Discontinuous Galerkin method for direct numerical simulation of the Navier Stokes equation

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Overview

Objective

- Understanding the Stokes and the Navier-Stokes formulation and perspectives
- Derivation from conservation equation
- Discretisation and function spaces for unknowns
- Discontinuous-Galerkin formulation for the Stokes and the Navier-Stokes equation
- Implementation and matrix assembly
- Sparsity pattern and solver selection
- Numerical experiment for the Stokes equation and the Navier-Stokes equation
- Future perspectives

Importance of the Navier Stokes equation

- Computational fluid dynamics $\implies$ One of the variants of the Navier Stokes equation
- The Navier Stokes equation involves state variables
- Incompressible condition $\implies$ state variables constant $\implies$ Equation of state not required
- Solved along with continuity equation
- Depends on time (Unsteady fluid flow) or independent of time (Steady fluid flow)
- Non linear coupled system of equations
- The Stokes equation is linearized form of the Navier Stokes equation

Governing equations

Reynolds transport theoreom (White F.M. [4])

$$\frac{dB'}{dt'}|_{cs} = \frac{d}{dt'} \int_{cv} b' \rho dV + \int_{cs} (b' \rho) u \cdot dA. \tag{1}$$

Momentum conservation equation

$$F = \frac{dM}{dt'} = \frac{d}{dt'} \int_{cv} u \rho dV + \int_{cs} (u \rho) u \cdot dA. \tag{2}$$

$$F = \int_{cs} \sigma \cdot dA + \int_{cv} \rho f dV. \tag{3}$$

Governing equations

Navier Stokes equation

$$-2\nabla \cdot (\nu \nabla^s u) + (1/\rho)\nabla p + (u \cdot \nabla)u = f \quad \text{in} \quad \Omega. \tag{4}$$

Dirichlet boundary:

$$u = u_D \quad \text{on} \quad \Gamma_D. \tag{5}$$

Neumann boundary:

$$-pn + 2\nu(n \cdot \nabla^s)u = t \quad \text{on} \quad \Gamma_N. \tag{6}$$

Continuity equation

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega. \tag{7}$$

Stokes equation

$$-2\nabla \cdot (\nu \nabla^s u) + (1/\rho)\nabla p = f \quad \text{in} \quad \Omega. \tag{8}$$

Flow classification (Kundu P.K [5])

Reynolds number

$$Re = \frac{uL}{\nu}. \tag{9}$$

Laminar flow

- Well defined velocity and pressure profile.
- Low Reynolds number.

Turbulent flow

- Fluctuations in velocity and pressure.
- Fluctuations are of the order of Kolmogrov scale.
- Low Reynolds number.

Grid

- Continuous domain $(\Omega) \implies$ Grid $(\mathcal{T})$
- Triangular element, τ_k, $\cup_{k=1}^{nel} \tau_k = \mathcal{T}$, *nel* is the total number of elements
- Grid boundary includes interelement boundaries.

$$\partial \mathcal{T} = \Gamma_D \cup \Gamma_N \cup \Gamma \tag{10}$$

Barycentric coordinate

For a triangle with vertices, r_1, r_2, r_3

$$r = \lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3. \tag{11}$$

Weights, $\lambda_1, \lambda_2, \lambda_3$ satisfy,

$$\lambda_1 + \lambda_2 + \lambda_3 = 1. \tag{12}$$

Discontinuous Galerkin method

- Multiply with test function and integrate
- Discontinuous at the interface of elements
- $P^D(\tau_k)$ denotes space of polynomials of degree at most D over τ_k.

Function space for velocity

$$\mathbb{V} = \{\phi \in (L^2(\mathcal{T}))^{d_u} | \quad \phi \in (P^D(\tau_k))^{d_u} \quad \forall \quad \tau_k \in \mathcal{T}\}. \tag{13}$$

Function space for pressure

$$\mathbb{Q} = \{\psi \in (L^2(\mathcal{T}))^{d_p} | \quad \psi \in (P^{D-1}(\tau_k))^{d_p} \quad \forall \quad \tau_k \in \mathcal{T}\}. \tag{14}$$

- We use Orthonormal basis function

Jump operator, Average operator

Jump operator

If p is scalar and u is vector,

$$\begin{aligned} [pn] &= p^+ n^+ + p^- n^- \quad \text{on} \quad \Gamma, \quad [pn] = pn \quad \text{on} \quad \Gamma_D. \\ [n \otimes u] &= n^+ \otimes u^+ + n^- \otimes u^- \quad \text{on} \quad \Gamma, \quad [n \otimes u] = n \otimes u \quad \text{on} \quad \Gamma_D. \\ [n \cdot u] &= n^+ \cdot u^+ + n^- \cdot u^- \quad \text{on} \quad \Gamma, \quad [n \cdot u] = n \cdot u \quad \text{on} \quad \Gamma_D. \end{aligned} \tag{15}$$

Average operator

The average operator is defined as,

$$\{u\} = \frac{u^+ + u^-}{2}. \tag{16}$$

Stokes equation

Strong form

$$
\begin{aligned}
-\nu \Delta u + \nabla p &= f \quad \text{in} \quad \Omega. \\
\nabla \cdot u &= 0 \quad \text{in} \quad \Omega.
\end{aligned}
\tag{17}
$$

Weak form (Montlaur et al. [1])

$$
\begin{aligned}
a_{IP}(u, \phi) + b(\phi, p) + (\{p\}, [n \cdot \phi])_{\Gamma \cup \Gamma_D} &= l_{IP}(\phi). \\
b(u, \psi) + (\{\psi\}, [n \cdot u])_{\Gamma \cup \Gamma_D} &= (q, n \cdot u_D)_{\Gamma_D}.
\end{aligned}
\tag{18}
$$

Stokes equation

Weak form

$$
\begin{aligned}
a_{IP}(u,\phi) &= (\nabla u, \nabla\phi) + C_{11}([n\otimes u],[n\otimes\phi])_{\Gamma\cup\Gamma_D} \\
&\quad -\nu(\{\nabla u\},[n\otimes\phi])_{\Gamma\cup\Gamma_D} - \nu([n\otimes u],\{\nabla\phi\})_{\Gamma\cup\Gamma_D}. \\
b(\phi,\psi) &= -\int_{\mathcal{T}} \psi\nabla\cdot\phi. \\
l_{IP}(\phi) &= (f,\phi) + (t,\phi)_{\Gamma_N} + C_{11}(u_D,\phi)_{\Gamma_D} - (n\otimes u_D, \nu\nabla\phi)_{\Gamma_D}.
\end{aligned}
\tag{19}
$$

Discrete form

$$
\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \tag{20}
$$

Stiffness matrix

Matrix A

$$A_{ij} = \sum_{k=1}^{d} (\frac{\partial \phi_i}{\partial x_k}, \frac{\partial \phi_j}{\partial x_k}) + C_{11} \sum_{k=1}^{d} ([\phi_i n_k], [\phi_j n_k])_{\Gamma \cup \Gamma_D}$$
$$-\nu \sum_{k=1}^{d} ([\phi_i n_k], \{\frac{\partial \phi_j}{\partial x_k}\})_{\Gamma \cup \Gamma_D} - \nu \sum_{k=1}^{d} (\{\frac{\partial \phi_i}{\partial x_k}\}, [\phi_j n_k])_{\Gamma \cup \Gamma_D} \tag{21}$$

Matrix B

$$B_{ij} = -\int_{\mathcal{T}} \frac{\partial \phi_i}{\partial x_i} \psi_j + (\{\psi_j\}, [n \cdot \phi_i])_{\Gamma \cup \Gamma_D} \tag{22}$$

Navier Stokes equation : Upwinding

- If n_τ is the unit normal from τ_1 to τ_2 and if we denote the upwind value of function g as g^{up} [3],

$$\begin{aligned} g^{up} &= g|_{\tau_1} \quad \text{if} \quad g \cdot n_\tau \geq 0 \\ g^{up} &= g|_{\tau_2} \quad \text{if} \quad g \cdot n_\tau < 0 \end{aligned} \tag{23}$$

- For the weak form of the Navier Stokes equation (Montlaur et al. [2]),

$$\begin{aligned} c(g; u, \phi) = \sum_{i=1}^{nel} \int_{\partial\Omega_i \backslash \Gamma_N} \frac{1}{2} [[(g \cdot n_i)(u^{ext} + u) - |g \cdot n_i|(u^{ext} - u)]] \cdot \phi \\ + \int_{\Gamma_N} (g \cdot n) u \cdot \phi - ((g \cdot \nabla)\phi, u). \end{aligned} \tag{24}$$

$$u^{ext} = \lim_{\epsilon \to 0} u(x + \epsilon n_i), \quad \epsilon \to 0^+ \quad \text{for} \quad x \in \partial\mathcal{T}_i \tag{25}$$

Newton method

Algotrithm for the Newton method is as follow:
1. Calculate $u^{iter} \in \mathbb{V}$ at iteration $iter$,
2. Verify $DS_{u^{iter}}(h^{iter}) = -S(u^{iter})$,
3. Set $u^{iter+1} := u^{iter} + h^{iter}$ till $||u^{iter+1} - u^{iter}|| < tol$.

We use the solution from the Stokes equation as initial guess. In discrete form the Newton method means, solving the equation,

$$\underbrace{\begin{pmatrix} A + C(U^{iter}) & B \\ B^T & 0 \end{pmatrix}}_{\text{Stiffness matrix}^{iter}} \underbrace{\begin{pmatrix} U^{iter+1} \\ P^{iter+1} \end{pmatrix}}_{\text{Solution vector}^{iter+1}} = \underbrace{\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}}_{\text{Right hand side}} . \tag{26}$$

Schur complement method

STEP 1:

$$U = A^{-1}(F_1 - BP), \tag{27}$$

STEP 2 :

$$-B^T A^{-1} BP = F_2 - B^T A^{-1} F_1, \tag{28}$$

STEP 3 :

Backsubstitution in equation (27).

Data types

- `params` : Structure with fields u_{npe}, u_{ndofs}, d_u, D, U.
- `paramsP` : Structure with fields p_{npe}, p_{ndofs}, d_p, $D{-}1$, P.
- `grid` : Structure containing informations related to grid.

Basis function in RBmatlab

$$\phi \in \mathbb{R}^{u_{npe} \times d_u}, \tag{29}$$

- The derivative of basis function $(\phi)_i$, where $1 \leq i \leq u_{npe}$ is cell.

Basis function in RBmatlab

$$\nabla(\phi)_i \in \mathbb{R}^{d_u \times d}. \tag{30}$$

Matrix assemblies : Jump operator and average operator

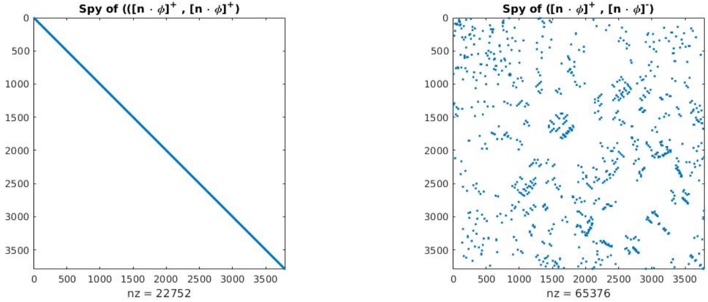
Jump operator

$$[A_h \cdot n], [B_h \cdot n] = A_h^+ n^+ B_h^+ n^+ + A_h^+ n^+ B_h^- n^- + A_h^- n^- B_h^+ n^+ + A_h^- n^- B_h^- n^- . \tag{31}$$

Average operator

$$\{A_h\} = \frac{(A_h^+ + A_h^-)}{2}. \tag{32}$$

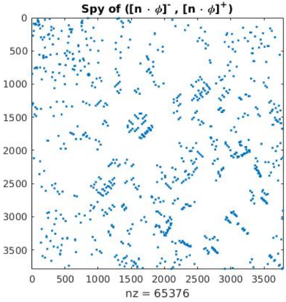
Sparsity pattern



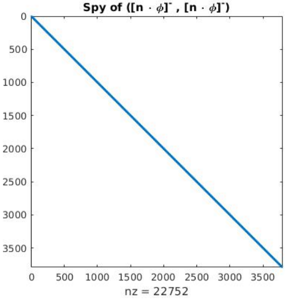
(a) $((n \otimes \phi)^+, (n \otimes \phi)^+)_{\Gamma \cup \Gamma_D}$ (b) $((n \otimes \phi)^+, (n \otimes \phi)^-)_{\Gamma \cup \Gamma_D}$

Figure: Sparsity pattern of constituents of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Sparsity pattern



(a) $((n \otimes \phi)^-, (n \otimes \phi)^+)_{\Gamma \cup \Gamma_D}$



(b) $((n \otimes \phi)^-, (n \otimes \phi)^-)_{\Gamma \cup \Gamma_D}$

Figure: Sparsity pattern of constituents of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Sparsity pattern

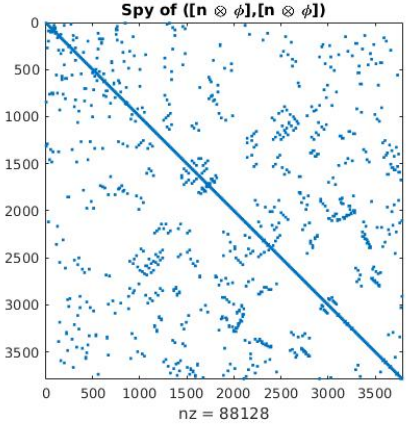


Figure: Sparsity pattern of constituents of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Error definition

- If P_h is the computed solution and P is the true solution, we define following errors:

L^2 error

$$P_{error,L^2} = (\int_{\Omega} |P - P_h|^2)^{\frac{1}{2}}. \tag{33}$$

H_0 error

$$P_{error,H_0} = \sum_{k=1}^{nel} (\int_{\tau_k} |\nabla P - \nabla P_h|^2)^{\frac{1}{2}}. \tag{34}$$

Stokes equation: Convergence test

- Domain: unit square [0,1] $\times$ [0,1].
- $x = 0$ is dirichlet boundary with inflow velocity at point $(0, y)$ as $u = (y(1-y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (2\nu - 1, 0)$.

Analytical solution

$$p = (1 - x), \tag{35}$$

$$u = (y(1-y), 0). \tag{36}$$

Stokes equation: Convergence test

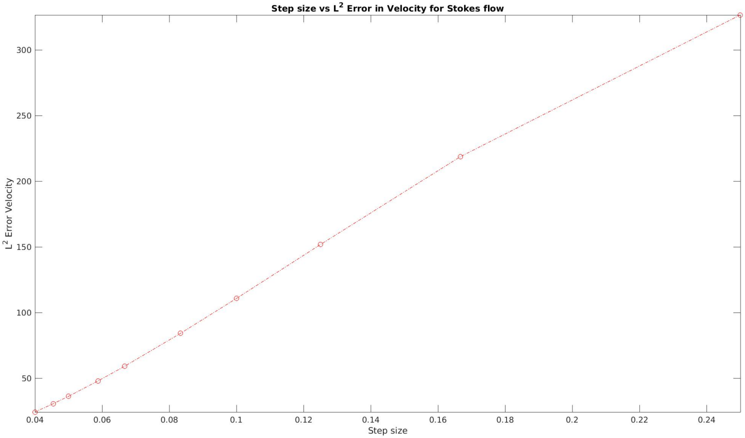


Figure: $h-$convergence test for velocity in L^2 error

Stokes equation: Convergence test

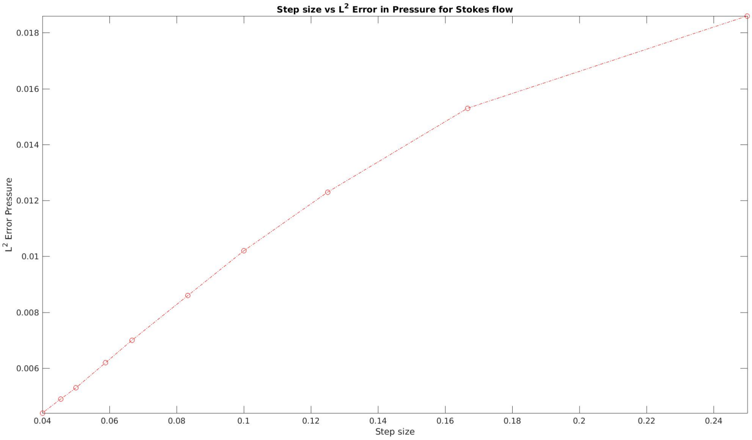


Figure: $h-$convergence test for pressure in L^2 error

Stokes equation: Convergence test

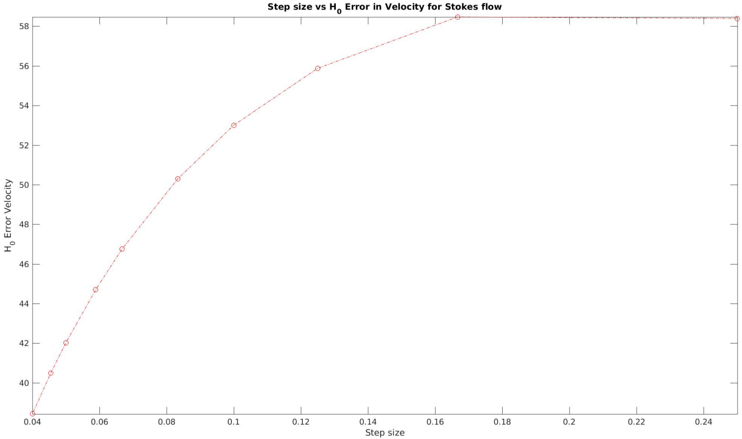


Figure: $h-$convergence test for velocity in H_0 error

Stokes equation: Convergence test

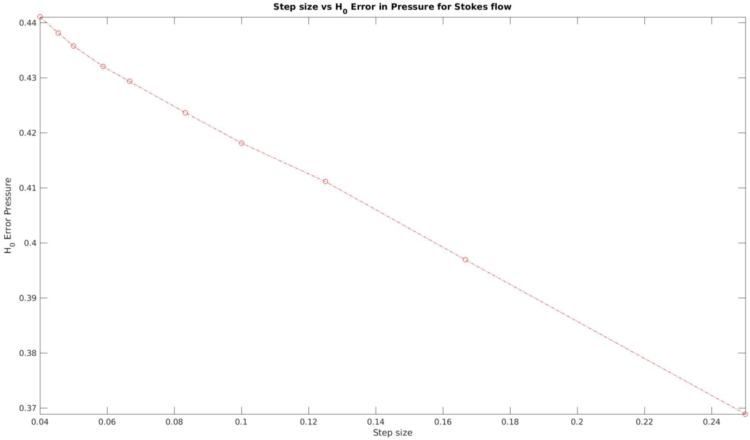
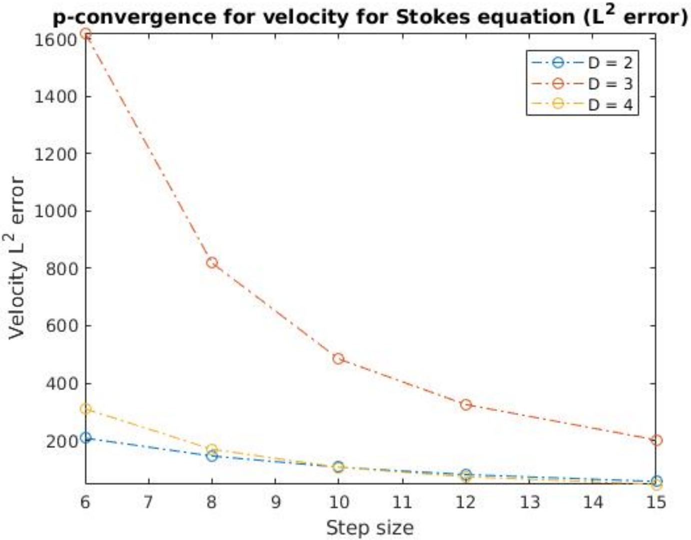
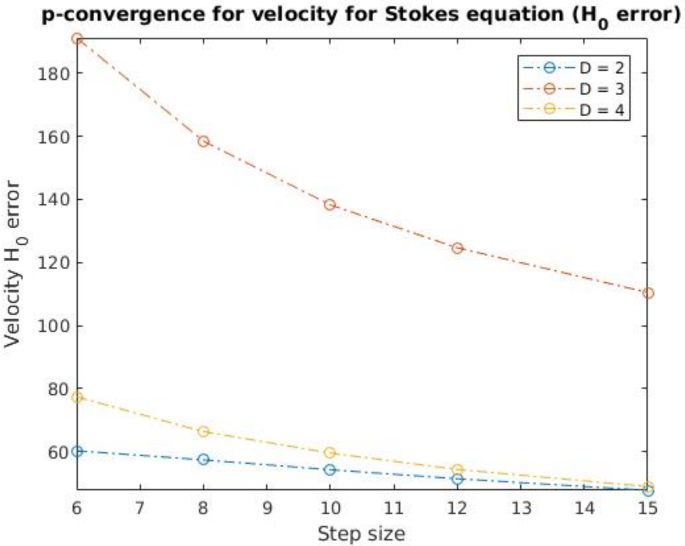


Figure: $h-$convergence test for pressure in H_0 error



(a) $p-$convergence for velocity in L^2 norm



(a) $p-$convergence for velocity in H_0 semi norm

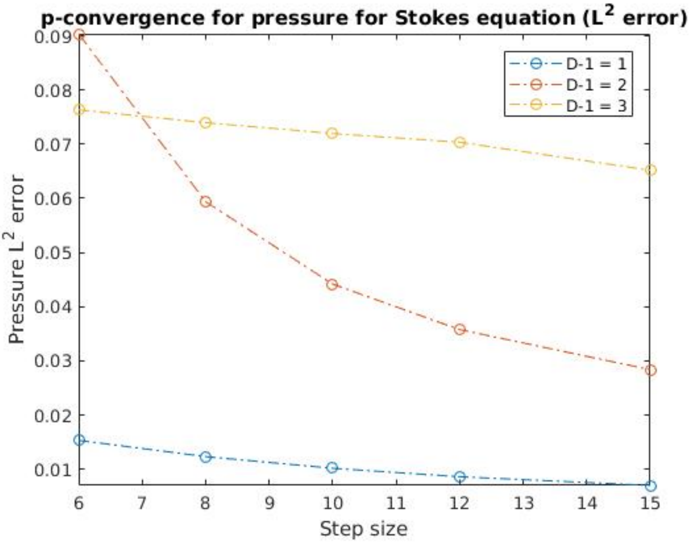
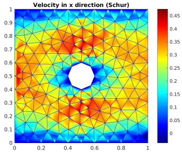


Figure: $p-$convergence for pressure in L^2 norm

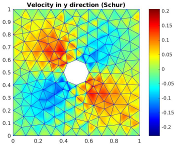
Stokes equation: Flow around cylinder

- Domain: [0,1] $\times$ [0,1] with a cut out cylinder of diameter 0.2 centered at $(0.5, 0.5)$.
- The boundary $x = 0$ is Dirichlet boundary with inflow velocity at point $(0, y)$ as $u = (y(1 - y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (0, 0)$.

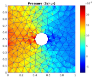
Stokes equation: Flow around cylinder



(a) $x-$velocity



(b) $y-$velocity



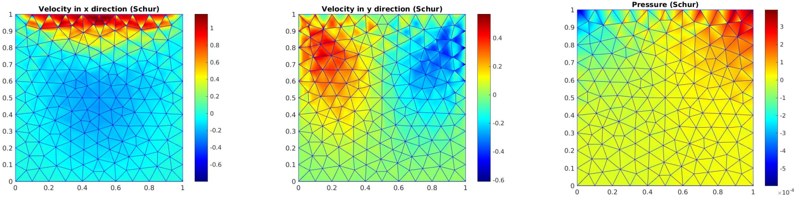
(a) Pressure

Figure: Flow over cylinder

Stokes equation: Lid driven cavity

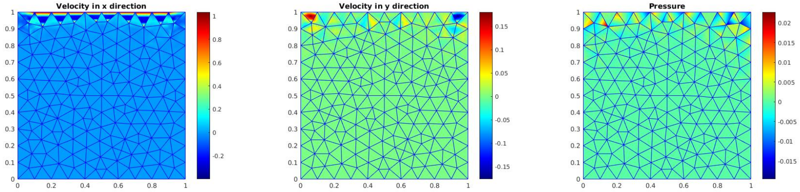
- Domain: unit square [0,1] $\times$ [0,1].
- Boundaries $x = 0, x = 1$ and $y = 0$, we impose no slip or zero velocity Dirichlet condition.
- On $y = 1$, we impose Dirichlet condition with Dirichlet velocity,

$$\begin{aligned} u &= (10x, 0) \quad \text{for} \quad 0 \leq x \leq 0.1, \\ u &= (1, 0) \quad \text{for} \quad 0.1 \leq x \leq 0.9, \\ u &= (10 - 10x, 0) \quad \text{for} \quad 0.9 \leq x \leq 1. \end{aligned} \tag{37}$$



(a) $x-$velocity (b) $y-$velocity (c) Pressure

Figure: Lid driven cavity problem (Schur complement method)



(a) $x-$velocity (b) $y-$velocity (c) Pressure

Figure: Lid driven cavity problem (*bicgstab* solver)

Penalty parameter

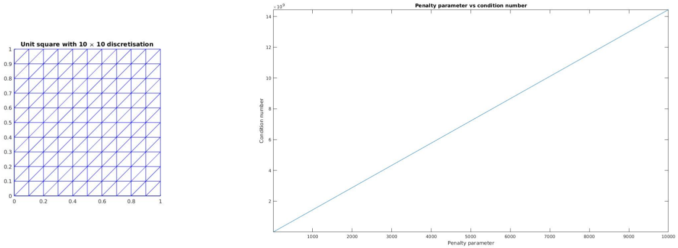


Figure: Effect of penalty parameter

Solver performance

Relative residual is measured as $\frac{||B-AX||_2}{||B||_2}$ for equation $AX = B$.

Solver/Method	**Relative residual**	**Run time**
Schur complement method	2.4436e-08	6.6253 Seconds
minres	2.4618e-05	35.7372 Seconds
bicgstab	9.0071e-05	58.3472 Seconds

Navier-Stokes equation: Convergence test

- Domain: unit square [0,1] $\times$ [0,1].
- $x = 0$ is dirichlet boundary with inflow velocity at point $(0, y)$ as $u = a * (y(1-y), 0)$.
- The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- Specified source term

Analytical solution

$$p = x(1-x), \tag{38}$$

$$\begin{aligned} u = (&x^2(1-y)^2(2y - 6y^2 + 4y^3), \\ &-y^2(1-y)^2(2x - 6x^2 + 4x^3)). \end{aligned} \tag{39}$$

Navier-Stokes equation: Convergence test

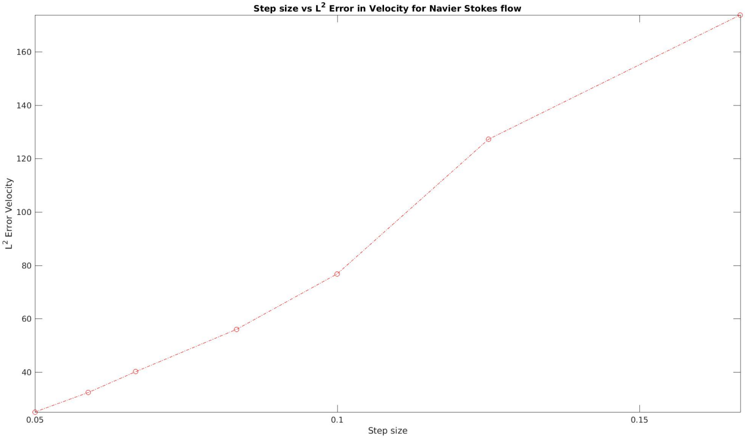


Figure: $h-$convergence test for velocity L^2 error

Navier-Stokes equation: Convergence test

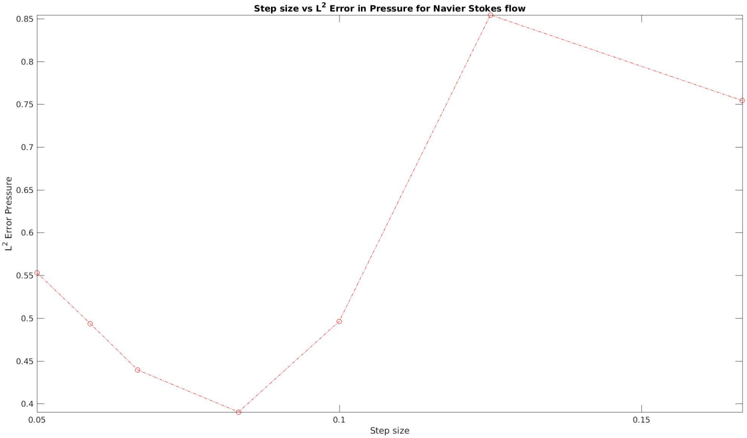


Figure: $h-$convergence test for pressure in L^2 error

Navier-Stokes equation: Convergence test

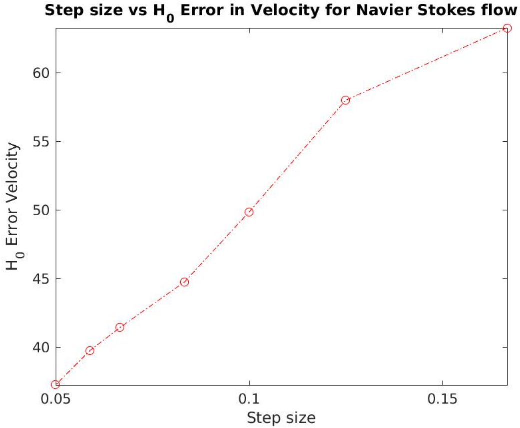


Figure: $h-$convergence test for velocity H_0 error

Navier-Stokes equation: Convergence test

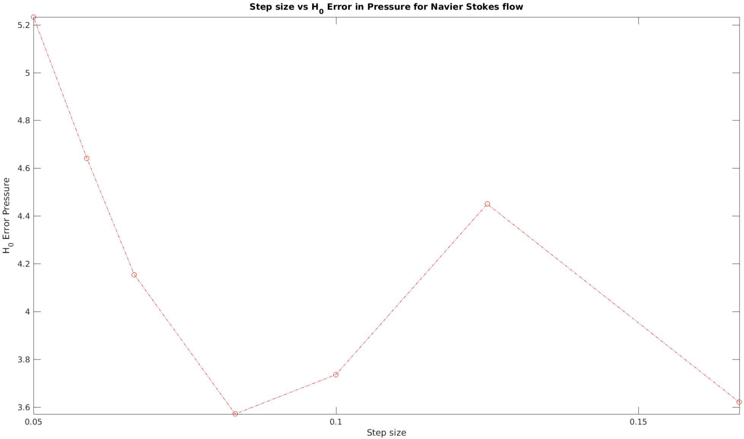
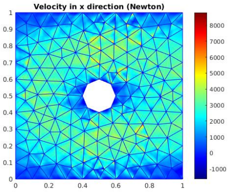
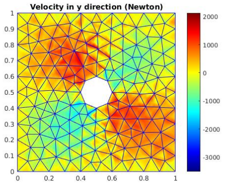


Figure: $h-$convergence test for pressure in H_0 error

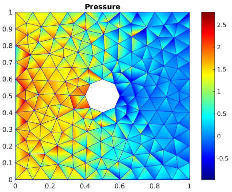
Navier-Stokes equation: Flow around cylinder



(a) $x-$velocity



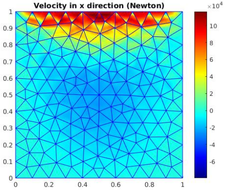
(b) $y-$velocity



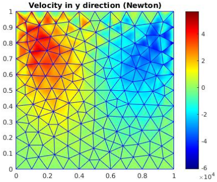
(c) Pressure

Figure: Flow over cylinder (Initial guess by Schur complement method)

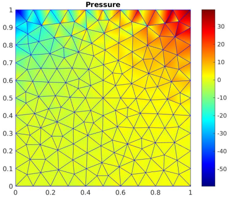
Navier-Stokes equation: Lid driven cavity



(a) $x-$velocity



(b) $y-$velocity



(c) Pressure

Figure: Lid driven cavity flow (Initial guess by Schur complement method)

Conclusions

Numerical considerations

- The higher polynomial degree does not always guarantee better accuracy. However, the convergence rate increases with increase in polynomial degree.
- The initial guess is crucial for success of the Newton method.
- The solution of the Stokes equation and the Navier Stokes equation show close to linear convergence in L^2 norm.

Solution method

- The Schur complement method: Efficient and accurate.
- The *minres* solver : Slow convergence
- The *bicgstab* : Convergence failure.

Outlook

Discontinuous Galerkin method

- Test for higher Reynold's number.
- Further solvers/methods.
- Time dependent cases.

Model order reduction

- Parametrization.
- Reduced order modelling.

References

[1] Montlaur A., Fernandez-Mendez S., and Huerta A. Discontinuous Galerkin methods for the Stokes equations using divergence-free approximations. *International Journal for Numerical Methods in Fluids*, 57(9):1071–1092, 2008.

[2] Montlaur A., Fernandez-Mendez S., Peraire J., and Huerta A. Discontinuous Galerkin methods for the Navier Stokes equations using solenoidal approximations. *International Journal for Numerical Methods in Fluids*, 64(5):549–564, 2010.

[3] Riviere B. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*. Frontiers in Applied Mathematics. Cambridge University Press, 2008.

[4] White F.M. *Fluid mechanics*. Second edition, 97-99, 2002. Academic press.

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Notations

Ω = Continuous domain,
Γ_D = Dirichlet boundary,
Γ_N = Neumann boundary,
cv = Control volume,
cs = Control surface,
B' = Extensive quantity under consideration,
b' = Intensive quantity corresponding to B',
u = flow velocity and $u : \Omega \to \mathbb{R}^d$,
p = pressure and $p : \Omega \to \mathbb{R}$,
ν = kinematic viscocity (fluid property) and $\nu : \Omega \to \mathbb{R}$,
f = external force and $f : \Omega \to \mathbb{R}^d$,
u_D = specified flow velocity at Dirichlet boundary and $u_D : \Gamma_D \to \mathbb{R}^d$,
n = normal unit vector and $n : \partial\Omega \to \mathbb{R}^d$,
ρ = density (fluid property) and $\rho : \Omega \to \mathbb{R}$,
t = specified Neumann flux and $t : \Gamma_N \to \mathbb{R}^d$.

Grid

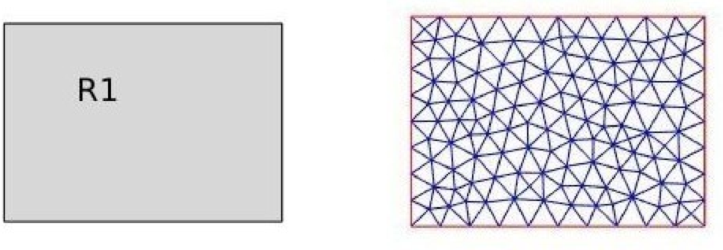


Figure: Continuous domain (left) and discretised domain or grid (right)

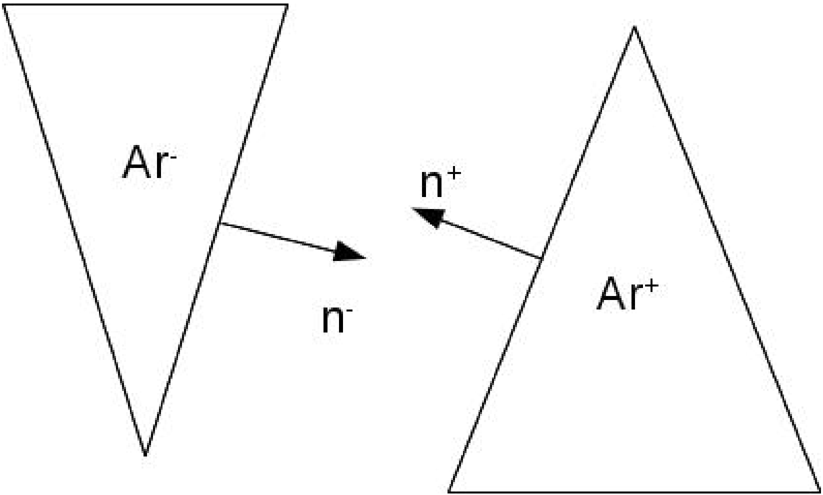
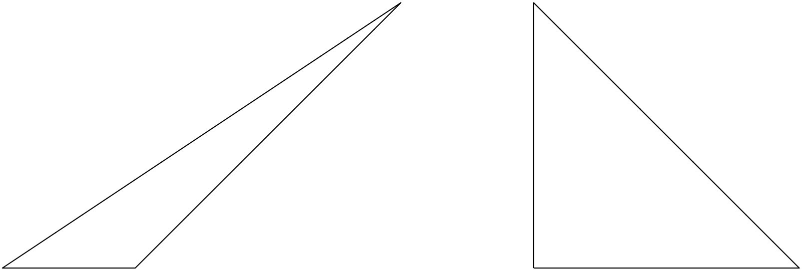


Figure: Element self (+) and neighbouring element (-)

Global and local coordinate system

- The transformation from local coordinate $\hat{X}$ to global coordinate X is defined by the mapping,

$$\begin{aligned} F_k : \hat{X} \mapsto X \quad \forall \quad \hat{X} \in \hat{T} \quad \text{and} \quad X \in \mathcal{T} \\ F_k(\hat{X}) : X = J_k\hat{X} + C \end{aligned} \tag{40}$$



Global geometry (left) to Local geometry (right)

Matrix assemblies

Assembly of $([n \otimes \phi],[n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Step 1 :

$$\begin{aligned} res_1^{++} &= (n \otimes \hat{\phi})^{+}(n \otimes \hat{\phi})^{+}, \\ res_1^{+-} &= (n \otimes \hat{\phi})^{+}(n \otimes \hat{\phi})^{-}, \\ res_1^{-+} &= (n \otimes \hat{\phi})^{-}(n \otimes \hat{\phi})^{+}, \\ res_1^{--} &= (n \otimes \hat{\phi})^{-}(n \otimes \hat{\phi})^{-}. \end{aligned} \tag{41}$$

Step 2 :

$$\begin{aligned} res_2^{++} &= \int_\Gamma res_1^{++} EL(i,j), \\ res_2^{+-} &= \int_\Gamma res_1^{+-} EL(i,j), \end{aligned} \tag{42}$$

Matrix assemblies

Assembly of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

$$
\begin{aligned}
res_2^{-+} &= \int_\Gamma res_1^{-+} EL(i,j), \\
res_2^{--} &= \int_\Gamma res_1^{--} EL(i,j).
\end{aligned}
\tag{43}
$$

Step 3 :

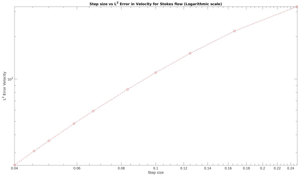
$$
\begin{aligned}
res_3^{++}[ids\_velocity\_self, ids\_velocity\_self] &= res_2^{++}, \\
res_3^{+-}[ids\_velocity\_self, ids\_velocity\_neighbour] &= res_2^{+-}, \\
res_3^{-+}[ids\_velocity\_neighbour, ids\_velocity\_self] &= res_2^{-+}, \\
res_3^{--}[ids\_velocity\_neighbour, ids\_velocity\_neighbour] &= res_2^{--}.
\end{aligned}
\tag{44}
$$

$$
res_3 = res_3^{++} + res_3^{+-} + res_3^{-+} + res_3^{--}. \tag{45}
$$

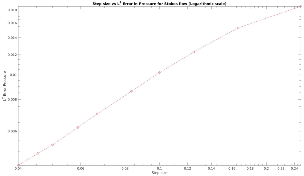
Program flow

- Grid preparation
- Fomrulating function space
- Matrix assembly
- Solving assembled form
- Post processing
- Newton method

Stokes equation: Convergence test



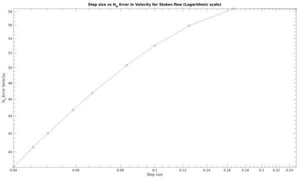
(a) $h-$convergence test for velocity L^2 error (Logarithmic scale)



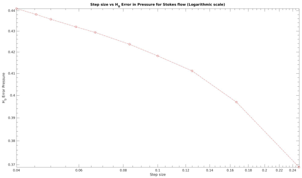
(b) $h-$convergence test for pressure in L^2 error (Logarithmic scale)

Figure: $h-$convergence in L^2 norm for the Stokes flow (Logarithmic scale)

Stokes equation: Convergence test



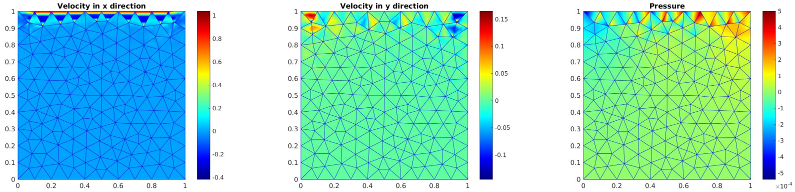
(a) $h-$convergence test for velocity H_0 error (Logarithmic scale)



(b) $h-$convergence test for pressure in H_0 error (Logarithmic scale)

Figure: $h-$convergence in H_0 norm for the Stokes flow (Logarithmic scale)

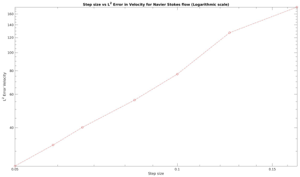
Stokes equation: Lid driven cavity



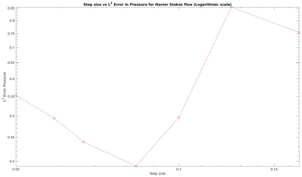
(a) $x-$velocity (b) $y-$velocity (c) Pressure

Figure: Lid driven cavity problem (*minres* solver)

Navier-Stokes equation: Convergence test



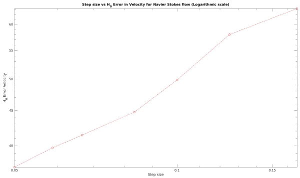
(a) $h-$convergence test for velocity L^2 error (Logarithmic scale)



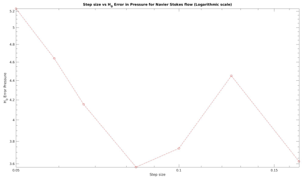
(b) $h-$convergence test for pressure in L^2 error (Logarithmic scale)

Figure: $h-$convergence for the Navier Stokes flow in L^2 error (Logarithmic scale)

Navier-Stokes equation: Convergence test



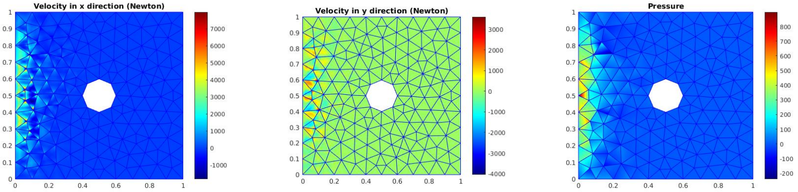
(a) $h-$convergence test for velocity H_0 error (logarithmic scale)



(b) $h-$convergence test for pressure in H_0 error (Logarithmic scale)

Figure: $h-$convergence for the Navier Stokes flow in H_0 error (Logarithmic scale)

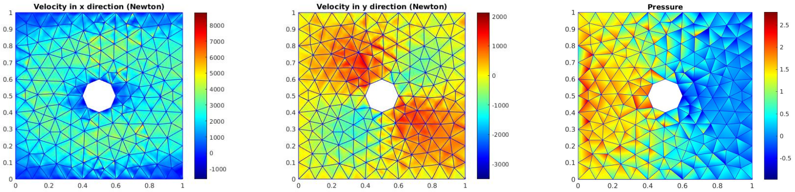
Navier-Stokes equation: Flow around cylinder



(a) $x-$velocity (b) $y-$velocity (c) Pressure

Figure: Flow over cylinder (Initial guess by *bicgstab* solver)

Navier-Stokes equation: Flow around cylinder



(a) $x-$velocity (b) $y-$velocity (c) Pressure

Figure: Flow over cylinder (Initial guess by minres solver)

Navier-Stokes equation: Lid driven cavity

- Domain: unit square [0,1] $\times$ [0,1].
- Boundaries $x = 0, x = 1$ and $y = 0$, we impose no slip or zero velocity Dirichlet condition.
- On $y = 1$, we impose Dirichlet condition with Dirichlet velocity,

$$\begin{aligned} u = a * (10x, 0) \quad &\text{for} \quad 0 \leq x \leq 0.1, \\ u = a * (1, 0) \quad &\text{for} \quad 0.1 \leq x \leq 0.9, \\ u = a * (10 - 10x, 0) \quad &\text{for} \quad 0.9 \leq x \leq 1. \end{aligned} \tag{46}$$

Navier-Stokes equation: Flow around cylinder

- Domain: [0,1] $\times$ [0,1] with a cut out cylinder of diameter 0.2 centered at $(0.5, 0.5)$.
- The boundary $x = 0$ is Dirichlet boundary with inflow velocity at point $(0, y)$ as $u = a * (y(1 - y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (0, 0)$.