Discontinuous Galerkin method for direct numerical simulation of the Navier Stokes equation

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March 21, 2018

Overview

Objective

- Understanding the Stokes and the Navier-Stokes formulation and perspectives
- Derivation from conservation equation
- Discretisation and function spaces for unknowns
- Discontinuous-Galerkin formulation for the Stokes and the Navier-Stokes equation
- Implementation and matrix assembly
- Sparsity pattern and solver selection
- Numerical experiment for the Stokes equation and the Navier-Stokes equation
- Future perspectives

Importance of the Navier Stokes equation

- Computational fluid dynamics $\implies$ One of the variants of the Navier Stokes equation
- Navier Stokes equation involves state variables
- Incompressible condition $\implies$ state variables constant $\implies$ Equation of state not required
- Solved along with continuity equation
- Depends on time (Unsteady fluid flow) or independent of time (Steady fluid flow)
- Non linear coupled system of equations
- Stokes equation is linearized form of the Navier Stokes equation

Notations

Ω = Continuous domain,
Γ_D = Dirichlet boundary,
Γ_N = Neumann boundary,
cv = Control volume,
cs = Control surface,
B' = Extensive quantity under consideration,
b' = Intensive quantity corresponding to B',
u = flow velocity and $u : \Omega \to \mathbb{R}^d$,
p = pressure and $p : \Omega \to \mathbb{R}$,
ν = kinematic viscocity (fluid property) and $\nu : \Omega \to \mathbb{R}$,
f = external force and $f : \Omega \to \mathbb{R}^d$,
u_D = specified flow velocity at Dirichlet boundary and $u_D : \Gamma_D \to \mathbb{R}^d$,
n = normal unit vector and $n : \partial\Omega \to \mathbb{R}^d$,
ρ = density (fluid property) and $\rho : \Omega \to \mathbb{R}$,
t = specified Neumann flux and $t : \Gamma_N \to \mathbb{R}^d$.

Governing equations

Reynolds transport theoreom

$$\frac{dB'}{dt'}|_{cs} = \frac{d}{dt'}\int_{cv} b'\rho dV + \int_{cs} (b'\rho) u \cdot dA \quad (1)$$

Momentum conservation equation

$$F = \frac{dM}{dt'} = \frac{d}{dt'}\int_{cv} u\rho dV + \int_{cs} (u\rho) u \cdot dA. \quad (2)$$

$$F = \int_{cs} \sigma \cdot dA + \int_{cv} \rho f dV, \quad (3)$$

Governing equations

Momentum conservation equation

$$F = \frac{dM}{dt'} = \frac{d}{dt'} \int_{cv} u \rho dV + \int_{cs} (u\rho) u \cdot dA. \tag{4}$$

$$F = \int_{cs} \sigma \cdot dA + \int_{cv} \rho f dV, \tag{5}$$

Governing equations

Navier Stokes equation

$$-2\nabla \cdot (\nu \nabla^s u) + (1/\rho)\nabla p + (u \cdot \nabla) u = f \quad \text{in} \quad \Omega. \tag{6}$$

Dirichlet boundary:

$$u = u_D \quad \text{on} \quad \Gamma_D. \tag{7}$$

Neumann boundary:

$$-pn + 2\nu(n \cdot \nabla^s) u = t \quad \text{on} \quad \Gamma_N. \tag{8}$$

Continuity equation

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega. \tag{9}$$

Stokes equation

$$-2\nabla \cdot (\nu \nabla^s u) + (1/\rho)\nabla p = f \quad \text{in} \quad \Omega. \tag{10}$$

Flow classification

Reynolds number

$$Re = \frac{uL}{\nu}. \tag{11}$$

Laminar flow

- Well defined velocity and pressure profile.
- Low Reynolds number.

Turbulent flow

- Fluctuations in velocity and pressure.
- Fluctuations are of the order of Kolmogrov scale.
- Low Reynolds number.

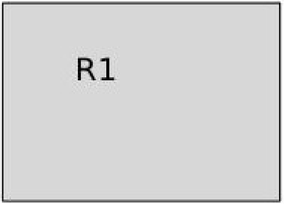
Grid

- Continuous domain $(\Omega) \implies$ Grid $(\mathcal{T})$
- Triangular element, τ_k, $\cup_{k=1}^{nel} \tau_k = \mathcal{T}$, *nel* is total number of elements
- Grid boundary includes interelement boundaries.

$$\partial \mathcal{T} = \Gamma_D \cup \Gamma_N \cup \Gamma \tag{12}$$

- Grid parameters : Parameters for formulation of problem
- Unstructured grid : Connectivity of vertices, Element

Grid



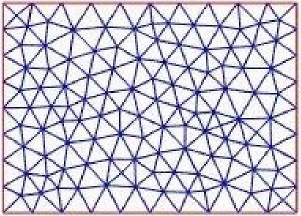


Figure: Continuous domain (left) and discretised domain or grid (right)

Grid

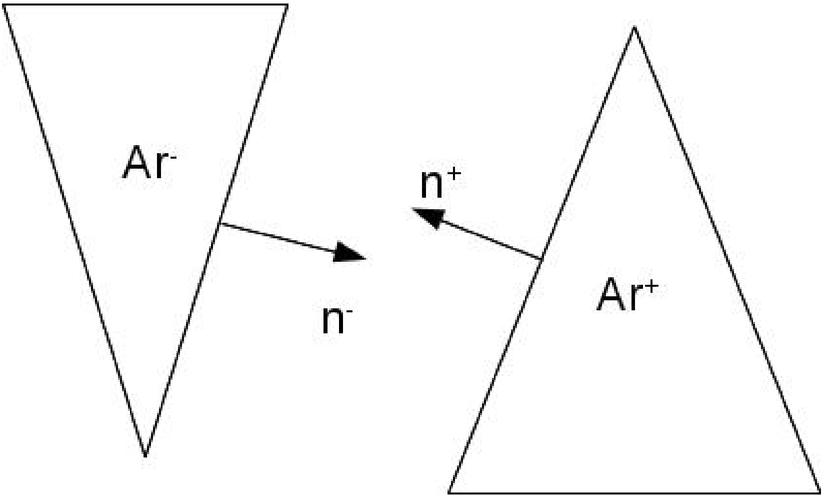


Figure: Element self (+) and neighbouring element (-)

Grid

Barycentric coordinate

For a triangle with vertices, r_1, r_2, r_3

$$r = \lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 \quad . \tag{13}$$

Barycentric coordinate

Weights, $\lambda_1, \lambda_2, \lambda_3$ satisfy,

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 \quad . \tag{14}$$

Discontinuous Galerkin method

- Multiply with test function and integrate
- Discontinuous at the interface of elements
- $P^D(\tau_k)$ denotes space of polynomials of degree at most D over τ_k.

Function space for velocity

$$\mathbb{V} = \{\phi \in (L^2(\mathcal{T}))^{d_u} | \quad \phi \in (P^D(\tau_k))^{d_u} \quad \forall \quad \tau_k \in \mathcal{T}\} \tag{15}$$

Function space for pressure

$$\mathbb{Q} = \{\psi \in (L^2(\mathcal{T}))^{d_p} | \quad \psi \in (P^{D-1}(\tau_k))^{d_p} \quad \forall \quad \tau_k \in \mathcal{T}\} \tag{16}$$

Basis function

- 2 Types of basis functions : Nodal basis function and Orthonormal basis function
- The number of degrees of freedom per element *npe* can be calculated as,

$$u_{npe} = d_u \frac{(D+1)(D+2)}{2} \quad \text{for velocity} \tag{17}$$

$$p_{npe} = d_p \frac{(D)(D+1)}{2} \quad \text{for pressure} \tag{18}$$

Orthonormal basis function

- Basis functions are orthonormal to each other with respect to suitable inner product.

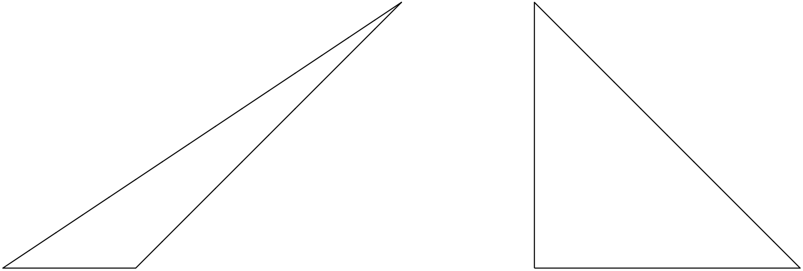
$$
\begin{aligned}
(\hat{\phi}_i, \hat{\phi}_j) &= \int_{\hat{T}} \hat{\phi}_i \hat{\phi}_j = 1 \quad \text{if} \quad i = j \\
(\hat{\phi}_i, \hat{\phi}_j) &= \int_{\hat{T}} \hat{\phi}_i \hat{\phi}_j = 0 \quad \text{if} \quad i \neq j
\end{aligned}
\tag{19}
$$

Global and local coordinate system

- Integral terms are evaluated on a reference triangle instead of the element itself.
- The transformation from local coordinate $\hat{X}$ to global coordinate X is defined by the mapping,

$$F_k : \hat{X} \mapsto X \quad \forall \quad \hat{X} \in \hat{T} \quad \text{and} \quad X \in \mathcal{T} \tag{20}$$

$$F_k(\hat{X}) : X = J_k \hat{X} + C \tag{21}$$



Global geometry (left) to Local geometry (right)

- The volume integral of a function $g(x)$ in global coordinates is related to volume integral on reference geometry as

$$\int_{\Omega} g(x)dx = \sum_{k=1}^{nel} \int_{\tau_k} g(x)dx = \sum_{k=1}^{nel} \int_{\hat{T}} g(\hat{x})|\det(J_k)|d\hat{x} \qquad (22)$$

- The linear boundary integral of a function $g(x)$ on global coordinates is related to boundary integral on reference geometry as,

$$\int_{\Gamma} g(x)ds = \int_{\hat{\Gamma}} g(\hat{x})ld\hat{s} \qquad (23)$$

- Also, the following holds,

$$\nabla g = JIT_k \quad \hat{\nabla}\hat{g} \qquad (24)$$

Jump operator, Average operator and L^2 scalar product

Jump operator

$$[pn] = p^+ n^+ + p^- n^- \text{ on } \Gamma$$
$$[pn] = pn \text{ on } \Gamma_D$$
where p is scalar.

$$[n \otimes u] = n^+ \otimes u^+ + n^- \otimes u^- \text{ on } \Gamma$$
$$[n \otimes u] = n \otimes u \text{ on } \Gamma_D$$
$$[n \cdot u] = n^+ \cdot u^+ + n^- \cdot u^- \text{ on } \Gamma$$
$$[n \cdot u] = n \cdot u \text{ on } \Gamma_D$$
where u is vector and $n \otimes u = u_i n_j$, $1 \leq i \leq d_u$, $1 \leq j \leq d$

Average operator

The average operator is defined as,

$$\{u\} = \frac{u^+ + u^-}{2} \quad . \tag{25}$$

Jump operator, Average operator and L^2 scalar product

L^2 scalar product

If p and q are scalars,

$$(p, q) = \int_\Omega pq \quad . \tag{26}$$

If p and q are vectors,

$$(p, q) = \int_\Omega p \cdot q \quad . \tag{27}$$

If p and q are tensors,

$$(p, q) = \int_\Omega p : q \quad \text{where} \quad p : q = Tr(pq^T) \quad . \tag{28}$$

Stokes equation

Strong form

$$-\nu\Delta u + \nabla p = f \quad \text{in} \quad \Omega \quad . \tag{29}$$

Weak form

$$a_{IP}(u,\phi) + b(\phi,p) + (\{p\},[n\cdot\phi])_{\Gamma\cup\Gamma_D} = l_{IP}(\phi) \quad . \tag{30}$$

$$\begin{aligned} a_{IP}(u,\phi) &= (\nabla u,\nabla\phi) + C_{11}([n\otimes u],[n\otimes\phi])_{\Gamma\cup\Gamma_D} \\ -\nu(\{\nabla u\},[n\otimes\phi])_{\Gamma\cup\Gamma_D} &- \nu([n\otimes u],\{\nabla\phi\})_{\Gamma\cup\Gamma_D} \quad . \end{aligned} \tag{31}$$

$$b(\phi,\psi) = -\int_{\mathcal{T}} \psi\nabla\cdot\phi \tag{32}$$

Stokes equation

Weak form

$$l_{IP}(\phi) = (f, \phi) + (t, \phi)_{\Gamma_N} + C_{11}(u_D, \phi)_{\Gamma_D} - (n \otimes u_D, \nu \nabla \phi)_{\Gamma_D} \tag{34}$$

Discrete form

$$AU + BP = F_1 \quad . \tag{35}$$

Continuity equation

Strong form

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega \quad . \tag{36}$$

Weak form

$$b(u, \psi) + (\{\psi\}, [n \cdot u])_{\Gamma \cup \Gamma_D} = (q, n \cdot u_D)_{\Gamma_D} \quad . \tag{37}$$

Discrete form

$$B^T U = F_2 \quad . \tag{38}$$

Saddle point problem

$$\underbrace{\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}}_{\text{Stiffness matrix}} \underbrace{\begin{pmatrix} U \\ P \end{pmatrix}}_{\text{Solution vector}} = \underbrace{\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}}_{\text{Right hand side (Known)}} \tag{39}$$

Here, $(\cdot, \cdot)$ is L^2 inner product, $\{\cdot\}$ is average operator, $[\cdot]$ is jump operator.

Stiffness matrix

Matrix A

$$A_{ij} = \sum_{k=1}^{d}(\frac{\partial\phi_i}{\partial x_k}, \frac{\partial\phi_j}{\partial x_k}) + C_{11}\sum_{k=1}^{d}([\phi_i n_k], [\phi_j n_k])_{\Gamma\cup\Gamma_D}$$
$$-\nu\sum_{k=1}^{d}([\phi_i n_k], \{\frac{\partial\phi_j}{\partial x_k}\})_{\Gamma\cup\Gamma_D} - \nu\sum_{k=1}^{d}(\{\frac{\partial\phi_i}{\partial x_k}\}, [\phi_j n_k])_{\Gamma\cup\Gamma_D} \quad (40)$$

Matrix B

$$B_{ij} = -\int_{\mathcal{T}} \frac{\partial\phi_i}{\partial x_i}\psi_j + (\{\psi_j\}, [n\cdot\phi_i])_{\Gamma\cup\Gamma_D} \quad (41)$$

Navier Stokes equation

Strong form

$$-\nu\Delta u + \nabla p + (u\cdot\nabla)u = f \quad \text{in} \quad \Omega \quad . \tag{42}$$

Weak form

$$a_{IP}(u,\phi) + c(u;u,\phi) + b(\phi,p) + (\{p\},[n\cdot\phi])_{\Gamma\cup\Gamma_D} = l_{IP}(\phi) \tag{43}$$

Discrete form

$$AU + C(U)U + BP = F \tag{44}$$

Navier Stokes equation : Upwinding

- Upwinding is the method used to discretise the convective term.
- If n_τ is the unit normal from τ_1 to τ_2 and if we denote the upwind value of function g as g^{up} [1],

$$\begin{aligned} g^{up} &= g|_{\tau_1} \quad \text{if} \quad g \cdot n_\tau \geq 0 \\ g^{up} &= g|_{\tau_2} \quad \text{if} \quad g \cdot n_\tau < 0 \end{aligned} \tag{45}$$

- For the weak form of the Navier Stokes equation,

$$c(g; u, \phi) = \sum_{i=1}^{nel} \int_{\partial\Omega_i \setminus \Gamma_N} \frac{1}{2}[[(g \cdot n_i)(u^{ext} + u) - |g \cdot n_i|(u^{ext} - u)]] \cdot \phi + \int_{\Gamma_N} (g \cdot n)u \cdot \phi - ((g \cdot \nabla)\phi, u) \quad . \tag{46}$$

$$u^{ext} = \lim_{\epsilon \to 0} u(x + \epsilon n_i) \quad \text{for} \quad x \in \partial\mathcal{T}_i \tag{47}$$

Newton method

$$S(u) = a(u,\phi) + b(\phi,p) + (\{p\},[n\cdot\phi])_{\Gamma\cup\Gamma_D} - l_{IP}(\phi) \tag{48}$$

$$\begin{aligned} S(u+h) - S(u) = (a(u+\delta h,\phi) + c(u+\delta h; u+\delta h,\phi) \\ +b(\phi,p+\delta h') + (\{p+\delta h'\},[n\cdot\phi])_{\Gamma\cup\Gamma_D} - l_{IP}(\phi)) - (a(u,\phi) \\ +c(u,u,\phi) + b(\phi,p) + (\{p\},[n\cdot\phi])_{\Gamma\cup\Gamma_D} - l_{IP}(\phi)) \end{aligned} \tag{49}$$

$$\begin{aligned} S(u+h) - S(u) = 2\delta c(u,h,\cdot) + \delta^2 c(h,h,\cdot) + \delta a(h,\cdot) \\ +\delta b(h',\cdot) + \delta(\{h'\},[n\cdot\phi])_{\Gamma\cup\Gamma_D} \end{aligned} \tag{50}$$

Newton method

$$DS(u) = \lim_{\delta \to 0} \frac{S(u+h) - S(u)}{\delta} \tag{51}$$

$$DS(u) = 2c(u, h, \cdot) + a(h, \cdot) + b(h', \cdot) + (\{h'\}, [n \cdot \phi])_{\Gamma \cup \Gamma_D} \tag{52}$$

Following similar procedure we write for continuity equation:

$$S'(u) = b(u, \psi) + (\{\psi\}, [n \cdot u])_{\Gamma \cup \Gamma_D} - (\psi, n \cdot u_D)_{\Gamma_D} \tag{53}$$

$$S'(u + \delta h) = b(u + \delta h, \psi) + (\{\psi\}, [n \cdot u + \delta h])_{\Gamma \cup \Gamma_D} - (\psi, n \cdot u_D)_{\Gamma_D} \tag{54}$$

$$DS'(u) = b(\delta h, \psi) + (\{\psi\}, [n \cdot \delta h])_{\Gamma \cup \Gamma_D} \tag{55}$$

Newton method

Algotrithm for the Newton method is as follow:

1. Select $u^{iter} \in \mathbb{V}$ at iteration *iter*,
2. Verify $DS_{u^{iter}}(h^{iter}) = -S(u^{iter})$,
3. Set $u^{iter+1} := u^{iter} + h^{iter}$ till $||u^{iter+1} - u^{iter}|| < tol$ where *tol* is specified tolerance.

We use the solution from the Stokes equation as initial guess. In discrete form the Newton method means, solving the equation (at iteration $=$ *iter*)

$$\underbrace{\begin{pmatrix} A + C(U^{iter}) & B \\ B^T & 0 \end{pmatrix}}_{\text{Stiffness matrix}^{iter}} \underbrace{\begin{pmatrix} U^{iter+1} \\ P^{iter+1} \end{pmatrix}}_{\text{Solution vector}^{iter+1}} = \underbrace{\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}}_{\text{Right hand side (Known functi}} \tag{56}$$

Schur complement method

STEP 1:

$$U = A^{-1}(F_1 - BP) \tag{57}$$

STEP 2 :

$$-B^T A^{-1} BP = F_2 - B^T A^{-1} F_1 \tag{58}$$

STEP 3 :

Backsubstituion in equation (57)

Data types

- `params` : Structure with fields u_{npe}, u_{ndofs}, d_u, D, U.
- `paramsP` : Structure with fields p_{npe}, p_{ndofs}, d_p, $D{-}1$, P.
- `grid` : Structure containing informations related to grid.

Basis function in RBmatlab

$$\phi \in \mathbb{R}^{u_{npe} \times d_u}, \tag{59}$$

$$\psi \in \mathbb{R}^{p_{npe} \times d_p}. \tag{60}$$

- The derivative of basis function $(\phi)_i$, where $1 \leq i \leq u_{npe}$ and $(\psi)_i$, where $1 \leq i \leq p_{npe}$ is cell.

Basis function in RBmatlab

$$\nabla(\phi)_i \in \mathbb{R}^{d_u \times d}. \tag{61}$$

$$\nabla(\psi)_i \in \mathbb{R}^{d_p \times d}. \tag{62}$$

Matrix assemblies : Steps

- Function evaluation
- Integration and transform from local to global geometry
- Allocation in global matrix / vector

Assembly of $(\nabla\phi, \nabla\phi)$

Step 1 :

$$res_1[i,j] = \nabla\phi_i \nabla\phi_j{}^T \quad \text{for} \quad 1 \leq i,j \leq u_{npe}. \tag{63}$$

Step 2 :

$$res_2 = \int_{\hat{T}} (res_1)(2Ar(k)). \tag{64}$$

Step 3 :

$$res_3[ids\_velocity, ids\_velocity] = res_2. \tag{65}$$

Matrix assemblies : Jump operator and average operator

Jump operator

$$[A_h \cdot n], [B_h \cdot n] = A_h^+ n^+ B_h^+ n^+ + A_h^+ n^+ B_h^- n^- + A_h^- n^- B_h^+ n^+ + A_h^- n^- B_h^- n^-, \tag{66}$$

Average operator

$$\{A_h\} = \frac{(A_h^+ + A_h^-)}{2}. \tag{67}$$

Matrix assemblies

Assembly of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Step 1 :

$$\begin{aligned} res_1^{++} &= (n \otimes \hat{\phi})^+ (n \otimes \hat{\phi})^+, \\ res_1^{+-} &= (n \otimes \hat{\phi})^+ (n \otimes \hat{\phi})^-, \\ res_1^{-+} &= (n \otimes \hat{\phi})^- (n \otimes \hat{\phi})^+, \\ res_1^{--} &= (n \otimes \hat{\phi})^- (n \otimes \hat{\phi})^-. \end{aligned} \tag{68}$$

Step 2 :

$$\begin{aligned} res_2^{++} &= \int_\Gamma res_1^{++} EL(i,j), \\ res_2^{+-} &= \int_\Gamma res_1^{+-} EL(i,j), \end{aligned} \tag{69}$$

Matrix assemblies

Assembly of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

$$
\begin{aligned}
res_2^{-+} &= \int_\Gamma res_1^{-+} EL(i,j), \\
res_2^{--} &= \int_\Gamma res_1^{--} EL(i,j).
\end{aligned} \tag{70}
$$

Step 3 :

$$
\begin{aligned}
res_3^{++}[ids\_velocity\_self, ids\_velocity\_self] &= res_2^{++}, \\
res_3^{+-}[ids\_velocity\_self, ids\_velocity\_neighbour] &= res_2^{+-}, \\
res_3^{-+}[ids\_velocity\_neighbour, ids\_velocity\_self] &= res_2^{-+}, \\
res_3^{--}[ids\_velocity\_neighbour, ids\_velocity\_neighbour] &= res_2^{--}.
\end{aligned} \tag{71}
$$

$$
res_3 = res_3^{++} + res_3^{+-} + res_3^{-+} + res_3^{--}. \tag{72}
$$

Program flow

- Grid preparation
- Fomrulating function space
- Matrix assembly
- Solving assembled form
- Post processing
- Newton method

Sparsity pattern

Table: Size and sparsity pattern of different terms

Matrix term	**Size**	**Sparsity pattern**
$(\nabla\phi, \nabla\phi)$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figure 3
$([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figure 5
$(\{\nabla\phi\}, [n \otimes \phi])_{\Gamma \cup \Gamma_D}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figure **??**
$(\{\psi\}, [n \cdot \phi])_{\Gamma \cup \Gamma_D}$	$\mathbb{R}^{p_{ndofs} \times u_{ndofs}}$	Figure **??**
$(-\int_{\hat{T}} \psi \nabla \cdot \phi)$	$\mathbb{R}^{p_{ndofs} \times u_{ndofs}}$	Figure **??**
$-((u_k \cdot \nabla)\phi, \phi)$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**
$((u_k \cdot n)\phi, \phi)_{\Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**
$((u_k \cdot n)\phi, \phi^{ext})_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**
$(\lvert u_k \cdot n \rvert \phi, \phi^{ext})_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**
$((u_k \cdot n)\phi, \phi)_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**
$(\lvert u_k \cdot n \rvert \phi, \phi)_{\partial T \setminus \Gamma_N}$	$\mathbb{R}^{u_{ndofs} \times u_{ndofs}}$	Figures **??**

Sparsity pattern

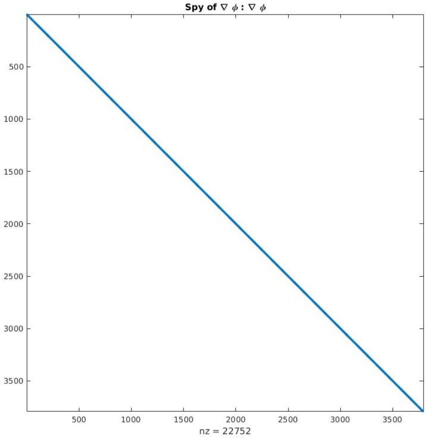
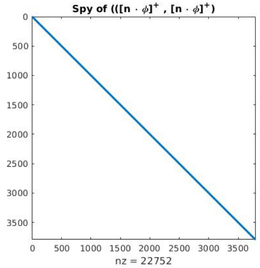
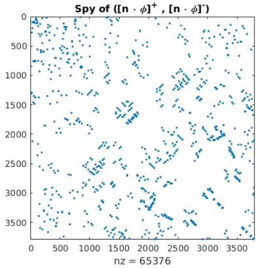


Figure: Sparsity pattern of $(\nabla\phi, \nabla\phi)$

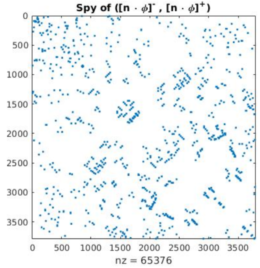
Sparsity pattern

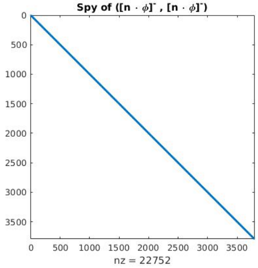


(a) $((n \otimes \phi)^+, (n \otimes \phi)^+)_{\Gamma \cup \Gamma_D}$

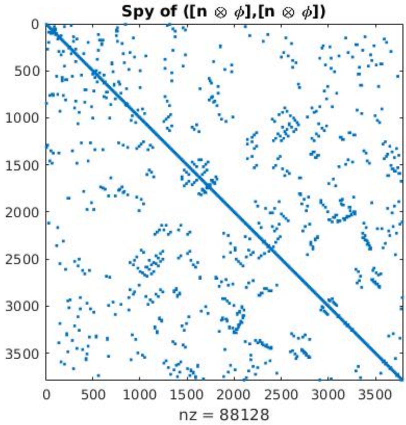


(b) $((n \otimes \phi)^+, (n \otimes \phi)^-)_{\Gamma \cup \Gamma_D}$





Sparsity pattern



(a) $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Figure: Sparsity pattern of constituents of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Error definition

- If P_h is the computed solution and P is the true solution, we define following errors:

L^2 error

$$P_{error,L^2} = \left(\int_{\Omega} |P - P_h|^2\right)^{\frac{1}{2}}. \tag{73}$$

H_0 error

$$P_{error,H_0} = \sum_{k=1}^{nel}\left(\int_{\tau_k} |\nabla P - \nabla P_h|^2\right)^{\frac{1}{2}}. \tag{74}$$

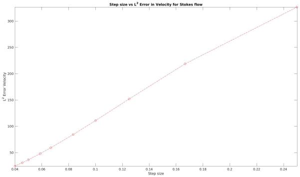
Stokes equation: Convergence test

- Domain: unit square [0,1] $\times$ [0,1].
- $x = 0$ is dirichlet boundary with inflow velocity at point $(0, y)$ as $u = (y(1-y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (2\nu - 1, 0)$.

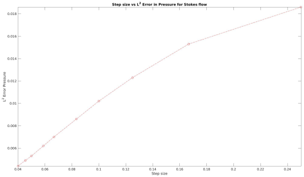
Analytical solution

$$p = (1 - x), \tag{75}$$

$$u = (y(1-y), 0). \tag{76}$$



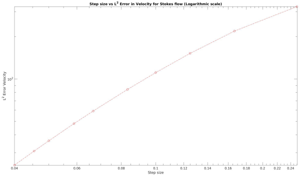
(a) $h-$convergence test for velocity L^2 error



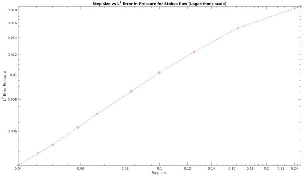
(b) $h-$convergence test for pressure in L^2 error

Figure: $h-$convergence test for pressure in L^2 error

Stokes equation: Convergence test



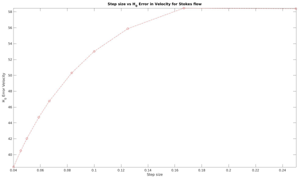
(a) $h-$convergence test for velocity L^2 error (Logarithmic scale)



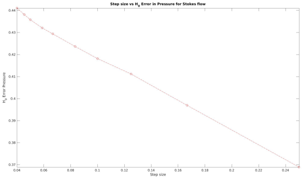
(b) $h-$convergence test for pressure in L^2 error (Logarithmic scale)

Figure: $h-$convergence in L^2 norm for the Stokes flow (Logarithmic scale)

Stokes equation: Convergence test



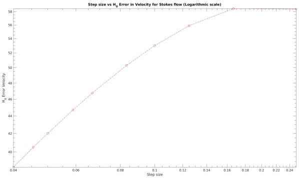
(a) $h-$convergence test for velocity H_0 error



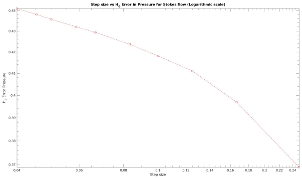
(b) $h-$convergence test for pressure in H_0 error

Figure: $h-$convergence in H_0 norm for the Stokes flow

Stokes equation: Convergence test

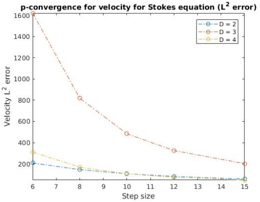


(a) $h-$convergence test for velocity H_0 error (Logarithmic scale)

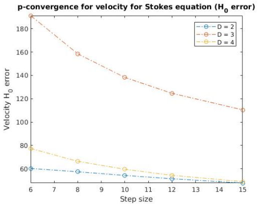


(b) $h-$convergence test for pressure in H_0 error (Logarithmic scale)

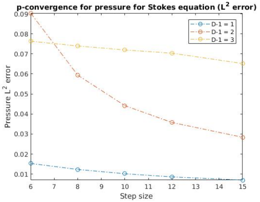
Figure: $h-$convergence in H_0 norm for the Stokes flow (Logarithmic scale)



(a) $p-$convergence for velocity in L^2 norm for Stokes flow



(b) $p-$convergence for velocity in H_0 semi norm for Stokes flow



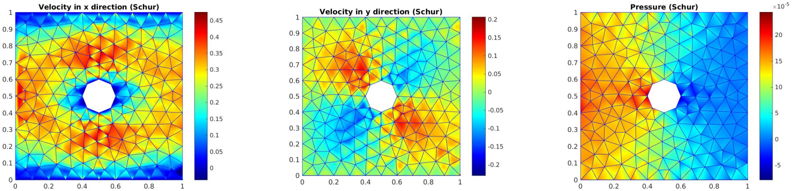
(c) $p-$convergence for pressure in L^2 norm for Stokes flow

Figure: $p-$convergence for the Stokes flow

Stokes equation: Flow around cylinder

- Domain: [0,1] $\times$ [0,1] with a cut out cylinder of diameter 0.2 centered at $(0.5, 0.5)$.
- The boundary $x = 0$ is Dirichlet boundary with inflow velocity at point $(0, y)$ as $u = (y(1-y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (0, 0)$.

Stokes equation: Flow around cylinder



(a) $x-$velocity (b) $y-$velocity (c) Pressure

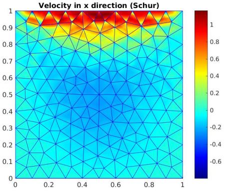
Figure: Flow over cylinder

Stokes equation: Lid driven cavity

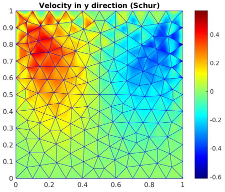
- Domain: unit square [0,1] $\times$ [0,1].
- Boundaries $x = 0, x = 1$ and $y = 0$, we impose no slip or zero velocity Dirichlet condition.
- On $y = 1$, we impose Dirichlet condition with Dirichlet velocity,

$$\begin{aligned} u &= (10x, 0) \quad \text{for} \quad 0 \leq x \leq 0.1, \\ u &= (1, 0) \quad \text{for} \quad 0.1 \leq x \leq 0.9, \\ u &= (10 - 10x, 0) \quad \text{for} \quad 0.9 \leq x \leq 1. \end{aligned} \tag{77}$$

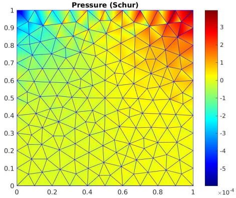
Stokes equation: Lid driven cavity



(a) $x-$velocity

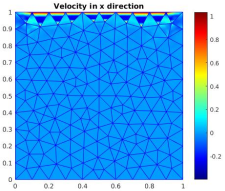


(b) $y-$velocity

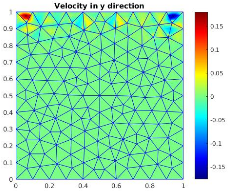


(c) Pressure

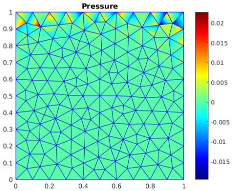
Figure: Lid driven cavity problem (Schur complement method)



(a) $x-$velocity

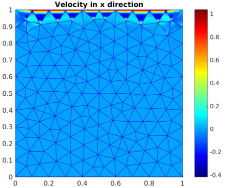


(b) $y-$velocity

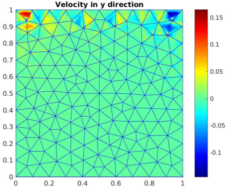


(c) Pressure

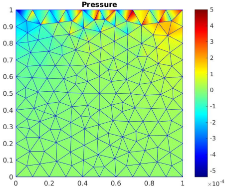
Figure: Lid driven cavity problem (*bicgstab* solver)



(a) $x-$velocity



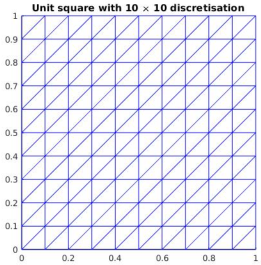
(b) $y-$velocity



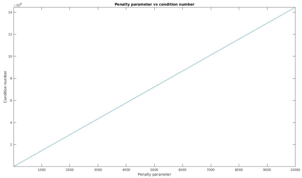
(c) Pressure

Figure: Lid driven cavity problem (*minres* solver)

Penalty parameter



(a) Unit square with 10×10 discretisation



(b) Penalty parameter vs Condition number

Figure: Effect of penalty parameter on condition number of the stiffness matrix

Solver performance

Relative residual is measured as $\frac{||B-AX||_2}{||B||_2}$ for equation $AX = B$.

Solver/Method	**Relative residual**	**Run time**
Schur complement method	2.4436e-08	6.6253 Seconds
minres	2.4618e-05	35.7372 Seconds
bicgstab	9.0071e-05	58.3472 Seconds

Navier-Stokes equation: Convergence test

- Domain: unit square [0,1] $\times$ [0,1].
- $x = 0$ is dirichlet boundary with inflow velocity at point $(0, y)$ as $u = a * (y(1 - y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.

Navier-Stokes equation: Convergence test

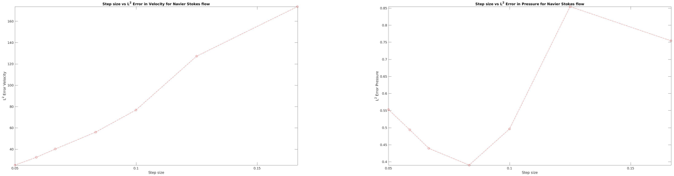
Source term

$$f = \begin{pmatrix} -4\nu(1+2y)(y^2 - 6xy^2 + 6x^2y^2 - y + 6xy \\ -6x^2y + 3x^2 - 6x^3 + 3x^4) + 1 - 2x \\ +4x^3y^2(2y^2 - 2y + 1)(y-1)^2(-1+2x)(x-1)^3, \\ 4\nu(-1+2x)(x^2 - 6x^2y + 6x^2y^2 - x + 6xy \\ -6xy^2 + 3y^2 - 6y^3 + 3y^4) + \\ 4x^2y^3(-1+2y)(y-1)^3(2x^2 - 2x + 1)(x-1)^2 \end{pmatrix}. \tag{78}$$

Analytical solution

$$p = x(1-x), \tag{79}$$

$$u = \begin{pmatrix} x^2(1-y)^2(2y - 6y^2 + 4y^3), \\ -y^2(1-y)^2(2x - 6x^2 + 4x^3) \end{pmatrix}. \tag{80}$$

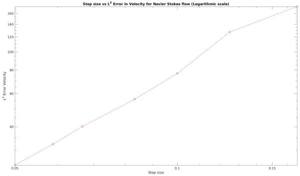


(a) $h-$convergence test for velocity L^2 error

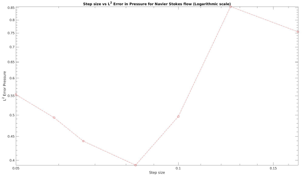
(b) $h-$convergence test for pressure in L^2 error

Figure: $h-$convergence for the Navier Stokes flow in L^2 error

Navier-Stokes equation: Convergence test



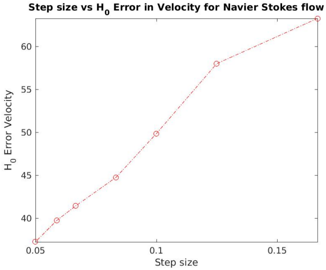
(a) $h-$convergence test for velocity L^2 error (Logarithmic scale)



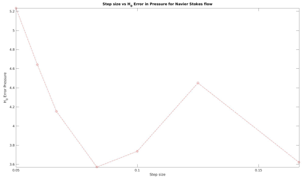
(b) $h-$convergence test for pressure in L^2 error (Logarithmic scale)

Figure: $h-$convergence for the Navier Stokes flow in L^2 error (Logarithmic scale)

Navier-Stokes equation: Convergence test



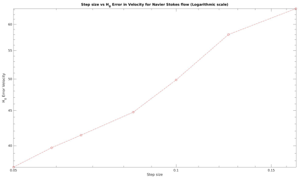
(a) $h-$convergence test for velocity H_0 error



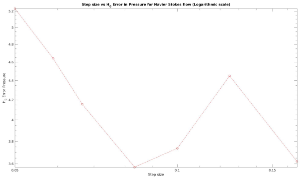
(b) $h-$convergence test for pressure in H_0 error

Figure: $h-$convergence for the Navier Stokes flow in H_0 error

Navier-Stokes equation: Convergence test



(a) $h-$convergence test for velocity H_0 error (logarithmic scale)



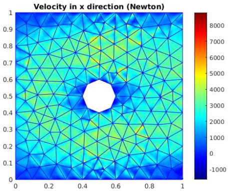
(b) $h-$convergence test for pressure in H_0 error (Logarithmic scale)

Figure: $h-$convergence for the Navier Stokes flow in H_0 error (Logarithmic scale)

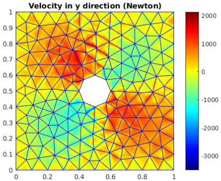
Navier-Stokes equation: Flow around cylinder

- Domain: [0,1] $\times$ [0,1] with a cut out cylinder of diameter 0.2 centered at $(0.5, 0.5)$.
- The boundary $x = 0$ is Dirichlet boundary with inflow velocity at point $(0, y)$ as $u = a * (y(1 - y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (0, 0)$.

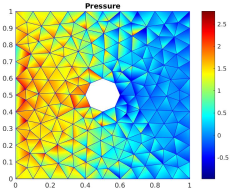
Navier-Stokes equation: Flow around cylinder



(a) $x-$velocity

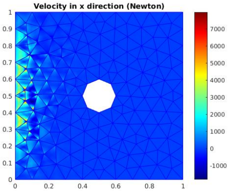


(b) $y-$velocity

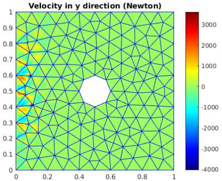


(c) Pressure

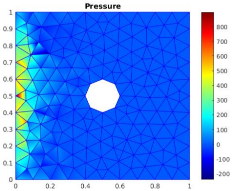
Figure: Flow over cylinder (Initial guess by Schur complement method)



(a) $x-$velocity

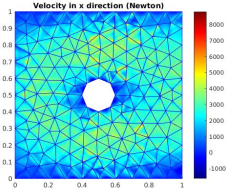


(b) $y-$velocity

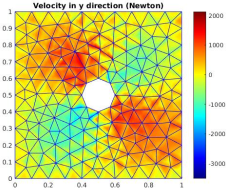


(c) Pressure

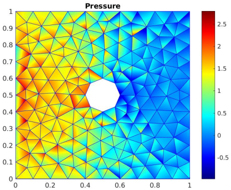
Figure: Flow over cylinder (Initial guess by *bicgstab* solver)



(a) $x-$velocity



(b) $y-$velocity



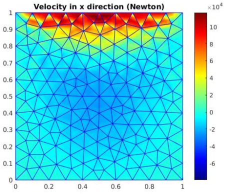
(c) Pressure

Figure: Flow over cylinder (Initial guess by minres solver)

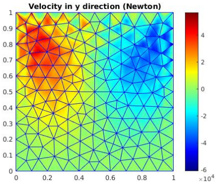
Navier-Stokes equation: Lid driven cavity

- Domain: unit square [0,1] $\times$ [0,1].
- Boundaries $x = 0, x = 1$ and $y = 0$, we impose no slip or zero velocity Dirichlet condition.
- On $y = 1$, we impose Dirichlet condition with Dirichlet velocity,

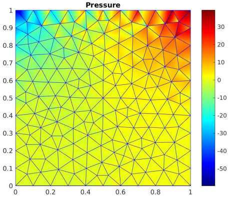
$$\begin{aligned} u = a * (10x, 0) \quad &\text{for} \quad 0 \leq x \leq 0.1, \\ u = a * (1, 0) \quad &\text{for} \quad 0.1 \leq x \leq 0.9, \\ u = a * (10 - 10x, 0) \quad &\text{for} \quad 0.9 \leq x \leq 1. \end{aligned} \tag{81}$$



(a) $x-$velocity



(b) $y-$velocity



(c) Pressure

Figure: Lid driven cavity flow (Initial guess by Schur complement method)

Conclusions

- The Schur complement : Efficient and accurate
- *minres* slow convergence and *bicgstab* does not converge
- Penalty parameter
- The solvers/methods which are applicable for the Saddle point problems should be used for solving the weak form of the Stokes equation.
- The initial guess is crucial for success of the Newton method.
- The solution of the Stokes equation and the Navier Stokes equation show close to linear convergence in L^2 norm.
- The higher polynomial degree does not always guarantee better accuracy. However, the convergence rate increases with increase in polynomial degree.

Outlook

- Test for higher Reynold's number
- Further solvers/methods
- Time dependent cases
- Parametrization
- Reduced order modelling

[1] Riviere B. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*. Frontiers in Applied Mathematics. Cambridge University Press, 2008.