Linear Algebra & Random Processes

Programming Assignment #2

Team #30

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Question 1 on Expectation and Variance

Let $X_1, \ldots\ldots, X_N$ denote a sequence of i.i.d. random variables (r.v.s), each with mean μ and variance σ^2.

Let $\overline{X}_n = \frac{1}{N}\sum_{i=1}^{N} X_i$ denote sample mean.

1. What is $E(\overline{X}_N)$ and $Var(\overline{X}_N)$?

Sol. a) $E(\overline{X}_N) = E\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right)$

$$= \frac{1}{N} E(X_1 + X_2 \text{----} X_N)$$

$$= \frac{1}{N}\left\{E(X_1) + E(X_2) \text{----} + E(X_N)\right\}$$

as, all the r.v.s have mean μ. ie, $E(X_i) = \mu$

So, $$= \frac{1}{N}\Big\{\underbrace{\mu + \mu + \mu \text{ --- } \mu}_{N\text{-times}}\Big\}$$

$$= \frac{1}{N} \cdot N\mu$$

$$\boxed{E(\bar{X}_N) = \mu}$$

Sol. b). $Var(\bar{X}_N) = Var\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right)$

as, $\boxed{Var(pY) = p^2 Var(Y)}$ so,

$$= \left(\frac{1}{N}\right)^2 Var\left(\sum_{i=1}^{N} X_i\right)$$

$$= \left(\frac{1}{N}\right)^2 Var(X_1 + X_2 \text{----} + X_N)$$

$$= \left(\frac{1}{N}\right)^2 \left[Var(X_1) + Var(X_2) \text{----} + Var(X_N)\right]$$

as, given that variance of each r.v. is σ^2, ie, $Var(X_i) = \sigma^2$ so,

$$= \left(\frac{1}{N}\right)^2 [\underbrace{\sigma^2 + \sigma^2 \text{------} + \sigma^2}_{N\text{-times}}]$$

$$= \left(\frac{1}{N}\right)^2 . N\sigma^2$$

$$\boxed{Var(\bar{X}_N) = \frac{\sigma^2}{N}}$$

Question 2 on Hoeffding inequality

2. While a concentration inequality was derived for Bernoulli r.v.s. in the class, a similar result holds for bounded r.v.s and we present the well-known Hoeffding inequality below:-

Theorem 1:- Let $X_1, \dots X_N$ denote a sequence of i.i.d. r.v.s with $X_i \in [a,b]$ for all i, where $-\infty < a \leq b < \infty$, letting $\bar{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i$ and μ denotes EX_i for all i, we have.

$$P(\bar{X}_n - \mu \geq \epsilon) \leq \exp\left(-\frac{2N\epsilon^2}{(b-a)^2}\right) \quad \text{—} (1)$$

$$P(\bar{X}_n - \mu \leq -\epsilon) \leq \exp\left(-\frac{2N\epsilon^2}{(b-a)^2}\right) \quad \text{—} (2)$$

Now for proving,

$$P(\mu \in [\bar{X}_n - \epsilon', \bar{X}_n + \epsilon']) \geq 1 - \delta \quad \text{—} (3)$$

Sol^n

Take L.H.S of equation (3):-

$$= P(\mu \in [\bar{X}_n - \epsilon', \bar{X}_n + \epsilon'])$$

$$= P(\mu \geq \bar{X}_n - \epsilon' \;\&\; \mu \leq \bar{X}_n + \epsilon')$$

writing in form of $\bar{X}_n$ & μ

$$= P(\bar{X}_n - \mu \leq \epsilon' \;\&\; \bar{X}_n - \mu \geq -\epsilon')$$

So, this is the portion with $[\bar{X}_n - \epsilon \;\&\; \bar{X}_n + \epsilon']$

So we will write in form of area outside this range

$$= 1 - P(\bar{X}_n - \mu \geq \epsilon') - P(\bar{X}_n - \mu \leq -\epsilon') \quad \text{--- (4)}$$

Now sum (1) & (2). in ϵ'

$$P(\bar{X}_n - \mu \geq \epsilon') + P(\bar{X}_n - \mu \leq -\epsilon') \leq 2\exp\left(-\frac{2N\epsilon'^2}{(b-a)^2}\right)$$

do, (1- data) both sides

$$1 - \left(P(\bar{X}_n - \mu \geq \epsilon') + P(\bar{X}_n - \mu \leq -\epsilon')\right) \cancel{\geq} 1 - 2\exp\left(\frac{-2N\epsilon'^2}{(b-a)^2}\right) \quad \text{--- (5)}$$

Now in (5) LHS is equivalent to (4) hence we will write.

$$P(\mu \in [\bar{X} - \epsilon', \bar{X}_n + \epsilon']) \cancel{\geq} 1 - 2\exp\left(\frac{-2\epsilon'^2}{(b-a)^2}\right) \quad \text{--- (6)}$$

Compare (6) with (3)

so, δ compes out to be,

$$\delta = 2\exp\left(-\frac{2N\epsilon'^2}{(b-a)^2}\right)$$

$$\log\frac{\delta}{2} = -\frac{2N\epsilon'^2}{(b-a)^2} \Rightarrow (b-a)^2\log\frac{2}{\delta} = 2N\epsilon'^2$$

$$\epsilon'^2 = \frac{(b-a)^2}{2N}\log\frac{2}{\delta}$$

$$\boxed{\epsilon' = \sqrt{\frac{(b-a)^2}{2N}\log\frac{2}{\delta}}}$$

Question 3 on Poisson distribution

Write a program to obtain N samples from a Poisson distribution with parameter = 10 and plot the histogram of the sample mean, with 1000 bars.

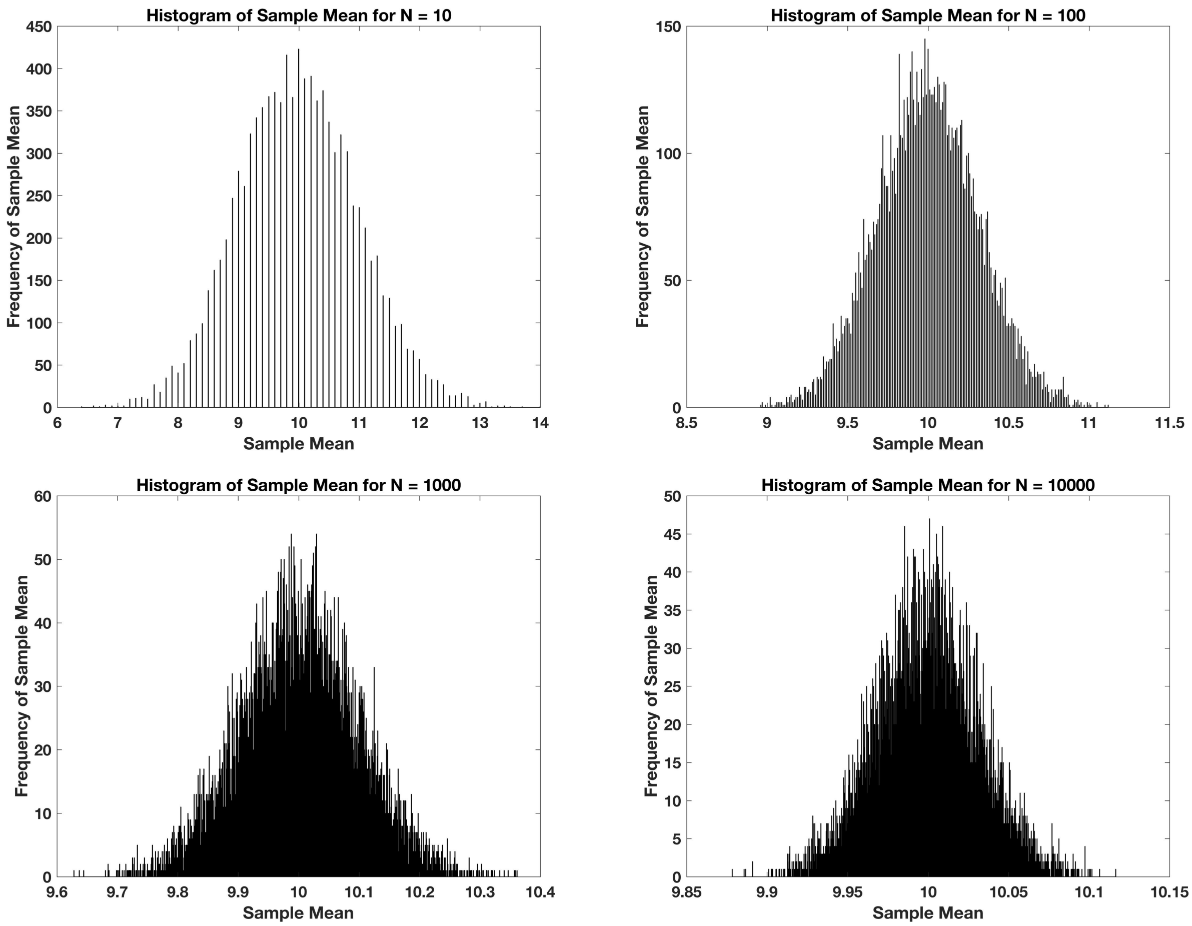
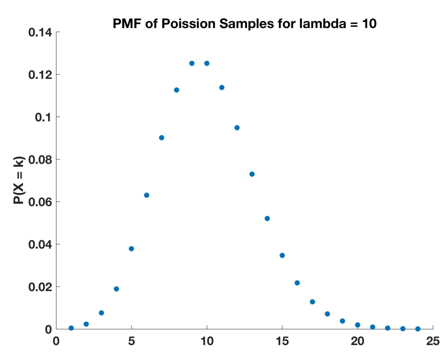


Figure 1.1: Histogram of Sample Mean with 1000 bars



Interpretation from the numerical results:

(3A) Is the sample mean close to the true mean? Why is this expected? Justify your answer.
Answer: Yes,the sample mean is close to the true mean. Because as we have generated samples of Poisson distribution with parameter = 10, so as we will take more and more number of samples, our sample means will also fall close to true mean. Which we can observe from above histogram plots that as the number of samples taken increases, variance in sample means decreases and sample means accumulating near to true mean increases.

(3B) How many times was the sample mean in the interval [9.99, 10.01] and [9.9, 10.1]?

Interval	**N = 10**	**N = 100**	**N = 1000**	**N = 10000**
[9.99,10.01]	376	389	821	2542
[9.9,10.1]	1177	2642	6911	9990

(3C) Calculate a 95% confidence interval for the sample mean using the numerical results. How many times did the true mean fall outside the confidence interval?

	N = 10	**N = 100**	**N = 1000**	**N = 10000**
95% Confidence Interval	[8.05,11.95]	[9.38,10.62]	[9.80,10.20]	[9.94,10.06]
True Mean outside CI	495	482	476	495

(3D)

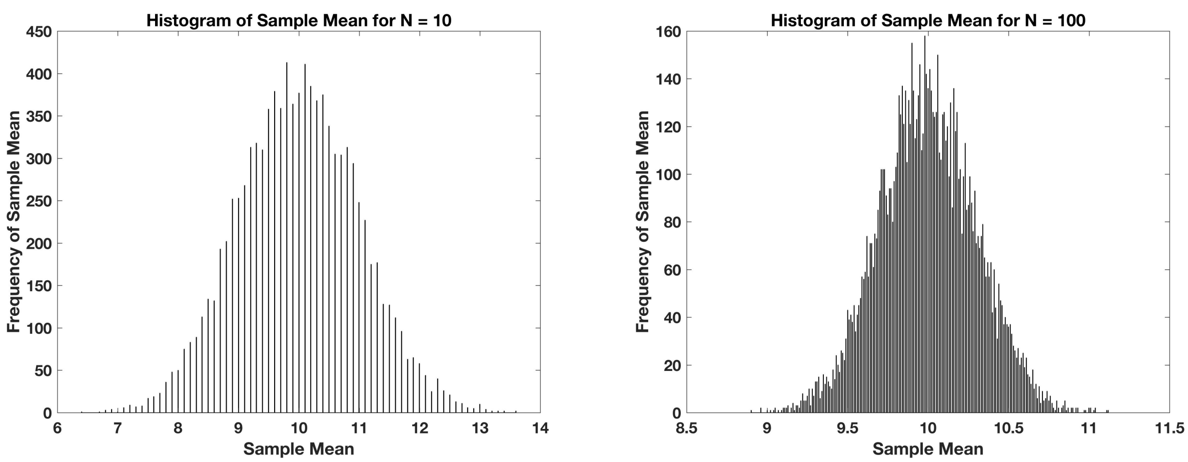


Figure 1.2: Histogram of Sample Mean with 1000 bars for n = 10000

3d) Why isn't Theorem 1 applicable for Poisson?

Sol^n. As we know from Hoeffding theorem that a r.v X_i takes value from a range $[a,b]$ where, $-\infty < a \leq b < \infty$.

But Poisson r.v. can take value from 0, 1, 2 ....., so we don't know the upper limit of the random variable X_i.

Poisson Approximation By Binomial

Let, $\lambda = np$ $\boxed{p = \frac{\lambda}{n}}$ where, $n = \#$ trials
$p =$ prob. of success for each trial

So, $P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$ Binomial distribution

ie, $P(X=x) = \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$ as, $\boxed{p = \frac{\lambda}{n}}$

$$= \frac{\lambda^x}{x!} \cdot \frac{n!}{(n-x)!} \cdot \left(\frac{1}{n^x}\right)\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

for large n, ie, $n \to \infty$

$$P(X=x) = \frac{\lambda^x}{x!} \cdot \lim_{n\to\infty} \frac{n!}{(n-x)!} \left(\frac{1}{n^x}\right)\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \left(\frac{\lambda^x}{x!}\right)(1)(e^{-\lambda})(1)$$

using exponential series ie, $\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$

$$\boxed{P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}}$$

So, from here we see that the poisson distribution is just a special case of binomial in which the chance of success is approximate to zero as the no. of trials 'n' grows to infinity.

* From the equation of ϵ is as,

$$\sqrt{\frac{1}{2N} \cdot \log \frac{2}{\delta}}$$

where N = No. of Samples

As we here asked for 95% confidence interval so, $\underline{\delta = 0.05}$

For that,

$\epsilon = 0.42946$ (for $N = 10$)

$\epsilon = 0.13581$ (for $N = 100$)

No. of Samples (N)	95% Confidence Interval (Theoretically)	95% confidence Interval (numerically)
10	9.570, 10.42946	8, 12
100	9.864, 10.135	9.4, 10.6

3e) For this we have used following inequality:-

Chebyshev's Inequality

X is a r.v $E(X^2) < \infty$ then

$$P(|X - E(X)| > \epsilon) \leq \frac{\text{Var } X}{\epsilon^2} \quad \text{--- (1)}$$

As the LHS of equation say the probability of sample mean falling outside of true mean with in range of ϵ.

So, as we want probability of sample mean falling outside ϵ to be 0.05 & ϵ is 0.1

ie, $P(|X - E(X)| > 0.1) \leq 0.05$ --- (2)

Comparing RHS of (1) & (2)

$$\frac{\text{Var}X}{\epsilon^2} = 0.05$$

as $\text{Var}X = \frac{\sigma^2}{N}$

and $\sigma^2 = 10$

$N = ?$

$\epsilon = 0.1$

$$\frac{\sigma^2}{N} \cdot \frac{1}{(0.1)^2} = 0.05$$

$$N = \frac{10}{(0.1)^2 \times (0.05)}$$

$$\boxed{N = 2 \times 10^4}$$

Similarly for $\epsilon = 0.01$

$$\boxed{N = 2 \times 10^6}$$

So from here we observe that

For a given λ the corresponding jump in N to increase the accuracy by decimal point is 100 times.

Question 4

4a) Consider a random variable X that takes values $\pm 1, \pm 2 \ldots$ with p.m.f f defined as,

$$f(k) = \frac{A}{k^2} \quad \text{for } k = \pm 1, \pm 2 \ldots$$

For what value (choice) of A would f be a valid pmf. ie, $\sum_{k \neq 0} f(k) = 1$? Justify your answer?

Sol^n. For this we will use $f(x) = x^2$ with $-\pi \leq x \leq \pi$ and find its expansion into a trignometric Fourier series.

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin x),$$

which is periodic and converges to $f(x)$ in $[-\pi, \pi]$

Observing that $f(x)$ is even, it is enough to determine the coefficients;

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) . \cos nx \qquad n = 0, 1, 2 \ldots$$

& $$\boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0} \qquad n = 1, 2 \ldots$$

For $n=0$ we have,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(0)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx$$

as, $\boxed{\int_{-a}^{a} f(x) = 2\int_{0}^{a} f(x) \quad \text{if, } f(x) \text{ is even}}$

so, $$= \frac{2}{\pi}\int_{0}^{\pi} x^2\, dx$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

And For $n = 1, 2, 3 \ldots$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos(nx)\, dx$$

$$= \frac{2}{\pi}\left[\frac{2x}{n^2}\cos nx + \left(\frac{x^2}{n} - \frac{2}{n^3}\right)\sin nx\right]_0^{\pi}$$

$$= \frac{2}{\pi} \times \frac{2\pi}{n^2} \cdot (-1)^n$$

$$\boxed{a_n = (-1)^n \frac{4}{n^2}}$$

Thus,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left((-1)^n \frac{4}{n^2} \cos nx\right)$$

Since, $f(\pi) = \pi^2$, we obtain,

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left((-1)^n \frac{4}{n^2} \cos(nx)\right)$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left((-1)^n (-1)^n \frac{1}{n^2}\right)$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Therefore, $\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$

Now. our question starts as,

$$f(k) = \frac{A^2}{k^2}$$

& $\displaystyle\sum_{k \neq 0} f(k) = 1$

so, $\displaystyle\sum_{k=\pm 1 \ldots} \frac{A}{k} = 1$

so, $A\left[\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} + \sum_{k=-\infty}^{-1}\frac{1}{k^2}\right] = 1$

As from the result derived

that, $$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}} \quad \text{— ①}$$

Further as $\left(1/k^2\right)$ is a even function

So, $$\boxed{\sum_{k=-\infty}^{-1} 1/k^2 = \sum_{k=1}^{\infty} 1/k^2} \quad \text{— ②}$$

Using ① & ②

$$A\left[\frac{\pi^2}{6} + \frac{\pi^2}{6}\right] = 1$$

$$A \times \frac{\pi^2}{3} = 1$$

$$\boxed{A = \frac{3}{\pi^2}}$$

So finaly function is,

$$\boxed{f(k) = \frac{3}{\pi^2} \times \frac{1}{k^2}}$$

(4B) Plot the histogram of the sample mean and calculate Confidence Interval.

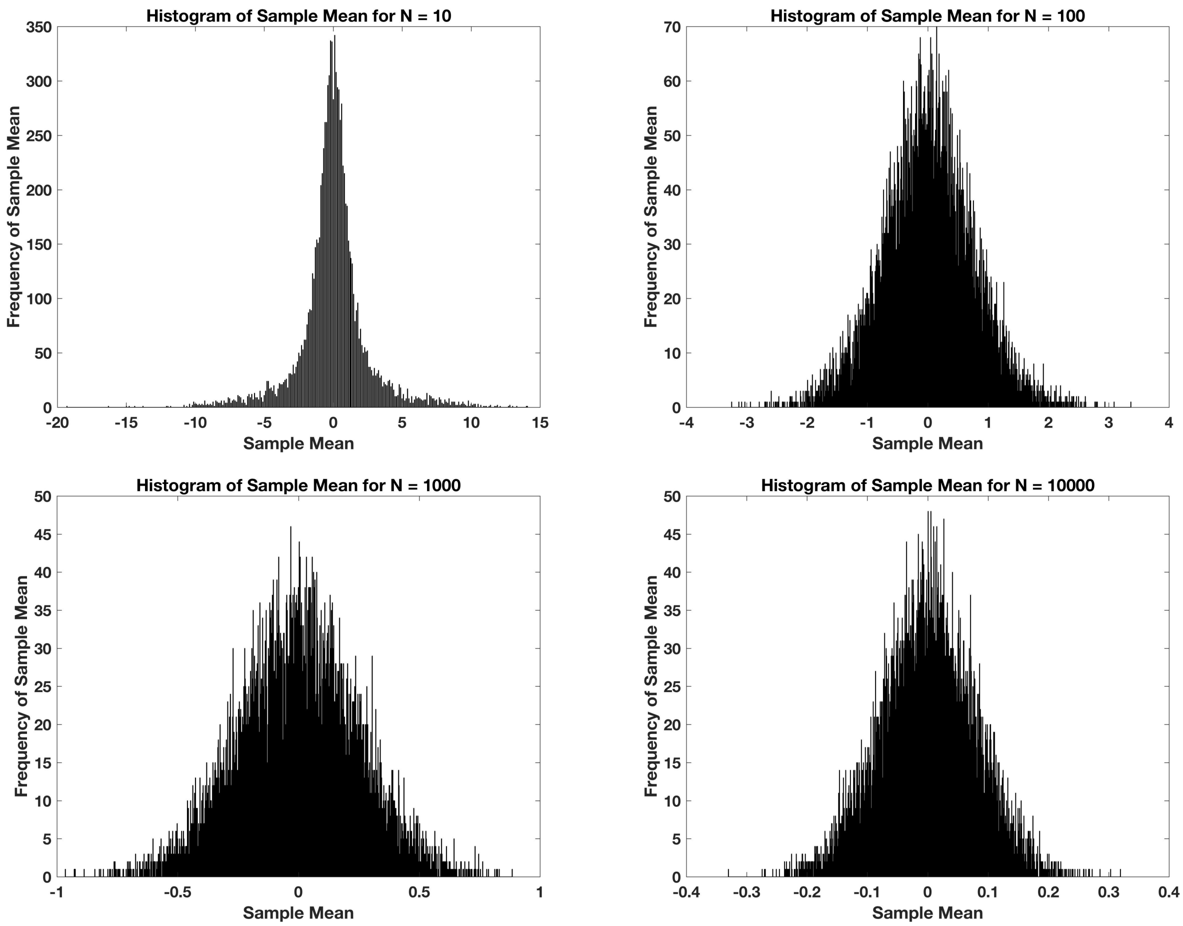
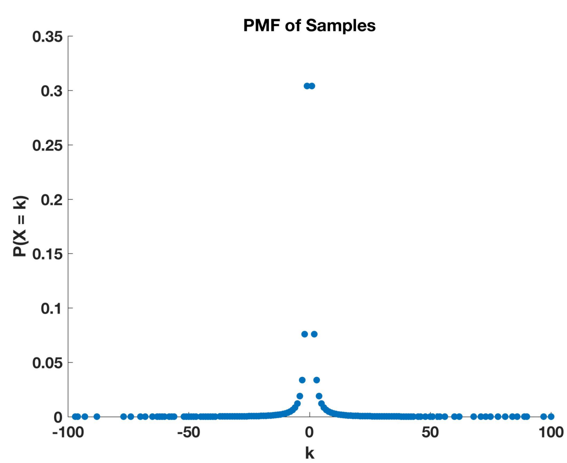


Figure 1.3: Histogram of Sample Mean with 1000 bars



Comparing Confidence Interval for N = 1000 & 10000

	N = 1000	**N = 10000**
95% Confidence Interval	[-0.5,0.5]	[-0.155,0.155]
True Mean outside CI	438	447

Table 1.1: Confidence interval and True mean falling outside CI for N = 1000 & 10000

Observation: We observe that the sample means stays close to **0**. Even thought the expectation i.e. mean of given p.m.f does not converge, but since we have generated samples of positive sign and negative sign with equal probability, the expectation will cancel out the same samples values with opposite sign, therefore the sample means is concentrated close to 0.