Tutorial - 2

(1) Evaluate $\int_0^2 \int_0^x e^{x+y}\,dy\,dx$

Sol: $$\int_{x=0}^{2} \int_{y=0}^{x} e^{x+y}\,dy\,dx$$

$$\int_{x=0}^{2} \left[e^{x+y}\right]_0^x \,dx$$

$$\int_{x=0}^{2} \left(e^{2x} - e^{x}\right) dx$$

$$\left[\frac{e^{2x}}{2} - e^{x}\right]_0^2 + c$$

$$\frac{e^4}{2} - e^2 - \left[\frac{e^0}{2} - e^0\right] + c$$

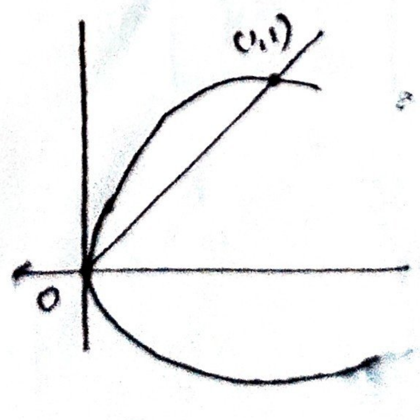
$$\frac{e^4}{2} - e^2 - \frac{1}{2} + 1 + c$$

$$\frac{e^4 + 1 - 2e^2}{2} + c$$

$$= 20.410 + c$$

(2) Given,

$y^2 = x$ and $y = x$ curves



Area = $\int_R y \cdot dx$

$y^2 = x$ & $y = x$.

$x^2 = x \Rightarrow x = 1,\ y = 1$

$\therefore$ intersection points are $(0,0)$ & $(1,1)$

lower limit is '0' & upper '1'

At curve
$y^2 = x$
$y = \sqrt{x}$
$y' = x$

required area $= \int_0^1 (y - y')\, dx$

$= \int_0^1 (\sqrt{x} - x)\, dx$

$= \int_0^1 (x)^{1/2} - x\, dx$

$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^1$

$= \left[\frac{2x^{3/2}}{3} - \frac{x^2}{2}\right]_0^1$

$= \left[\frac{2(1)^{3/2}}{3} - \frac{1}{2}\right] - \left[0 - 0\right]$

$= \left[\frac{2}{3} - \frac{1}{2}\right]$

$= \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$ sq units.

(3) Change the order & evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dx\,dy$.

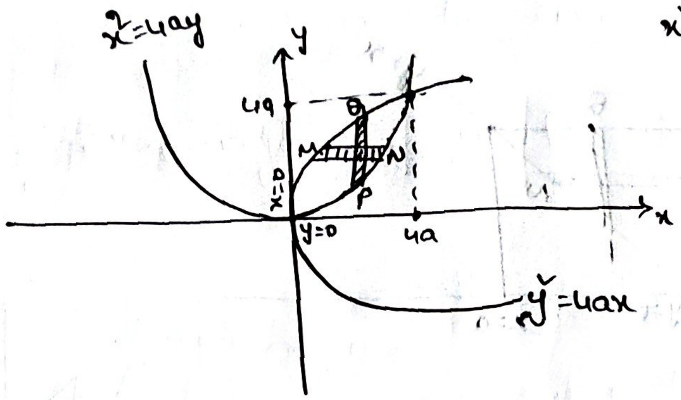
Sol.

Given,

$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dx\,dy$ which implies, limits of P Q

$y \to x^2/4a$ to $2\sqrt{ax}$

$x \to 0$ to $4a$.



$x^2=4ay$ & $y^2=4ax$.

After changing the order limits are:

$x \to y^2/4a$ to $2\sqrt{ay}$

$y \to 0$ to $4a$.

$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a} [x]_{y^2/4a}^{2\sqrt{ay}}\,dy = \int_0^{4a}\left(2\sqrt{a}\cdot(y)^{1/2} - y^2/4a\right)dy$$

$$= \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a} = \left[\frac{4a^{1/2}(4a)^{3/2}}{3} - \frac{(4a)^3}{12a}\right]$$

$$\left[\frac{32\cdot a^{1/2+3/2}}{3} - \frac{64a^{3}}{12a}\right] = \left[\frac{32a^2}{3} - \frac{16a^2}{3}\right] = \frac{16a^2}{3} \text{ sq.units}$$

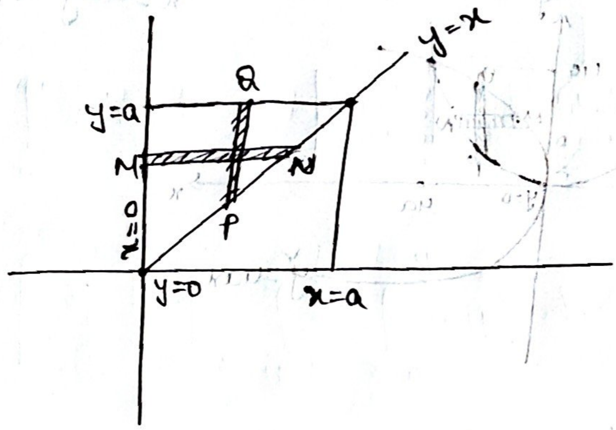
(4) Change the order and evaluate $\int_0^a \int_x^a (x^2+y^2)\,dx\,dy$.

Sol: given,

$$\int_0^a \int_x^a (x^2+y^2)\,dx\,dy$$ which implies, limits of y:

$y \to x$ to a $\quad\therefore y=x$

$x \to 0$ to a. $\quad y=a$.



After changing the order the limits are

$x \to 0$ to y

$y \to 0$ to a

$$\int_{y=0}^{a} \int_{x=0}^{y} (x^2+y^2)\,dx\,dy = \int_0^a \left[\frac{x^3}{3}+y^2x\right]_0^y dy$$

$$= \int_0^a \left[\frac{y^3}{3}+y^3\right] dy = \left[\frac{y^4}{12}+\frac{y^4}{4}\right]_0^a = \frac{a^4}{12}+\frac{a^4}{4}$$

$$= \frac{a^4+3a^4}{12} = \frac{\cancel{4}a^4}{\cancel{12}_{3}} = \frac{a^4}{3} \text{ sq.units.}$$

(5) Evaluate $\int_0^a \int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy$ by changing to polar co-ordinates.

Sol: given,

$$\int_{x=0}^{a} \int_{y=0}^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy$$

limits are

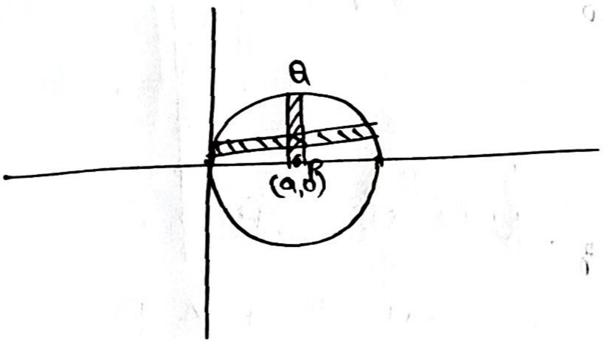
$y=0$ to $\sqrt{2ax-x^2}$ $\Rightarrow$ $y^2=2ax-x^2 \Rightarrow x^2+y^2=2ax$

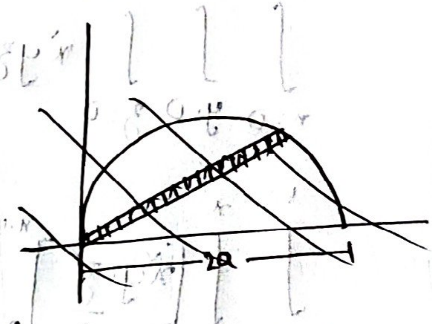
$x=0$ to a

$x^2-2ax+y^2+a^2-a^2=0$

$x^2-2ax+a^2+y^2=a^2$

$(x-a)^2+y^2=a^2.$

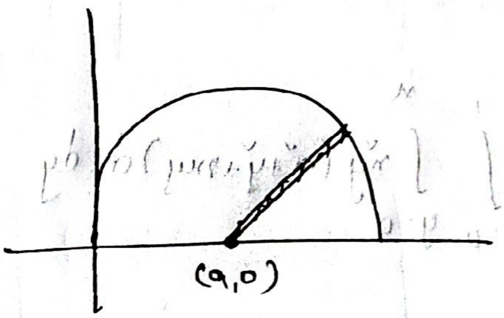




$\cancel{r=0 \text{ to } 2a}$

$\theta = 0$ ~~to~~

$x=r\cos\theta$, $y=r\sin\theta$



$r=0$ to a.

$\theta=0$ to π.

whereas, $x^2+y^2=r^2$

$dx\,dy = r\cdot dr\,d\theta$

$$\therefore \int_0^a \int_0^\pi (r^2)\cdot r\,dr\,d\theta$$

$$\int_0^a r^3 [\theta]_0^\pi \, dr$$

$$\int_0^a r^3 \pi \, dr = \left[\frac{r^4\pi}{4}\right]_0^a = \frac{a^4\pi}{4} \text{ sq. units.}$$

⑦ Evaluate $\int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{x+y} x^2 y z \, dx\,dy\,dz$.

Sol: Given,

$$\int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{x+y} x^2 y z \, dx\,dy\,dz.$$

$$\int_{x=0}^{1} \int_{y=0}^{x} \left[x^2 y \cdot \frac{z^2}{2}\right]_0^{x+y} dy\,dx$$

$$\frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{x} x^2 y (x+y)^2 \, dy\,dx. = \frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{x} x^2 y (x^2+y^2+2xy)\, dx\,dy$$

$$\frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{x} x^4 y + x^2 y^3 + 2x^3 y^2 \, dy\,dx$$

$$\frac{1}{2}\int_{x=0}^{1} \left[\frac{x^4 y^2}{2} + \frac{x^2 y^4}{4} + \frac{2x^3 \cdot y^3}{3}\right]_0^x dx = \frac{1}{2}\int_{x=0}^{1} \left[\frac{x^6}{2} + \frac{x^6}{4} + x^5\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \frac{3x^6}{4} + x^5 \, dx = \frac{1}{2}\left[\frac{3x^7}{28} + \frac{x^6}{6}\right]_0^1 = \frac{1}{2}\left[\frac{3}{28} + \frac{1}{6}\right] = \frac{23}{168}.$$

(8) Evaluate $\int_{x=0}^{1}\int_{y=0}^{x}\int_{z=0}^{x+y}(x+y+z)\,dz\,dy\,dx.$

Sol: given,

$$\int_{x=0}^{1}\int_{y=0}^{x}\int_{z=0}^{x+y}(x+y+z)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{x}\left[(x+y)z + \frac{z^2}{2}\right]_0^{x+y} dy\,dz = \int_{x=0}^{1}\int_{y=0}^{x}(x+y)(x+y) + \frac{(x+y)^2}{2}\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{x}(x+y)^2 + \frac{(x+y)^2}{2}\,dy\,dx = \frac{3}{2}\int_{x=0}^{1}\int_{y=0}^{x}(x+y)^2\,dy\,dx = \frac{\cancel{3}}{2}\int_{x=0}^{1}\left[\frac{(x+y)^3}{\cancel{3}}\right]_0^x dx$$

$$= \frac{1}{2}\int_0^1\left[(2x)^3 - x^3\right]dx = \frac{1}{2}\int_0^1\left[7x^3\right]dx = \frac{1}{2}\left[\frac{7x^4}{4}\right]_0^1 = \frac{7}{8} + C.$$

(9) Evaluate $\int_{y=0}^{9}\int_{x=0}^{y/3}\int_{z=0}^{\sqrt{y^2-9x^2}} z\,dz\,dx\,dy$

Sol: Given,

$$\int_{y=0}^{9}\int_{x=0}^{y/3}\int_{z=0}^{\sqrt{y^2-9x^2}} z\,dz\,dx\,dy$$

$$= \frac{1}{2}\int_{y=0}^{9}\int_{x=0}^{y/3}\left[z^2\right]_0^{\sqrt{y^2-9x^2}} dx\,dy = \frac{1}{2}\int_{y=0}^{9}\int_{x=0}^{y/3}(y^2-9x^2)\,dx\,dy = \cancel{\frac{1}{2}\int_{y=0}^{9}\left[y^3 - 9x^2y\right]}.$$

$$= \frac{1}{2}\int_{y=0}^{9}\left[y^2x - \frac{\cancel{9}^{3}x^3}{\cancel{3}}\right]_0^{y/3} dy = \frac{1}{2}\int_{y=0}^{9} y^2\frac{y}{3} - 3\left(\frac{y}{3}\right)^3 dy = \frac{1}{2}\int_{y=0}^{9}\frac{3y^3}{9} - \frac{y^3}{9}\,dy$$

$$= \frac{1}{2} \cdot \int_{y=0}^{9} \frac{2y^3}{9} dy = \frac{1}{9}\left[\frac{y^4}{4}\right]_0^9 = \frac{1}{9\times 4}\left[9^{\cancel{4}\,3}\right] = \frac{729}{4}$$

(10) Evaluate $\iiint_B 8xyz\,dv$, where $B: 2 \le x \le 3, 1 \le y \le 2 \,\&\, 0 \le z \le 1$

Sol: Given

$x = 2$ to 3, $y = 1$ to 2, $z = 0$ to 1

$$\int_{x=2}^{3}\int_{y=1}^{2}\int_{z=0}^{1} 8xyz\,dx\,dy\,dz = \int_{x=2}^{3}\int_{y=1}^{2} \overset{4}{\cancel{8}}xy\left[\frac{z^2}{2}\right]_0^1 dy\,dx = \int_{x=2}^{3}\int_{y=1}^{2} 4xy\,dy\,dx$$

$$= \int_{x=2}^{3} \cancel{4}x \left[\frac{y^2}{\cancel{2}}\right]_1^2 dx = \int_{x=2}^{3} 2x\,[4-1]\,dx = \int_{x=2}^{3} 2x(3)\,dx = \left[x^2(3)\right]_2^3$$

$$= 3[9-4] = 3 \times 5 = 15 + C$$