Discontinuous Galerkin method for direct numerical simulation of the Navier Stokes equation

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Overview

Objective

- Understanding the Stokes and the Navier-Stokes formulation and perspectives
- Derivation from conservation equation
- Discretisation and function spaces for unknowns
- Discontinuous-Galerkin formulation for the Stokes and the Navier-Stokes equation
- Implementation and matrix assembly
- Sparsity pattern and solver selection
- Numerical experiment for the Stokes equation and the Navier-Stokes equation
- Future perspectives

Importance of the Navier Stokes equation

- Computational fluid dynamics $\implies$ One of the variants of the Navier Stokes equation
- The Navier Stokes equation involves state variables
- Incompressible condition $\implies$ state variables constant $\implies$ Equation of state not required
- Solved along with continuity equation
- Depends on time (Unsteady fluid flow) or independent of time (Steady fluid flow)
- Non linear coupled system of equations
- The Stokes equation is linearized form of the Navier Stokes equation

Notations

Ω = Continuous domain,
Γ_D = Dirichlet boundary,
Γ_N = Neumann boundary,
cv = Control volume,
cs = Control surface,
B' = Extensive quantity under consideration,
b' = Intensive quantity corresponding to B',
u = flow velocity and $u : \Omega \to \mathbb{R}^d$,
p = pressure and $p : \Omega \to \mathbb{R}$,
ν = kinematic viscocity (fluid property) and $\nu : \Omega \to \mathbb{R}$,
f = external force and $f : \Omega \to \mathbb{R}^d$,
u_D = specified flow velocity at Dirichlet boundary and $u_D : \Gamma_D \to \mathbb{R}^d$,
n = normal unit vector and $n : \partial\Omega \to \mathbb{R}^d$,
ρ = density (fluid property) and $\rho : \Omega \to \mathbb{R}$,
t = specified Neumann flux and $t : \Gamma_N \to \mathbb{R}^d$.

Governing equations

Reynolds transport theoreom (White F.M. [4])

$$\frac{dB'}{dt'}|_{cs} = \frac{d}{dt'}\int_{cv} b'\rho dV + \int_{cs} (b'\rho) u \cdot dA. \tag{1}$$

Momentum conservation equation

$$F = \frac{dM}{dt'} = \frac{d}{dt'}\int_{cv} u\rho dV + \int_{cs} (u\rho) u \cdot dA. \tag{2}$$

$$F = \int_{cs} \sigma \cdot dA + \int_{cv} \rho f dV. \tag{3}$$

Governing equations

Navier Stokes equation

$$-2\nabla \cdot (\nu \nabla^s u) + (1/\rho)\nabla p + (u \cdot \nabla)u = f \quad \text{in} \quad \Omega. \tag{4}$$

Dirichlet boundary:

$$u = u_D \quad \text{on} \quad \Gamma_D. \tag{5}$$

Neumann boundary:

$$-pn + 2\nu(n \cdot \nabla^s)u = t \quad \text{on} \quad \Gamma_N. \tag{6}$$

Continuity equation

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega. \tag{7}$$

Stokes equation

$$-2\nabla \cdot (\nu \nabla^s u) + (1/\rho)\nabla p = f \quad \text{in} \quad \Omega. \tag{8}$$

Flow classification (Kundu P.K [5])

Reynolds number

$$Re = \frac{uL}{\nu}. \tag{9}$$

Laminar flow

- Well defined velocity and pressure profile.
- Low Reynolds number.

Turbulent flow

- Fluctuations in velocity and pressure.
- Fluctuations are of the order of Kolmogrov scale.
- Low Reynolds number.

Grid

- Continuous domain $(\Omega) \implies$ Grid $(\mathcal{T})$
- Triangular element, τ_k, $\cup_{k=1}^{nel} \tau_k = \mathcal{T}$, *nel* is the total number of elements
- Grid boundary includes interelement boundaries.

$$\partial \mathcal{T} = \Gamma_D \cup \Gamma_N \cup \Gamma \tag{10}$$

Barycentric coordinate

For a triangle with vertices, r_1, r_2, r_3

$$r = \lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3. \tag{11}$$

Weights, $\lambda_1, \lambda_2, \lambda_3$ satisfy,

$$\lambda_1 + \lambda_2 + \lambda_3 = 1. \tag{12}$$

Discontinuous Galerkin method

- Multiply with test function and integrate
- Discontinuous at the interface of elements
- $P^D(\tau_k)$ denotes space of polynomials of degree at most D over τ_k.

Function space for velocity

$$\mathbb{V} = \{\phi \in (L^2(\mathcal{T}))^{d_u} | \quad \phi \in (P^D(\tau_k))^{d_u} \quad \forall \quad \tau_k \in \mathcal{T}\}. \tag{13}$$

Function space for pressure

$$\mathbb{Q} = \{\psi \in (L^2(\mathcal{T}))^{d_p} | \quad \psi \in (P^{D-1}(\tau_k))^{d_p} \quad \forall \quad \tau_k \in \mathcal{T}\}. \tag{14}$$

- We use Orthonormal basis function

Jump operator, Average operator

Jump operator

$$[pn] = p^+ n^+ + p^- n^- \text{ on } \Gamma,$$
$$[pn] = pn \text{ on } \Gamma_D,$$
where p is scalar.

$$[n \otimes u] = n^+ \otimes u^+ + n^- \otimes u^- \text{ on } \Gamma,$$
$$[n \otimes u] = n \otimes u \text{ on } \Gamma_D,$$
$$[n \cdot u] = n^+ \cdot u^+ + n^- \cdot u^- \text{ on } \Gamma,$$
$$[n \cdot u] = n \cdot u \text{ on } \Gamma_D,$$
where u is vector and $n \otimes u = u_i n_j$, $1 \leq i \leq d_u$, $1 \leq j \leq d$.

Average operator

The average operator is defined as,

$$\{u\} = \frac{u^+ + u^-}{2}. \tag{15}$$

Stokes equation

Strong form

$$\begin{aligned} -\nu \Delta u + \nabla p &= f \quad \text{in} \quad \Omega. \\ \nabla \cdot u &= 0 \quad \text{in} \quad \Omega. \end{aligned} \tag{16}$$

Weak form (Montlaur et al. [1])

$$\begin{aligned} a_{IP}(u, \phi) + b(\phi, p) + (\{p\}, [n \cdot \phi])_{\Gamma \cup \Gamma_D} &= l_{IP}(\phi). \\ b(u, \psi) + (\{\psi\}, [n \cdot u])_{\Gamma \cup \Gamma_D} &= (q, n \cdot u_D)_{\Gamma_D}. \end{aligned} \tag{17}$$

Stokes equation

Weak form

$$
\begin{aligned}
a_{IP}(u,\phi) = (\nabla u, \nabla \phi) + C_{11}([n \otimes u],[n \otimes \phi])_{\Gamma \cup \Gamma_D} \\
-\nu(\{\nabla u\},[n \otimes \phi])_{\Gamma \cup \Gamma_D} - \nu([n \otimes u],\{\nabla \phi\})_{\Gamma \cup \Gamma_D}.
\end{aligned} \tag{18}
$$

$$
b(\phi,\psi) = -\int_{\mathcal{T}} \psi \nabla \cdot \phi. \tag{19}
$$

$$
l_{IP}(\phi) = (f,\phi) + (t,\phi)_{\Gamma_N} + C_{11}(u_D,\phi)_{\Gamma_D} - (n \otimes u_D, \nu \nabla \phi)_{\Gamma_D}. \tag{20}
$$

Discrete form

$$
\begin{aligned}
AU + BP = F_1. \\
B^T U = F_2.
\end{aligned} \tag{21}
$$

Stiffness matrix

$$\underbrace{\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}}_{\text{Stiffness matrix}} \underbrace{\begin{pmatrix} U \\ P \end{pmatrix}}_{\text{Solution vector}} = \underbrace{\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}}_{\text{Right hand side (Known)}} \tag{22}$$

Matrix A

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{d} \left(\frac{\partial \phi_i}{\partial x_k}, \frac{\partial \phi_j}{\partial x_k}\right) + C_{11} \sum_{k=1}^{d} ([\phi_i n_k], [\phi_j n_k])_{\Gamma \cup \Gamma_D} \\ &- \nu \sum_{k=1}^{d} \left([\phi_i n_k], \left\{\frac{\partial \phi_j}{\partial x_k}\right\}\right)_{\Gamma \cup \Gamma_D} - \nu \sum_{k=1}^{d} \left(\left\{\frac{\partial \phi_i}{\partial x_k}\right\}, [\phi_j n_k]\right)_{\Gamma \cup \Gamma_D} \end{aligned} \tag{23}$$

Matrix B

$$B_{ij} = -\int_{\mathcal{T}} \frac{\partial \phi_i}{\partial x_i} \psi_j + (\{\psi_j\}, [n \cdot \phi_i])_{\Gamma \cup \Gamma_D} \tag{24}$$

Navier Stokes equation : Upwinding

- If n_τ is the unit normal from τ_1 to τ_2 and if we denote the upwind value of function g as g^{up} [3],

$$\begin{aligned} g^{up} &= g|_{\tau_1} \quad \text{if} \quad g \cdot n_\tau \geq 0 \\ g^{up} &= g|_{\tau_2} \quad \text{if} \quad g \cdot n_\tau < 0 \end{aligned} \tag{25}$$

- For the weak form of the Navier Stokes equation (Montlaur et al. [2]),

$$\begin{aligned} c(g; u, \phi) = \sum_{i=1}^{nel} \int_{\partial\Omega_i \setminus \Gamma_N} \frac{1}{2} [[(g \cdot n_i)(u^{ext} + u) - |g \cdot n_i|(u^{ext} - u)]] \cdot \phi \\ + \int_{\Gamma_N} (g \cdot n) u \cdot \phi - ((g \cdot \nabla)\phi, u). \end{aligned} \tag{26}$$

$$u^{ext} = \lim_{\epsilon \to 0} u(x + \epsilon n_i), \quad \epsilon \to 0^+ \quad \text{for} \quad x \in \partial\mathcal{T}_i \tag{27}$$

Newton method

Algotrithm for the Newton method is as follow:

1. Calculate $u^{iter} \in \mathbb{V}$ at iteration *iter*,
2. Verify $DS_{u^{iter}}(h^{iter}) = -S(u^{iter})$,
3. Set $u^{iter+1} := u^{iter} + h^{iter}$ till $||u^{iter+1} - u^{iter}|| < tol$ where *tol* is specified tolerance.

We use the solution from the Stokes equation as initial guess. In discrete form the Newton method means, solving the equation,

$$\underbrace{\begin{pmatrix} A + C(U^{iter}) & B \\ B^T & 0 \end{pmatrix}}_{\text{Stiffness matrix}^{iter}} \underbrace{\begin{pmatrix} U^{iter+1} \\ P^{iter+1} \end{pmatrix}}_{\text{Solution vector}^{iter+1}} = \underbrace{\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}}_{\text{Right hand side}} . \tag{28}$$

Schur complement method

STEP 1:

$$U = A^{-1}(F_1 - BP) \tag{29}$$

STEP 2 :

$$-B^T A^{-1} BP = F_2 - B^T A^{-1} F_1 \tag{30}$$

STEP 3 :

Backsubstitution in equation (29).

Data types

- `params` : Structure with fields u_{npe}, u_{ndofs}, d_u, D, U.
- `paramsP` : Structure with fields p_{npe}, p_{ndofs}, d_p, $D{-}1$, P.
- `grid` : Structure containing informations related to grid.

Basis function in RBmatlab

$$\phi \in \mathbb{R}^{u_{npe} \times d_u}, \tag{31}$$

$$\psi \in \mathbb{R}^{p_{npe} \times d_p}. \tag{32}$$

- The derivative of basis function $(\phi)_i$, where $1 \leq i \leq u_{npe}$ and $(\psi)_i$, where $1 \leq i \leq p_{npe}$ is cell.

Basis function in RBmatlab

$$\nabla(\phi)_i \in \mathbb{R}^{d_u \times d}. \tag{33}$$

$$\nabla(\psi)_i \in \mathbb{R}^{d_p \times d}. \tag{34}$$

Matrix assemblies : Jump operator and average operator

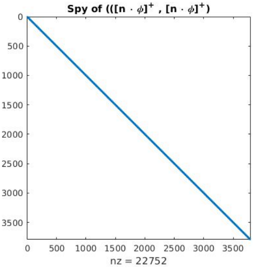
Jump operator

$$[A_h \cdot n], [B_h \cdot n] = A_h^+ n^+ B_h^+ n^+ + A_h^+ n^+ B_h^- n^- + A_h^- n^- B_h^+ n^+ + A_h^- n^- B_h^- n^- . \tag{35}$$

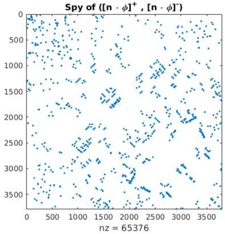
Average operator

$$\{A_h\} = \frac{(A_h^+ + A_h^-)}{2}. \tag{36}$$

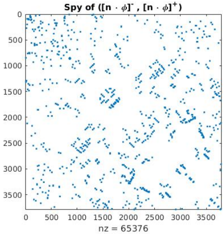
Sparsity pattern



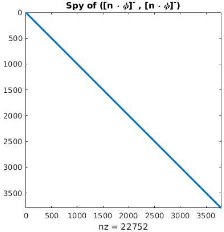
(a) $((n \otimes \phi)^{+}, (n \otimes \phi)^{+})_{\Gamma \cup \Gamma_D}$



(b) $((n \otimes \phi)^{+}, (n \otimes \phi)^{-})_{\Gamma \cup \Gamma_D}$



(c) $((n \otimes \phi)^{-}, (n \otimes \phi)^{+})_{\Gamma \cup \Gamma_D}$



(d) $((n \otimes \phi)^{-}, (n \otimes \phi)^{-})_{\Gamma \cup \Gamma_D}$

Sparsity pattern

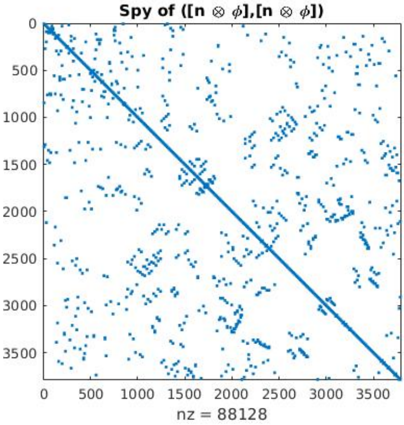


Figure: Sparsity pattern of constituents of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Error definition

- If P_h is the computed solution and P is the true solution, we define following errors:

L^2 error

$$P_{error,L^2} = (\int_{\Omega} |P - P_h|^2)^{\frac{1}{2}}. \tag{37}$$

H_0 error

$$P_{error,H_0} = \sum_{k=1}^{nel}(\int_{\tau_k} |\nabla P - \nabla P_h|^2)^{\frac{1}{2}}. \tag{38}$$

Stokes equation: Convergence test

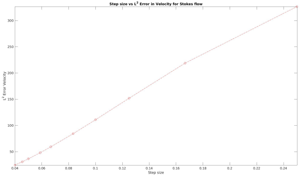
- Domain: unit square [0,1] × [0,1].
- $x = 0$ is dirichlet boundary with inflow velocity at point $(0, y)$ as $u = (y(1-y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (2\nu - 1, 0)$.

Analytical solution

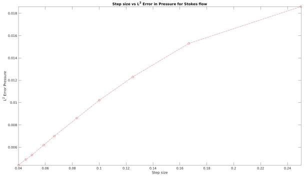
$$p = (1 - x), \tag{39}$$

$$u = (y(1-y), 0). \tag{40}$$

Stokes equation: Convergence test



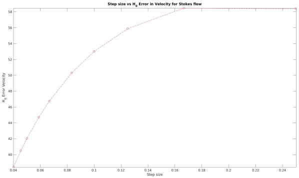
(a) $h-$convergence test for velocity L^2 error



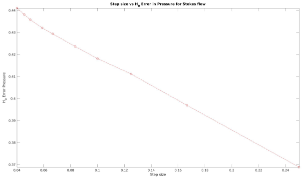
(b) $h-$convergence test for pressure in L^2 error

Figure: $h-$convergence test for pressure in L^2 error

Stokes equation: Convergence test

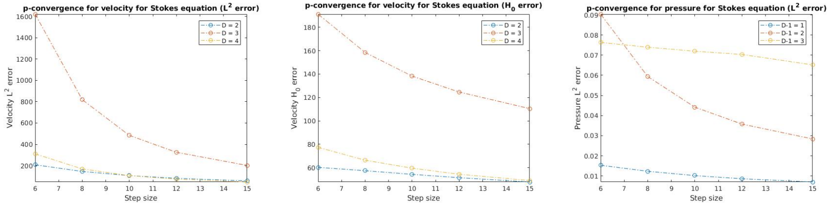


(a) $h-$convergence test for velocity H_0 error



(b) $h-$convergence test for pressure in H_0 error

Figure: $h-$convergence in H_0 norm for the Stokes flow



(a) $p-$convergence for velocity in L^2 norm for Stokes flow

(b) $p-$convergence for velocity in H_0 semi norm for Stokes flow

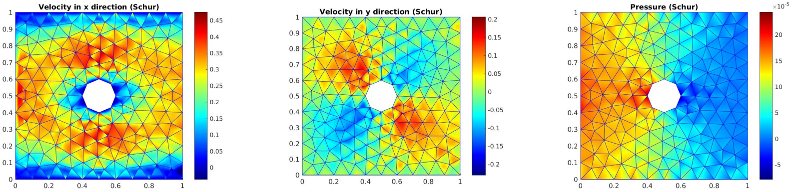
(c) $p-$convergence for pressure in L^2 norm for Stokes flow

Figure: $p-$convergence for the Stokes flow

Stokes equation: Flow around cylinder

- Domain: [0,1] $\times$ [0,1] with a cut out cylinder of diameter 0.2 centered at $(0.5, 0.5)$.
- The boundary $x = 0$ is Dirichlet boundary with inflow velocity at point $(0, y)$ as $u = (y(1 - y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (0, 0)$.

Stokes equation: Flow around cylinder



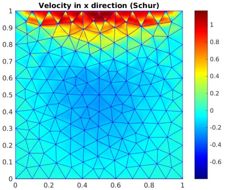
(a) $x-$velocity (b) $y-$velocity (c) Pressure

Figure: Flow over cylinder

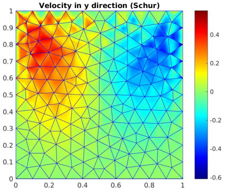
Stokes equation: Lid driven cavity

- Domain: unit square [0,1] $\times$ [0,1].
- Boundaries $x = 0, x = 1$ and $y = 0$, we impose no slip or zero velocity Dirichlet condition.
- On $y = 1$, we impose Dirichlet condition with Dirichlet velocity,

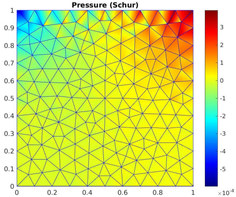
$$
\begin{aligned}
u &= (10x, 0) \quad \text{for} \quad 0 \leq x \leq 0.1, \\
u &= (1, 0) \quad \text{for} \quad 0.1 \leq x \leq 0.9, \\
u &= (10 - 10x, 0) \quad \text{for} \quad 0.9 \leq x \leq 1.
\end{aligned} \tag{41}
$$



(a) $x-$velocity

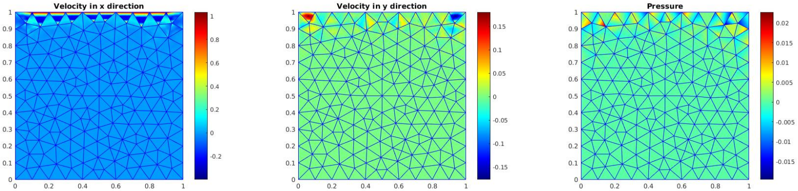


(b) $y-$velocity



(c) Pressure

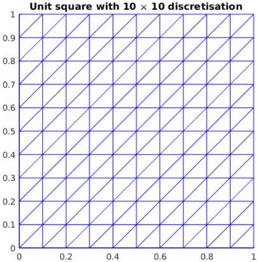
Figure: Lid driven cavity problem (Schur complement method)



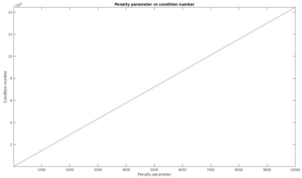
(a) $x-$velocity (b) $y-$velocity (c) Pressure

Figure: Lid driven cavity problem (*bicgstab* solver)

Penalty parameter



(a) Unit square with 10×10 discretisation



(b) Penalty parameter vs Condition number

Figure: Effect of penalty parameter on condition number of the stiffness matrix

Solver performance

Relative residual is measured as $\frac{||B - AX||_2}{||B||_2}$ for equation $AX = B$.

Solver/Method	**Relative residual**	**Run time**
Schur complement method	2.4436e-08	6.6253 Seconds
minres	2.4618e-05	35.7372 Seconds
bicgstab	9.0071e-05	58.3472 Seconds

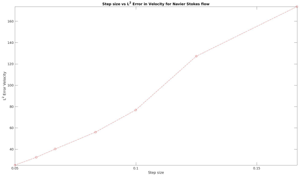
- Domain: unit square $[0,1] \times [0,1]$.
- $x = 0$ is dirichlet boundary with inflow velocity at point $(0, y)$ as $u = a * (y(1-y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- Specified source term

Analytical solution

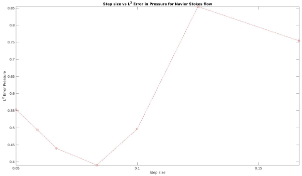
$$p = x(1-x), \tag{42}$$

$$\begin{aligned} u = (&x^2(1-y)^2(2y - 6y^2 + 4y^3), \\ &-y^2(1-y)^2(2x - 6x^2 + 4x^3)). \end{aligned} \tag{43}$$

Navier-Stokes equation: Convergence test



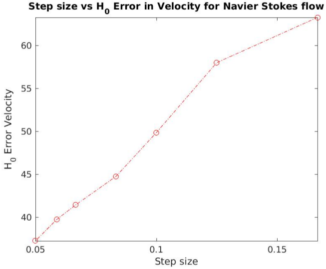
(a) $h-$convergence test for velocity L^2 error



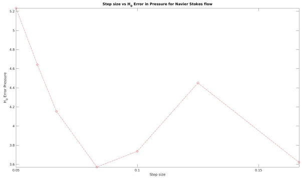
(b) $h-$convergence test for pressure in L^2 error

Figure: $h-$convergence for the Navier Stokes flow in L^2 error

Navier-Stokes equation: Convergence test



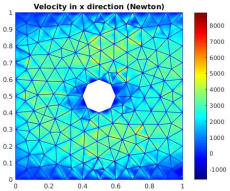
(a) $h-$convergence test for velocity H_0 error



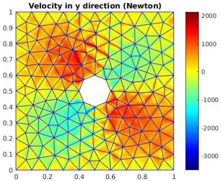
(b) $h-$convergence test for pressure in H_0 error

Figure: $h-$convergence for the Navier Stokes flow in H_0 error

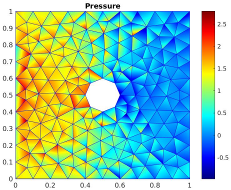
Navier-Stokes equation: Flow around cylinder



(a) $x-$velocity

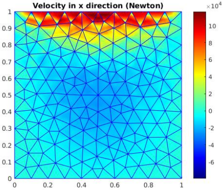


(b) $y-$velocity

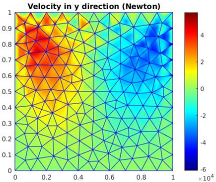


(c) Pressure

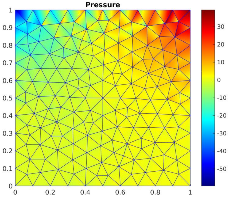
Figure: Flow over cylinder (Initial guess by Schur complement method)



(a) $x-$velocity



(b) $y-$velocity



(c) Pressure

Figure: Lid driven cavity flow (Initial guess by Schur complement method)

Conclusions

- The Schur complement : Efficient and accurate.
- *minres* slow convergence and *bicgstab* does not converge.
- Penalty parameter should be properly adjusted.
- The solvers/methods which are applicable for the Saddle point problems should be used for solving the weak form of the Stokes equation.
- The initial guess is crucial for success of the Newton method.
- The solution of the Stokes equation and the Navier Stokes equation show close to linear convergence in L^2 norm.
- The higher polynomial degree does not always guarantee better accuracy. However, the convergence rate increases with increase in polynomial degree.

Outlook

- Test for higher Reynold's number.
- Further solvers/methods.
- Time dependent cases.
- Parametrization.
- Reduced order modelling.

[1] Montlaur A., Fernandez-Mendez S., and Huerta A. Discontinuous Galerkin methods for the Stokes equations using divergence-free approximations. *International Journal for Numerical Methods in Fluids*, 57(9):1071–1092, 2008.

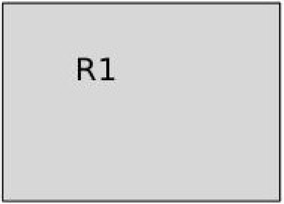
[2] Montlaur A., Fernandez-Mendez S., Peraire J., and Huerta A. Discontinuous Galerkin methods for the Navier Stokes equations using solenoidal approximations. *International Journal for Numerical Methods in Fluids*, 64(5):549–564, 2010.

[3] Riviere B. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*. Frontiers in Applied Mathematics. Cambridge University Press, 2008.

[4] White F.M. *Fluid mechanics*. Second edition, 97-99, 2002. Academic press.

[5] Kundu P. K. and Cohen I. M. *Fluid Mechanics*. Academic Press, 2002.

Grid



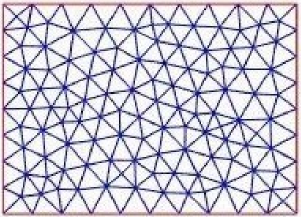


Figure: Continuous domain (left) and discretised domain or grid (right)

Grid

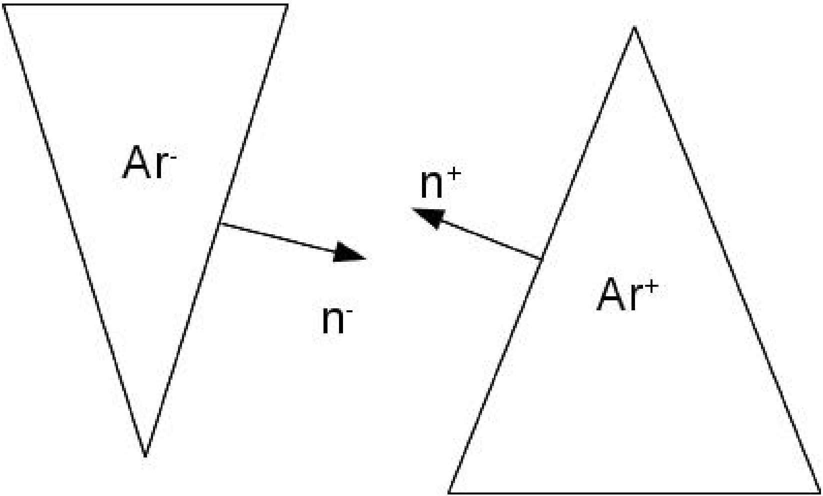


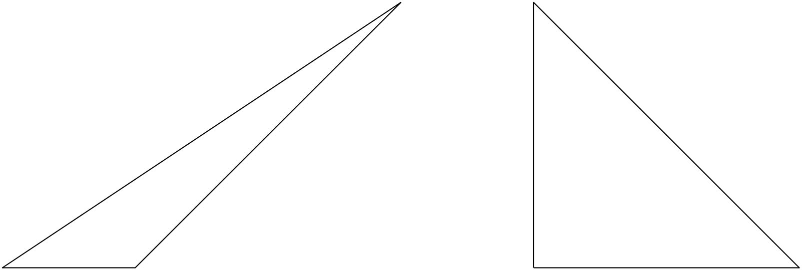
Figure: Element self (+) and neighbouring element (-)

Global and local coordinate system

- Integral terms are evaluated on a reference triangle instead of the element itself.
- The transformation from local coordinate $\hat{X}$ to global coordinate X is defined by the mapping,

$$F_k : \hat{X} \mapsto X \quad \forall \quad \hat{X} \in \hat{T} \quad \text{and} \quad X \in \mathcal{T} \tag{44}$$

$$F_k(\hat{X}) : X = J_k\hat{X} + C \tag{45}$$



Global geometry (left) to Local geometry (right)

Matrix assemblies

Assembly of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Step 1 :

$$
\begin{aligned}
res_1^{++} &= (n \otimes \hat{\phi})^{+}(n \otimes \hat{\phi})^{+}, \\
res_1^{+-} &= (n \otimes \hat{\phi})^{+}(n \otimes \hat{\phi})^{-}, \\
res_1^{-+} &= (n \otimes \hat{\phi})^{-}(n \otimes \hat{\phi})^{+}, \\
res_1^{--} &= (n \otimes \hat{\phi})^{-}(n \otimes \hat{\phi})^{-}.
\end{aligned}
\tag{46}
$$

Step 2 :

$$
\begin{aligned}
res_2^{++} &= \int_{\Gamma} res_1^{++} EL(i,j), \\
res_2^{+-} &= \int_{\Gamma} res_1^{+-} EL(i,j),
\end{aligned}
\tag{47}
$$

Assembly of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

$$\begin{aligned} res_2^{-+} &= \int_\Gamma res_1^{-+} EL(i,j), \\ res_2^{--} &= \int_\Gamma res_1^{--} EL(i,j). \end{aligned} \tag{48}$$

Step 3 :

$$\begin{aligned} res_3^{++}[ids\_velocity\_self, ids\_velocity\_self] &= res_2^{++}, \\ res_3^{+-}[ids\_velocity\_self, ids\_velocity\_neighbour] &= res_2^{+-}, \\ res_3^{-+}[ids\_velocity\_neighbour, ids\_velocity\_self] &= res_2^{-+}, \\ res_3^{--}[ids\_velocity\_neighbour, ids\_velocity\_neighbour] &= res_2^{--}. \end{aligned} \tag{49}$$

$$res_3 = res_3^{++} + res_3^{+-} + res_3^{-+} + res_3^{--}. \tag{50}$$

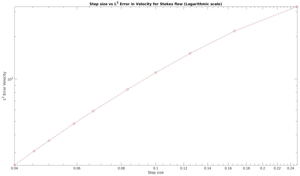
Program flow

- Grid preparation
- Fomrulating function space
- Matrix assembly
- Solving assembled form
- Post processing
- Newton method

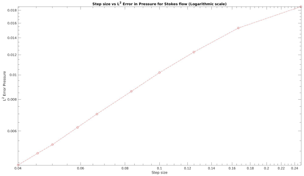
Sparsity pattern

Table: Size and sparsity pattern of different terms

Matrix term	Size	Sparsity pattern
$(\nabla\phi, \nabla\phi)$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figure **??**
$([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figure 2
$(\{\nabla\phi\}, [n \otimes \phi])_{\Gamma \cup \Gamma_D}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figure **??**
$(\{\psi\}, [n \cdot \phi])_{\Gamma \cup \Gamma_D}$	$\mathbb{R}^{p_{ndofs} \times u_{ndofs}}$	Figure **??**
$(-\int_{\hat{T}} \psi \nabla \cdot \phi)$	$\mathbb{R}^{p_{ndofs} \times u_{ndofs}}$	Figure **??**
$-((u_k \cdot \nabla)\phi, \phi)$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**
$((u_k \cdot n)\phi, \phi)_{\Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**
$((u_k \cdot n)\phi, \phi^{ext})_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**
$(\lvert u_k \cdot n \rvert \phi, \phi^{ext})_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**
$((u_k \cdot n)\phi, \phi)_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**
$(\lvert u_k \cdot n \rvert \phi, \phi)_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**



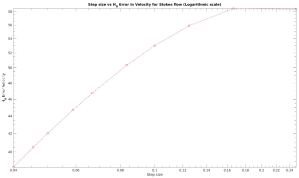
(a) $h-$convergence test for velocity L^2 error (Logarithmic scale)



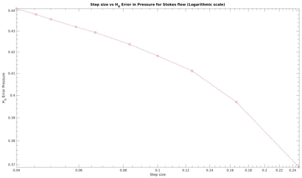
(b) $h-$convergence test for pressure in L^2 error (Logarithmic scale)

Figure: $h-$convergence in L^2 norm for the Stokes flow (Logarithmic scale)

Stokes equation: Convergence test



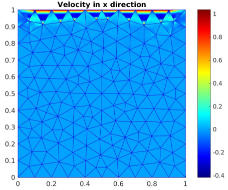
(a) $h-$convergence test for velocity H_0 error (Logarithmic scale)



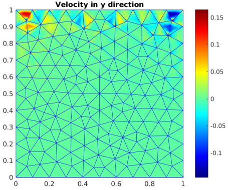
(b) $h-$convergence test for pressure in H_0 error (Logarithmic scale)

Figure: $h-$convergence in H_0 norm for the Stokes flow (Logarithmic scale)

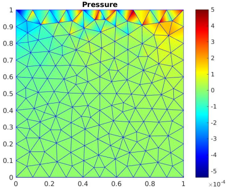
Stokes equation: Lid driven cavity



(a) $x-$velocity

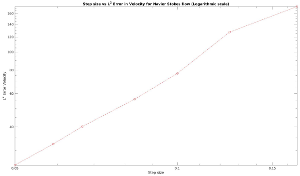


(b) $y-$velocity

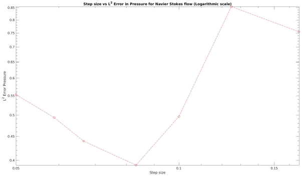


(c) Pressure

Figure: Lid driven cavity problem (*minres* solver)



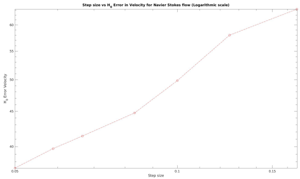
(a) $h-$convergence test for velocity L^2 error (Logarithmic scale)



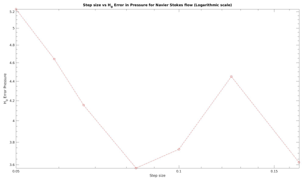
(b) $h-$convergence test for pressure in L^2 error (Logarithmic scale)

Figure: $h-$convergence for the Navier Stokes flow in L^2 error (Logarithmic scale)

Navier-Stokes equation: Convergence test

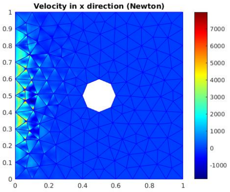


(a) $h-$convergence test for velocity H_0 error (logarithmic scale)

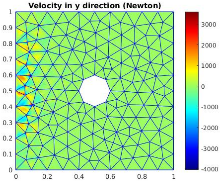


(b) $h-$convergence test for pressure in H_0 error (Logarithmic scale)

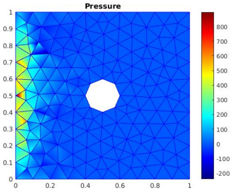
Figure: $h-$convergence for the Navier Stokes flow in H_0 error (Logarithmic scale)



(a) $x-$velocity

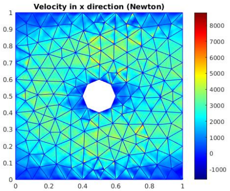


(b) $y-$velocity

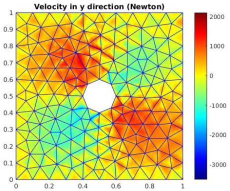


(c) Pressure

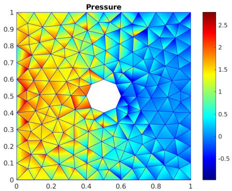
Figure: Flow over cylinder (Initial guess by *bicgstab* solver)



(a) $x-$velocity



(b) $y-$velocity



(c) Pressure

Figure: Flow over cylinder (Initial guess by minres solver)

Navier-Stokes equation: Lid driven cavity

- Domain: unit square [0,1] $\times$ [0,1].
- Boundaries $x = 0, x = 1$ and $y = 0$, we impose no slip or zero velocity Dirichlet condition.
- On $y = 1$, we impose Dirichlet condition with Dirichlet velocity,

$$
\begin{aligned}
u = a * (10x, 0) \quad &\text{for} \quad 0 \leq x \leq 0.1, \\
u = a * (1, 0) \quad &\text{for} \quad 0.1 \leq x \leq 0.9, \\
u = a * (10 - 10x, 0) \quad &\text{for} \quad 0.9 \leq x \leq 1.
\end{aligned}
\tag{51}
$$

Navier-Stokes equation: Flow around cylinder

- Domain: [0,1] $\times$ [0,1] with a cut out cylinder of diameter 0.2 centered at $(0.5, 0.5)$.
- The boundary $x = 0$ is Dirichlet boundary with inflow velocity at point $(0, y)$ as $u = a * (y(1 - y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (0, 0)$.