Chapter 6
Stokes flow

This chapter and the following one deal with CFD applications. The fluid flow is characterized by either the Stokes equations or the Navier–Stokes equations. The domain is two-dimensional, but the numerical methods and the analysis can be generalized to three-dimensional domains.

6.1 Preliminaries

6.1.1 Vector notation

The gradient of a vector function $\boldsymbol{v} : \mathbb{R}^d \to \mathbb{R}^d$ is a matrix, and the divergence of a matrix function $\boldsymbol{A} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a vector:

$$\nabla \boldsymbol{v} = \left(\frac{\partial v_i}{\partial x_j}\right)_{1\le i,j\le d}, \quad \nabla\cdot\boldsymbol{A} = \left(\sum_{j=1}^{d}\frac{\partial a_{ij}}{\partial x_j}\right)_{1\le i\le d}.$$

Consequently, we have for a vector function $\boldsymbol{v} = (v_i)_{1\le i\le d}$

$$\Delta\boldsymbol{v} = \nabla\cdot\nabla\boldsymbol{v} = (\Delta v_i)_{1\le i\le d}.$$

The L^2 inner product of two matrix functions $\boldsymbol{A}, \boldsymbol{B}$ is defined by

$$(\boldsymbol{A}, \boldsymbol{B})_\Omega = \int_\Omega \boldsymbol{A} : \boldsymbol{B} = \int_\Omega \sum_{1\le i,j\le d} A_{ij} B_{ij}.$$

6.1.2 Barycentric coordinates

Let E be a triangle with vertices A_1, A_2, A_3 and let $\lambda_1, \lambda_2, \lambda_3$ be the corresponding barycentric coordinates of a point M in E. The point M is the common vertex to three triangles E_1, E_2, E_3 whose union forms E (see Fig. 6.1). For instance, the vertices of triangle E_1 are the points M, A_2, A_3. Let $|E|$ denote the area of E. The barycentric coordinates are

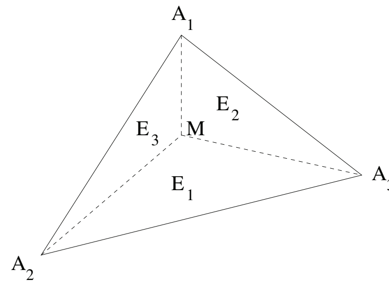

Figure 6.1. *Barycentric coordinates.*

defined by the ratio of two areas:

$$\lambda_i(M) = \frac{|E_i|}{|E|}.$$

Denote the edges of E by e_i such that $e_1 = [A_2, A_3]$, $e_2 = [A_3, A_1]$, $e_3 = [A_1, A_2]$ and denote the midpoint of e_i by B_i. Clearly, we have

$$\lambda_i(A_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases} \qquad \lambda_i(B_j) = \begin{cases} 0 & \text{for } i = j, \\ \frac{1}{2} & \text{for } i \neq j. \end{cases}$$

The next result is an application of the Gauss quadrature rule for $Q_G = 1$ given in Appendix A.

Lemma 6.1. *Let E be a triangle and let e denote one edge of E with midpoint B. Then, for all $v \in \mathbb{P}_1(E)$, we have*

$$\int_e v = v(B)|e|. \tag{6.1}$$

We now construct a basis of $\mathbb{P}_1(E)$ from the barycentric coordinates.

Lemma 6.2.

$$\mathbb{P}_1(E) = \text{span}(1 - 2\lambda_1, 1 - 2\lambda_2, 1 - 2\lambda_3).$$

Proof. Since $\dim(\mathbb{P}_1) = 3$, it suffices to show that these functions are linearly independent. Assume that there are coefficients $\alpha_1, \alpha_2, \alpha_3$ such that

$$\sum_{i=1}^{3} \alpha_i (1 - 2\lambda_i) = 0.$$

Fix an edge e_k and integrate over e_k:

$$0 = \int_{e_k} \sum_{i=1}^{3} \alpha_i (1 - 2\lambda_i) = \sum_{i=1}^{3} \alpha_i |e_k| (1 - 2\lambda_i)(B_k) = \alpha_k |e_k|.$$

Thus, $\alpha_k = 0$ for $k = 1, \ldots, 3$. $\square$

Therefore, any linear v can be written as

$$v = \sum_{i=1}^{3} v_i(1 - 2\lambda_i), \quad v_i \in \mathbb{R}.$$

The coefficients v_i are obtained by evaluating v at the midpoint B_i. Hence, we have

$$\forall v \in \mathbb{P}_1(E), \quad v = \sum_{i=1}^{3} v(B_i)(1 - 2\lambda_i). \tag{6.2}$$

6.1.3 An approximation operator of degree one

Let $\mathcal{E}_h$ denote a triangular mesh of a bounded domain Ω (see Section 2.3). We first define a local operator $\pi : H^1(E) \to \mathbb{P}_1(E)$ for any given mesh element.

We fix a triangle E with edges e_1, e_2, e_3, and we define, for any $v \in H^1(E)$, the polynomial $\pi v \in \mathbb{P}_1(E)$ such that

$$\int_{e_k} \pi v = \int_{e_k} v, \quad k = 1, 2, 3.$$

This uniquely defines πv because, from (6.2), it suffices to determine $\pi v(B_k)$ for $k = 1, 2, 3$ (with B_k being the midpoint of the edge e_k). From (6.1), we obtain

$$\int_{e_k} v = \int_{e_k} \pi v = |e_k| \pi v(B_k).$$

Equivalently,

$$\pi v(B_k) = \frac{1}{|e_k|} \int_{e_k} v.$$

The degrees of freedom of the linear πv are associated with the midpoints of the edges of E and defined by $\frac{1}{|e_k|} \int_{e_k} v$.

Furthermore, we have

$$\forall v \in \mathbb{P}_1(E), \quad \pi v = v.$$

Indeed, from (6.2), we have

$$\pi v - v = \sum_{i=1}^{3} (\pi v(B_i) - v(B_i))(1 - 2\lambda_i)$$

and

$$\pi v(B_i) = \frac{1}{|e_i|} \int_{e_i} v = \frac{1}{|e_i|} |e_i| v(B_i) = v(B_i).$$

Let us now define the approximation operator $\boldsymbol{R} : H^1(\Omega)^2 \to \mathcal{D}_1(\mathcal{E}_h)$ such that

$$\forall \boldsymbol{v} = (v_1, v_2) \in H^1(\Omega)^2, \quad \forall E \in \mathcal{E}_h, \quad \boldsymbol{R}\boldsymbol{v}|_E = (\pi v_1, \pi v_2).$$

In other words, if e denotes any edge in the mesh, we have

$$\int_e \boldsymbol{R}\boldsymbol{v} = \int_e \boldsymbol{v}. \tag{6.3}$$

The operator $\boldsymbol{R}$ is called the Crouzeix–Raviart operator [39]. The function $\boldsymbol{R}\boldsymbol{v}$ is discontinuous across the edges except at the midpoints.

Lemma 6.3. *The operator $\boldsymbol{R}$ satisfies*

$$\begin{aligned}
\forall \boldsymbol{v} \in H^1(\Omega)^2, \quad & \int_E \nabla \cdot (\boldsymbol{R}\boldsymbol{v} - \boldsymbol{v}) = 0,\\
\forall \boldsymbol{v} \in H^1(\Omega)^2, \quad \forall e \in \Gamma_h, \quad & \int_e [\boldsymbol{R}(\boldsymbol{v})] = \boldsymbol{0},\\
\forall \boldsymbol{v} \in H^1_0(\Omega)^2, \quad \forall e \in \partial\Omega, \quad & \int_e \boldsymbol{R}\boldsymbol{v} = \boldsymbol{0},\\
\forall \boldsymbol{v} \in H^2(\Omega)^2, \quad & \|\nabla(\boldsymbol{v} - \boldsymbol{R}\boldsymbol{v})\|_{L^2(E)} \le C h_E \|\nabla^2 \boldsymbol{v}\|_{L^2(E)}.
\end{aligned}$$

Proof. The first equality is proved by Green's formula and by (6.3):

$$\int_E \nabla(\boldsymbol{R}\boldsymbol{v} - \boldsymbol{v}) = \int_{\partial E} (\boldsymbol{R}\boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_E = \sum_{e \in \partial E} \left(\int_e (\boldsymbol{R}\boldsymbol{v} - \boldsymbol{v}) \right) \cdot \boldsymbol{n}_E = 0.$$

The second result is trivial:

$$\int_e [\boldsymbol{R}\boldsymbol{v}] = \int_e [\boldsymbol{R}\boldsymbol{v} - \boldsymbol{v}] = \int_e (\boldsymbol{R}\boldsymbol{v} - \boldsymbol{v})|_{E_1} - \int_e (\boldsymbol{R}\boldsymbol{v} - \boldsymbol{v})|_{E_2} = 0.$$

The result is similar for the third equality:

$$\int_e \boldsymbol{R}\boldsymbol{v} = \int_e (\boldsymbol{R}\boldsymbol{v} - \boldsymbol{v}) = \boldsymbol{0}.$$

Finally, the last result holds true because $\boldsymbol{R}\boldsymbol{v} = \boldsymbol{v}$ if $\boldsymbol{v} \in \mathbb{P}_1(E)^2$ (see [59]). □

6.1.4 An approximation operator of higher degree

There exists a similar operator $\boldsymbol{R}$ such that $\boldsymbol{R}\boldsymbol{v}|_E \in \mathbb{P}_k(E)^2$ for all triangles E and for $k = 2$ and $k = 3$ (see [56, 38, 63]). This operator satisfies for any E

$$\forall \boldsymbol{v} \in H^1(\Omega)^2, \quad \forall q \in \mathbb{P}_{k-1}(E), \quad \int_E q \nabla \cdot (\boldsymbol{R}\boldsymbol{v} - \boldsymbol{v}) = 0, \tag{6.4}$$

$$\forall \boldsymbol{v} \in H^1(\Omega)^2, \quad \forall e \in \Gamma_h, \quad \forall \boldsymbol{q} \in \mathbb{P}_{k-1}(e)^2, \quad \int_e \boldsymbol{q} \cdot [\boldsymbol{R}\boldsymbol{v}] = 0, \tag{6.5}$$

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^2, \quad \forall e \in \partial\Omega, \quad \forall \boldsymbol{q} \in \mathbb{P}_{k-1}(e)^2, \quad \int_e \boldsymbol{q} \cdot \boldsymbol{R}\boldsymbol{v} = 0, \tag{6.6}$$

$$\forall s \in [1, k+1], \quad \forall \boldsymbol{v} \in H^s(\Omega)^2, \quad \|\boldsymbol{v} - \boldsymbol{R}\boldsymbol{v}\|_{L^2(\Omega)} + h_E \|\nabla(\boldsymbol{v} - \boldsymbol{R}\boldsymbol{v})\|_{L^2(E)} \leq C h_E^s |\boldsymbol{v}|_{H^s(\Delta_E)}, \tag{6.7}$$

where Δ_E is a suitable macroelement containing E. Furthermore, each triangle $E \in E_h^i$ has at least one side e such that

$$\forall \boldsymbol{v} \in H^1(\Omega)^2, \quad \int_e (\boldsymbol{R}\boldsymbol{v} - \boldsymbol{v}) = \boldsymbol{0}. \tag{6.8}$$

6.1.5 Local L^2 projection

We fix a mesh element E. Let $p \in H^s(E)$ and let $\tilde{p}$ denote the L^2 projection of p onto $\mathbb{P}_k(E)$ defined by

$$\forall v \in \mathbb{P}_k(E), \quad \int_E (p - \tilde{p}) v = 0.$$

Then, there is a constant C independent of h_E such that

$$\|p - \tilde{p}\|_{L^2(E)} + h_E \|\nabla(p - \tilde{p})\|_{L^2(E)} \leq C h_E^{\min(k+1,s)} \|p\|_{H^s(E)}.$$

6.1.6 General inf-sup condition

We present the inf-sup condition in a general setting first [59]. Let $b : X \times M \to \mathbb{R}$ be a continuous bilinear form defined on two Hilbert spaces X and M. Let $\|\cdot\|_X$ (resp., $\|\cdot\|_Y$) and $(\cdot,\cdot)_X$ (resp., $(\cdot,\cdot)_Y$) denote the norm and inner product on X (resp., Y). The spaces X and M satisfy an inf-sup condition [7, 18] if there is a constant $\beta > 0$ such that

$$\inf_{q \in M} \sup_{v \in X} \frac{b(v,q)}{\|q\|_M \|v\|_X} \geq \beta. \tag{6.9}$$

We denote by X' and M' the dual spaces of X and M. We define the mappings $B : X \to M'$ and $B' : M \to X'$ by

$$\forall v \in X, \quad \forall q \in M, \quad Bv(q) = B'q(v) = b(v,q).$$

We also define the kernel of B, its orthogonal set, and its polar set:

$$V = \operatorname{Ker}(B) = \{v \in X : \forall q \in M,\ b(v,q) = 0\},$$
$$V^\perp = \{w \in X : \forall v \in V,\ (w,v)_X = 0\},$$
$$V^\circ = \{\phi \in X' : \forall v \in V,\ \phi(v) = 0\}.$$

Lemma 6.4. *The following statements are equivalent.*

(i) *The* inf-sup *condition* (6.9) *holds true.*

(ii) *The mapping B is an isomorphism from $V^\perp$ onto M' and*

$$\forall v \in V^\perp, \quad \|Bv\|_{M'} \geq \beta \|v\|_X.$$

(iii) *The mapping B' is an isomorphism from M onto V° and*

$$\forall q \in M, \quad \|B'q\|_{X'} \geq \beta \|q\|_M.$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. We now give an example of spaces that satisfy an inf-sup condition, namely the spaces $H_0^1(\Omega)$ and $L_0^2(\Omega)$. The space $L_0^2(\Omega)$ is the space of square-integrable functions with zero average:

$$L_0^2(\Omega) = \left\{ v \in L^2(\Omega) : \quad \int_\Omega v = 0 \right\}.$$

There exists a positive constant β such that

$$\inf_{q \in L_0^2(\Omega)} \sup_{v \in H_0^1(\Omega)^2} \frac{(\nabla \cdot v, q)_\Omega}{\|q\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}} \geq \beta. \tag{6.10}$$

Equivalently, from statement (ii), for any $q \in L_0^2(\Omega)$, there is a function $v \in H_0^1(\Omega)^2$ such that

$$\nabla \cdot v = q, \quad \|\nabla v\|_{L^2(\Omega)} \leq \frac{1}{\beta} \|q\|_{L^2(\Omega)}.$$

6.2 Model problem and weak solution

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. The Stokes equations for an incompressible viscous fluid confined in Ω are

$$-\mu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{6.11}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \tag{6.12}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega. \tag{6.13}$$

The unknown variables are the fluid velocity $\boldsymbol{u}$ and the fluid pressure p. The constant $\mu > 0$ is the fluid viscosity; the function $\boldsymbol{f}$ is a body force acting on the fluid. Equation (6.11) is referred to as the momentum equation, whereas (6.12) is the incompressibility equation (or continuity equation). Since p is uniquely defined up to an additive constant, we also assume that $\int_\Omega p = 0$. If we assume that $\boldsymbol{f} \in L^2(\Omega)^2$, a weak solution to (6.11)–(6.13) is the pair $(\boldsymbol{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ satisfying

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^2, \quad \mu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega - (\nabla \cdot \boldsymbol{v}, p)_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega, \tag{6.14}$$

$$\forall q \in L_0^2(\Omega), \quad (\nabla \cdot \boldsymbol{u}, q)_\Omega = 0. \tag{6.15}$$

The space of divergence-free vector functions is defined by

$$\boldsymbol{V} = \{\boldsymbol{v} \in H_0^1(\Omega)^2 : \forall q \in L_0^2(\Omega), (\nabla \cdot \boldsymbol{v}, q)_\Omega = 0\}.$$

The space $\boldsymbol{V}$ is equipped with the norm $\boldsymbol{v} \mapsto \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}$. Clearly, if $(\boldsymbol{u}, p)$ is a weak solution satisfying (6.14), (6.15), then $\boldsymbol{u}$ is a solution to the following problem:

$$\forall \boldsymbol{v} \in \boldsymbol{V}, \quad \mu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega. \tag{6.16}$$

One can check that the bilinear form $(\boldsymbol{v}, \boldsymbol{w}) \mapsto \mu(\nabla \boldsymbol{v}, \nabla \boldsymbol{w})_\Omega$ is continuous and coercive and that the linear form $\boldsymbol{v} \mapsto (\boldsymbol{f}, \boldsymbol{v})_\Omega$ is continuous. Lax–Milgram's theorem (Theorem 2.8) implies that there is a unique $\boldsymbol{u} \in \boldsymbol{V}$ satisfying (6.16). Next, we consider the mapping

$$\Phi : \boldsymbol{v} \mapsto \mu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega - (\boldsymbol{f}, \boldsymbol{v})_\Omega.$$

The mapping Φ belongs to the dual space of $H_0^1(\Omega)^2$ and vanishes on the space $\boldsymbol{V}$ since $\boldsymbol{u}$ satisfies (6.16). Therefore, Φ belongs to the polar space $\boldsymbol{V}^\circ$. From the inf-sup condition (6.10) and from (ii) in Lemma 6.4, there is a unique $p \in L_0^2(\Omega)$ satisfying

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^2, \quad (\nabla \cdot \boldsymbol{v}, p) = \Phi(\boldsymbol{v}).$$

This is equivalent to (6.14). Thus, we have proved that there is a unique weak solution to (6.11)–(6.13).

6.3 DG scheme

Let $\mathcal{E}_h$ be a mesh of Ω as defined in Section 2.3. For any integer $k \geq 1$, we define the discrete velocity and pressure spaces:

$$\begin{aligned} \boldsymbol{X}_h &= \{\boldsymbol{v} \in L^2(\Omega)^2 : \forall E \in \mathcal{E}_h, \boldsymbol{v} \in (\mathbb{P}_k(E))^2\}, \\ M_h &= \{q \in L_0^2(\Omega) : \forall E \in \mathcal{E}_h, q \in \mathbb{P}_{k-1}(E)\}. \end{aligned}$$

We introduce the bilinear forms $a_\epsilon : \boldsymbol{X}_h \times \boldsymbol{X}_h \to \mathbb{R}$ and $b : \boldsymbol{X}_h \times M_h \to \mathbb{R}$ corresponding to DG discretizations of the diffusive term $-\Delta \boldsymbol{u}$ and the pressure term ∇p, respectively:

$$\begin{aligned} a_\epsilon(\boldsymbol{w}, \boldsymbol{v}) = \sum_{E \in \mathcal{E}_h} \int_E \nabla \boldsymbol{w} : \nabla \boldsymbol{v} + \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{\sigma_e^0}{|e|} \int_e [\boldsymbol{w}] \cdot [\boldsymbol{v}] \\ - \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\nabla \boldsymbol{w}\}\boldsymbol{n}_e \cdot [\boldsymbol{v}] + \epsilon \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{\nabla \boldsymbol{v}\}\boldsymbol{n}_e \cdot [\boldsymbol{w}], \end{aligned} \tag{6.17}$$

$$b(\boldsymbol{v}, q) = -\sum_{E \in \mathcal{E}_h} \int_E q \nabla \cdot \boldsymbol{v} + \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{q\}[\boldsymbol{v}] \cdot \boldsymbol{n}_e. \tag{6.18}$$

As usual, the choice of the parameter ϵ will yield the NIPG method ($\epsilon = 1$), the SIPG method ($\epsilon = -1$), or the IIPG method ($\epsilon = 0$). The penalty parameter is denoted by σ_e^0 for an edge e and is strictly positive. For simplicity, we do not assume superpenalization. The derivation of the bilinear form a_ϵ is similar to the one for the elliptic problem. We now give some details on the form b. Using Green's theorem on one mesh element E, we have

$$\int_E \nabla p \cdot \boldsymbol{v} = -\int_E p \nabla \cdot \boldsymbol{v} + \int_{\partial E} p \boldsymbol{v} \cdot \boldsymbol{n}_E.$$

We sum over all mesh elements and use the normal vector $\boldsymbol{n}_e$ fixed for each edge:

$$\sum_{E \in \mathcal{E}_h} \int_E \nabla p \cdot \boldsymbol{v} = -\sum_{E \in \mathcal{E}_h} \int_E p \nabla \cdot \boldsymbol{v} + \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e [p \boldsymbol{v} \cdot \boldsymbol{n}_e].$$

Finally, since p is continuous, we have $p|_e = \{p\}|_e$, and we can write

$$\sum_{E \in \mathcal{E}_h} \int_E \nabla p \cdot \boldsymbol{v} = -\sum_{E \in \mathcal{E}_h} \int_E p \nabla \cdot \boldsymbol{v} + \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{p\}[\boldsymbol{v} \cdot \boldsymbol{n}_e],$$

which is exactly the expression $b(\boldsymbol{v}, p)$.

With these spaces and bilinear forms, the numerical method is as follows: Find $(\boldsymbol{U}_h, P_h) \in \boldsymbol{X}_h \times M_h$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad \mu a_\epsilon(\boldsymbol{U}_h, \boldsymbol{v}) + b(\boldsymbol{v}, P_h) = (\boldsymbol{f}, \boldsymbol{v})_\Omega, \tag{6.19}$$

$$\forall q \in M_h, \quad b(\boldsymbol{U}_h, q) = 0. \tag{6.20}$$

Next, we state a consistency result and a coercivity result. The proofs are omitted, as they are similar to the proofs given in the previous chapters.

Lemma 6.5. *Assume that the weak solution* $(\boldsymbol{u}, p)$ *also belongs to* $H^2(\mathcal{E}_h)^2 \times H^1(\mathcal{E}_h)$. *Then, it satisfies the scheme* (6.19)–(6.20).

We define the energy norm for the Stokes problem:

$$\forall \boldsymbol{v} \in H^1(\mathcal{E}_h)^2, \quad \|\boldsymbol{v}\|_{\mathcal{E}} = \left(\sum_{E\in\mathcal{E}_h} \|\nabla \boldsymbol{v}\|^2_{L^2(E)} + \sum_{e\in\Gamma_h\cup\partial\Omega} \frac{\sigma_e^0}{|e|} \|[\boldsymbol{v}]\|^2_{L^2(e)} \right)^{1/2}. \tag{6.21}$$

Lemma 6.6. *Assume that* σ_0^e *is sufficiently large if* $\epsilon = -1$ *or* $\epsilon = 0$. *Then, there is a constant* $\kappa > 0$ *independent of h such that*

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad a_\epsilon(\boldsymbol{v}, \boldsymbol{v}) \geq \kappa \|\boldsymbol{v}\|^2_{\mathcal{E}}.$$

6.3.1 Existence and uniqueness of solution

Since problem (6.19), (6.20) results in a square system of linear equations in finite dimension, it suffices to prove uniqueness of the solution. Let $\boldsymbol{W}_h$ denote the difference of two solutions. Set the data $\boldsymbol{f} = \boldsymbol{0}$ and choose $\boldsymbol{v} = \boldsymbol{W}_h$ in (6.19). Since $\boldsymbol{W}_h$ satisfies (6.20), we are left with

$$a_\epsilon(\boldsymbol{W}_h, \boldsymbol{W}_h) = 0.$$

The coercivity of a_ϵ yields that $\boldsymbol{W}_h = 0$. Thus, (6.19) is reduced to

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad b(\boldsymbol{v}, P_h) = 0.$$

At this point, one cannot conclude that $P_h = 0$. This result is a consequence of the inf-sup condition established in Section 6.4.

6.3.2 Local mass conservation

In this context, local mass conservation is a consequence of the discretization of the incompressibility equation. We fix an element $E \in \mathcal{E}_h$ and choose a function $q = 1$ on E and zero elsewhere. Equation (6.20) becomes

$$-\int_E \nabla \cdot \boldsymbol{U}_h + \frac{1}{2} \sum_{e\in\partial E\setminus\partial\Omega} \int_e [\boldsymbol{U}_h] \cdot \boldsymbol{n}_e + \sum_{e\in\partial E\cap\partial\Omega} \int_e \boldsymbol{U}_h \cdot \boldsymbol{n}_e = 0.$$

Denoting by $\boldsymbol{n}_E$ the outward normal to E and using Green's formula, we have

$$-\int_{\partial E} \boldsymbol{U}_h \cdot \boldsymbol{n}_E + \frac{1}{2} \sum_{e \in \partial E \setminus \partial\Omega} \int_e [\boldsymbol{U}_h] \cdot \boldsymbol{n}_e + \sum_{e \in \partial E \cap \partial\Omega} \int_e \boldsymbol{U}_h \cdot \boldsymbol{n}_e = 0,$$

or equivalently

$$\sum_{e \in \partial E \setminus \partial\Omega} \int_e \{\boldsymbol{U}_h\} \cdot \boldsymbol{n}_E = 0.$$

We remark that this equation is comparable to the local mass balance satisfied by the exact solution

$$\sum_{e \in \partial E \setminus \partial\Omega} \int_e \boldsymbol{u} \cdot \boldsymbol{n}_E = 0.$$

6.4 Discrete inf-sup condition

In this section, we prove an inf-sup condition for the spaces $\tilde{\boldsymbol{X}}_h$ and M_h, where $\tilde{\boldsymbol{X}}_h$ is a subspace of $\boldsymbol{X}_h$:

$$\tilde{\boldsymbol{X}}_h = \{\boldsymbol{v}_h \in \boldsymbol{X}_h : \forall e \in \Gamma_h \cup \partial\Omega,\, [\boldsymbol{v}_h]|_e \cdot \boldsymbol{n}_e = 0\}.$$

The proof relies on the Raviart–Thomas interpolant [88, 59, 89] defined in the following lemma.

Lemma 6.7. *The Raviart–Thomas interpolant $\pi : H^1(\Omega)^2 \to \boldsymbol{X}_h$ satisfies for all $\boldsymbol{v} \in H^1(\Omega)^2$*

$$\forall E \in \mathcal{E}_h,\ \forall q \in \mathbb{P}_{k-1}(E),\ \int_E q \nabla \cdot (\pi \boldsymbol{v} - \boldsymbol{v}) = 0, \tag{6.22}$$

$$\forall e \in \Gamma_h \cup \partial\Omega,\ \forall q \in \mathbb{P}_{k-1}(e),\ \int_e q(\pi \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_e = 0, \tag{6.23}$$

$$\forall e \in \Gamma_h \cup \partial\Omega,\ \pi \boldsymbol{v}|_e \cdot \boldsymbol{n}_e \in \mathbb{P}_{k-1}(e), \tag{6.24}$$

$$\forall E \in \mathcal{E}_h,\ \|\pi \boldsymbol{v} - \boldsymbol{v}\|_{L^2(E)} + h_E \|\nabla(\pi \boldsymbol{v} - \boldsymbol{v})\|_{L^2(E)} \leq C h_E \|\nabla \boldsymbol{v}\|_{L^2(E)}, \tag{6.25}$$

$$\|\pi \boldsymbol{v}\|_{\mathcal{E}} \leq C \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \tag{6.26}$$

with a constant C independent of h_E and h.

Theorem 6.8. *There exists a constant $\beta^* > 0$, independent of h, such that*

$$\inf_{q \in M_h} \sup_{\boldsymbol{v} \in \tilde{\boldsymbol{X}}_h} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\mathcal{E}} \|q\|_{L^2(\Omega)}} \geq \beta^*. \tag{6.27}$$

Proof. We shall prove that for any $q \in M_h$ there exists $\boldsymbol{v}$ in $\tilde{\boldsymbol{X}}_h$ such that

$$b(\boldsymbol{v}, q) \geq \beta_1^* \|q\|^2_{L^2(\Omega)}, \tag{6.28}$$

$$\|\boldsymbol{v}\|_{\mathcal{E}} \leq \beta_2^* \|q\|_{L^2(\Omega)} \tag{6.29}$$

with constants $\beta_1^* > 0$ and $\beta_2^* > 0$ independent of h, q, and $\boldsymbol{v}$. Clearly, this will imply (6.27) because

$$\frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\mathcal{E}}\|q\|_{L^2(\Omega)}} \geq \beta_1^* \frac{\|q\|^2_{L^2(\Omega)}}{\|\boldsymbol{v}\|_{\mathcal{E}}\|q\|_{L^2(\Omega)}} = \beta_1^* \frac{\|q\|_{L^2(\Omega)}}{\|\boldsymbol{v}\|_{\mathcal{E}}} \geq \frac{\beta_1^*}{\beta_2^*}.$$

Let $q \in M_h$. Since $q \in L^2_0(\Omega)$ and the spaces $H^1_0(\Omega)^2$, $L^2_0(\Omega)$ satisfy the inf-sup condition (6.10), there exists $\tilde{\boldsymbol{v}} \in H^1_0(\Omega)^2$ such that

$$-\nabla \cdot \tilde{\boldsymbol{v}} = q, \quad \|\nabla \tilde{\boldsymbol{v}}\|_{L^2(\Omega)} \leq \frac{1}{\beta}\|q\|_{L^2(\Omega)}. \tag{6.30}$$

Define the Raviart–Thomas interpolant $\tilde{\boldsymbol{v}}_h = \pi \tilde{\boldsymbol{v}}$. Then, $\tilde{\boldsymbol{v}}_h \in \tilde{\boldsymbol{X}}_h$ from (6.23) and (6.24). We also have from (6.22) and (6.25)

$$-\sum_{E\in\mathcal{E}_h}\int_E (\nabla \cdot \tilde{\boldsymbol{v}}_h) q = -\sum_{E\in\mathcal{E}_h}\int_E (\nabla \cdot \tilde{\boldsymbol{v}}) q = \|q\|^2_{L^2(\Omega)}.$$

Therefore, it follows that

$$b(\tilde{\boldsymbol{v}}_h, q) = \|q\|^2_{L^2(\Omega)}.$$

This implies (6.28) with the constant $\beta_1^* = 1$. Property (6.26) and inequality (6.30) imply that

$$\|\tilde{\boldsymbol{v}}_h\|_{\mathcal{E}} \leq C\|q\|_{L^2(\Omega)}.$$

This concludes the proof. $\square$

Remark: Define the subspace of $\boldsymbol{X}_h$:

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{X}_h; \forall q \in M_h, b(\boldsymbol{v}, q) = 0\}.$$

We say that $\boldsymbol{V}_h$ is the space of discretely divergence-free functions. An immediate consequence of Lemma 6.4 is that for a given q in M_h there exists a unique $\boldsymbol{v}$ in $\tilde{\boldsymbol{X}}_h$ such that

$$\forall \boldsymbol{w} \in \boldsymbol{V}_h, \quad \sum_{E\in\mathcal{E}_h}\int_E \nabla\boldsymbol{v} : \nabla\boldsymbol{w} + \sum_{e\in\Gamma_h\cup\partial\Omega}\frac{\sigma_e^0}{|e|}\int_e [\boldsymbol{v}]\cdot[\boldsymbol{w}] = 0,$$

$$b(\boldsymbol{v}, q) = -\|q\|^2_{L^2(\Omega)}, \quad \|\boldsymbol{v}\|_{\mathcal{E}} \leq \frac{1}{\beta^*}\|q\|_{L^2(\Omega)}.$$

6.5 Error estimates

We prove optimal error estimates for the velocity and pressure. First, we need a lemma on the approximation operator $\boldsymbol{R}$.

Lemma 6.9.

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^2, \quad \forall q \in M_h, \quad b(\boldsymbol{R}\boldsymbol{v} - \boldsymbol{v}, q) = 0. \tag{6.31}$$

Proof. Let $\boldsymbol{v} \in H_0^1(\Omega)^2$ and $q \in M_h$. Then,

$$b(\boldsymbol{Rv} - \boldsymbol{v}, q) = -\sum_{E\in\mathcal{E}_h} \int_E q\nabla \cdot (\boldsymbol{Rv} - \boldsymbol{v}) + \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \{q\}[\boldsymbol{Rv} - \boldsymbol{v}] \cdot \boldsymbol{n}_e.$$

The result is then a consequence of (6.4)–(6.6). □

Theorem 6.10. *Assume that the exact solution* $(\boldsymbol{u}, p)$ *belongs to* $H^{k+1}(\Omega)^2 \times H^k(\Omega)$. *Then, the solution* $(\boldsymbol{U}_h, P_h)$ *of* (6.19), (6.20) *satisfies the error estimate*

$$\|\boldsymbol{u} - \boldsymbol{U}_h\|_{\mathcal{E}} \le Ch^k \left(|\boldsymbol{u}|_{H^{k+1}(\Omega)} + \frac{1}{\mu}|p|_{H^k(\Omega)} \right), \tag{6.32}$$

where C is independent of h and μ.

Proof. Denote $\boldsymbol{\chi} = \boldsymbol{U}_h - \boldsymbol{Ru}$ and $\xi = P_h - \tilde{p}$, where $\tilde{p}$ is the L^2 projection of p. The errors $\boldsymbol{\chi}$ and ξ satisfy the equations

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad \mu a_\epsilon(\boldsymbol{\chi}, \boldsymbol{v}) + b(\boldsymbol{v}, \xi) = \mu a_\epsilon(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{v}) + b(\boldsymbol{v}, p - \tilde{p}),$$
$$\forall q \in M_h, \quad b(\boldsymbol{\chi}, q) = b(\boldsymbol{u} - \tilde{\boldsymbol{u}}, q).$$

Choosing $\boldsymbol{v} = \boldsymbol{\chi}$, $q = \xi$ and using (6.31), we obtain

$$a_\epsilon(\boldsymbol{\chi}, \boldsymbol{\chi}) = a_\epsilon(\boldsymbol{u} - \boldsymbol{Ru}, \boldsymbol{\chi}) + \frac{1}{\mu} b(\boldsymbol{\chi}, p - \tilde{p}), \tag{6.33}$$

which yields by coercivity of a_ϵ

$$\kappa \|\boldsymbol{\chi}\|_{\mathcal{E}}^2 \le a_\epsilon(\boldsymbol{u} - \boldsymbol{Ru}, \boldsymbol{\chi}) + \frac{1}{\mu} b(\boldsymbol{\chi}, p - \tilde{p}).$$

Then, we need only bound the two terms on the right-hand side. Throughout this chapter, the generic constant C is independent of the mesh size h and the fluid viscosity μ. By definition, we have

$$\begin{aligned} a_\epsilon(\boldsymbol{u} - \boldsymbol{Ru}, \boldsymbol{\chi}) &= \sum_{E\in\mathcal{E}_h} \int_E \nabla(\boldsymbol{u} - \boldsymbol{Ru}) : \nabla\boldsymbol{\chi} - \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \{\nabla(\boldsymbol{u} - \boldsymbol{Ru})\}\boldsymbol{n}_e \cdot [\boldsymbol{\chi}] \\ &\quad + \epsilon \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \{\nabla\boldsymbol{\chi}\}\boldsymbol{n}_e \cdot [\boldsymbol{u} - \boldsymbol{Ru}] + \sum_{e\in\Gamma_h\cup\partial\Omega} \frac{\sigma_e^0}{|e|} \int_e [\boldsymbol{u} - \boldsymbol{Ru}] \cdot [\boldsymbol{\chi}] \\ &= T_1 + T_2 + T_3 + T_4. \end{aligned}$$

Using Cauchy–Schwarz's inequality and the bound (6.7), we have

$$\begin{aligned} T_1 &\le \frac{\kappa}{10} \|\!|\nabla\boldsymbol{\chi}|\!\|_{L^2(\Omega)}^2 + C \sum_{E\in\mathcal{E}_h} \|\nabla(\boldsymbol{u} - \boldsymbol{Ru})\|_{L^2(E)}^2 \\ &\le \frac{\kappa}{10} \|\!|\nabla\boldsymbol{\chi}|\!\|_{L^2(\Omega)}^2 + Ch^k |\boldsymbol{u}|_{H^{k+1}(\Omega)}, \\ T_4 &\le \frac{\kappa}{10} \|\!|\nabla\boldsymbol{\chi}|\!\|_{L^2(\Omega)}^2 + Ch^k |\boldsymbol{u}|_{H^{k+1}(\Omega)}. \end{aligned}$$

Similarly,

$$T_2 \leq \frac{\kappa}{10} \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{1}{|e|} \|[\boldsymbol{\chi}]\|_{L^2(e)}^2 + C \sum_{e \in \Gamma_h \cup \partial\Omega} |e| \|\{\nabla(\boldsymbol{u} - \boldsymbol{R}\boldsymbol{u})\}\boldsymbol{n}_e\|_{L^2(e)}^2.$$

The term $\|\{\nabla(\boldsymbol{u} - \boldsymbol{R}\boldsymbol{u})\}\boldsymbol{n}_e\|_{L^2(e)}$ is bounded using trace inequality (2.2). For instance, if the edge e belongs to the element E_e, we have

$$\|\nabla(\boldsymbol{u} - \boldsymbol{R}\boldsymbol{u})\boldsymbol{n}_e\|_{L^2(e)} \leq C h_E^{-1/2} \Big(\|\nabla(\boldsymbol{u} - \boldsymbol{R}\boldsymbol{u})\|_{L^2(E)} + h_E \|\nabla^2(\boldsymbol{u} - \boldsymbol{R}\boldsymbol{u})\|_{L^2(E)} \Big).$$

Now since we do not have an estimate of $\|\nabla^2(\boldsymbol{u} - \boldsymbol{R}\boldsymbol{u})\|_{L^2(E)}$, we introduce an approximation $\tilde{\boldsymbol{u}}$ of $\boldsymbol{u}$ satisfying (2.10), and we use an inverse inequality (3.6):

$$\begin{aligned} \|\nabla^2(\boldsymbol{u} - \boldsymbol{R}\boldsymbol{u})\|_{L^2(E} &\leq \|\nabla^2(\boldsymbol{u} - \tilde{\boldsymbol{u}})\|_{L^2(E)} + \|\nabla^2(\tilde{\boldsymbol{u}} - \boldsymbol{R}\boldsymbol{u})\|_{L^2(E)} \\ &\leq \|\nabla^2(\boldsymbol{u} - \tilde{\boldsymbol{u}})\|_{L^2(E)} + C h_E^{-1} \|\nabla(\tilde{\boldsymbol{u}} - \boldsymbol{R}\boldsymbol{u})\|_{L^2(E)}. \end{aligned}$$

We skip many details (left to the reader), and we obtain

$$T_2 \leq \frac{\kappa}{10} \sum_{e \in \Gamma_h \cup \partial\Omega} \frac{1}{|e|} \|[\boldsymbol{\chi}]\|_{L^2(e)}^2 + C h^{2k} |\boldsymbol{u}|_{H^{k+1}(\Omega)}^2.$$

The third term is simply bounded using (2.5) and (6.7):

$$T_3 \leq \frac{\kappa}{10} \|\nabla \boldsymbol{\chi}\|_{L^2(\Omega)}^2 + C h^{2k} |\boldsymbol{u}|_{H^{k+1}(\Omega)}^2.$$

Finally, since $\nabla \cdot \boldsymbol{\chi} \in \mathbb{P}_{k-1}(E)^2$, the term involving the pressure reduces to

$$\frac{1}{\mu} b(\boldsymbol{\chi}, p - \tilde{p}) = \frac{1}{\mu} \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{p - \tilde{p}\}[\boldsymbol{\chi}] \cdot \boldsymbol{n}_e \leq \frac{\kappa}{10} \|\boldsymbol{\chi}\|_{\mathcal{E}}^2 + \frac{C}{\mu^2} h^{2k} |p|_{H^k(\Omega)}^2.$$

Combining all the bounds above and using the triangle inequality

$$\|\boldsymbol{u} - \boldsymbol{U}_h\|_{\mathcal{E}} \leq \|\boldsymbol{u} - \boldsymbol{R}\boldsymbol{u}\|_{\mathcal{E}} + \|\boldsymbol{\chi}\|_{\mathcal{E}},$$

we obtain the final result. □

Theorem 6.11. *Under the assumptions and notation of Theorem* 6.10*, there is a constant* C *independent of* h *and* μ *such that*

$$\|p - P_h\|_{L^2(\Omega)} \leq C h^k (\mu |\boldsymbol{u}|_{H^{k+1}(\Omega)} + |p|_{H^k(\Omega)}).$$

Proof. Let $\tilde{p}$ be the L^2 projection of p. We can write the error equation as follows:

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad a_\epsilon(\boldsymbol{U}_h - \boldsymbol{u}, \boldsymbol{v}) + \frac{1}{\mu} b(\boldsymbol{v}, P_h - \tilde{p}) = \frac{1}{\mu} b(\boldsymbol{v}, p - \tilde{p}). \tag{6.34}$$

From the remark in Section 6.4, there exists $\boldsymbol{v} \in \tilde{\boldsymbol{X}}_h$ such that

$$b(\boldsymbol{v}, P_h - \tilde{p}) = -\|P_h - \tilde{p}\|_{L^2(\Omega)}^2, \quad \|\boldsymbol{v}\|_{\mathcal{E}} \leq \frac{1}{\beta^*} \|P_h - \tilde{p}\|_{L^2(\Omega)}, \tag{6.35}$$

and in particular

$$\sum_{E\in\mathcal{E}_h}\int_E \nabla(\boldsymbol{U}_h-\boldsymbol{Ru}):\nabla\boldsymbol{v}+\sum_{e\in\Gamma_h\cup\partial\Omega}\frac{\sigma_e^0}{|e|}\int_e[\boldsymbol{U}_h-\boldsymbol{Ru}]\cdot[\boldsymbol{v}]=0.$$

Therefore, (6.34) becomes

$$\begin{aligned}\frac{1}{\mu}\|P_h-\tilde{p}\|^2_{L^2(\Omega)} &= a_\epsilon(\boldsymbol{U}_h-\boldsymbol{u},\boldsymbol{v})-\frac{1}{\mu}b(\boldsymbol{v},p-\tilde{p})\\ &=\sum_{E\in\mathcal{E}_h}\int_E\nabla(\boldsymbol{Ru}-\boldsymbol{u}):\nabla\boldsymbol{v}+\sum_{e\in\Gamma_h\cup\partial\Omega}\frac{\sigma_e^0}{|e|}\int_e[\boldsymbol{Ru}-\boldsymbol{u}]\cdot[\boldsymbol{v}]-\frac{1}{\mu}b(\boldsymbol{v},p-\tilde{p})\\ &\quad-\sum_{e\in\Gamma}\int_e\{\nabla(\boldsymbol{U}-\boldsymbol{u})\}\boldsymbol{n}_e\cdot[\boldsymbol{v}]+\epsilon\sum_{e\in\Gamma}\int_e\{\nabla\boldsymbol{v}\}\boldsymbol{n}_e\cdot[\boldsymbol{U}-\boldsymbol{u}].\end{aligned}$$

The terms on the right-hand side can be easily bounded, and we obtain the final result. □

We now state the a priori error estimate for the velocity in the L^2 norm. The proof uses the Aubin–Nitsche lift technique as in the proof of Theorem 2.14, and thus it is omitted. The estimate is optimal for the SIPG method and suboptimal for the IIPG and NIPG methods.

Theorem 6.12. *Assume that Ω is convex. Then, under the hypotheses of Theorem* 6.10*, there exists a constant C independent of h and μ such that*

$$\|\boldsymbol{u}-\boldsymbol{U}\|_{L^2(\Omega)}\le Ch^{k+1-\delta}\left(|\boldsymbol{u}|_{H^{k+1}(\Omega)}+\frac{1}{\mu}|p|_{H^k(\Omega)}\right), \tag{6.36}$$

where $\delta=0$ for SIPG and $\delta=0$ for IIPG and NIPG.

6.6 Numerical results

We solve (6.19)–(6.20) in the case of a known smooth function. We vary the polynomial degree k from 1 to 3. The domain $\Omega=(0,1)^2$ is subdivided into 2048 triangles. We assume that the penalty parameter takes the same value for all edges e. Numerical errors for the velocity and pressure are given in Table 6.1 and Table 6.2, respectively. Rates are computed from errors obtained on two successive meshes. In the SIPG case, we obtain optimal error estimates as predicted by the theory. In the NIPG case with positive penalty, we obtain optimal error estimates for the velocity in the energy norm and the pressure in the L^2 norm. We also observe, as for the elliptic problem, optimal convergence rates for the velocity in the L^2 norm if the polynomial degree is odd and suboptimal if the polynomial degree is even. Finally, we add convergence rates for the NIPG 0 method. This method numerically converges for $k\ge 2$.

6.7 Bibliographical remarks

Primal DG methods for Stokes are introduced in [107, 61]. A method using discontinuous divergence-free approximations of the velocity and continuous approximations of the pressure is described in [11]. Mixed DG methods for Stokes are analyzed and studied in [34, 100]: an additional unknown, namely the gradient of velocity, is introduced.

Table 6.1. *Numerical errors and convergence rates for Stokes velocity.*

Method	k	σ_e^0	$\Vert\nabla(\boldsymbol{u}-\boldsymbol{U}_h)\Vert_{H^0(\mathcal{E}_h)}$	Rate	$\Vert\boldsymbol{u}-\boldsymbol{U}_h\Vert_{L^2(\Omega)}$	Rate
NIPG	1	1	5.8810×10^{-03}	1.0259	7.6486×10^{-05}	2.0013
	2	1	1.3406×10^{-04}	2.0041	4.7542×10^{-06}	1.9699
	3	1	3.6084×10^{-06}	2.9843	1.1940×10^{-08}	3.9685
SIPG	1	10	4.1955×10^{-03}	1.0345	6.8338×10^{-05}	1.8462
	2	10	1.3995×10^{-04}	2.0251	3.8299×10^{-07}	3.0411
	3	10	7.4763×10^{-06}	3.4034	1.7002×10^{-08}	4.4039
IIPG	1	10	4.1446×10^{-03}	1.0159	4.8448×10^{-05}	1.8866
	2	10	1.2701×10^{-04}	2.0012	1.8436×10^{-06}	2.0620
	3	10	3.2272×10^{-06}	3.0023	9.2767×10^{-09}	3.9536
NIPG	2	0	1.4465×10^{-04}	2.0093	5.8801×10^{-06}	1.9718
	3	0	3.9253×10^{-06}	2.9767	1.3147×10^{-08}	3.9715

Table 6.2. *Numerical errors and convergence rates for Stokes pressure.*

Method	k	σ_e^0	$\Vert p-P_h\Vert_{L^2(\Omega)}$	Rate
NIPG	1	1	9.8746×10^{-3}	1.0248
	2	1	5.1239×10^{-5}	2.2044
	3	1	2.9978×10^{-6}	2.9778
SIPG	1	10	2.5143×10^{-2}	0.9874
	2	10	5.7527×10^{-5}	1.8978
	3	10	1.7230×10^{-6}	3.3224
IIPG	1	10	2.3386×10^{-2}	0.9863
	2	10	4.7975×10^{-5}	1.9923
	3	10	1.5089×10^{-6}	2.9383
NIPG	2	0	6.5898×10^{-5}	2.1872
	3	0	3.6788×10^{-6}	2.9465

Exercises

6.1. Prove Lemma 6.1.

6.2. Prove Lemma 6.5.

6.3. Let $(\phi_1,\dots,\phi_{N_u})$ be a basis of $\boldsymbol{X}_h$ and let $(\psi_1,\dots,\psi_{N_p})$ be a basis of M_h. Let ξ_i's and η_i's denote the coefficients of the solutions $\boldsymbol{U}_h$ and P_h, respectively, with respect to the basis functions ϕ_i's and ψ_i's. Derive the linear system resulting from (6.19)–(6.20) of the form $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ if the unknown vector is

$$\boldsymbol{x}=(\xi_1,\dots,\xi_{N_u},\eta_1,\dots,\eta_{N_p})^T.$$

6.4. Prove Theorem 6.12.