

Discontinuous Galerkin Method for Direct Numerical Simulation of the Navier Stokes Equation

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Overview

- 1 Introduction
- 2 Engineering perspectives and mathematical formulation
- 3 Discretisation and function spaces
- 4 Implementation aspects
- 5 Numerical experiments
- 6 Conclusions and outlook

Importance of the Navier Stokes equation

- Computational fluid dynamics \implies One of the variants of the Navier Stokes equation
- The Navier Stokes equation involves state variables
- Incompressible condition \implies state variables constant \implies Equation of state not required
- Solved along with continuity equation
- Depends on time (Unsteady fluid flow) or independent of time (Steady fluid flow)
- Non linear coupled system of equations
- The Stokes equation is linearized form of the Navier Stokes equation

Governing equations

Reynolds transport theorem (White F.M. [4])

$$\left. \frac{dB'}{dt'} \right|_{cs} = \frac{d}{dt'} \int_{cv} b' \rho dV + \int_{cs} (b' \rho) u \cdot dA. \quad (1)$$

Momentum conservation equation

$$F = \frac{dM}{dt'} = \frac{d}{dt'} \int_{cv} u \rho dV + \int_{cs} (u \rho) u \cdot dA. \quad (2)$$

$$F = \int_{cs} \sigma \cdot dA + \int_{cv} \rho f dV. \quad (3)$$

Governing equations

Navier Stokes equation

$$-2\nabla \cdot (\nu \nabla^s u) + (1/\rho) \nabla p + (u \cdot \nabla) u = f \quad \text{in } \Omega. \quad (4)$$

Dirichlet boundary:

$$u = u_D \quad \text{on } \Gamma_D. \quad (5)$$

Neumann boundary:

$$-pn + 2\nu(n \cdot \nabla^s)u = t \quad \text{on } \Gamma_N. \quad (6)$$

Continuity equation

$$\nabla \cdot u = 0 \quad \text{in } \Omega. \quad (7)$$

Stokes equation

$$-2\nabla \cdot (\nu \nabla^s u) + (1/\rho) \nabla p = f \quad \text{in } \Omega. \quad (8)$$

Flow classification (Kundu P.K [5])

Reynolds number

$$Re = \frac{uL}{\nu}. \quad (9)$$

Laminar flow

- Well defined velocity and pressure profile.
- Low Reynolds number.

Turbulent flow

- Fluctuations in velocity and pressure.
- Fluctuations are of the order of Kolmogorov scale.
- High Reynolds number.

- Continuous domain $(\Omega) \implies \text{Grid } (\mathcal{T})$.
- Triangular element, $\tau_k, \cup_{k=1}^{nel} \tau_k = \mathcal{T}$.
- Grid boundary includes interelement boundaries.

$$\partial\mathcal{T} = \Gamma_D \cup \Gamma_N \cup \Gamma \quad (10)$$

Barycentric coordinate

For a triangle with vertices, r_1, r_2, r_3

$$r = \lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3. \quad (11)$$

Weights, $\lambda_1, \lambda_2, \lambda_3$ satisfy,

$$\lambda_1 + \lambda_2 + \lambda_3 = 1. \quad (12)$$

Discontinuous Galerkin method

- Multiply with test function and integrate.
- Discontinuous at the interface of elements.
- $P^D(\tau_k)$ denotes space of polynomials of degree at most D over τ_k .

Function space for velocity

$$\mathbb{V} = \{\phi \in (L^2(\mathcal{T}))^{d_u} \mid \phi \in (P^D(\tau_k))^{d_u} \quad \forall \quad \tau_k \in \mathcal{T}\}. \quad (13)$$

Function space for pressure

$$\mathbb{Q} = \{\psi \in (L^2(\mathcal{T}))^{d_p} \mid \psi \in (P^{D-1}(\tau_k))^{d_p} \quad \forall \quad \tau_k \in \mathcal{T}\}. \quad (14)$$

- We use Orthonormal basis function

Jump operator, Average operator

Jump operator

If p is scalar and u is vector,

$$\begin{aligned}[pn] &= p^+ n^+ + p^- n^- \quad \text{on } \Gamma, \quad [pn] = pn \quad \text{on } \Gamma_D. \\ [n \otimes u] &= n^+ \otimes u^+ + n^- \otimes u^- \quad \text{on } \Gamma, \quad [n \otimes u] = n \otimes u \quad \text{on } \Gamma_D. \\ [n \cdot u] &= n^+ \cdot u^+ + n^- \cdot u^- \quad \text{on } \Gamma, \quad [n \cdot u] = n \cdot u \quad \text{on } \Gamma_D.\end{aligned}\tag{15}$$

Average operator

The average operator is defined as,

$$\{u\} = \frac{u^+ + u^-}{2}.\tag{16}$$

Stokes equation

Strong form

$$\begin{aligned} -\nu \Delta u + \nabla p &= f \quad \text{in } \Omega. \\ \nabla \cdot u &= 0 \quad \text{in } \Omega. \end{aligned} \tag{17}$$

Weak form (Montlaur et al. [1])

$$\begin{aligned} a_{IP}(u, \phi) + b(\phi, p) + (\{p\}, [n \cdot \phi])_{\Gamma \cup \Gamma_D} &= l_{IP}(\phi). \\ b(u, \psi) + (\{\psi\}, [n \cdot u])_{\Gamma \cup \Gamma_D} &= (q, n \cdot u_D)_{\Gamma_D}. \end{aligned} \tag{18}$$

Stokes equation

Weak form

$$\begin{aligned}a_{IP}(u, \phi) &= (\nabla u, \nabla \phi) + C_{11}([n \otimes u], [n \otimes \phi])_{\Gamma \cup \Gamma_D} \\&\quad - \nu(\{\nabla u\}, [n \otimes \phi])_{\Gamma \cup \Gamma_D} - \nu([n \otimes u], \{\nabla \phi\})_{\Gamma \cup \Gamma_D}. \\b(\phi, \psi) &= - \int_{\mathcal{T}} \psi \nabla \cdot \phi. \\l_{IP}(\phi) &= (f, \phi) + (t, \phi)_{\Gamma_N} + C_{11}(u_D, \phi)_{\Gamma_D} \\&\quad - (n \otimes u_D, \nu \nabla \phi)_{\Gamma_D}.\end{aligned}\tag{19}$$

Discrete form

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}\tag{20}$$

Stiffness matrix

Matrix A

$$A_{ij} = \sum_{k=1}^d \left(\frac{\partial \phi_i}{\partial x_k}, \frac{\partial \phi_j}{\partial x_k} \right) + C_{11} \sum_{k=1}^d ([\phi_i n_k], [\phi_j n_k])_{\Gamma \cup \Gamma_D} \\ - \nu \sum_{k=1}^d ([\phi_i n_k], \{ \frac{\partial \phi_j}{\partial x_k} \})_{\Gamma \cup \Gamma_D} - \nu \sum_{k=1}^d (\{ \frac{\partial \phi_i}{\partial x_k} \}, [\phi_j n_k])_{\Gamma \cup \Gamma_D} \quad (21)$$

Matrix B

$$B_{ij} = - \int_{\mathcal{T}} \frac{\partial \phi_i}{\partial x_i} \psi_j + (\{ \psi_j \}, [n \cdot \phi_i])_{\Gamma \cup \Gamma_D} \quad (22)$$

Navier Stokes equation : Upwinding

- If n_τ is the unit normal from τ_1 to τ_2 and if we denote the upwind value of function g as g^{up} [3],

$$\begin{aligned} g^{up} &= g|_{\tau_1} & \text{if } g \cdot n_\tau \geq 0 \\ g^{up} &= g|_{\tau_2} & \text{if } g \cdot n_\tau < 0 \end{aligned} \quad (23)$$

- For the weak form of the Navier Stokes equation (Montlaur et al. [2]),

$$\begin{aligned} c(g; u, \phi) &= \sum_{i=1}^{nel} \int_{\partial\Omega_i \setminus \Gamma_N} \frac{1}{2} [[(g \cdot n_i)(u^{ext} + u) - |g \cdot n_i|(u^{ext} - u)]] \cdot \phi \\ &\quad + \int_{\Gamma_N} (g \cdot n) u \cdot \phi - ((g \cdot \nabla)\phi, u). \end{aligned} \quad (24)$$

$$u^{ext} = \lim_{\epsilon \rightarrow 0} u(x + \epsilon n_i), \quad \epsilon \rightarrow 0^+ \quad \text{for } x \in \partial\mathcal{T}_i \quad (25)$$

Newton method

Algorithm for the Newton method is as follow:

1. Calculate $u^{iter} \in \mathbb{V}$ at iteration $iter$,
2. Rewrite weak form as $S = 0$ and verify $DS_{u^{iter}}(h^{iter}) = -S(u^{iter})$,
3. Set $u^{iter+1} := u^{iter} + h^{iter}$ till $\|u^{iter+1} - u^{iter}\| < tol$.

We use the solution from the Stokes equation as initial guess. In discrete form the Newton method means, solving the equation,

$$\begin{pmatrix} A + C(U^{iter}) & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U^{iter+1} \\ P^{iter+1} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}. \quad (26)$$

Stiffness matrix ^{$iter$} Solution vector ^{$iter+1$} Right hand side

STEP 1:

$$U = A^{-1}(F_1 - BP), \quad (27)$$

STEP 2 :

$$-B^T A^{-1}BP = F_2 - B^T A^{-1}F_1, \quad (28)$$

STEP 3 :

Backsubstitution in equation (27).

- `params` : Structure with fields u_{npe} , u_{ndofs} , d_u , D , U .
- `paramsP` : Structure with fields p_{npe} , p_{ndofs} , d_p , $D-1$, P .
- `grid` : Structure containing informations related to grid.

Basis function in RBmatlab

$$\phi \in \mathbb{R}^{u_{npe} \times d_u}, \quad (29)$$

- The derivative of basis function $(\phi)_i$, where $1 \leq i \leq u_{npe}$ is cell.

Derivative of basis function in RBmatlab

$$\nabla(\phi)_i \in \mathbb{R}^{d_u \times d}. \quad (30)$$

Matrix assemblies : Jump operator and average operator

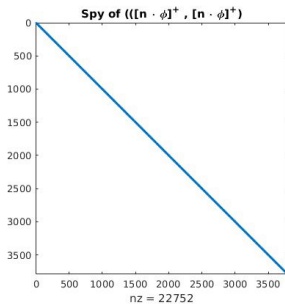
Jump operator

$$[A_h \cdot n], [B_h \cdot n] = A_h^+ n^+ B_h^+ n^+ + A_h^+ n^+ B_h^- n^- + A_h^- n^- B_h^+ n^+ + A_h^- n^- B_h^- n^-. \quad (31)$$

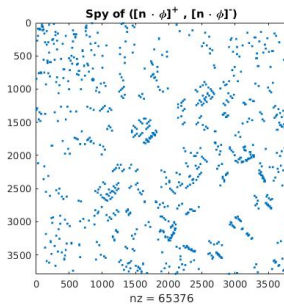
Average operator

$$\{A_h\} = \frac{(A_h^+ + A_h^-)}{2}. \quad (32)$$

Sparsity pattern



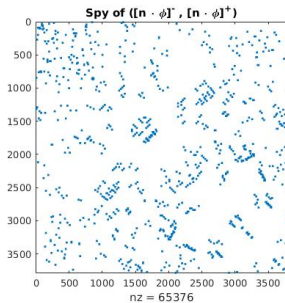
(a) $((n \otimes \phi)^+, (n \otimes \phi)^+)_{\Gamma \cup \Gamma_D}$



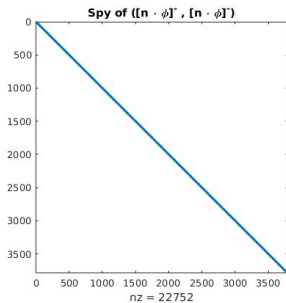
(b) $((n \otimes \phi)^+, (n \otimes \phi)^-)_{\Gamma \cup \Gamma_D}$

Figure: Sparsity pattern of constituents of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Sparsity pattern



(a) $((n \otimes \phi)^-, (n \otimes \phi)^+)_{\Gamma \cup \Gamma_D}$



(b) $((n \otimes \phi)^-, (n \otimes \phi)^-)_{\Gamma \cup \Gamma_D}$

Figure: Sparsity pattern of constituents of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Sparsity pattern

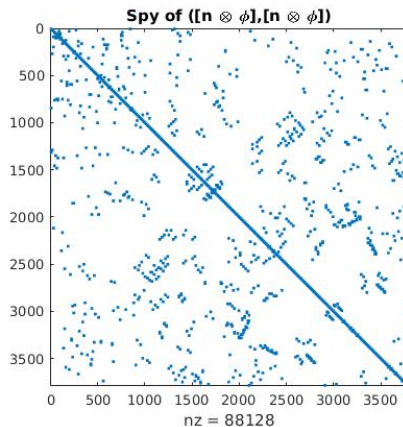


Figure: Sparsity pattern of constituents of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Error definition

- If P_h is the computed solution and P is the true solution, we define following errors:

L^2 error

$$P_{error,L^2} = \left(\int_{\Omega} |P - P_h|^2 \right)^{\frac{1}{2}}. \quad (33)$$

H_0 error

$$P_{error,H_0} = \sum_{k=1}^{nel} \left(\int_{\tau_k} |\nabla P - \nabla P_h|^2 \right)^{\frac{1}{2}}. \quad (34)$$

Stokes equation: Convergence test

- Domain: unit square $[0,1] \times [0,1]$.
- $x = 0$ is dirichlet boundary with inflow velocity at point $(0, y)$ as $u = (y(1 - y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (2\nu - 1, 0)$.

Analytical solution

$$p = (1 - x), \quad (35)$$

$$u = (y(1 - y), 0). \quad (36)$$

Stokes equation: Convergence test

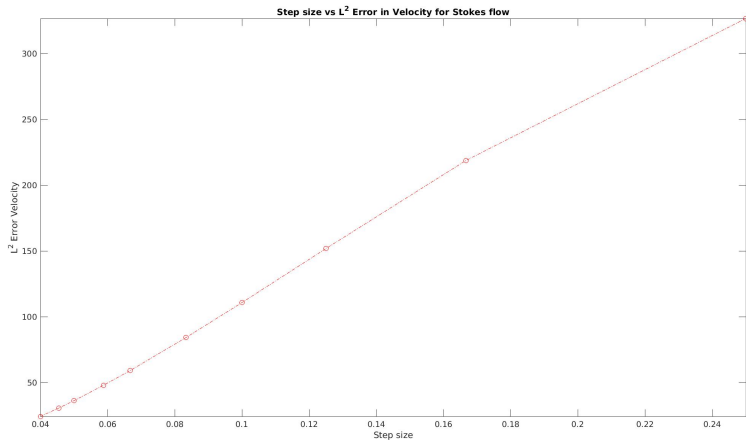


Figure: h -convergence test for velocity in L^2 error

Stokes equation: Convergence test

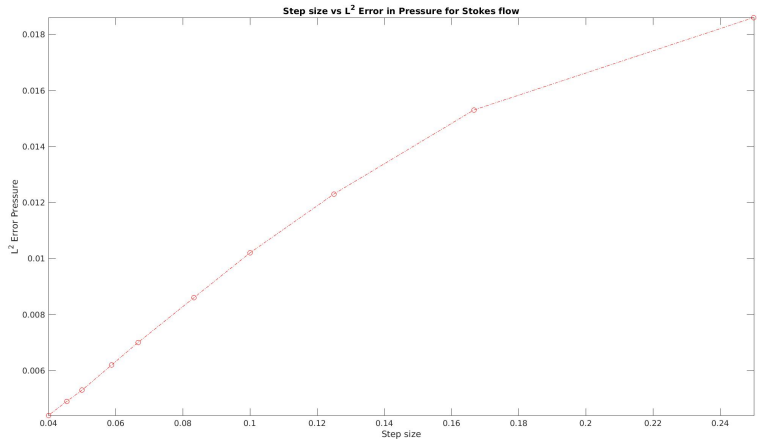


Figure: h -convergence test for pressure in L^2 error

Stokes equation: Convergence test

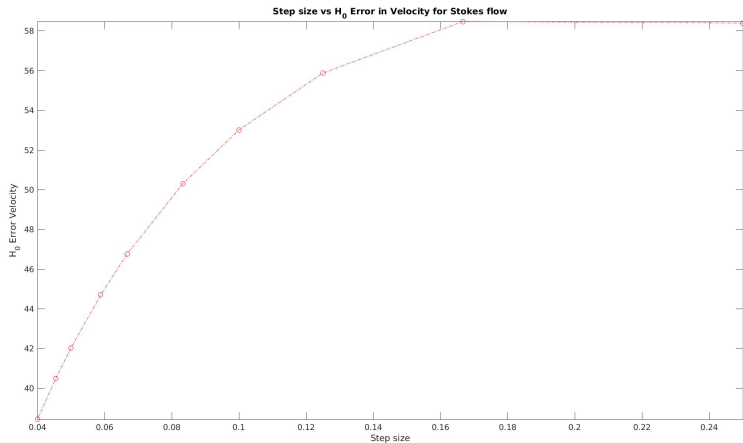


Figure: h -convergence test for velocity in H_0 error

Stokes equation: Convergence test

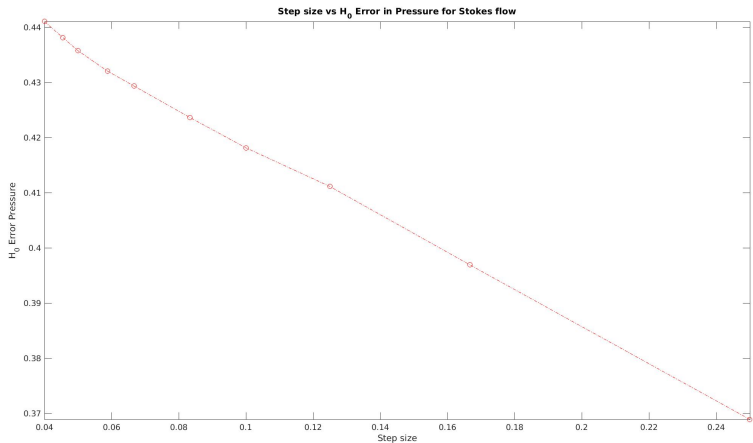
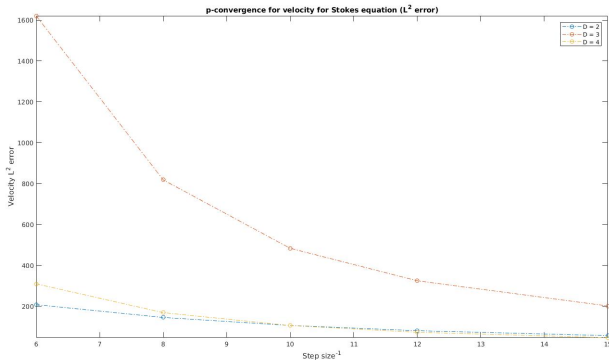


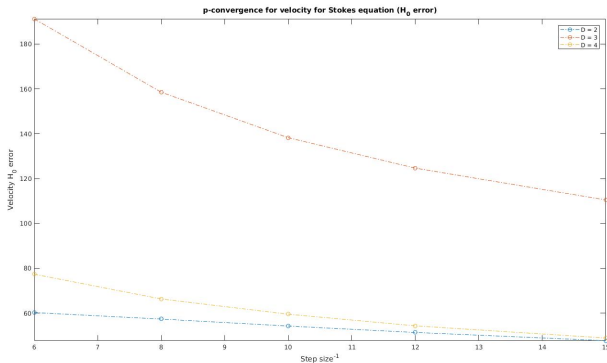
Figure: h -convergence test for pressure in H_0 error

Stokes equation: Convergence test



(a) p -convergence for velocity in L^2 error

Stokes equation: Convergence test



(a) p -convergence for velocity in H_0 error

Stokes equation: Convergence test

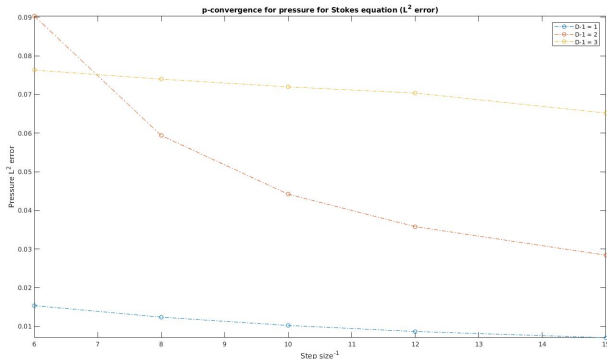
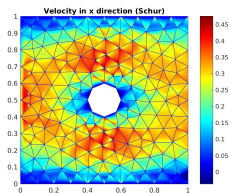


Figure: p -convergence for pressure in L^2 error

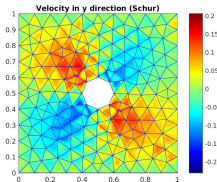
Stokes equation: Flow around cylinder

- Domain: $[0,1] \times [0,1]$ with a cut out cylinder of diameter 0.2 centered at $(0.5, 0.5)$.
- The boundary $x = 0$ is Dirichlet boundary with inflow velocity at point $(0, y)$ as $u = (y(1 - y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (0, 0)$.

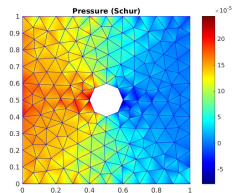
Stokes equation: Flow around cylinder



(a) x-velocity



(b) y-velocity



(c) Pressure

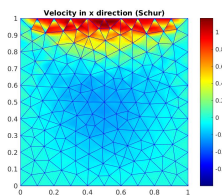
Figure: Flow over cylinder

Stokes equation: Lid driven cavity

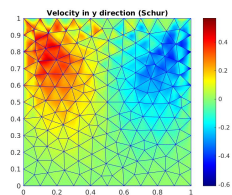
- Domain: unit square $[0,1] \times [0,1]$.
- Boundaries $x = 0, x = 1$ and $y = 0$, we impose no slip or zero velocity Dirichlet condition.
- On $y = 1$, we impose Dirichlet condition with Dirichlet velocity,

$$\begin{aligned}u &= (10x, 0) \quad \text{for } 0 \leq x \leq 0.1, \\u &= (1, 0) \quad \text{for } 0.1 \leq x \leq 0.9, \\u &= (10 - 10x, 0) \quad \text{for } 0.9 \leq x \leq 1.\end{aligned} \tag{37}$$

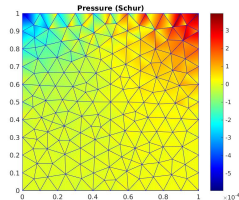
Stokes equation: Lid driven cavity



(a) x-velocity



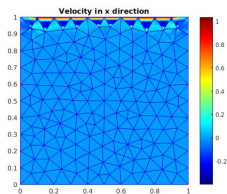
(b) y-velocity



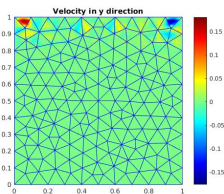
(c) Pressure

Figure: Lid driven cavity problem (Schur complement method)

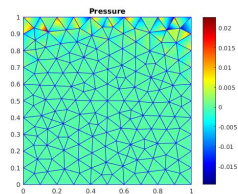
Stokes equation: Lid driven cavity



(a) x-velocity



(b) y-velocity



(c) Pressure

Figure: Lid driven cavity problem (*bicgstab* solver)

Penalty parameter

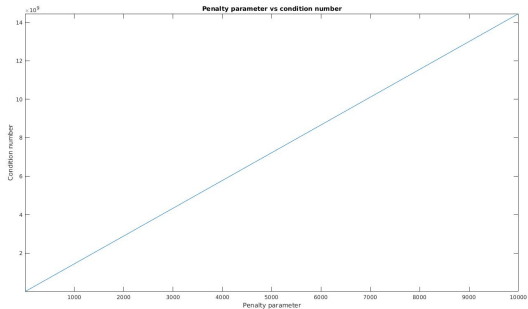
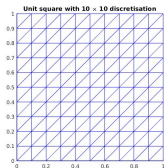


Figure: Effect of penalty parameter

Solver performance

Relative residual is measured as $\frac{\|B-AX\|_2}{\|B\|_2}$ for equation $AX = B$.

Solver/Method	Relative residual	Run time
Schur complement method	2.4436e-08	6.6253 Seconds
<i>minres</i>	2.4618e-05	35.7372 Seconds
<i>bicgstab</i>	9.0071e-05	58.3472 Seconds

Navier-Stokes equation: Convergence test

- Domain: unit square $[0,1] \times [0,1]$.
- $x = 0$ is dirichlet boundary with inflow velocity at point $(0, y)$ as $u = a * (y(1 - y), 0)$.
- The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- Specified source term

Analytical solution

$$p = x(1 - x), \quad (38)$$

$$u = (x^2(1 - y)^2(2y - 6y^2 + 4y^3), \quad (39) \\ -y^2(1 - y)^2(2x - 6x^2 + 4x^3)).$$

Navier-Stokes equation: Convergence test

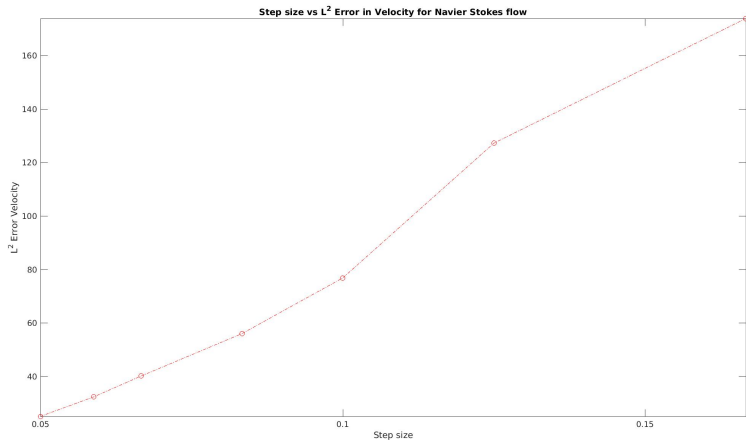


Figure: h -convergence test for velocity L^2 error

Navier-Stokes equation: Convergence test

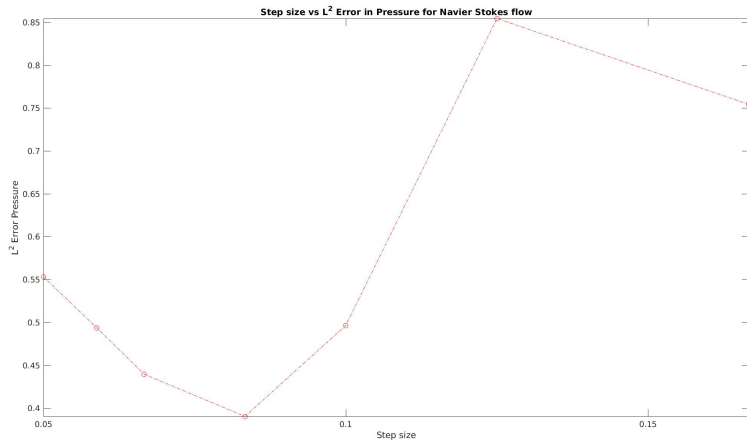


Figure: h -convergence test for pressure in L^2 error

Navier-Stokes equation: Convergence test

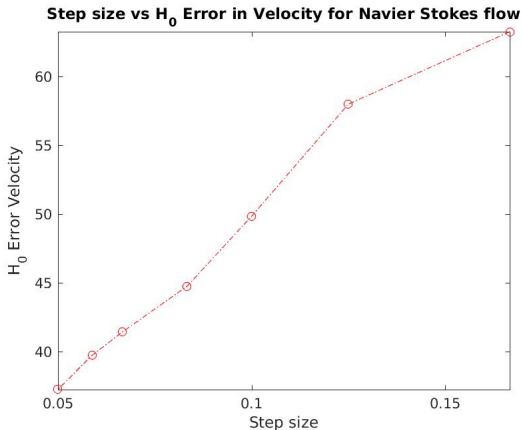


Figure: h -convergence test for velocity H_0 error

Navier-Stokes equation: Convergence test

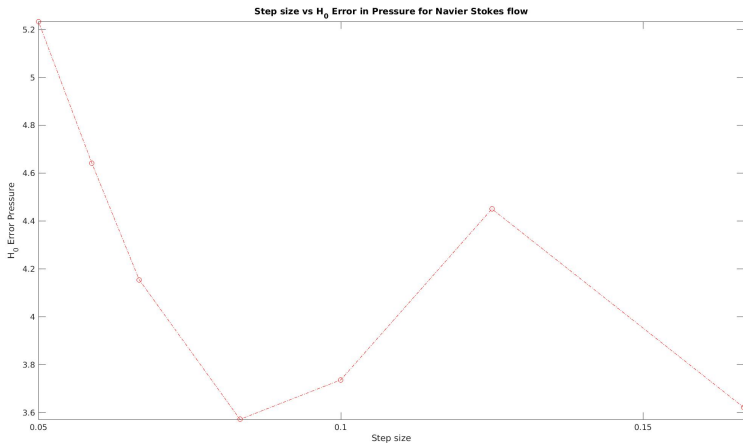
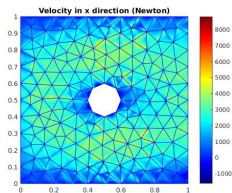
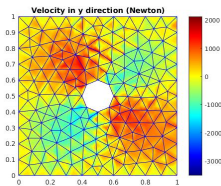


Figure: h -convergence test for pressure in H_0 error

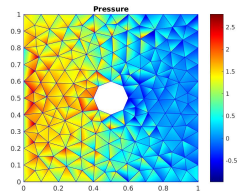
Navier-Stokes equation: Flow around cylinder



(a) x-velocity



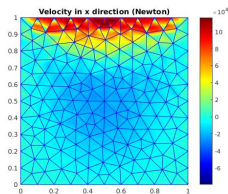
(b) y-velocity



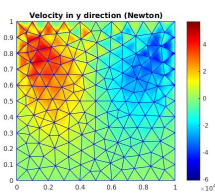
(c) Pressure

Figure: Flow over cylinder (Initial guess by Schur complement method)

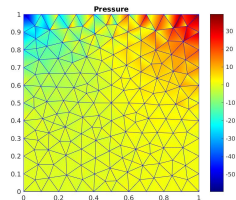
Navier-Stokes equation: Lid driven cavity



(a) x-velocity



(b) y-velocity



(c) Pressure

Figure: Lid driven cavity flow (Initial guess by Schur complement method)

Conclusions

Numerical considerations

- The higher polynomial degree does not always guarantee better accuracy. However, the convergence rate increases with increase in polynomial degree.
- The initial guess, here, solution of the Stokes equation, is crucial for success of the Newton method for the Navier Stokes equation.
- Convergence of solution.

Solvers performance

- The Schur complement method: Efficient and accurate.
- The *minres* solver : Slow convergence
- The *bicgstab* : Convergence failure.

Discontinuous Galerkin method

- Test for higher Reynold's number.
- Further solvers/methods.
- Time dependent cases.

Model order reduction

- Parametrization.
- Reduced order modelling.

- [1] Montlaur A., Fernandez-Mendez S., and Huerta A. Discontinuous Galerkin methods for the Stokes equations using divergence-free approximations. *International Journal for Numerical Methods in Fluids*, 57(9):1071–1092, 2008.
- [2] Montlaur A., Fernandez-Mendez S., Peraire J., and Huerta A. Discontinuous Galerkin methods for the Navier Stokes equations using solenoidal approximations. *International Journal for Numerical Methods in Fluids*, 64(5):549–564, 2010.
- [3] Riviere B. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*. Frontiers in Applied Mathematics. Cambridge University Press, 2008.
- [4] White F.M. *Fluid mechanics*. Second edition, 97-99, 2002. Academic press.
- [5] Kundu P. K. and Cohen I. M. *Fluid Mechanics*. Academic Press, 2002.

Notations

Ω = Continuous domain,

Γ_D = Dirichlet boundary,

Γ_N = Neumann boundary,

cv = Control volume,

cs = Control surface,

B' = Extensive quantity under consideration,

b' = Intensive quantity corresponding to B' ,

u = flow velocity and $u : \Omega \rightarrow \mathbb{R}^d$,

p = pressure and $p : \Omega \rightarrow \mathbb{R}$,

ν = kinematic viscosity (fluid property) and $\nu : \Omega \rightarrow \mathbb{R}$,

f = external force and $f : \Omega \rightarrow \mathbb{R}^d$,

u_D = specified flow velocity at Dirichlet boundary and $u_D : \Gamma_D \rightarrow \mathbb{R}^d$,

n = normal unit vector and $n : \partial\Omega \rightarrow \mathbb{R}^d$,

ρ = density (fluid property) and $\rho : \Omega \rightarrow \mathbb{R}$,

t = specified Neumann flux and $t : \Gamma_N \rightarrow \mathbb{R}^d$.

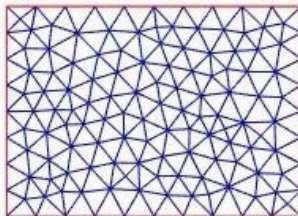
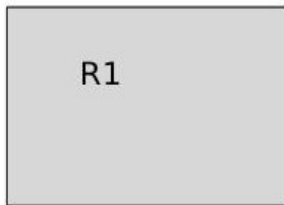


Figure: Continuous domain (left) and discretised domain or grid (right)

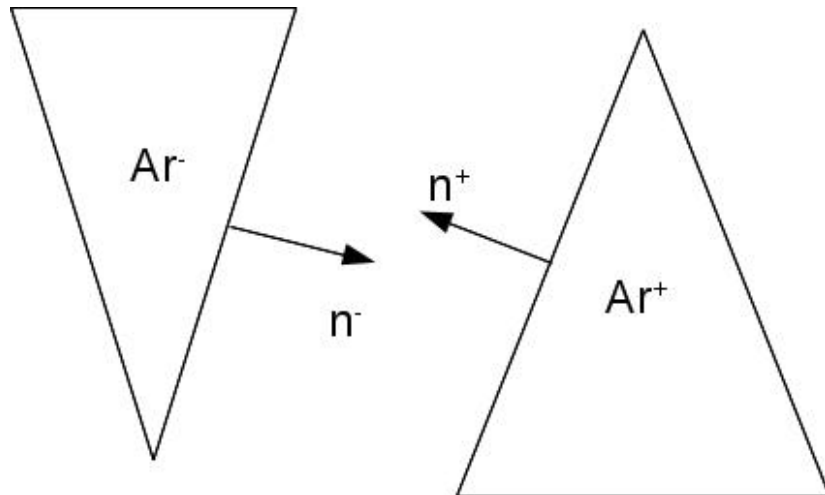
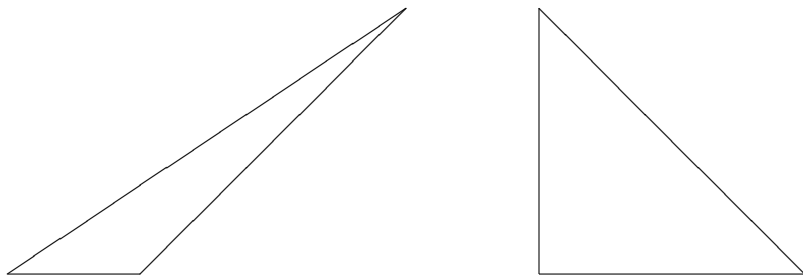


Figure: Element self (+) and neighbouring element (-)

Global and local coordinate system

- The transformation from local coordinate \hat{X} to global coordinate X is defined by the mapping,

$$\begin{aligned} F_k : \hat{X} \mapsto X \quad \forall \quad \hat{X} \in \hat{\mathcal{T}} \quad \text{and} \quad X \in \mathcal{T} \\ F_k(\hat{X}) : X = J_k \hat{X} + C \end{aligned} \tag{40}$$



Global geometry (left) to Local geometry (right)

Assembly of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

Step 1 :

$$\begin{aligned} res_1^{++} &= (n \otimes \hat{\phi})^+ (n \otimes \hat{\phi})^+, \\ res_1^{+-} &= (n \otimes \hat{\phi})^+ (n \otimes \hat{\phi})^-, \\ res_1^{-+} &= (n \otimes \hat{\phi})^- (n \otimes \hat{\phi})^+, \\ res_1^{--} &= (n \otimes \hat{\phi})^- (n \otimes \hat{\phi})^-. \end{aligned} \tag{41}$$

Step 2 :

$$\begin{aligned} res_2^{++} &= \int_{\Gamma} res_1^{++} EL(i, j), \\ res_2^{+-} &= \int_{\Gamma} res_1^{+-} EL(i, j), \end{aligned} \tag{42}$$

Assembly of $([n \otimes \phi], [n \otimes \phi])_{\Gamma \cup \Gamma_D}$

$$\begin{aligned} res_2^{-+} &= \int_{\Gamma} res_1^{-+} EL(i, j), \\ res_2^{--} &= \int_{\Gamma} res_1^{--} EL(i, j). \end{aligned} \tag{43}$$

Step 3 :

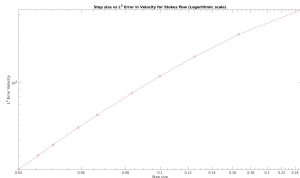
$$\begin{aligned} res_3^{++}[ids_velocity_self, ids_velocity_self] &= res_2^{++}, \\ res_3^{+-}[ids_velocity_self, ids_velocity_neighbour] &= res_2^{+-}, \\ res_3^{-+}[ids_velocity_neighbour, ids_velocity_self] &= res_2^{-+}, \\ res_3^{--}[ids_velocity_neighbour, ids_velocity_neighbour] &= res_2^{--}. \end{aligned} \tag{44}$$

$$res_3 = res_3^{++} + res_3^{+-} + res_3^{-+} + res_3^{--}. \tag{45}$$

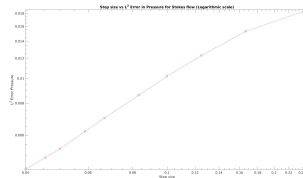
Program flow

- Grid preparation
- Formulating function space
- Matrix assembly
- Solving assembled form
- Post processing
- Newton method

Stokes equation: Convergence test



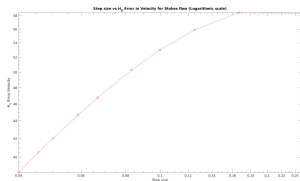
(a) h –convergence test for velocity L^2 error (Logarithmic scale)



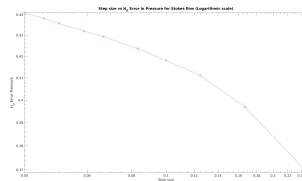
(b) h –convergence test for pressure in L^2 error (Logarithmic scale)

Figure: h –convergence in L^2 norm for the Stokes flow (Logarithmic scale)

Stokes equation: Convergence test



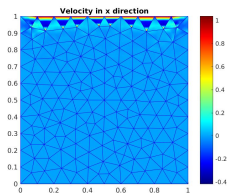
(a) h -convergence test for velocity H_0 error (Logarithmic scale)



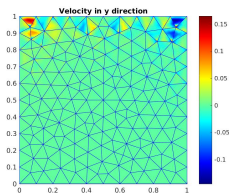
(b) h -convergence test for pressure in H_0 error (Logarithmic scale)

Figure: h -convergence in H_0 norm for the Stokes flow (Logarithmic scale)

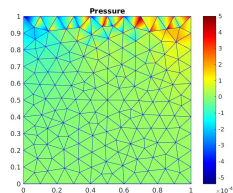
Stokes equation: Lid driven cavity



(a) x-velocity



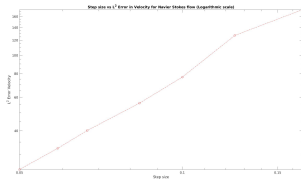
(b) y-velocity



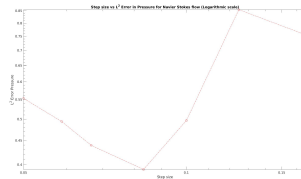
(c) Pressure

Figure: Lid driven cavity problem (*minres* solver)

Navier-Stokes equation: Convergence test



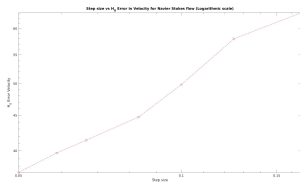
(a) h -convergence test for velocity L^2 error (Logarithmic scale)



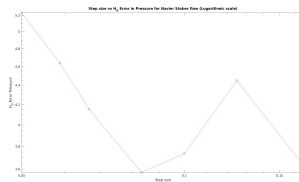
(b) h -convergence test for pressure in L^2 error (Logarithmic scale)

Figure: h -convergence for the Navier Stokes flow in L^2 error (Logarithmic scale)

Navier-Stokes equation: Convergence test



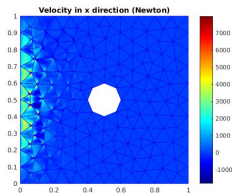
(a) h -convergence test for velocity H_0 error (logarithmic scale)



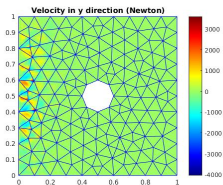
(b) h -convergence test for pressure in H_0 error (Logarithmic scale)

Figure: h -convergence for the Navier Stokes flow in H_0 error (Logarithmic scale)

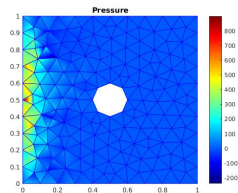
Navier-Stokes equation: Flow around cylinder



(a) x-velocity



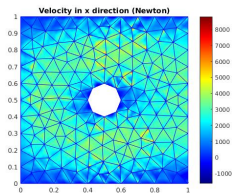
(b) y-velocity



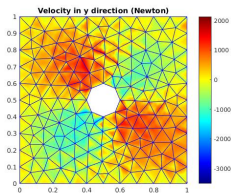
(c) Pressure

Figure: Flow over cylinder (Initial guess by *bicgstab* solver)

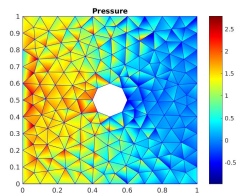
Navier-Stokes equation: Flow around cylinder



(a) x-velocity



(b) y-velocity



(c) Pressure

Figure: Flow over cylinder (Initial guess by minres solver)

Navier-Stokes equation: Lid driven cavity

- Domain: unit square $[0,1] \times [0,1]$.
- Boundaries $x = 0, x = 1$ and $y = 0$, we impose no slip or zero velocity Dirichlet condition.
- On $y = 1$, we impose Dirichlet condition with Dirichlet velocity,

$$\begin{aligned}u &= a * (10x, 0) \quad \text{for } 0 \leq x \leq 0.1, \\u &= a * (1, 0) \quad \text{for } 0.1 \leq x \leq 0.9, \\u &= a * (10 - 10x, 0) \quad \text{for } 0.9 \leq x \leq 1.\end{aligned} \tag{46}$$

Navier-Stokes equation: Flow around cylinder

- Domain: $[0,1] \times [0,1]$ with a cut out cylinder of diameter 0.2 centered at $(0.5, 0.5)$.
- The boundary $x = 0$ is Dirichlet boundary with inflow velocity at point $(0, y)$ as $u = a * (y(1 - y), 0)$.
- The boundaries $y = 0$ and $y = 1$ are Dirichlet boundaries with no slip or zero velocity condition. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$.
- The source term is $f = (0, 0)$.