Chapter 1
Discontinuous Galerkin model order reduction of geometrically parametrized Stokes equation

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Abstract The present work focuses on the geometrical parametrization and the reduced order modeling of the Stokes equation. We discuss the concept of a parametrized geometry and its application within a reduced order modeling technique. The full order model is based on the discontinuous Galerkin method with an interior penalty formulation. We introduce the broken Sobolev spaces as well as the weak formulation required for an affine parameter dependency. The operators are transformed from a fixed domain to a parameter dependent domain using the affine parameter dependency. The proper orthogonal decomposition is used to obtain the basis of functions of the reduced order model. By using the Galerkin projection the linear system is projected onto the reduced space. During this process, offline-online decomposition is used to separate parameter dependent operation from parameter independent operations. Finally this technique is applied to an obstacle test problem. The numerical outcomes presented include experimental error analysis, eigenvalue decay and measurement of online simulation time.
Keywords Discontinuous Galerkin method, Stokes flow, Geometric parametrization, Proper orthogonal decomposition

1.1 Introduction

The subject of the mathematical applications in fluid mechanics starts with one of the variants of the Navier-Stokes equation. In case of laminar flow, i.e. when

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fluctuations are negligible, this linearized form of the Navier-Stokes equation is the Stokes equation.

Discontinuous Galerkin Method (DGM) has found traction as numerical method for the elliptic problems [6] as well as the hyperbolic problems [2]. DGM uses polynomial approximation of a suitable degree providing higher accuracy as well as allows discontinuity at the interface, by the concept of numerical flux, allowing greater flexibility. This fact makes DGM naturally attractive to problems, such as shock capturing, due to presence of steep gradients or discontinuities. Additionally, since the Dirichlet conditions are applied as boundary penalty, it avoids the necessity to construct a subspace of Sobolev space.

Geometric parametrization has emerged as a important application of the Parametric Partial Differential Equations (PPDEs) and as an alternative to the shape optimization. The concept of geometric parametrization allows to transfer operator evaluated on one geometric domain to another geometric domain efficiently. Model Order Reduction (MOR) on the other hand allows reducing the size of the system to be solved by working with the smaller system containing only dominant components. It is pertinent to mention that identifying the "dominant" components is critical to the success of the model order reduction strategy. The faster computations obtained by MOR has helped in many query context, real time computation and quick transfer of computational results to industrial problems.

As evident from above advantages, the application of geometric parametrization and reduced order modeling to discontinuous Galerkin method will remain at the forefront of scientific work. The present work is aimed to contribute to this emerging field.

1.2 Geometric parametrization

Consider $\Omega = \Omega(\mu) \in \mathbb{R}^d$ as an open bounded domain. The parameter set $\mu \in \mathbb{P}$, where $\mathbb{P}$ is parameter space, completely characterizes the domain. Also, consider a parameter set $\bar{\mu} \in \mathbb{P}$, as the known parameter set and $\Omega(\bar{\mu})$ as the reference domain, whose configuration is completely known. The invertible mapping $\boldsymbol{F}(\cdot, \mu) : \Omega(\bar{\mu}) \to \Omega(\mu)$ links the reference domain and the parametrized domain. In the case of affine transformation, $\boldsymbol{F}$ is of the form,

$$\begin{aligned} x = \boldsymbol{F}(\hat{x}, \mu) = \boldsymbol{G_F}(\mu)\hat{x} + c_F(\mu) \ ; \forall x \in \Omega \, , \ \hat{x} \in \Omega(\bar{\mu}) \, , \\ \boldsymbol{G_F}(\mu) \in \mathbb{R}^{d\times d} \ , \ c_F \in \mathbb{R}^{d\times 1} \ . \end{aligned} \tag{1.1}$$

The boundary of $\Omega(\mu)$, that is $\partial\Omega(\mu)$ is divided into Neumann boundary $\Gamma_N(\mu)$ and Dirichlet boundary $\Gamma_D(\mu)$ i.e. $\partial\Omega(\mu) = \Gamma_N(\mu) \cup \Gamma_D(\mu)$. In order to have $\boldsymbol{F}(\hat{x}, \mu)$ affine form, the domain $\Omega(\mu)$ is divided into n_{su} triangular subdomains such that $\Omega(\mu) = \bigcup_{i=1}^{n_{su}} \Omega_i(\mu) \ , \ \Omega_i(\mu) \bigcap \Omega_j(\mu) = \emptyset \ , \ \text{for } i \neq j$.

1.3 Discontinuous Galerkin formulation

The domain Ω is divided into N_{el} number of triangular elements τ_k such that $\Omega = \bigcup_{k=1}^{N_{el}} \tau_k$. The triangulation $\mathcal{T}$ is the set of all triangular elements i.e. $\mathcal{T} = \{\tau_k\}_{k=1}^{N_{el}}$. The internal boundary is denoted by $\Gamma = \bigcup_{k=1}^{N_{el}} \partial\tau_k \backslash \partial\Omega$. The vertices of triangles $\{\tau_k\}_{k=1}^{N_{el}}$ are called nodes and $\overrightarrow{n}$ is the outward pointing normal to an edge of element.

The governing equations in strong form can be stated as,

$$\begin{aligned} \text{Stokes equation: } & -\nu\Delta\overrightarrow{u} + \nabla p = \overrightarrow{f}\,, \text{ in } \Omega\,, \\ \text{Continuity equation: } & \nabla \cdot \overrightarrow{u} = 0\,, \text{ in } \Omega\,, \\ \text{Dirichlet condition: } & \overrightarrow{u} = \overrightarrow{u}_D\,, \text{ on } \Gamma_D\,, \\ \text{Neumann condition: } & -p\overrightarrow{n} + \nu\overrightarrow{n} \cdot \nabla\overrightarrow{u} = \overrightarrow{t}\,, \text{ on } \Gamma_N\,. \end{aligned} \tag{1.2}$$

The velocity vector field $\overrightarrow{u}$ and pressure scalar field p are the unknowns. ν is the material property known as kinematic viscosity. Vector $\overrightarrow{f}$ is the external force term or source term. $\overrightarrow{u}_D$ is the Dirichlet velocity and vector $\overrightarrow{t}$ is the Neumann value.

Before introducing the weak form let us introduce the broken Sobolev spaces for the unknowns.

$$\begin{aligned} \text{For velocity: } & \mathbb{V} = \{\overrightarrow{\phi} \in (L^2(\Omega))^d |\, \overrightarrow{\phi}|_{\tau_k} \in (P^D(\tau_k))^d\,,\ \tau_k \in \mathcal{T}\}\,, \\ \text{For pressure: } & \mathbb{Q} = \{\psi \in (L^2(\Omega))|\, \psi|_{\tau_k} \in (P^{D-1}(\tau_k))\,,\ \tau_k \in \mathcal{T}\}\,. \end{aligned}$$

Here, $P^D(\tau_k)$ denotes space of polynomials of degree at most D, $D \geq 2$ over τ_k.

In finite dimensional or discrete system, velocity approximation $\overrightarrow{u}_h(x)$ and pressure approximation $p_h(x)$ at any point $x \in \Omega$ are given by,

$$\overrightarrow{u}_h(x) = \sum_{i=1}^{u_{ndofs}} \overrightarrow{\phi}_i \hat{u}_i\,,\ p_h(x) = \sum_{i=1}^{p_{ndofs}} \psi_i \hat{p}_i\,, \tag{1.3}$$

where $\hat{u}_i$'s and $\hat{p}_i$'s are coefficients of velocity basis functions and pressure basis functions respectively.

We expect that $\overrightarrow{u}_h \rightarrow \overrightarrow{u}$ and $p_h \rightarrow p$ as $u_{ndofs} \rightarrow \infty$ and $p_{ndofs} \rightarrow \infty$ respectively. Considering scope of present work, the convergence analysis will not be discussed here. The readers are advised to refer to [1], [7].

In the subsequent sections, $(\cdot)\,, (\cdot)_{\Gamma_D}\,, (\cdot)_{\Gamma_N}\,, (\cdot)_\Gamma$ represent the L^2 scalar product over $\Omega, \Gamma_D, \Gamma_N, \Gamma$ respectively. The jump operator $[\cdot]$ and the average operator $\{\cdot\}$ are important concepts in DGM formulation and are required to approximate the numerical flux. Different definitions are used in literatures for jump and mean operators. We use the jump and average operators as defined in [5].

The weak form of the Stokes equation is given by,

$$a_{IP}(\vec{u},\vec{\phi}) + b(\vec{\phi},p) + \left(\{p\},[\vec{n}\cdot\vec{\phi}]\right)_{\Gamma\cup\Gamma_D} = l_{IP}(\vec{\phi})\,, \tag{1.4}$$

$$\begin{aligned} a_{IP}(\vec{u},\vec{\phi}) &= \left(\nabla\vec{u},\nabla\vec{\phi}\right) + C_{11}\left([\vec{u}],[\vec{\phi}]\right)_{\Gamma\cup\Gamma_D} \\ -\nu\left(\{\nabla\vec{u}\},[\vec{n}\otimes\vec{\phi}]\right)_{\Gamma\cup\Gamma_D} &- \nu\left([\vec{n}\otimes\vec{u}],\{\nabla\vec{\phi}\}\right)_{\Gamma\cup\Gamma_D}\,, \end{aligned} \tag{1.5}$$

$$b(\vec{\phi},\psi) = -\int_\Omega \psi\nabla\cdot\vec{\phi}\,, \tag{1.6}$$

$$l_{IP}(\vec{\phi}) = \left(\vec{f},\vec{\phi}\right) + \left(\vec{t},\vec{\phi}\right)_{\Gamma_N} + C_{11}\left(\vec{u}_D,\vec{\phi}\right)_{\Gamma_D} - \left(\vec{n}\otimes\vec{u}_D,\nu\nabla\vec{\phi}\right)_{\Gamma_D}\,. \tag{1.7}$$

The penalty parameter $C_{11} > 0$ is an empirical constant to be kept large enough to maintain coercivity of $a_{IP}(\vec{u},\vec{\phi})$.

The weak form of the continuity equation is as follow,

$$b(\vec{u},\psi) + (\psi,[\vec{n}\cdot\vec{u}])_{\Gamma\cup\Gamma_D} = (\psi,\vec{n}\cdot\vec{u}_D)_{\Gamma_D}\,. \tag{1.8}$$

In discrete form the system of equations can be written as,

$$\underbrace{\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{B}^T & 0\end{pmatrix}}_{\text{Stiffness matrix}} \underbrace{\begin{pmatrix}U\\ P\end{pmatrix}}_{\text{Solution vector}} = \underbrace{\begin{pmatrix}F_1\\ F_2\end{pmatrix}}_{\text{Right hand side (Known)}}\,. \tag{1.9}$$

Here, $\boldsymbol{A}_{ij} = a_{IP}(\vec{\phi}_i,\vec{\phi}_j)$, $\boldsymbol{B}_{ij} = b(\vec{\phi}_i,\psi_j) + \left(\{\psi_j\},[n\cdot\vec{\phi}_i]\right)_{\Gamma\cup\Gamma_D}$, $F_1 = l_{IP}(\vec{\phi}_i)$ and $F_2 = \left(\psi_j,\vec{n}\cdot\vec{u}_D\right)_{\Gamma_D}$ for $i = 1,\ldots,u_{ndofs}$ and $j = 1,\ldots,p_{ndofs}$. The column vectors U and P are coefficients $\hat{u}_i$'s and $\hat{p}_i$'s respectively (equation (1.3)).

1.4 Affine expansion

We evaluate and solve the Stokes equation weak formulation on the reference domain $\Omega(\bar{\mu})$. Given a parameter set $\mu \neq \bar{\mu}$, we need to evaluate the linear systems of equation (1.9) on new domain $\Omega(\mu)$. To accomplish this we use affine expansion using linear nature of equation and diving $\Omega(\bar{\mu})$ into triangular subdomains $\Omega_i(\bar{\mu})$, $i = \{1,2,\ldots,n_{su}\}$ as explained earlier in the section geometric parametrization [Section 1.2]. The affine expansion of operators is essentially change of variable and has been explained in literatures such as [4]. However it is pertinent to explain two expansions as specific to DGM formulation.

- In order to transfer the terms containing jump and average operator following approach is used in present analysis.

$$\left(\{\nabla\vec{\phi}\}, \left[\vec{n} \otimes \vec{\phi}\right]\right) = \left(\nabla\vec{\phi}^+, \vec{n}^+ \otimes \vec{\phi}^+\right) + \left(\nabla\vec{\phi}^+, \vec{n}^- \otimes \vec{\phi}^-\right) + \left(\nabla\vec{\phi}^-, \vec{n}^+ \otimes \vec{\phi}^+\right) + \left(\nabla\vec{\phi}^-, \vec{n}^- \otimes \vec{\phi}^-\right) .$$

Each term on the right hand side of the above equation can be transformed using the affine map.

- The coercivity term $C_{11}\left([\vec{\phi}], [\vec{u}]\right)_{\Gamma \cup \Gamma_D}$ is not transformed but used as evaluated on reference domain $\Omega(\bar{\mu})$. The affine transformation is given by,

$$C_{11}\left([\vec{\phi}(\hat{x}), \vec{u}(\hat{x})]\right)_{\Gamma(\mu)\cup\Gamma_D(\mu)} = C_{11}\alpha\left([\vec{\phi}(\boldsymbol{F}(\hat{x})), \vec{u}(\boldsymbol{F}(\hat{x}))]\right)_{\Gamma(\bar{\mu})\cup\Gamma_D(\bar{\mu})} ,$$
$$\alpha = \frac{\text{length of } (\Gamma(\mu) \cup \Gamma_D(\mu))}{\text{length of } (\Gamma(\bar{\mu}) \cup \Gamma_D(\bar{\mu}))} , \ \hat{x} \in \Omega(\bar{\mu}) , \ x \in \Omega(\mu) .$$

Since, C_{11} is empirical coefficient replacing $C_{11}\alpha$ with C_{11} will not change the formulation as long as coercivity of a_{IP} over parameter space $\mathbb{P}$ is maintained.

1.5 Reduced basis method

In this section, the snapshot proper orthogonal decomposition method and the offline-online decomposition are briefly described. For detailed explanation, we refer to [4]. In first step, the solutions based on $\mu_n, n \in \{1, ..., n_s\}$ are calculated i.e. n_s snapshots are generated. The velocity snapshots and the pressure snapshots are stored in $\boldsymbol{S}_v \in \mathbb{R}^{u_{ndofs}\times n_s}$ and $\boldsymbol{S}_p \in \mathbb{R}^{p_{ndofs}\times n_s}$ respectively. Let us also introduce inner product matrices $\boldsymbol{M}_v \in \mathbb{R}^{u_{ndofs}\times u_{ndofs}}$ and $\boldsymbol{M}_p \in \mathbb{R}^{p_{ndofs}\times p_{ndofs}}$.

$$\boldsymbol{M}_{v,ij} = \int_\Omega \vec{\phi}_i \cdot \vec{\phi}_j + \sum_{k=1}^{N_{el}} \int_{\tau_k} \nabla\vec{\phi}_i : \nabla\vec{\phi}_j , \ i, j = 1, \ldots, u_{ndofs} ,$$
$$\boldsymbol{M}_{p,ij} = \int_\Omega \psi_i \psi_j , \ i, j = 1, \ldots, p_{ndofs} .$$

The dimension of reduced basis is denoted as N and it is asserted that $N << u_{ndofs}$, $N < n_s$. Next, the spectral decomposition of the snapshots is performed.

$$\boldsymbol{S}_v^T \boldsymbol{M}_v \boldsymbol{S}_v = \boldsymbol{V}\boldsymbol{\Theta}\boldsymbol{V}^T . \tag{1.10}$$

The columns of $\boldsymbol{V}$ are eigenvectors and Θ has eigenvalues θ_i , $1 \le i, j \le n_s$ in sorted order ($\theta_1 \ge \ldots \ge \theta_{n_s}$) such that, $\Theta_{ij} = \theta_i \delta_{ij}$.

The projection matrix $\boldsymbol{B}_v \in \mathbb{R}^{N\times N}$, used for the projection from the space of full order model to the space of reduced order model, is given by,

$$\boldsymbol{B}_v = \boldsymbol{S}_v \boldsymbol{V} \boldsymbol{\Theta}^{-\frac{1}{2}} \boldsymbol{R} , \ \boldsymbol{R} = [\boldsymbol{I}_{N\times N}; \boldsymbol{0}_{(n_s - N)\times N}] , \tag{1.11}$$

where, $\boldsymbol{I}_{N\times N}$ is the identity matrix of size $N\times N$. The reduced basis space $\boldsymbol{B}_p$ can be generated in similar manner using the pressure snapshots $\boldsymbol{S}_p$ and the inner product matrix $\boldsymbol{M}_p$. Above procedure is performed during offline phase.

The discrete system of equations is projected onto the reduced basis space by Galerkin projection as,

$$\underbrace{\begin{pmatrix} \boldsymbol{B}_v^T \boldsymbol{A}(\mu)\boldsymbol{B}_v & \boldsymbol{B}_v^T \boldsymbol{B}(\mu)\boldsymbol{B}_p \\ \boldsymbol{B}_p^T \boldsymbol{B}(\mu)^T \boldsymbol{B}_v & \boldsymbol{0} \end{pmatrix}}_{\tilde{K}} \underbrace{\begin{pmatrix} U_N \\ P_N \end{pmatrix}}_{\zeta} = \underbrace{\begin{pmatrix} \boldsymbol{B}_v^T F_1(\mu) \\ \boldsymbol{B}_p^T F_2(\mu) \end{pmatrix}}_{\tilde{F}} . \tag{1.12}$$

The solution vectors U and P (equation (1.9)) are then computed as, $U = \boldsymbol{B}_v U_N$, $P = \boldsymbol{B}_p P_N$.

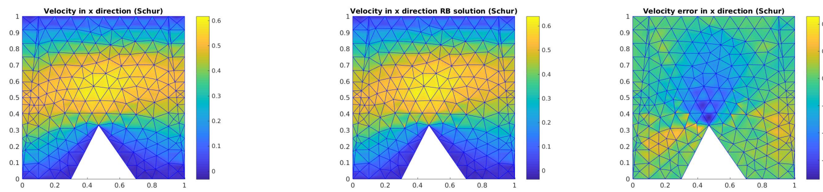
The projection onto reduced basis, solution of smaller system of equations and computation of U and P are steps performed during online phase.

1.6 Numerical example

We perform the POD-Galerkin method as mentioned in the section 1.5. The numerical experiments were performed using RBmatlab [3], [8]. The reference domain $\Omega(\bar{\mu})$ is the unit square domain $[0,1] \times [0,1]$ with triangle with vertices $(0.3,0), (0.5,0.3), (0.7,0)$ as obstacle. The geometric parameters are the coordinates of the tip of the obstacle i.e. $\bar{\mu} = (0.5, 0.3)$. The boundary $x = 0$ is Dirichlet boundary with inflow velocity at point $(0, y)$ as $u = (y(1-y), 0)$. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $\overrightarrow{t} = (0,0)$. Other boundaries are Dirichlet boundary with no slip condition. The source term is $\overrightarrow{f} = (0,0)$.

The training set was generated by random generation of 100 parameters within the interval $[0.4, 0.6]\times[0.4, 0.6]$. The test set contained 10 random parameters within the interval $[0.4, 0.6] \times [0.4, 0.6]$. For velocity basis function polynomial of degree $P^D = 2$ and for pressure basis function polynomial of degree $P^{D-1} = 1$ was used. The number of velocity degrees of freedom and pressure degrees of freedom were $u_{ndofs} = 4704$ and $p_{ndofs} = 1176$ respectively.

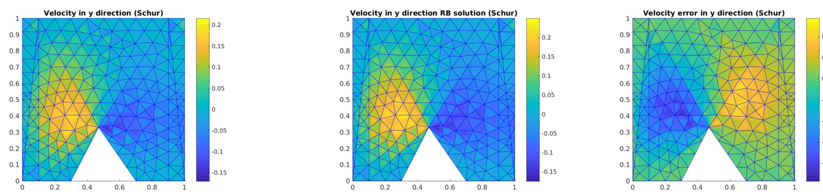
Figure 1.1 shows the solution computed by DGM and Reduced Basis (RB) at parameter value $\mu = (0.47, 33)$. Figure 1.2 shows error vs size of the reduced basis space. The drop in error w.r.t to the increased size of reduced basis is inline with the expectation based on the eigenvalue decay (Figure 1.3). The online simulation time for the reduced basis with 10 basis functions was 2.25 seconds resulting in the speedup of 19.5 as compared to the discontinuous Galerkin model.



(a) Velocity $x-$direction DGM solution

(b) Velocity $x-$direction RB solution

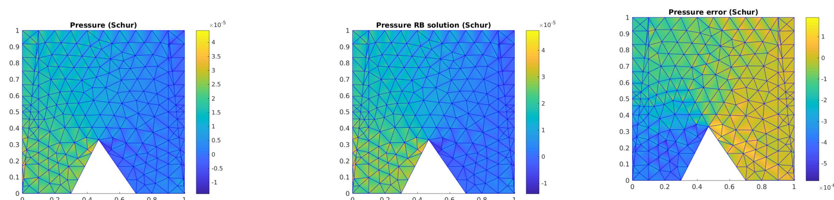
(c) $x-$component of Velocity absolute error $\vec{u}_h - \vec{u}_N$



(d) Velocity $y-$direction DGM solution

(e) Velocity $y-$direction RB solution

(f) $y-$component of Velocity absolute error $\vec{u}_h - \vec{u}_N$

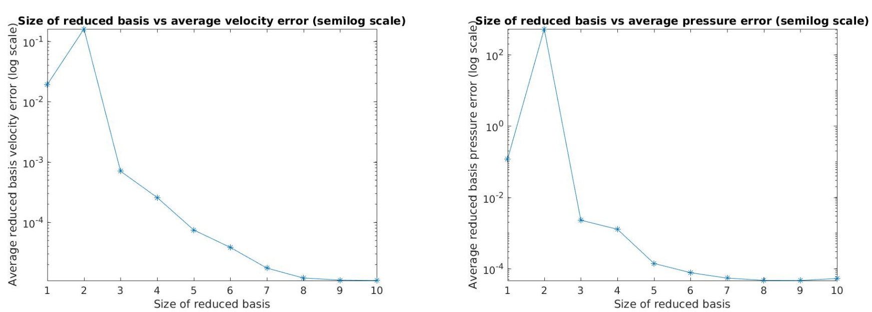


(g) Pressure DGM solution

(h) Pressure RB solution

(i) Pressure absolute error $p_h - p_N$

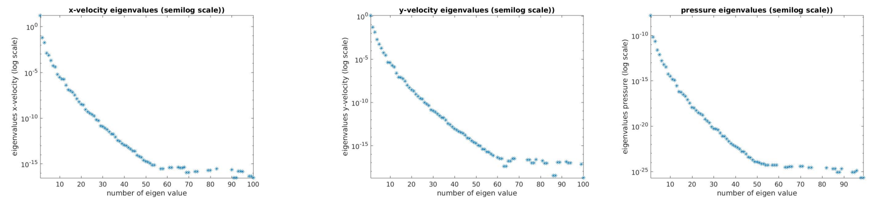
Fig. 1.1: DGM and RB solution $\mu = [\mu_x \ \mu_y] = [0.47 \ 0.33]$



(a) Size of reduced basis space vs. Average relative error in velocity with inner product induced by $\boldsymbol{M}_v$

(b) Size of reduced basis space vs. Average relative error in pressure with inner product induced by $\boldsymbol{M}_p$

Fig. 1.2: Size of reduced basis vs Average relative error



(a) $x-$Velocity eigenvalues (semilog scale)

(b) $y-$Velocity eigenvalues (semilog scale)

(c) Pressure eigenvalues (semilog scale)

Fig. 1.3: Eigenvalue decay

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