L. Fackler, Applied Computational Economics and Finance, MIT Press, Cambridge, MA, 2002.
[31] A. Ron, Introduction to Numerical Analysis, 2010.
[32] J.R. Philip, The function inverfc θ, Aust. J. Phys. 13 (1960) 13–20.
[33] A. Strecok, On the calculation of the inverse of the error function, Math. Comp. 22 (1968) 144–158.
[34] H.E. Fettis, A stable algorithm for computing the inverse error function in the tail-end region, Math. Comp. 28 (1974) 585–587;
H.E. Fettis, A stable algorithm for computing the inverse error function in the tail-end region, Math. Comp. 29 (1975) 673. Corrigendum.