Chapter 8

Change of Variables, Parametrizations, Surface Integrals

8.1 The transformation formula

In evaluating any integral, if the integral depends on an auxiliary function of the variables involved, it is often a good idea to change variables and try to simplify the integral. The formula which allows one to pass from the original integral to the new one is called the **transformation formula** (or **change of variables formula**). It should be noted that certain conditions need to be met before one can achieve this, and we begin by reviewing the one variable situation.

Let $\mathcal{D}$ be an open interval, say (a, b), in $\mathbb{R}$, and let $\varphi : \mathcal{D} \to \mathbb{R}$ be a 1-1 , $\mathcal{C}^1$ mapping (function) such that $\varphi' \neq 0$ on $\mathcal{D}$. Put $\mathcal{D}^* = \varphi(\mathcal{D})$. By the hypothesis on φ, it's either increasing or decreasing everywhere on $\mathcal{D}$. In the former case $\mathcal{D}^* = (\varphi(a), \varphi(b))$, and in the latter case, $\mathcal{D}^* = (\varphi(b), \varphi(a))$. Now suppose we have to evaluate the integral

$$I = \int_a^b f(\varphi(u))\varphi'(u)\ \mathrm{d}u,$$

for a nice function f. Now put $x = \varphi(u)$, so that $\mathrm{d}x = \varphi'(u)\ \mathrm{d}u$. This change of variable allows us to express the integral as

$$I = \int_{\varphi(a)}^{\varphi(b)} f(x)\ \mathrm{d}x = \mathrm{sgn}(\varphi') \int_{\mathcal{D}^*} f(x)\ \mathrm{d}x,$$

where $\mathrm{sgn}(\varphi')$ denotes the sign of φ' on $\mathcal{D}$. We then get the transformation formula

$$\int_{\mathcal{D}} f(\varphi(u))|\varphi'(u)|\ \mathrm{d}u = \int_{\mathcal{D}^*} f(x)\ \mathrm{d}x$$

This generalizes to higher dimensions as follows:

Theorem 1. *Let $\mathcal{D}$ be a bounded open set in $\mathbb{R}^n$, $\varphi : \mathcal{D} \to \mathbb{R}^n$ a $\mathcal{C}^1$, 1-1 mapping whose Jacobian determinant* $\det(D\varphi)$ *is everywhere non-vanishing on $\mathcal{D}$, $\mathcal{D}^* = \varphi(\mathcal{D})$, and f an integrable function on $\mathcal{D}^*$. Then we have the* **transformation formula**

$$\int \underset{\mathcal{D}}{\cdots} \int f(\varphi(u)) |\det D\varphi(u)| \; du_1 \dots \; du_n = \int \underset{\mathcal{D}^*}{\cdots} \int f(x) \; dx_1 \dots \; dx_n.$$

Of course, when $n = 1$, $\det D\varphi(u)$ is simply $\varphi'(u)$, and we recover the old formula. This theorem is quite hard to prove, and we will discuss the 2-dimensional case in detail in the next section. In any case, this is one of the most useful things one can learn in Calculus, and one should feel free to (properly) use it as much as possible.

8.2 The formula in the plane

Let $\mathcal{D}$ be an open set in $\mathbb{R}^2$. We will call a mapping $\varphi : \mathcal{D} \to \mathbb{R}^2$ as above **primitive** if it is either of the form

$$(P1) \qquad \tilde{g} : (u, v) \mapsto (u, g(u, v))$$

or of the form

$$(P2) \qquad \tilde{h} : (u, v) \mapsto (h(u, v), v),$$

with g, h continuously differentiable (i.e. $\mathcal{C}^1$) and $\partial g/\partial v, \partial h/\partial u$ nowhere vanishing on $\mathcal{D}$. The condition $\partial g/\partial v \neq 0$ ensures that $\tilde{g}$ is bijective and similarly for $\tilde{h}$. If φ is a linear map then it can be written as the composite of primitive linear maps. Indeed a linear map of the form (P1) looks like

$$(u, v) \mapsto (u, v) \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

and a linear map of the form (P2) like

$$(u, v) \mapsto (u, v) \begin{pmatrix} a' & 0 \\ b' & 1 \end{pmatrix}$$

and we have

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} a' & 0 \\ b' & 1 \end{pmatrix} = \begin{pmatrix} a' + ab' & a \\ bb' & b \end{pmatrix}, \quad \begin{pmatrix} a' & 0 \\ b' & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} a' & aa' \\ b' & b + ab' \end{pmatrix}.$$

Now given *any* 2×2 matrix M one can always solve for a, b, a', b' for at least one order of the factors. Hence M will be a product of primitive linear maps.

We will now prove the transformation formula when φ is a composition of two primitive transformations, one of type (P1) and the other of type (P2), i.e.

$$\phi = \tilde{g} \circ \tilde{h}$$

and for f a *continuous* function.

For simplicity, let us assume that the functions $\partial g/\partial v(u,v)$ and $\partial h/\partial u(u,v)$ are always positive. (If either of them is negative, it is elementary to modify the argument.) Put $\mathcal{D}_1 = \{(h(u,v),v)|(u,v) \in \mathcal{D}\}$ and $\mathcal{D}^* = \{(x, g(x,v))|(x,v) \in \mathcal{D}_1\}$. so that we have bijective maps

$$\mathcal{D} \xrightarrow{\tilde{h}} \mathcal{D}_1 \xrightarrow{\tilde{g}} \mathcal{D}^*$$

whose composite is ϕ.

Enclose $\mathcal{D}_1$ in a closed rectangle R, and look at the intersection P of $\mathcal{D}_1$ with a partition of R, which is bounded by the lines $x = x_m, m = 1, 2, ...,$ and $v = v_r, r = 1, 2, ...,$ with the subrectangles R_{mr} being of sides $\triangle x = l$ and $\triangle v = k$. Let R^*, respectively R^*_{mr}, denote the image of R, respectively R_{mr}, under $(u,v) \to (u, g(u,v))$. Then each R^*_{mr} is bounded by the parallel lines $x = x_m$ and $x = x_m + l$ and by the arcs of the two curves $y = g(x, v_r)$ and $y = g(x, v_r + k)$ (it is a region of type I in previous terminology). Then we have

$$\mathrm{area}(R^*_{mr}) = \int_{x_m}^{x_m+l} (g(x, v_r + k) - g(x, v_r))\ \mathrm{d}x.$$

By the mean value theorem of 1-variable integral calculus and continuity of g, we can write

$$\mathrm{area}(R^*_{mr}) = l[g(x'_m, v_r + k) - g(x'_m, v_r)],$$

for some point $x'_m \in (x_m, x_m + l)$. By the mean value theorem of 1-variable differential calculus, we get

$$\mathrm{area}(R^*_{mr}) = lk \frac{\partial g}{\partial v}(x'_m, v'_r),$$

for some $v'_r \in (v_r, v_r + k)$. Now for any continuous function f on $\mathcal{D}^*$ we have

$$\iint_{\mathcal{D}^*} f(x,y)\ \mathrm{d}x\ \mathrm{d}y = \lim_P \iint_{\mathcal{D}^*} s_P\ \mathrm{d}x\ \mathrm{d}y = \lim_P \sum_{m,r} f(x'_m, g(x'_m, v'_r))\mathrm{area}(R^*_{mr})$$

where s_P is the step function with constant value $f(x'_m, g(x'_m, v'_r))$ on R^*_{mr}. Indeed, given $\epsilon > 0$, by the small span theorem applied to the continuous function $(x,v) \mapsto f(x, g(x,v))$ we can find a partition P so that

$$|f(x', g(x', v')) - f(x, g(x,v))| < \frac{\epsilon}{\mathrm{area}(\mathcal{D}^*)}$$

for $(x, v), (x', v')$ contained in the same subrectangle R_{mr} of P. But then

$$\left| \iint_{\mathcal{D}^*} (f(x, y) - s_P) \, \mathrm{d}x \, \mathrm{d}y \right| \leq \sum_{m,r} \iint_{R^*_{mr}} |f(x, y) - f(x'_m, g(x'_m, v'_r))| \, \mathrm{d}x \, \mathrm{d}y < \epsilon.$$

Hence we obtain

$$\iint_{\mathcal{D}^*} f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \lim_P \sum_{m,r} klf(x'_m, g(x'_m, v'_r)) \frac{\partial g}{\partial v}(x'_m, v'_r).$$

The expression on the right tends to the integral $\iint_{\mathcal{D}_1} f(x, g(x, v)) \frac{\partial g}{\partial v}(x, v) \, \mathrm{d}x \, \mathrm{d}v$. Thus we get the identity

$$\boxed{\iint_{\mathcal{D}^*} f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \iint_{\mathcal{D}_1} f(x, g(x, v)) \frac{\partial g}{\partial v}(x, v) \, \mathrm{d}x \, \mathrm{d}v}$$

By applying the same argument, with the roles of x, y (respectively g, h) switched, we obtain

$$\boxed{\iint_{\mathcal{D}_1} f(x, g(x, v)) \frac{\partial g}{\partial v}(x, v) \, \mathrm{d}x \, \mathrm{d}y = \iint_{\mathcal{D}} f(h(u, v), g(h(u, v), v)) \frac{\partial g}{\partial v}(h(u, v), v) \frac{\partial h}{\partial u}(u, v) \, \mathrm{d}u \, \mathrm{d}v}$$

Since $\varphi = \tilde{g} \circ \tilde{h}$ we get by the chain rule that

$$D\varphi(u, v) = D\tilde{g}(h(u, v), v) \cdot D\tilde{h}(u, v) = \begin{pmatrix} 1 & \frac{\partial g}{\partial u}(h(u, v), v) \\ 0 & \frac{\partial g}{\partial v}(h(u, v), v) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial h}{\partial u}(u, v) & 0 \\ \frac{\partial h}{\partial v}(u, v) & 1 \end{pmatrix}$$

and

$$\det D\varphi(u, v) = \frac{\partial g}{\partial v}(h(u, v), v) \cdot \frac{\partial h}{\partial u}(u, v)$$

which is by hypothesis > 0. Thus we get

$$\boxed{\iint_{\mathcal{D}^*} f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \iint_{\mathcal{D}} f(\varphi(u, v)) |\det D\varphi(u, v)| \, \mathrm{d}u \, \mathrm{d}v}$$

as asserted in the Theorem.

How to do the general case of φ? The fact is, we can subdivide $\mathcal{D}$ into a finite union of subregions, on each of which φ can be realized as a composition of primitive transformations. We refer the interested reader to chapter 3, volume 2, of "Introduction to Calculus and Analysis" by R. Courant and F. John.

8.3 Examples

(1) Let $\Phi = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq a^2\}$, with $a > 0$. We know that $A = \text{area}(\Phi) = \pi a^2$. But let us now do this by using **polar coordinates.** Put

$$\mathcal{D} = \{(r, \theta) \in \mathbb{R}^2 | 0 \leq r \leq a, 0 \leq \theta < 2\pi\},$$

and define $\varphi : \mathcal{D} \to \Phi$ by

$$\varphi(r, \theta) = (r\cos\theta, r\sin\theta).$$

Then it is easy to check that φ is 1-1, onto, and moreover,

$$\frac{\partial\varphi}{\partial r} = (\cos\theta, \sin\theta) \text{ and } \frac{\partial\varphi}{\partial\theta} = (-r\sin\theta, r\cos\theta).$$

Hence

$$\det(D\varphi) = \det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} = r,$$

which is non-negative, and is non-zero iff $r > 0$. The integral will not change if we integrate over $\{r \geq 0, 0 \leq \theta \leq 2\pi\}$ or over $\{r > 0, 0 \leq \theta \leq 2\pi\}$ as the set $\{r = 0, 0 \leq \theta \leq 2\pi\}$ has content zero.
By the transformation formula, we get

$$A = \iint\limits_{\Phi} \mathrm{d}x\ \mathrm{d}y = \iint\limits_{\mathcal{D}} |\det D\varphi(r, \theta)|\ \mathrm{d}r\ \mathrm{d}\theta = \int\limits_0^a \int\limits_0^{2\pi} r\ \mathrm{d}r\ \mathrm{d}\theta =$$

$$= \int\limits_0^a r\ \mathrm{d}r \int\limits_0^{2\pi} \mathrm{d}\theta = 2\pi \int\limits_0^a r\ \mathrm{d}r = 2\pi \left.\frac{r^2}{2}\right]_0^a = \pi a^2.$$

(Note that $\mathcal{D}$ is a rectangular strip, which is transformed by φ into a circular region.)

(2) Compute the integral $I = \int\int_R x dx dy$ where R is the region $\{(r, \phi) \mid 1 \leq r \leq 2,\ 0 \leq \phi \leq \pi/4\}$.

We have $I = \int_1^2 \int_0^{\pi/4} r\cos(\phi) r dr d\phi = \sin(\phi)]_0^{\pi/4} \cdot \left.\frac{r^3}{3}\right]_1^2 = \frac{\sqrt{2}}{2}(8/3 - 1/3) = \frac{7\sqrt{2}}{6}$.

(3) Let Φ be the region inside the parallelogram bounded by $y = 2x, y = 2x - 2, y = x$ and $y = x + 1$. Evaluate $I = \iint\limits_{\Phi} xy\ \mathrm{d}x\ \mathrm{d}y$.

The parallelogram is spanned by the vectors $(1, 2)$ and $(2, 2)$, so it seems reasonable to make the change of variable

$$\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}1 & 2 \\ 2 & 2\end{pmatrix}\begin{pmatrix}u \\ v\end{pmatrix}$$

since then we'll have to integrate over the box $[0,1] \times [0,1]$ in the $u-v$-region. The Jacobi matrix of this transformation is of course constant and has determinant -2. So we get

$$\begin{aligned} I &= \int_0^1 \int_0^1 (u+2v)(2u+2v)|-2|dudv = 2\int_0^1 \int_0^1 (2u^2+6uv+4v^2)dudv \\ &= 2\int_0^1 2u^3/3 + 3u^2v + 4v^2u\Big]_0^1 dv = 2\int_0^1 (2/3+3v+4v^2)dv \\ &= 2(2v/3 + 3v^2/2 + 4v^3/3\Big]_0^1) = 2(2/3+3/2+4/3) = 7. \end{aligned}$$

(4) Find the volume of the cylindrical region in $\mathbb{R}^3$ defined by

$$W = \{(x,y,z)|x^2+y^2 \le a^2, 0 \le z \le h\},$$

where a, h are positive real numbers. We need to compute

$$I = \text{ vol}(W) = \iiint\limits_W \, \mathrm{d}x \; \mathrm{d}y \; \mathrm{d}z.$$

It is convenient here to introduce the **cylindrical coordinates** given by the transformation

$$\varphi : \mathcal{D} \to \mathbb{R}^3$$

given by

$$\varphi(r,\theta,z) = (r\cos\theta, r\sin\theta, z), \text{ where } \mathcal{D} = \{0 \le r \le a^2, 0 \le \theta \le 2\pi, 0 \le z \le h\}.$$

It is easy to see φ is 1-1, $\mathcal{C}^1$ and onto W. Moreover,

$$|\det D\varphi| = |\det \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} | = r$$

Again, the set $\{(0,\theta,z)\}$ where $|\det D\varphi| = 0$ has content 0, and the integral will be unchanged if this set is included. By the transformation formula,

$$I = \int\limits_0^h \int\limits_0^a \int\limits_0^{2\pi} r \; \mathrm{d}r \; \mathrm{d}\theta \; \mathrm{d}z = \pi a^2 h,$$

as expected.

(5) Let W be the **unit ball** $\overline{B}_0(1)$ in $\mathbb{R}^3$ with center at the origin and radius 1. Evaluate

$$I = \iiint_W e^{(x^2+y^2+z^2)^{3/2}} \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}z.$$

Here it is best to use the **spherical coordinates** given by the transformation

$$\psi : \mathcal{D} \to \mathbb{R}^3, \ (\rho, \theta, \phi) \to (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi),$$

where $\mathcal{D} = \{0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$. Then ψ is $\mathcal{C}^1$, 1-1 and onto W. Moreover

$$\det(D\psi) = \det \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix} =$$

$$= \cos\phi \det \begin{pmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{pmatrix} - \rho\sin\phi \det \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{pmatrix} =$$

$$= -\rho^2(\cos\phi)^2\sin\phi - \rho^2(\sin\phi)^3 = -\rho^2\sin\phi.$$

Note that $\sin\phi$ is ≥ 0 on $[0, \pi]$. Hence $|\det(D\psi)| = \rho^2\sin\phi$, and we get (by the transformation formula):

$$I = \int_0^1 \int_0^\pi \int_0^{2\pi} e^{\rho^3}\rho^2 \sin\phi \ \mathrm{d}\rho \ \mathrm{d}\phi \ \mathrm{d}\theta = \int_0^1 e^{\rho^3}\rho^2 \ \mathrm{d}\rho \int_0^\pi \sin\phi \ \mathrm{d}\phi \int_0^{2\pi} \ \mathrm{d}\theta =$$

$$= 2\pi \int_0^1 e^{\rho^3}\rho^2 \ \mathrm{d}\rho \ (-\cos\phi)|_0^\pi = \frac{4\pi}{3} \int_0^1 e^u \ \mathrm{d}u,$$

where $u = \rho^3$

$$= \frac{4\pi}{3} e^u \Big]_0^1 = \frac{4\pi}{3}(e-1).$$

8.4 Parametrizations

Let n, k be positive integers with $k \leq n$. A subset Φ of $\mathbb{R}^n$ is called a **parametrized k-fold** iff there exist a bounded, connected region T in $\mathbb{R}^k$ together with a $\mathcal{C}^1$ injective mapping

$$\varphi : T \to \mathbb{R}^n, \ u \to (x_1(u), x_2(u), ..., x_n(u)),$$

such that $\varphi(T) = \Phi$.

It is called a **parametrized surface** when $k = 2$, and a parametrized curve when $k = 1$.

Example: We restrict the spherical coordinates of the above example to the sphere $S_0(a)$ of radius $a > 0$, i.e. we set $\rho = a$. Then the parametrizing domain is

$$T = \{(\theta, \phi) \in \mathbb{R}^2 | 0 \leq \theta < 2\pi, 0 < \phi < \pi\} = [0, 2\pi) \times (0, \pi)$$

and the parametrization is given by

$$(\theta, \phi) \to (a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi).$$

Note that we omitted the endpoints $\theta = 2\pi$ and $\phi = 0, \pi$ so that the parametrization is injective. This means we are really parametrizing the unit sphere minus its "north" and "south" pole (corresponding to $\phi = 0$ and $\phi = \pi$ and arbitrary θ). If we want to integrate over the sphere, removing those two points doesn't make a difference because they form a set of content zero.

8.5 Surface integrals in $\mathbb{R}^3$

8.5.1 Integrals of scalar valued functions

Let Φ be a parametrized surface in $\mathbb{R}^3$, given by a $\mathcal{C}^1$ injective mapping

$$\varphi : T \to \mathbb{R}^3, \quad T \subset \mathbb{R}^2, \quad \varphi(u, v) = (x(u, v), y(u, v), z(u, v)).$$

When we say that φ is C^1 we mean that it is so on an open set containing T. Let $\xi = \varphi(u, v) = (x, y, z)$ be a point on Φ. Consider the curve C_1 on Φ passing through ξ on which v is constant. Then the tangent vector to C_1 at ξ is simply given by $\frac{\partial\varphi}{\partial u}(u, v)$. Similarly, we may consider the curve C_2 on Φ passing through ξ on which u is constant. Then the tangent vector to C_2 at ξ is given by $\frac{\partial\varphi}{\partial v}(u, v)$. So the surface Φ has a **tangent plane** $\mathcal{J}_\Phi(\xi)$ at ξ iff $\frac{\partial\varphi}{\partial u}$ and $\frac{\partial\varphi}{\partial v}$ are linearly independent there. From now on we will assume this to be the case (see also the discussion of tangent spaces in Ch. 3 of the class notes). When this happens at every point, we call Φ **smooth.** In fact, for integration purposes, it suffices to know that $\frac{\partial\varphi}{\partial u}$ and $\frac{\partial\varphi}{\partial v}$ are independent except at a set $\{\varphi(u, v)\}$ of content zero.

By the definition of the cross product, there is a natural choice for a **normal vector to Φ at $\xi = \varphi(u, v)$** given by:

$$\frac{\partial\varphi}{\partial u}(u, v) \times \frac{\partial\varphi}{\partial v}(u, v).$$

Definition : Let f be a bounded scalar field on the parametrized surface Φ. The **surface integral of f over Φ,** denoted $\iint\limits_{\Phi} f \, \mathrm{d}S$, is given by the formula

$$\iint\limits_{\Phi} f \, \mathrm{d}S = \iint\limits_{T} f(\varphi(u, v)) \left\| \frac{\partial\varphi}{\partial u}(u, v) \times \frac{\partial\varphi}{\partial v}(u, v) \right\| \, \mathrm{d}u \, \mathrm{d}v.$$

We say that f is **integrable** on Φ if $f(u,v)\left\|\frac{\partial\varphi}{\partial u}(u,v)\times\frac{\partial\varphi}{\partial v}(u,v)\right\|$ is integrable on T. An important special case is when $f=1$. In this case, we get

$$\boxed{\operatorname{area}(\Phi)=\iint_T ||\tfrac{\partial\varphi}{\partial u}(u,v)\times\tfrac{\partial\varphi}{\partial v}(u,v)||\ \mathrm{d}u\ \mathrm{d}v}$$

Note that this formula is similiar to that for a line integral in that we have to put in a scaling factor which measures how the parametrization changes the (infinitesimal) length of the curve, resp. area of the surface. Note also that unlike the case of curves this formula only covers the case of surfaces in $\mathbb{R}^3$ rather than in a general $\mathbb{R}^n$ (more about the general case in the last section).

Note further that the definition of a surface integral is **independent** of the parametrization. Indeed, suppose $\alpha: T_1\to T$ is a bijective $\mathcal{C}^1$-map whose total derivative $D\alpha$ is everywhere bijective, i.e. $\det(D\alpha)(u_1,v_1)\neq 0$ for all $(u_1,v_1)\in T_1$. Write $\alpha(u_1,v_1)=(u(u_1,v_1),v(u_1,v_1))$ and $\varphi_1(u_1,v_1):=\varphi(\alpha(u_1,v_1))$. Then the chain rule implies

$$\begin{aligned}\frac{\partial\varphi_1}{\partial u_1}(u_1,v_1)\times\frac{\partial\varphi_1}{\partial v_1}(u_1,v_1)&=\left(\frac{\partial\varphi}{\partial u}(\alpha(u_1,v_1))\frac{\partial u}{\partial u_1}(u_1,v_1)+\frac{\partial\varphi}{\partial v}(\alpha(u_1,v_1))\frac{\partial v}{\partial u_1}(u_1,v_1)\right)\\&\quad\times\left(\frac{\partial\varphi}{\partial u}(\alpha(u_1,v_1))\frac{\partial u}{\partial v_1}(u_1,v_1)+\frac{\partial\varphi}{\partial v}(\alpha(u_1,v_1))\frac{\partial v}{\partial v_1}(u_1,v_1)\right)\\=\left(\frac{\partial\varphi}{\partial u}(\alpha(u_1,v_1))\times\frac{\partial\varphi}{\partial v}(\alpha(u_1,v_1))\right)&\cdot\left(\frac{\partial u}{\partial u_1}(u_1,v_1)\frac{\partial v}{\partial v_1}(u_1,v_1)-\frac{\partial v}{\partial u_1}(u_1,v_1)\frac{\partial u}{\partial v_1}(u_1,v_1)\right)\\&=\left(\frac{\partial\varphi}{\partial u}(\alpha(u_1,v_1))\times\frac{\partial\varphi}{\partial v}(\alpha(u_1,v_1))\right)\cdot\det(D\alpha)(u_1,v_1)\end{aligned}\tag{8.1}$$

and the transformation formula of Theorem 1 implies

$$\begin{aligned}\iint_\Phi f\ \mathrm{d}S&=\iint_T f(u,v)\left\|\frac{\partial\varphi}{\partial u}(u,v)\times\frac{\partial\varphi}{\partial v}(u,v)\right\|\ \mathrm{d}u\ \mathrm{d}v\\&=\iint_{T_1} f(\alpha(u_1,v_1))\left\|\frac{\partial\varphi_1}{\partial u_1}(u_1,v_1)\times\frac{\partial\varphi_1}{\partial v_1}(u_1,v_1)\right\|\ \mathrm{d}u_1\ \mathrm{d}v_1.\end{aligned}$$

Example: Find the area of the standard sphere $S=S_0(a)$ in $\mathbb{R}^3$ given by

$$x^2+y^2+z^2=a^2$$

with $a>0$. Recall the parametrization of S from above given by

$$T=\{(\theta,\phi)\in\mathbb{R}^2|0\le\theta<2\pi,0<\phi<\pi\}=[0,2\pi)\times(0,\pi)$$

and

$$\varphi(\theta,\phi) = (x(\theta,\phi), y(\theta,\phi), z(\theta,\phi)) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi).$$

So we have

$$\frac{\partial\varphi}{\partial\theta} = (-a\sin\phi\sin\theta, a\sin\phi\cos\theta, 0)$$

and

$$\frac{\partial\varphi}{\partial\phi} = (a\cos\phi\cos\theta, a\cos\phi\sin\theta, -a\sin\phi).$$

Hence

$$\begin{aligned}
\frac{\partial\varphi}{\partial\theta}\times\frac{\partial\varphi}{\partial\phi} &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \end{pmatrix} \\
&= \det\begin{pmatrix} a\sin\phi\cos\theta & 0 \\ a\cos\phi\sin\theta & -a\sin\phi \end{pmatrix}\mathbf{i} - \det\begin{pmatrix} -a\sin\phi\sin\theta & 0 \\ a\cos\phi\cos\theta & -a\sin\phi \end{pmatrix}\mathbf{j} + \\
&+ \det\begin{pmatrix} -a\sin\phi\sin\theta & a\sin\phi\cos\theta \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta \end{pmatrix}\mathbf{k} \\
&= -a^2\cos\theta\sin^2\phi\mathbf{i} - a^2\sin\theta\sin^2\phi\mathbf{j} - a^2(\sin^2\theta\sin\phi\cos\phi + \cos^2\theta\sin\phi\cos\phi)\mathbf{k} \\
&= -a^2\cos\theta\sin^2\phi\mathbf{i} - a^2\sin\theta\sin^2\phi\mathbf{j} - a^2\sin\phi\cos\phi\mathbf{k}.
\end{aligned}$$

Note that for a point in the upper hemisphere $z > 0$ we have $0 < \phi < \frac{\pi}{2}$ with our parametrisation, hence $\sin\phi > 0$ and $\cos\phi > 0$ which means that the k-component of $\frac{\partial\varphi}{\partial\theta}\times\frac{\partial\varphi}{\partial\phi}$ is negative. So what we've computed here is an **inward** pointing normal vector to the sphere. Moreover

$$\begin{aligned}
||\frac{\partial\varphi}{\partial\theta}\times\frac{\partial\varphi}{\partial\phi}|| &= a^2(\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \sin^2\phi\cos^2\phi)^{1/2} \\
&= a^2(\sin^4\phi + \sin^2\phi\cos^2\phi)^{1/2} = a^2|\sin\phi|.
\end{aligned}$$

So we find

$$\text{area}(S) = a^2\int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} |\sin\phi|\ \mathrm{d}\phi = 2\pi a^2 \int_0^{\pi} \sin\phi\ \mathrm{d}\phi = 4\pi a^2.$$

Example: Find the area of the cylinder $\Phi = \varphi(T)$ where $T = [0, 2\pi]\times[0,1]$ and $\varphi(u,v) = (\cos(u), \sin(u), v)$. We have

$$\frac{\partial\varphi}{\partial u} = (-\sin(u), \cos(u), 0), \quad \frac{\partial\varphi}{\partial v} = (0,0,1), \quad \frac{\partial\varphi}{\partial u}\times\frac{\partial\varphi}{\partial v} = (\cos(u), \sin(u), 0)$$

and

$$\left\|\frac{\partial\varphi}{\partial u}\times\frac{\partial\varphi}{\partial v}\right\| = 1.$$

Hence

$$\operatorname{area}(\Phi) = \int_0^1 \int_0^{2\pi} du dv = 2\pi.$$

Assume now that we are trying to approximate the area of a parametrized surface Φ by a sum over areas of little triangles with vertices lying on Φ, in the same way that we approximated the length of a parametrized curve by a sum over lengths of little line segments with vertices on the curve. This corresponds to a subdivision (partition) of the parameter space T into triangles. Continuing the example of the cylinder we subdivide $T = [0, 2\pi] \times [0, 1]$ into the $2km$ congruent triangles with vertex sets

$$S_{ij} := \left\{ \left(\frac{2j\pi}{m}, \frac{i}{k} \right), \left(\frac{(2j+2)\pi}{m}, \frac{i}{k} \right), \left(\frac{(2j+1)\pi}{m}, \frac{i+1}{k} \right) \right\}$$

and

$$S'_{ij} := \left\{ \left(\frac{(2j+1)\pi}{m}, \frac{i+1}{k} \right), \left(\frac{(2j+3)\pi}{m}, \frac{i+1}{k} \right), \left(\frac{(2j+2)\pi}{m}, \frac{i}{k} \right) \right\}$$

for $0 \leq j \leq m-1$ and $0 \leq i \leq k-1$ (and taking the first coordinate modulo 2π). Of course, the areas of these $2km$ triangles sum up to the area of T which is (also) 2π. Now let's look at the triangle Δ spanned by $\varphi(S_{ij})$ which has vertices

$$\left\{ \left(\cos\frac{2j\pi}{m}, \sin\frac{2j\pi}{m}, \frac{i}{k} \right), \left(\cos\frac{(2j+2)\pi}{m}, \sin\frac{(2j+2)\pi}{m}, \frac{i}{k} \right), \right.$$
$$\left. \left(\cos\frac{(2j+1)\pi}{m}, \sin\frac{(2j+1)\pi}{m}, \frac{i+1}{k} \right) \right\}$$

The baseline of Δ has length $b := 2\sin(\frac{\pi}{m})$ and the height of Δ is

$$h := \sqrt{\frac{1}{k^2} + \left(1 - \cos\frac{\pi}{m}\right)^2},$$

so the area of Δ is

$$\operatorname{area}(\Delta) = \frac{1}{2} \cdot b \cdot h = \sin(\frac{\pi}{m}) \cdot \sqrt{\frac{1}{k^2} + \left(1 - \cos\frac{\pi}{m}\right)^2}.$$

Summing over all $2km$ triangles, our approximation to the area of the cylinder then becomes

$$A_{k,m} := 2 \cdot k \cdot m \cdot \sin(\frac{\pi}{m}) \cdot \sqrt{\frac{1}{k^2} + \left(1 - \cos\frac{\pi}{m}\right)^2}.$$

The first thing to notice is that this set of numbers is **unbounded** as can be seen from fixing m and letting k tend to infinity. This is in **contrast** to the **curve case** where the length of a polygon inscribed into a curve is always bounded by the length of the curve (essentially because of the triangle inequality). The second thing to notice is that if we pick a sequence of subdivisions of T whose diameter tends to zero and so that the corresponding sequence of areas $A_{k,m}$ **does have a limit** (for example

$k = m^2 \to \infty$), this limit need **not** equal the area of Φ. Indeed, setting $x = \frac{1}{m} = \frac{1}{\sqrt{k}}$ we have

$$\begin{aligned}
\lim_{m\to\infty} A_{m^2,m} &= \lim_{x\to 0} \frac{2\sin(\pi x)}{x} \cdot \frac{\sqrt{x^4 + (1-\cos(\pi x))^2}}{x^2} \\
&= 2\pi \cdot \lim_{x\to 0} \sqrt{\frac{x^4 + (1-\cos(\pi x))^2}{x^4}} = 2\pi \cdot \sqrt{1 + \lim_{x\to 0} \frac{(1-\cos(\pi x))^2}{x^4}} \\
&= 2\pi \cdot \sqrt{1 + \frac{\pi^4}{4}}.
\end{aligned}$$

Moreover, this limit will usually depend on the particular sequence of subdivisions we choose. For example, for $k = m \to \infty$ the limit is 2π. So the idea of approximating areas by finer and finer subdivisions into triangles **fails** in all respects.

Here is a useful result:

Proposition 1. *Let Φ be a surface in $\mathbb{R}^3$ parametrized by a $\mathcal{C}^1$, 1-1 function*

$$\varphi : T \to \mathbb{R}^3 \text{ of the form } \varphi(u,v) = (u, v, h(u,v)).$$

In other words, Φ is the graph of $z = h(x,y)$. Then for any integrable scalar field f on Φ, we have

$$\iint_\Phi f \ dS = \iint_T f(u,v,h(u,v)) \sqrt{(\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2 + 1} \ du \ dv.$$

Proof.

$$\frac{\partial \varphi}{\partial u} = (1, 0, \frac{\partial h}{\partial u}) \text{ and } \frac{\partial \varphi}{\partial v} = (0, 1, \frac{\partial h}{\partial v}).$$

$$\Rightarrow \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial h}{\partial u} \\ 0 & 1 & \frac{\partial h}{\partial v} \end{pmatrix} L = -\frac{\partial h}{\partial u}\mathbf{i} - \frac{\partial h}{\partial v}\mathbf{j} + \mathbf{k}.$$

$$\Rightarrow ||\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}|| = \sqrt{(\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2 + 1}.$$

Now the assertion follows by the definition of $\iint\limits_\Phi f \ \mathrm{d}S$.

Example. Let Φ be the surface in $\mathbb{R}^3$ bounded by the triangle with vertices (1,0,0), (0,1,0) and (0,0,1). Evaluate the surface integral $\iint\limits_\Phi x \ \mathrm{d}S$.

Note that Φ is a triangular piece of the plane $x + y + z = 1$. Hence Φ is parametrized by

$$\varphi : T \to \mathbb{R}^3 \ , \ \varphi(u,v) = (u, v, h(u,v)),$$

where $h(u,v) = 1-u-v$, and $T = \{0 \le v \le 1-u, 0 \le u \le 1\}$.

$$\frac{\partial h}{\partial u} = -1 \ , \ \frac{\partial h}{\partial v} = -1, \text{ and } \sqrt{(\frac{\partial h}{\partial u})^2 + (\frac{\partial h}{\partial v})^2 + 1} = \sqrt{1+1+1} = \sqrt{3}.$$

By the Proposition above, we have:

$$\iint_{\Phi} x \ \mathrm{d}S = \sqrt{3} \int_0^1 \int_0^{1-u} u \ \mathrm{d}u \ \mathrm{d}v = \sqrt{3} \int_0^1 u(1-u) \ \mathrm{d}u = \frac{\sqrt{3}}{6}.$$

8.5.2 Integrals of vector valued functions

There is also a notion of an integral of a vector field over a surface. As in the case of line integrals this is in fact obtained by integrating a suitable projection of the vector field (which is then a scalar field) over the surface. Whereas for curves the natural direction to project on is the tangent direction, for a surface in $\mathbb{R}^3$ one uses the **normal** direction to the surface.

Note that a **unit normal vector** to Φ at $\xi = \varphi(u,v)$ is given by

$$\mathbf{n} = \frac{\frac{\partial \varphi}{\partial u}(u,v) \times \frac{\partial \varphi}{\partial v}(u,v)}{||\frac{\partial \varphi}{\partial u}(u,v) \times \frac{\partial \varphi}{\partial v}(u,v)||}$$

and that $\mathbf{n} = \mathbf{n}(u,v)$ varies with $(u,v) \in T$. This defines a unit vector field on Φ called the **unit normal field.**

Definition: Let F be a vector field on Φ. Then the **surface integral** of F over Φ, denoted $\iint_{\Phi} F \cdot \mathbf{n} \ \mathrm{d}S$, is defined by

$$\begin{aligned}\iint_{\Phi} F \cdot \mathbf{n} \ \mathrm{d}S &= \iint_{T} F(\varphi(u,v)) \cdot \mathbf{n}(u,v) \left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| \ \mathrm{d}u \ \mathrm{d}v \\ &= \iint_{T} F(\varphi(u,v)) \cdot (\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}) \ \mathrm{d}u \ \mathrm{d}v.\end{aligned}$$

Again, we say that F is **integrable** over Φ if $F(\varphi(u,v)) \cdot \mathbf{n}(u,v) ||\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}||$ is integrable over T. It is clear that this definition is **independent** of the parametrization if we choose a coordinate change map $\alpha : T_1 \to T$ with $\det(D\alpha)(u_1,v_1) > 0$ for all $(u_1,v_1) \in T_1$. Then we have

$$\frac{\frac{\partial \varphi_1}{\partial u}(u_1,v_1) \times \frac{\partial \varphi_1}{\partial v_1}(u_1,v_1)}{||\frac{\partial \varphi_1}{\partial u_1}(u_1,v_1) \times \frac{\partial \varphi_1}{\partial v_1}(u_1,v_1)||} = \frac{\frac{\partial \varphi}{\partial u}(\alpha(u_1,v_1)) \times \frac{\partial \varphi}{\partial v}(\alpha(u_1,v_1))}{||\frac{\partial \varphi}{\partial u}(\alpha(u_1,v_1)) \times \frac{\partial \varphi}{\partial v}(\alpha(u_1,v_1))||}$$

using (8.1) and $\det(D\alpha)(u_1,v_1) = |\det(D\alpha)(u_1,v_1)|$. So the surface integral we have defined associates a number to a vector field and an **oriented** surface. A change of orientation changes the sign of the integral, just as is the case for line integrals.

The notation $\iint\limits_{\Phi} F \cdot \mathbf{n}\, \mathrm{d}S$ does not make reference to any coordinates whereas the definition of the surface integral explicitly refers to a parametrization. As in the case of line integrals there is a **third notation** for this integral which makes explicit reference to coordinates (x, y, z) of the ambient space. If $F = (P, Q, R)$ we write

$$\iint\limits_{\Phi} F \cdot \mathbf{n}\, \mathrm{d}S = \iint\limits_{\Phi} P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy. \tag{8.2}$$

Here the notation $v \wedge w$ indicates a product of vectors which is bilinear (i.e. $(\lambda_1 v_1 + \lambda_2 v_2) \wedge w = \lambda_1 (v_1 \wedge w) + \lambda_2 (v_1 \wedge w)$ and similarly in the other variable) and antisymmetric (i.e. $v \wedge w = -w \wedge v$, in particular $v \wedge v = 0$). In this sense $\wedge$ is similar to $\times$ on $\mathbb{R}^3$ except that $v \wedge w$ does not lie in the same space where v and w lie. On the positive side $v \wedge w$ can be defined for vectors v, w in any vector space. After these lengthy remarks let's see how this formalism works out in practice. If the surface Φ is parametrized by a function $\varphi(u, v) = (x(u, v), y(u, v), z(u, v))$ then $dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$ and by the properties of $\wedge$ outlined above we have

$$dy \wedge dz = \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) du \wedge dv$$

which is the x-coordinate of $\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}$. So the equation (8.2) does indeed hold if we interpret an integral $\int_T f du \wedge dv$ (over some region T in $\mathbb{R}^2$) as an ordinary double integral $\int_T f du dv$ whereas $\int_T f dv \wedge du$ equals $-\int_T f du dv$. All of this suggests that one should give meaning to dx, dy, dz as elements of some vector space instead of being purely formal notation as in our definition of multiple integrals. This can be done and is in fact necessary to a complete development of integration in higher dimensions and on spaces more general than $\mathbb{R}^n$.

8.6 Integrals over k-folds in $\mathbb{R}^n$

If T is a connected open region of $\mathbb{R}^k$ with boundary of content zero, $1 \leq k \leq n$ and φ is a parametrization

$$\varphi : T \to \mathbb{R}^n, \; u \to (x_1(u), x_2(u), ..., x_n(u)),$$

such that $\Phi := \varphi(T)$ has a well defined tangent space at every point $\varphi(u)$, i.e. so that for any u the vectors $(D\varphi)(u)(e_i)$ are linearly independent for $i = 1, \ldots, k$, one might wonder what the correct way to integrate over Φ is. In particular, how do we compute the k-dimensional volume of Φ? For that we need a digression on linear algebra.

If $v_1, \ldots, v_n$ are linearly independent vectors in $\mathbb{R}^n$ then the volume of the parallelepiped P spanned by $v_1, \ldots, v_n$ is

$$\mathrm{vol}(P) = |\det(v_{ij})|$$

where v_{ij}, $j = 1, \ldots, n$ are the coordinates of v_i. Here the notion of volume is related to the standard inner product $<, >$ in $\mathbb{R}^n$. In fact, all geometric notions that have to do with measuring lengths of vectors, angles between vectors, areas and volumes can be expressed in terms of the inner product (and would correspondingly change if we would put a different positive definite bilinear form on $\mathbb{R}^n$ as the inner product). For the volume the relation is simply the following. If V is the matrix (v_{ij}) then $V \cdot V^t$ is the symmetric matrix with entries $\sum_{\iota=1}^n v_{i\iota}v_{j\iota} =< v_i, v_j >=< v_j, v_i >$. But since V and the transpose matrix V^t have the same determinant we get by multiplicativity of the determinant

$$\mathrm{vol}(P) = |\det(V)| = \sqrt{\det(< v_i, v_j >)}$$

and this last expression visibly only depends on the inner product. The length of a vector v has a similar expression $||v|| = \sqrt{< v, v >}$. If we now have k linearly independent vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ then the volume of the k-dimensional parallelepiped $P = P(v_1, \ldots, v_k)$ spanned by $v_1, \ldots, v_k$ is given by the same formula

$$\mathrm{vol}(P) = \sqrt{\det(< v_i, v_j >)}$$

where the determinant is that of a $k \times k$-matrix. One could say that we endow the k-dimensional space spanned by the v_i with the inner product of the ambient space $\mathbb{R}^n$.

Coming back to our k-fold $\Phi \subseteq \mathbb{R}^n$ the definition of an integral of a scalar valued function f over Φ is then

$$\begin{aligned}\iint_\Phi f \, \mathrm{d}V &= \iint_T f(u)\sqrt{\det(< D\varphi(u)(e_i), D\varphi(u)(e_j) >)} \, \mathrm{d}u_1 \cdots \, \mathrm{d}u_k \\ &= \iint_T f(u)\sqrt{\det\left(< \frac{\partial\varphi}{\partial u_i}(u), \frac{\partial\varphi}{\partial u_j}(u) >\right)} \, \mathrm{d}u_1 \cdots \, \mathrm{d}u_k. \qquad (8.3)\end{aligned}$$

When discussing the cross product in $\mathbb{R}^3$ we have remarked that the length of $v_1 \times v_2$ is the area of the parallelogram spanned by v_1 and v_2. Hence for $n = 3$ we have

$$\|v_1 \times v_2\| = \sqrt{\det\begin{pmatrix} < v_1, v_1 > & < v_1, v_2 > \\ < v_2, v_1 > & < v_2, v_2 > \end{pmatrix}} = \sqrt{||v_1||^2 \cdot ||v_2||^2 - < v_1, v_2 >^2},$$

a formula that can be checked directly.

So the case $k = 2$, $n = 3$ of formula (8.3) recovers the definition of surface integrals in $\mathbb{R}^3$ given in section 8.5. The case $k = 1$ of formula (8.3) recovers our definition of line integrals and the case $k = n$ of formula (8.3) recovers the transformation formula of Theorem 1.