

Reduced order modeling of geometrically parametrized discontinuous Galerkin formulation for the Stokes equation

Nirav Vasant Shah, Martin Hess and Gianluigi Rozza

Abstract The present work focuses on geometrical parametrization and reduced order modeling of Stokes flow. The importance of Stokes flow, advantages of discontinuous Galerkin method are introduced first. We also discuss the concept of geometric parametrization and its application along with importance of reduced order model technique. The full order model is based on discontinuous Galerkin method interior penalty formulation. The concepts of broken Sobolev spaces, relevant norms, jump and mean operator are introduced. The weak formulation is derived based in suitable space to obtain the full order model. We then introduce the concept of geometric parametrization. The operators are transformed from fixed domain to parameter dependent domain by exploring affine parameter dependence which results in efficient assembly of system matrix. Thereafter, proper orthogonal decomposition is applied to obtain basis for function space for reduced order model. By using Galerkin projection the linear system to be solved is projected onto reduced space. During the process, offline-online decomposition is used to separate computation of expensive parameter independent part and fast parameter independent part. Finally the technique is applied to test problem. The numerical outcomes presented include the experimental error analysis, eigenvalue computation and measurement of online simulation time. [13]

Nirav Vasant Shah

Scuola Internazionale Superiore di Studi Avanzati - via Bonomea, 265 - 34136 Trieste ITALY,
e-mail: snirav@sissa.it

Martin Hess

Scuola Internazionale Superiore di Studi Avanzati - via Bonomea, 265 - 34136 Trieste ITALY
e-mail: martin.hess@sissa.it

1 Introduction

The subject of mathematical applications in fluid mechanics starts with one of the variants of the Navier-Stokes equations, such as the Stokes equation. Almost all processes of fluid mechanics require considerations related to the Navier-Stokes equations. Navier-Stokes equation is non-linear, characterizing flow fluctuations. However, in case of laminar flow, i.e. when fluctuations are negligible, this linearized form of the Navier-Stokes equation is the Stokes equation.

Discontinuous Galerkin method (DGM) has found traction as numerical method for elliptic problems **pereire reference** as well as hyperbolic problems **Book on compressible flow reference**. This is due to its several advantages over Finite Element Method (FEM) and Finite Volume Method (FVM). In fact, DG method is considered as combination of FEM and FVM. DGM uses polynomial approximation of suitable degree providing higher accuracy as well as allows discontinuity at the interface, by the concept of numerical flux, allowing greater flexibility. This fact makes DGM naturally attractive to problems such as shock capturing due to presence of steep gradients or discontinuities. Additionally, since the Dirichlet conditions are applied as boundary penalty, it avoids necessity to work with subspace of FEM. Several variants of DGM exist based on computational advantages such as sparsity pattern or extension of computational stencil, complexity of numerical implementation etc.

Geometric parametrization has emerged as important application of Parametric Partial Differential Equations (PPDEs) and as alternative to shape optimization. The concept of geometric parametrization allows to transfer operator evaluated on one domain to another domain efficiently. For linear equations, this means exploiting affine parameter dependence as will be shown in later section. Model Order Reduction (MOR) on the other hand allows reducing the size of the system to be solved and working with the smaller system containing only dominant components and discarding the non-dominant components. It is pertinent to mention that identifying "dominant" components is critical to the success of model order reduction strategy. Optimization of engineering components using geometric parametrization combined with MOR for PPDEs has given quite useful results in the fields such as mechanical, naval and aeronautic designs. Also, the faster computations obtained by MOR has helped in many query context, real time computation and quick transfer of computational results to industrial problems.

In the present work, we first introduce Discontinuous Galerkin Interior Penalty Method (DG-IPM). We subsequently introduce notion of parametrization characterizing geometry of the domain under consideration, exploit affine parameter dependence and its application in the context of offline-online decomposition. We then apply Proper Orthogonal Decomposition (POD) for constructing reduced basis space and apply Galerkin projection to project the system of equations on the space constructed by POD. Finally we present a test problem to demonstrate the introduced method and present numerical result.

2 Section Heading

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Use the standard `equation` environment to typeset your equations, e.g.

$$a \times b = c, \quad (1)$$

however, for multiline equations we recommend to use the `eqnarray` environment¹.

$$|\nabla U_\alpha^\mu(y)| \leq \frac{1}{d-\alpha} \int \left| \nabla \frac{1}{|\xi-y|^{d-\alpha}} \right| d\mu(\xi) = \int \frac{1}{|\xi-y|^{d-\alpha+1}} d\mu(\xi) \quad (2)$$

$$= (d-\alpha+1) \int_{d(y)}^{\infty} \frac{\mu(B(y,r))}{r^{d-\alpha+2}} dr \leq (d-\alpha+1) \int_{d(y)}^{\infty} \frac{r^{d-\alpha}}{r^{d-\alpha+2}} dr \quad (3)$$

2.1 Subsection Heading

Instead of simply listing headings of different levels we recommend to let every heading be followed by at least a short passage of text. Further on please use the \LaTeX automatism for all your cross-references and citations as has already been described in Sect. 2.

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2.1.1 Subsubsection Heading

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Please note that the first line of text that follows a heading is not indented, whereas the first lines of all subsequent paragraphs are.

¹ In physics texts please activate the class option `vecphys` to depict your vectors in *boldface-italic* type - as is customary for a wide range of physical subjects

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1. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
 - a. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
 - b. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
2. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

Subparagraph Heading

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For unnumbered list we recommend to use the `itemize` environment – it will automatically be rendered in line with the preferred layout.

- Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development, cf. Table 1.

Fig. 1 If the width of the figure is less than 7.8 cm use the `sidecaption` command to flush the caption on the left side of the page. If the figure is positioned at the top of the page, align the sidecaption with the top of the figure – to achieve this you simply need to use the optional argument `[t]` with the `sidecaption` command

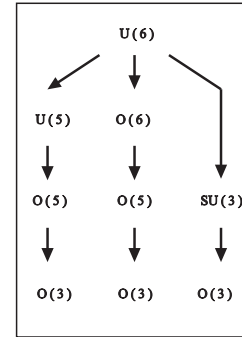


Fig. 2 If the width of the figure is less than 7.8 cm use the `sidecaption` command to flush the caption on the left side of the page. If the figure is positioned at the top of the page, align the sidecaption with the top of the figure – to achieve this you simply need to use the optional argument `[t]` with the `sidecaption` command

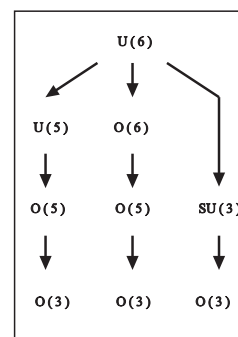


Table 1 Please write your table caption here

Classes	Subclass	Length	Action Mechanism
Translation	mRNA ^a	22 (19–25)	Translation repression, mRNA cleavage
Translation	mRNA cleavage	21	mRNA cleavage
Translation	mRNA	21–22	mRNA cleavage
Translation	mRNA	24–26	Histone and DNA Modification

^a Table foot note (with superscript)

- Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
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Run-in Heading Boldface Version Use the \LaTeX automatism for all your cross-references and citations as has already been described in Sect. 2.

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Run-in Heading Displayed Version

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3 Section Heading

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- Type 2 That addresses central themes pertaining to migration, health, and disease. In Sect. 2.1, Wilson discusses the role of human migration in infectious disease distributions and patterns.

3.1 Subsection Heading

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3.1.1 Subsubsection Heading

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Theorem 1 *Theorem text goes here.*

Definition 1 Definition text goes here.

Proof Proof text goes here. □

Paragraph Heading

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Theorem 2 *Theorem text goes here.*

Definition 2 Definition text goes here.

Proof Proof text goes here. □

Trailer Head

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```
\begin{trailer}{Trailer Head}
...
\end{trailer}
```

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If you want to emphasize complete paragraphs of texts in an `Questions` we recommend to use

```
\begin{question}{Questions}
...
\end{question}
```

> Important

If you want to emphasize complete paragraphs of texts in an **Important** we recommend to use

```
\begin{important}{Important}  
...  
\end{important}
```

! Attention

If you want to emphasize complete paragraphs of texts in an **Attention** we recommend to use

```
\begin{warning}{Attention}  
...  
\end{warning}
```

Program Code

If you want to emphasize complete paragraphs of texts in an **Program Code** we recommend to use

```
\begin{programcode}{Program Code}  
\begin{verbatim}...\end{verbatim}  
\end{programcode}
```

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```
\begin{tips}{Tips}  
...  
\end{tips}
```

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```
\begin{overview}{Overview}
...
\end{overview}
```

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```
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...
\end{backgroundinformation}
```

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```
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...
\end{legalttext}
```

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Appendix

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$$a \times b = c \tag{4}$$

References

References

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1. Broy, M.: Software engineering — from auxiliary to key technologies. In: Broy, M., Dener, E. (eds.) *Software Pioneers*, pp. 10-13. Springer, Heidelberg (2002)
2. Dod, J.: Effective substances. In: *The Dictionary of Substances and Their Effects*. Royal Society of Chemistry (1999) Available via DIALOG.
<http://www.rsc.org/dose/titleofsubordinatedocument>. Cited 15 Jan 1999
3. Geddes, K.O., Czapor, S.R., Labahn, G.: *Algorithms for Computer Algebra*. Kluwer, Boston (1992)
4. Hamburger, C.: Quasimonotonicity, regularity and duality for nonlinear systems of partial differential equations. *Ann. Mat. Pura. Appl.* **169**, 321–354 (1995)
5. Slifka, M.K., Whitton, J.L.: Clinical implications of dysregulated cytokine production. *J. Mol. Med.* (2000) doi: 10.1007/s001090000086
6. J. Dod, in *The Dictionary of Substances and Their Effects*, Royal Society of Chemistry. (Available via DIALOG, 1999), <http://www.rsc.org/dose/titleofsubordinatedocument>. Cited 15 Jan 1999
7. H. Ibach, H. Lüth, *Solid-State Physics*, 2nd edn. (Springer, New York, 1996), pp. 45-56
8. S. Preuss, A. Demchuk Jr., M. Stuke, *Appl. Phys. A* **61**

³ Make sure that all references from the list are cited in the text. Those not cited should be moved to a separate *Further Reading* section or chapter.

⁴ Always use the standard abbreviation of a journal's name according to the *ISSN List of Title Word Abbreviations*, see <http://www.issn.org/en/node/344>

9. M.K. Slifka, J.L. Whitton, J. Mol. Med., doi: 10.1007/s001090000086
10. S.E. Smith, in *Neuromuscular Junction*, ed. by E. Zaimis. Handbook of Experimental Pharmacology, vol 42 (Springer, Heidelberg, 1976), p. 593
11. Calfee, R. C., & Valencia, R. R. (1991). *APA guide to preparing manuscripts for journal publication*. Washington, DC: American Psychological Association.
12. Dod, J. (1999). Effective substances. In: The dictionary of substances and their effects. Royal Society of Chemistry. Available via DIALOG.
<http://www.rsc.org/dose/Effectivesubstances>. Cited 15 Jan 1999.
13. Harris, M., Karper, E., Stacks, G., Hoffman, D., DeNiro, R., Cruz, P., et al. (2001). Writing labs and the Hollywood connection. *J Film Writing*, 44(3), 213–245.
14. O’Neil, J. M., & Egan, J. (1992). Men’s and women’s gender role journeys: Metaphor for healing, transition, and transformation. In B. R. Wainrig (Ed.), *Gender issues across the life cycle* (pp. 107–123). New York: Springer.
15. Kreger, M., Brindis, C.D., Manuel, D.M., Sassoubre, L. (2007). Lessons learned in systems change initiatives: benchmarks and indicators. *American Journal of Community Psychology*, doi: 10.1007/s10464-007-9108-14.
16. Alber John, Daniel C. O’Connell, and Sabine Kowal. 2002. Personal perspective in TV interviews. *Pragmatics* 12:257–271
17. Cameron, Deborah. 1997. Theoretical debates in feminist linguistics: Questions of sex and gender. In *Gender and discourse*, ed. Ruth Wodak, 99–119. London: Sage Publications.
18. Cameron, Deborah. 1985. *Feminism and linguistic theory*. New York: St. Martin’s Press.
19. Dod, Jake. 1999. Effective substances. In: The dictionary of substances and their effects. Royal Society of Chemistry. Available via DIALOG.
[http://www.rsc.org/dose/title of subordinate document](http://www.rsc.org/dose/titleofsubordinate document). Cited 15 Jan 1999
20. Suleiman, Camelia, Daniel C. O’Connell, and Sabine Kowal. 2002. ‘If you and I, if we, in this later day, lose that sacred fire...’: Perspective in political interviews. *Journal of Psycholinguistic Research*. doi: 10.1023/A:1015592129296.
21. Brown B, Aaron M (2001) The politics of nature. In: Smith J (ed) The rise of modern genomics, 3rd edn. Wiley, New York
22. Dod J (1999) Effective Substances. In: The dictionary of substances and their effects. Royal Society of Chemistry. Available via DIALOG.
<http://www.rsc.org/dose/titleofsubordinatedocument>. Cited 15 Jan 1999
23. Slifka MK, Whitton JL (2000) Clinical implications of dysregulated cytokine production. *J Mol Med*, doi: 10.1007/s001090000086
24. Smith J, Jones M Jr, Houghton L et al (1999) Future of health insurance. *N Engl J Med* 341:325–329
25. South J, Blass B (2001) The future of modern genomics. Blackwell, London