

# Discontinuous Galerkin method for direct numerical simulation of the Navier Stokes equation

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- 1 Introduction
- 2 Engineering perspectives and mathematical formulation
- 3 Discretisation and function spaces
- 4 Implementation aspects
- 5 Numerical experiments
- 6 Conclusions and outlook

# Objective

- Understanding the Stokes and the Navier-Stokes formulation and perspectives
- Derivation from conservation equation
- Discretisation and function spaces for unknowns
- Discontinuous-Galerkin formulation for the Stokes and the Navier-Stokes equation
- Implementation and matrix assembly
- Sparsity pattern and solver selection
- Numerical experiment for the Stokes equation and the Navier-Stokes equation
- Future perspectives

# Importance of the Navier Stokes equation

- Computational fluid dynamics  $\implies$  One of the variants of the Navier Stokes equation
- Navier Stokes equation involves state variables
- Incompressible condition  $\implies$  state variables constant  $\implies$  Equation of state not required
- Solved along with continuity equation
- Depends on time (Unsteady fluid flow) or independent of time (Steady fluid flow)
- Non linear coupled system of equations
- Stokes equation is linearized form of the Navier Stokes equation

# Notations

$\Omega$  = Continuous domain,

$\Gamma_D$  = Dirichlet boundary,

$\Gamma_N$  = Neumann boundary,

$cv$  = Control volume,

$cs$  = Control surface,

$B'$  = Extensive quantity under consideration,

$b'$  = Intensive quantity corresponding to  $B'$ ,

$u$  = flow velocity and  $u : \Omega \rightarrow \mathbb{R}^d$ ,

$p$  = pressure and  $p : \Omega \rightarrow \mathbb{R}$ ,

$\nu$  = kinematic viscosity (fluid property) and  $\nu : \Omega \rightarrow \mathbb{R}$ ,

$f$  = external force and  $f : \Omega \rightarrow \mathbb{R}^d$ ,

$u_D$  = specified flow velocity at Dirichlet boundary and  $u_D : \Gamma_D \rightarrow \mathbb{R}^d$ ,

$n$  = normal unit vector and  $n : \partial\Omega \rightarrow \mathbb{R}^d$ ,

$\rho$  = density (fluid property) and  $\rho : \Omega \rightarrow \mathbb{R}$ ,

$t$  = specified Neumann flux and  $t : \Gamma_N \rightarrow \mathbb{R}^d$ .

# Governing equations

## Reynolds transport theorem (White F.M. [4])

$$\left. \frac{dB'}{dt'} \right|_{cs} = \frac{d}{dt'} \int_{cv} b' \rho dV + \int_{cs} (b' \rho) u \cdot dA \quad (1)$$

## Momentum conservation equation

$$F = \frac{dM}{dt'} = \frac{d}{dt'} \int_{cv} u \rho dV + \int_{cs} (u \rho) u \cdot dA. \quad (2)$$

$$F = \int_{cs} \sigma \cdot dA + \int_{cv} \rho f dV, \quad (3)$$

# Governing equations

## Navier Stokes equation

$$-2\nabla \cdot (\nu \nabla^s u) + (1/\rho) \nabla p + (u \cdot \nabla) u = f \quad \text{in } \Omega. \quad (4)$$

Dirichlet boundary:

$$u = u_D \quad \text{on } \Gamma_D. \quad (5)$$

Neumann boundary:

$$-pn + 2\nu(n \cdot \nabla^s)u = t \quad \text{on } \Gamma_N. \quad (6)$$

## Continuity equation

$$\nabla \cdot u = 0 \quad \text{in } \Omega. \quad (7)$$

## Stokes equation

$$-2\nabla \cdot (\nu \nabla^s u) + (1/\rho) \nabla p = f \quad \text{in } \Omega. \quad (8)$$

# Flow classification (Kundu P.K [5])

## Reynolds number

$$Re = \frac{uL}{\nu}. \quad (9)$$

## Laminar flow

- Well defined velocity and pressure profile.
- Low Reynolds number.

## Turbulent flow

- Fluctuations in velocity and pressure.
- Fluctuations are of the order of Kolmogorov scale.
- Low Reynolds number.



- Continuous domain  $(\Omega) \implies$  Grid  $(\mathcal{T})$
- Triangular element,  $\tau_k, \cup_{k=1}^{nel} \tau_k = \mathcal{T}$ ,  $nel$  is the total number of elements
- Grid boundary includes interelement boundaries.

$$\partial\mathcal{T} = \Gamma_D \cup \Gamma_N \cup \Gamma \quad (10)$$

## Barycentric coordinate

For a triangle with vertices,  $r_1, r_2, r_3$

$$r = \lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3. \quad (11)$$

Weights,  $\lambda_1, \lambda_2, \lambda_3$  satisfy,

$$\lambda_1 + \lambda_2 + \lambda_3 = 1. \quad (12)$$

# Discontinuous Galerkin method

- Multiply with test function and integrate
- Discontinuous at the interface of elements
- $P^D(\tau_k)$  denotes space of polynomials of degree at most  $D$  over  $\tau_k$ .

## Function space for velocity

$$\mathbb{V} = \{\phi \in (L^2(\mathcal{T}))^{d_u} \mid \phi \in (P^D(\tau_k))^{d_u} \quad \forall \quad \tau_k \in \mathcal{T}\} \quad (13)$$

## Function space for pressure

$$\mathbb{Q} = \{\psi \in (L^2(\mathcal{T}))^{d_p} \mid \psi \in (P^{D-1}(\tau_k))^{d_p} \quad \forall \quad \tau_k \in \mathcal{T}\} \quad (14)$$

- We use Orthonormal basis function

# Jump operator, Average operator

## Jump operator

$$[pn] = p^+ n^+ + p^- n^- \text{ on } \Gamma$$

$$[pn] = pn \text{ on } \Gamma_D$$

where  $p$  is scalar.

$$[n \otimes u] = n^+ \otimes u^+ + n^- \otimes u^- \text{ on } \Gamma$$

$$[n \otimes u] = n \otimes u \text{ on } \Gamma_D$$

$$[n \cdot u] = n^+ \cdot u^+ + n^- \cdot u^- \text{ on } \Gamma$$

$$[n \cdot u] = n \cdot u \text{ on } \Gamma_D$$

where  $u$  is vector and  $n \otimes u = u_i n_j$ ,  $1 \leq i \leq d_u$ ,  $1 \leq j \leq d$

## Average operator

The average operator is defined as,

$$\{u\} = \frac{u^+ + u^-}{2} \quad . \quad (15)$$

## Strong form

$$\begin{aligned} -\nu \Delta u + \nabla p &= f & \text{in } \Omega & . \\ \nabla \cdot u &= 0 & \text{in } \Omega & . \end{aligned} \tag{16}$$

## Weak form (Montlaur et al. [1])

$$\begin{aligned} a_{IP}(u, \phi) + b(\phi, p) + (\{p\}, [n \cdot \phi])_{\Gamma \cup \Gamma_D} &= l_{IP}(\phi) & . \\ b(u, \psi) + (\{\psi\}, [n \cdot u])_{\Gamma \cup \Gamma_D} &= (q, n \cdot u_D)_{\Gamma_D} & . \end{aligned} \tag{17}$$

# Stokes equation

## Weak form

$$a_{IP}(u, \phi) = (\nabla u, \nabla \phi) + C_{11}([n \otimes u], [n \otimes \phi])_{\Gamma \cup \Gamma_D} - \nu(\{\nabla u\}, [n \otimes \phi])_{\Gamma \cup \Gamma_D} - \nu([n \otimes u], \{\nabla \phi\})_{\Gamma \cup \Gamma_D}. \quad (18)$$

$$b(\phi, \psi) = - \int_{\mathcal{T}} \psi \nabla \cdot \phi. \quad (19)$$

$$l_{IP}(\phi) = (f, \phi) + (t, \phi)_{\Gamma_N} + C_{11}(u_D, \phi)_{\Gamma_D} - (n \otimes u_D, \nu \nabla \phi)_{\Gamma_D}. \quad (20)$$

## Discrete form

$$\begin{aligned} AU + BP &= F_1 \\ B^T U &= F_2 \end{aligned} \quad (21)$$

# Stiffness matrix

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \quad (22)$$

Stiffness matrix   Solution vector   Right hand side (Known)

## Matrix A

$$\begin{aligned} A_{ij} = & \sum_{k=1}^d \left( \frac{\partial \phi_i}{\partial x_k}, \frac{\partial \phi_j}{\partial x_k} \right) + C_{11} \sum_{k=1}^d ([\phi_i n_k], [\phi_j n_k])_{\Gamma \cup \Gamma_D} \\ & - \nu \sum_{k=1}^d ([\phi_i n_k], \{ \frac{\partial \phi_j}{\partial x_k} \})_{\Gamma \cup \Gamma_D} - \nu \sum_{k=1}^d (\{ \frac{\partial \phi_i}{\partial x_k} \}, [\phi_j n_k])_{\Gamma \cup \Gamma_D} \end{aligned} \quad (23)$$

## Matrix B

$$B_{ij} = - \int_{\mathcal{T}} \frac{\partial \phi_i}{\partial x_j} \psi_j + (\{ \psi_j \}, [n \cdot \phi_i])_{\Gamma \cup \Gamma_D} \quad (24)$$

# Navier Stokes equation : Upwinding

- If  $n_\tau$  is the unit normal from  $\tau_1$  to  $\tau_2$  and if we denote the upwind value of function  $g$  as  $g^{up}$  [3],

$$\begin{aligned} g^{up} &= g|_{\tau_1} & \text{if } g \cdot n_\tau \geq 0 \\ g^{up} &= g|_{\tau_2} & \text{if } g \cdot n_\tau < 0 \end{aligned} \quad (25)$$

- For the weak form of the Navier Stokes equation (Montlaur et al. [2]),

$$\begin{aligned} c(g; u, \phi) &= \sum_{i=1}^{nel} \int_{\partial\Omega_i \setminus \Gamma_N} \frac{1}{2} [[(g \cdot n_i)(u^{ext} + u) - |g \cdot n_i|(u^{ext} - u)]] \cdot \phi \\ &\quad + \int_{\Gamma_N} (g \cdot n) u \cdot \phi - ((g \cdot \nabla) \phi, u) \quad . \end{aligned} \quad (26)$$

$$u^{ext} = \lim_{\epsilon \rightarrow 0} u(x + \epsilon n_i), \quad \epsilon \rightarrow 0^+ \quad \text{for } x \in \partial\mathcal{T}_i \quad (27)$$

# Newton method

Algorithm for the Newton method is as follow:

1. Calculate  $u^{iter} \in \mathbb{V}$  at iteration  $iter$ ,
2. Verify  $DS_{u^{iter}}(h^{iter}) = -S(u^{iter})$ ,
3. Set  $u^{iter+1} := u^{iter} + h^{iter}$  till  $\|u^{iter+1} - u^{iter}\| < tol$  where  $tol$  is specified tolerance.

We use the solution from the Stokes equation as initial guess. In discrete form the Newton method means, solving the equation,

$$\begin{pmatrix} A + C(U^{iter}) & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U^{iter+1} \\ P^{iter+1} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \quad (28)$$

Stiffness matrix <sup>$iter$</sup>       Solution vector <sup>$iter+1$</sup>       Right hand side



STEP 1:

$$U = A^{-1}(F_1 - BP) \quad (29)$$

STEP 2 :

$$-B^T A^{-1}BP = F_2 - B^T A^{-1}F_1 \quad (30)$$

STEP 3 :

Backsubstitution in equation (29)

# Data types

- `params` : Structure with fields  $u_{npe}$ ,  $u_{ndofs}$ ,  $d_u$ ,  $D$ ,  $U$ .
- `paramsP` : Structure with fields  $p_{npe}$ ,  $p_{ndofs}$ ,  $d_p$ ,  $D-1$ ,  $P$ .
- `grid` : Structure containing informations related to grid.

## Basis function in RBmatlab

$$\phi \in \mathbb{R}^{u_{npe} \times d_u}, \quad (31)$$

$$\psi \in \mathbb{R}^{p_{npe} \times d_p}. \quad (32)$$

- The derivative of basis function  $(\phi)_i$ , where  $1 \leq i \leq u_{npe}$  and  $(\psi)_i$ , where  $1 \leq i \leq p_{npe}$  is cell.

## Basis function in RBmatlab

$$\nabla(\phi)_i \in \mathbb{R}^{d_u \times d}. \quad (33)$$

$$\nabla(\psi)_i \in \mathbb{R}^{d_p \times d}. \quad (34)$$

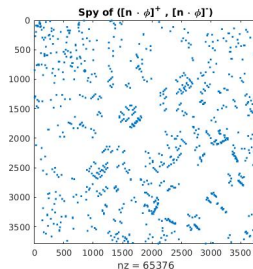
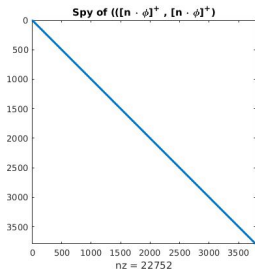
## Jump operator

$$[A_h \cdot n], [B_h \cdot n] = A_h^+ n^+ B_h^+ n^+ + A_h^+ n^+ B_h^- n^- + A_h^- n^- B_h^+ n^+ + A_h^- n^- B_h^- n^-, \quad (35)$$

## Average operator

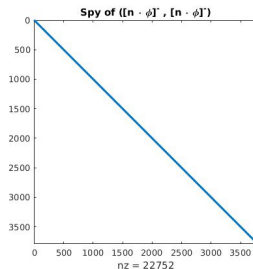
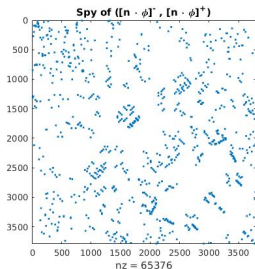
$$\{A_h\} = \frac{(A_h^+ + A_h^-)}{2}. \quad (36)$$

# Sparsity pattern

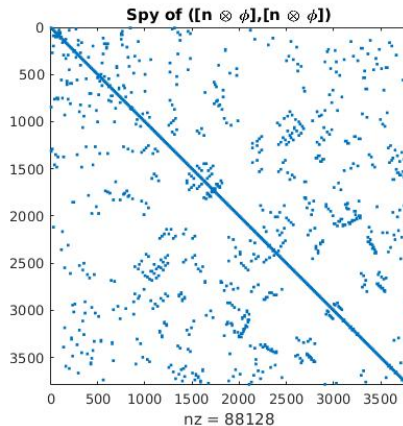


(a)  $((n \otimes \phi)^+, (n \otimes \phi)^+)_{\Gamma \cup \Gamma_D}$

(b)  $((n \otimes \phi)^+, (n \otimes \phi)^-)_{\Gamma \cup \Gamma_D}$



# Sparsity pattern



(a)  $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Figure: Sparsity pattern of constituents of  $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

# Error definition

- If  $P_h$  is the computed solution and  $P$  is the true solution, we define following errors:

$L^2$  error

$$P_{error,L^2} = \left( \int_{\Omega} |P - P_h|^2 \right)^{\frac{1}{2}}. \quad (37)$$

$H_0$  error

$$P_{error,H_0} = \sum_{k=1}^{nel} \left( \int_{\tau_k} |\nabla P - \nabla P_h|^2 \right)^{\frac{1}{2}}. \quad (38)$$

# Stokes equation: Convergence test

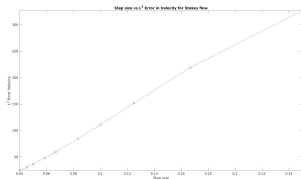
- Domain: unit square  $[0,1] \times [0,1]$ .
- $x = 0$  is dirichlet boundary with inflow velocity at point  $(0, y)$  as  $u = (y(1 - y), 0)$ .
- The boundaries  $y = 0$  and  $y = 1$  are Dirichlet boundaries with no slip or zero velocity condition. The boundary  $x = 1$  is a Neumann boundary with zero Neumann value i.e.  $t = (0, 0)$ .
- The source term is  $f = (2\nu - 1, 0)$ .

## Analytical solution

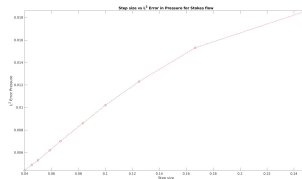
$$p = (1 - x), \quad (39)$$

$$u = (y(1 - y), 0). \quad (40)$$

# Stokes equation: Convergence test



(a)  $h$ -convergence test for velocity  $L^2$  error

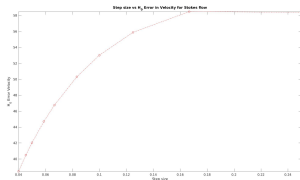


(b)  $h$ -convergence test for pressure in  $L^2$  error

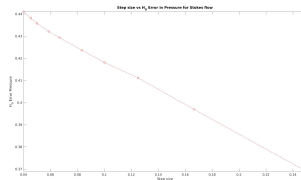
Figure:  $h$ -convergence test for pressure in  $L^2$  error



# Stokes equation: Convergence test

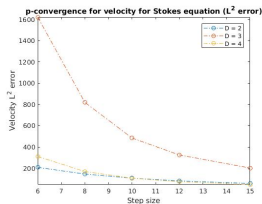


(a)  $h$ -convergence test for velocity  $H_0$  error

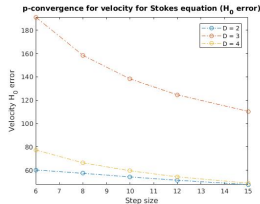


(b)  $h$ -convergence test for pressure in  $H_0$  error

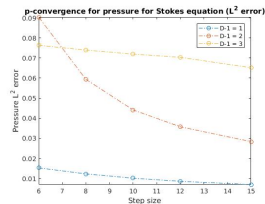
Figure:  $h$ -convergence in  $H_0$  norm for the Stokes flow



(a)  $p$ -convergence for velocity in  $L^2$  norm for Stokes flow



(b)  $p$ -convergence for velocity in  $H_0$  semi norm for Stokes flow



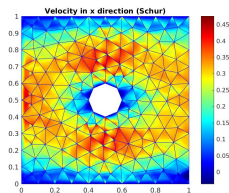
(c)  $p$ -convergence for pressure in  $L^2$  norm for Stokes flow

Figure:  $p$ -convergence for the Stokes flow

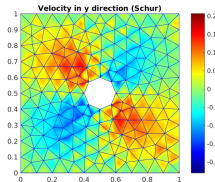
# Stokes equation: Flow around cylinder

- Domain:  $[0,1] \times [0,1]$  with a cut out cylinder of diameter 0.2 centered at  $(0.5, 0.5)$ .
- The boundary  $x = 0$  is Dirichlet boundary with inflow velocity at point  $(0, y)$  as  $u = (y(1 - y), 0)$ .
- The boundaries  $y = 0$  and  $y = 1$  are Dirichlet boundaries with no slip or zero velocity condition. The boundary  $x = 1$  is a Neumann boundary with zero Neumann value i.e.  $t = (0, 0)$ .
- The source term is  $f = (0, 0)$ .

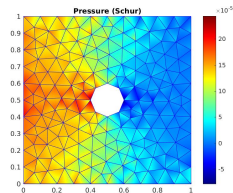
# Stokes equation: Flow around cylinder



(a) x-velocity



(b) y-velocity



(c) Pressure

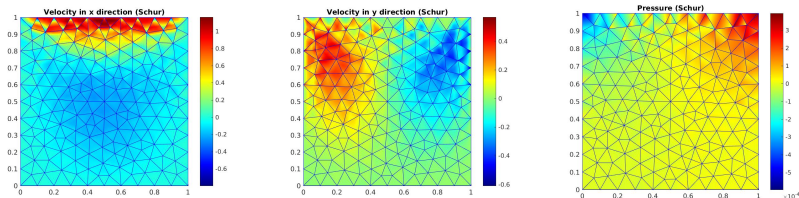
Figure: Flow over cylinder

# Stokes equation: Lid driven cavity

- Domain: unit square  $[0,1] \times [0,1]$ .
- Boundaries  $x = 0, x = 1$  and  $y = 0$ , we impose no slip or zero velocity Dirichlet condition.
- On  $y = 1$ , we impose Dirichlet condition with Dirichlet velocity,

$$\begin{aligned}u &= (10x, 0) \quad \text{for } 0 \leq x \leq 0.1, \\u &= (1, 0) \quad \text{for } 0.1 \leq x \leq 0.9, \\u &= (10 - 10x, 0) \quad \text{for } 0.9 \leq x \leq 1.\end{aligned} \tag{41}$$

# Stokes equation: Lid driven cavity



(a) x-velocity

(b) y-velocity

(c) Pressure

Figure: Lid driven cavity problem (Schur complement method)

# Stokes equation: Lid driven cavity

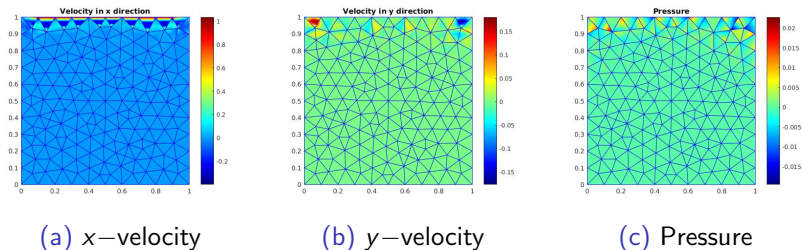
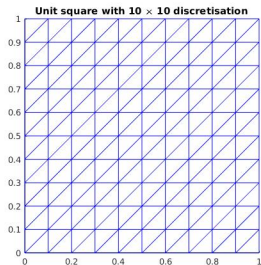
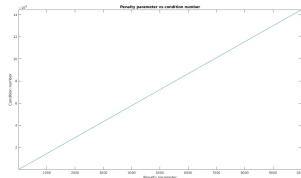


Figure: Lid driven cavity problem (*bicgstab* solver)

# Penalty parameter



(a) Unit square with  $10 \times 10$  discretisation



(b) Penalty parameter vs Condition number

**Figure:** Effect of penalty parameter on condition number of the stiffness matrix



# Solver performance

Relative residual is measured as  $\frac{\|B-AX\|_2}{\|B\|_2}$  for equation  $AX = B$ .

Solver/Method	Relative residual	Run time
Schur complement method	2.4436e-08	6.6253 Seconds
<i>minres</i>	2.4618e-05	35.7372 Seconds
<i>bicgstab</i>	9.0071e-05	58.3472 Seconds

# Navier-Stokes equation: Convergence test

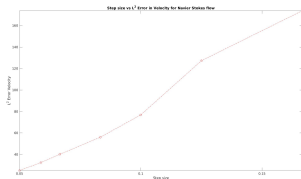
- Domain: unit square  $[0,1] \times [0,1]$ .
- $x = 0$  is dirichlet boundary with inflow velocity at point  $(0, y)$  as  $u = a * (y(1 - y), 0)$ .
- The boundaries  $y = 0$  and  $y = 1$  are Dirichlet boundaries with no slip or zero velocity condition. The boundary  $x = 1$  is a Neumann boundary with zero Neumann value i.e.  $t = (0, 0)$ .
- Specified source term

## Analytical solution

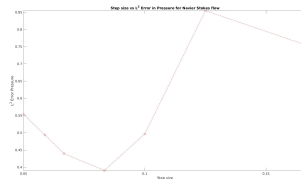
$$p = x(1 - x), \quad (42)$$

$$\begin{aligned} u = & (x^2(1 - y)^2(2y - 6y^2 + 4y^3), \\ & -y^2(1 - y)^2(2x - 6x^2 + 4x^3)). \end{aligned} \quad (43)$$

# Navier-Stokes equation: Convergence test



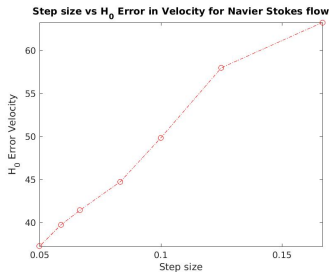
(a)  $h$ -convergence test for velocity  $L^2$  error



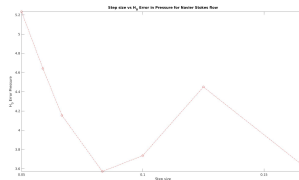
(b)  $h$ -convergence test for pressure in  $L^2$  error

Figure:  $h$ -convergence for the Navier Stokes flow in  $L^2$  error

# Navier-Stokes equation: Convergence test



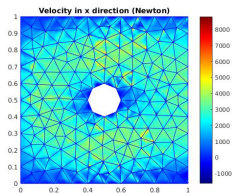
(a)  $h$ -convergence test for velocity  $H_0$  error



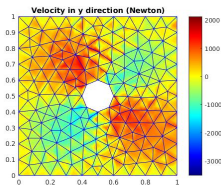
(b)  $h$ -convergence test for pressure in  $H_0$  error

Figure:  $h$ -convergence for the Navier Stokes flow in  $H_0$  error

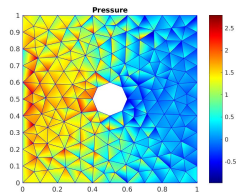
# Navier-Stokes equation: Flow around cylinder



(a) x-velocity



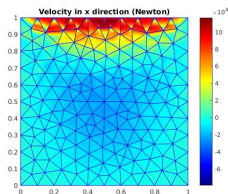
(b) y-velocity



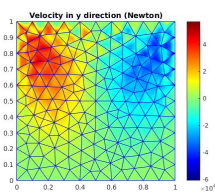
(c) Pressure

Figure: Flow over cylinder (Initial guess by Schur complement method)

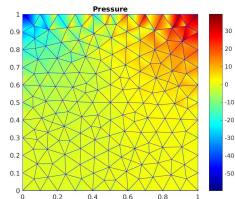
# Navier-Stokes equation: Lid driven cavity



(a) x-velocity



(b) y-velocity



(c) Pressure

Figure: Lid driven cavity flow (Initial guess by Schur complement method)

- The Schur complement : Efficient and accurate
- *minres* slow convergence and *bicgstab* does not converge
- Penalty parameter
- The solvers/methods which are applicable for the Saddle point problems should be used for solving the weak form of the Stokes equation.
- The initial guess is crucial for success of the Newton method.
- The solution of the Stokes equation and the Navier Stokes equation show close to linear convergence in  $L^2$  norm.
- The higher polynomial degree does not always guarantee better accuracy. However, the convergence rate increases with increase in polynomial degree.

- Test for higher Reynold's number
- Further solvers/methods
- Time dependent cases
- Parametrization
- Reduced order modelling



- [1] Montlaur A., Fernandez-Mendez S., and Huerta A. Discontinuous Galerkin methods for the Stokes equations using divergence-free approximations. *International Journal for Numerical Methods in Fluids*, 57(9):1071–1092, 2008.
- [2] Montlaur A., Fernandez-Mendez S., Peraire J., and Huerta A. Discontinuous Galerkin methods for the Navier Stokes equations using solenoidal approximations. *International Journal for Numerical Methods in Fluids*, 64(5):549–564, 2010.
- [3] Riviere B. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*. Frontiers in Applied Mathematics. Cambridge University Press, 2008.
- [4] White F.M. *Fluid mechanics*. Second edition, 97-99, 2002. Academic press.
- [5] Kundu P. K. and Cohen I. M. *Fluid Mechanics*. Academic Press, 2002.

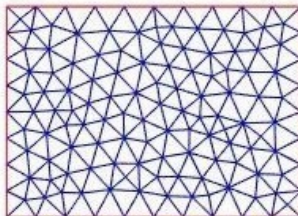
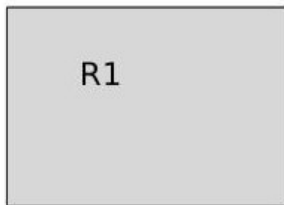


Figure: Continuous domain (left) and discretised domain or grid (right)

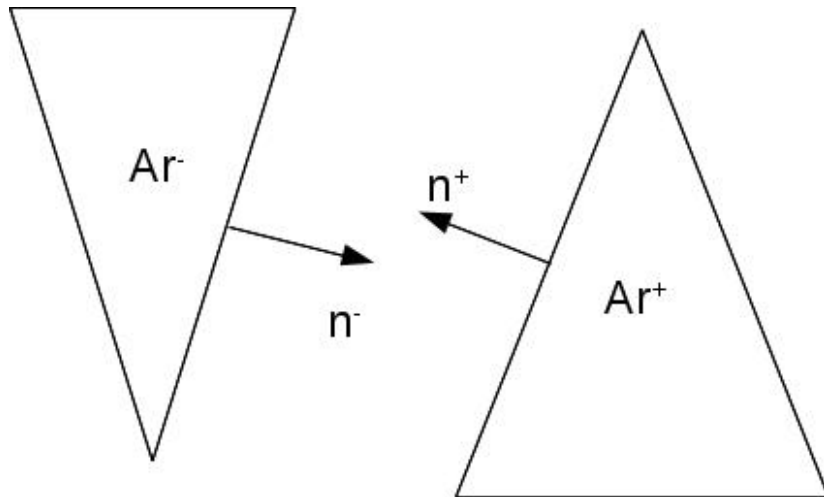


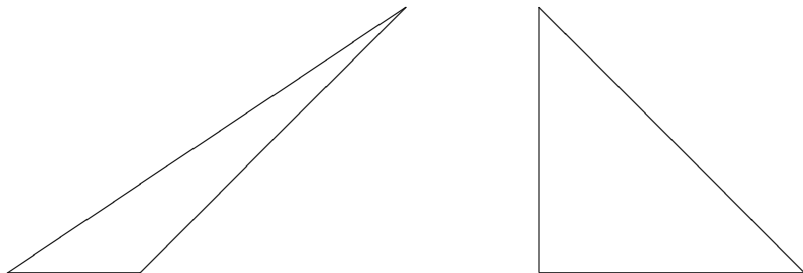
Figure: Element self (+) and neighbouring element (-)

# Global and local coordinate system

- Integral terms are evaluated on a reference triangle instead of the element itself.
- The transformation from local coordinate  $\hat{X}$  to global coordinate  $X$  is defined by the mapping,

$$F_k : \hat{X} \mapsto X \quad \forall \quad \hat{X} \in \hat{T} \quad \text{and} \quad X \in \mathcal{T} \quad (44)$$

$$F_k(\hat{X}) : X = J_k \hat{X} + C \quad (45)$$



Global geometry (left) to Local geometry (right)

## Assembly of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Step 1 :

$$\begin{aligned} res_1^{++} &= (n \otimes \hat{\phi})^+ (n \otimes \hat{\phi})^+, \\ res_1^{+-} &= (n \otimes \hat{\phi})^+ (n \otimes \hat{\phi})^-, \\ res_1^{-+} &= (n \otimes \hat{\phi})^- (n \otimes \hat{\phi})^+, \\ res_1^{--} &= (n \otimes \hat{\phi})^- (n \otimes \hat{\phi})^-. \end{aligned} \tag{46}$$

Step 2 :

$$\begin{aligned} res_2^{++} &= \int_{\Gamma} res_1^{++} EL(i, j), \\ res_2^{+-} &= \int_{\Gamma} res_1^{+-} EL(i, j), \end{aligned} \tag{47}$$

## Assembly of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

$$\begin{aligned} res_2^{-+} &= \int_{\Gamma} res_1^{-+} EL(i, j), \\ res_2^{--} &= \int_{\Gamma} res_1^{--} EL(i, j). \end{aligned} \tag{48}$$

Step 3 :

$$\begin{aligned} res_3^{++}[ids\_velocity\_self, ids\_velocity\_self] &= res_2^{++}, \\ res_3^{+-}[ids\_velocity\_self, ids\_velocity\_neighbour] &= res_2^{+-}, \\ res_3^{-+}[ids\_velocity\_neighbour, ids\_velocity\_self] &= res_2^{-+}, \\ res_3^{--}[ids\_velocity\_neighbour, ids\_velocity\_neighbour] &= res_2^{--}. \end{aligned} \tag{49}$$

$$res_3 = res_3^{++} + res_3^{+-} + res_3^{-+} + res_3^{--}. \tag{50}$$

- Grid preparation
- Formulating function space
- Matrix assembly
- Solving assembled form
- Post processing
- Newton method

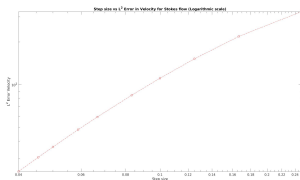
# Sparsity pattern

**Table:** Size and sparsity pattern of different terms

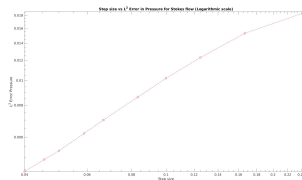
Matrix term	Size	Sparsity pattern
$(\nabla \phi, \nabla \phi)$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figure ??
$([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figure 2
$(\{\nabla \phi\}, [n \otimes \phi])_{\Gamma \cup \Gamma_D}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figure ??
$(\{\psi\}, [n \cdot \phi])_{\Gamma \cup \Gamma_D}$	$\mathbb{R}^{p_{ndofs} \times u_{ndofs}}$	Figure ??
$(-\int_{\hat{T}} \psi \nabla \cdot \phi)$	$\mathbb{R}^{p_{ndofs} \times u_{ndofs}}$	Figure ??
$-((u_k \cdot \nabla) \phi, \phi)$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures ??
$((u_k \cdot n) \phi, \phi)_{\Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures ??
$((u_k \cdot n) \phi, \phi^{ext})_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures ??
$( u_k \cdot n  \phi, \phi^{ext})_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures ??
$((u_k \cdot n) \phi, \phi)_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures ??
$( u_k \cdot n  \phi, \phi)_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures ??



# Stokes equation: Convergence test



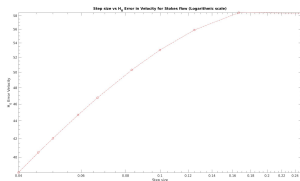
(a)  $h$ -convergence test for velocity  $L^2$  error (Logarithmic scale)



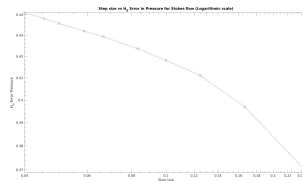
(b)  $h$ -convergence test for pressure in  $L^2$  error (Logarithmic scale)

Figure:  $h$ -convergence in  $L^2$  norm for the Stokes flow (Logarithmic scale)

# Stokes equation: Convergence test



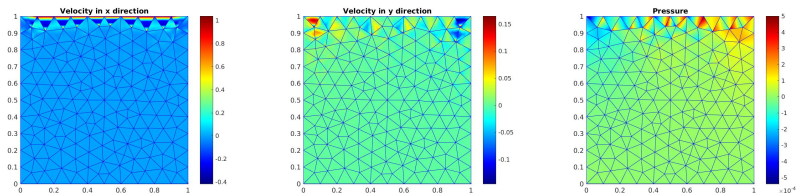
(a)  $h$ -convergence test for velocity  $H_0$  error (Logarithmic scale)



(b)  $h$ -convergence test for pressure in  $H_0$  error (Logarithmic scale)

Figure:  $h$ -convergence in  $H_0$  norm for the Stokes flow (Logarithmic scale)

# Stokes equation: Lid driven cavity



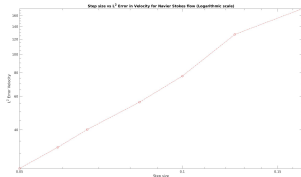
(a) x-velocity

(b) y-velocity

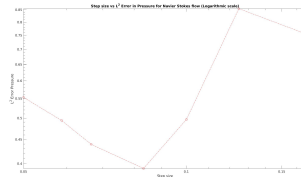
(c) Pressure

Figure: Lid driven cavity problem (*minres* solver)

# Navier-Stokes equation: Convergence test



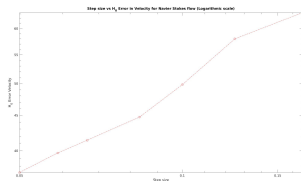
(a)  $h$ -convergence test for velocity  $L^2$  error (Logarithmic scale)



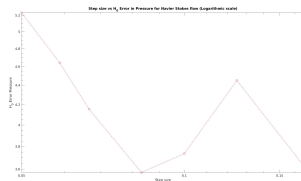
(b)  $h$ -convergence test for pressure in  $L^2$  error (Logarithmic scale)

Figure:  $h$ -convergence for the Navier Stokes flow in  $L^2$  error (Logarithmic scale)

# Navier-Stokes equation: Convergence test



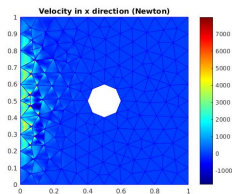
(a)  $h$ -convergence test for velocity  $H_0$  error (logarithmic scale)



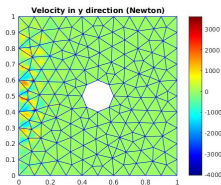
(b)  $h$ -convergence test for pressure in  $H_0$  error (Logarithmic scale)

Figure:  $h$ -convergence for the Navier Stokes flow in  $H_0$  error (Logarithmic scale)

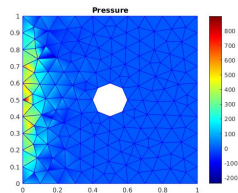
# Navier-Stokes equation: Flow around cylinder



(a) x-velocity



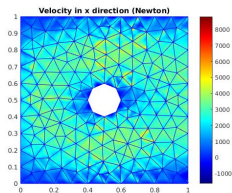
(b) y-velocity



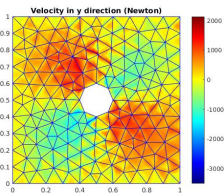
(c) Pressure

Figure: Flow over cylinder (Initial guess by *bicgstab* solver)

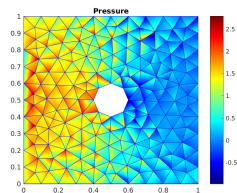
# Navier-Stokes equation: Flow around cylinder



(a) x-velocity



(b) y-velocity



(c) Pressure

Figure: Flow over cylinder (Initial guess by minres solver)

# Navier-Stokes equation: Lid driven cavity

- Domain: unit square  $[0,1] \times [0,1]$ .
- Boundaries  $x = 0, x = 1$  and  $y = 0$ , we impose no slip or zero velocity Dirichlet condition.
- On  $y = 1$ , we impose Dirichlet condition with Dirichlet velocity,

$$\begin{aligned}u &= a * (10x, 0) \quad \text{for } 0 \leq x \leq 0.1, \\u &= a * (1, 0) \quad \text{for } 0.1 \leq x \leq 0.9, \\u &= a * (10 - 10x, 0) \quad \text{for } 0.9 \leq x \leq 1.\end{aligned} \tag{51}$$



# Navier-Stokes equation: Flow around cylinder

- Domain:  $[0,1] \times [0,1]$  with a cut out cylinder of diameter 0.2 centered at  $(0.5, 0.5)$ .
- The boundary  $x = 0$  is Dirichlet boundary with inflow velocity at point  $(0, y)$  as  $u = a * (y(1 - y), 0)$ .
- The boundaries  $y = 0$  and  $y = 1$  are Dirichlet boundaries with no slip or zero velocity condition. The boundary  $x = 1$  is a Neumann boundary with zero Neumann value i.e.  $t = (0, 0)$ .
- The source term is  $f = (0, 0)$ .