Reduced order modeling of geometrically parametrized discontinuous Galerkin formulation for the Stokes equation

Nirav Vasant Shah, Martin Hess and Gianluigi Rozza

Abstract The present work focuses on geometrical parametrization and reduced order modeling of Stokes flow. The importance of Stokes flow, advantages of discontinuous Galerkin method are introduced first. We also discuss the concept of geometric parametrization and its application along with importance of reduced order model technique. The full order model is based on discontinuous Galerkin method interior penalty formulation. The concepts of broken Sobolev spaces, relevant norms, jump and mean operator are introduced. The weak formulation is derived based in suitable space to obtain the full order model. We then introduce the concept of geometric parametrization. The operators are transformed from fixed domain to parameter dependent domain by exploring affine parameter dependence which results in efficient assembly of system matrix. Thereafter, proper orthogonal decomposition is applied to obtain basis for function space for reduced order model. By using Galerkin projection the linear system to be solved is projected onto reduced space. During the process, offline-online decomposition is used to separate computation of expensive parameter independent part and fast parameter independent part. Finally the technique is applied to test problem. The numerical outcomes presented include the experimental error analysis, eigenvalue computation and measurement of online simulation time. [13]

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1 Introduction

The subject of mathematical applications in fluid mechanics starts with one of the variants of the Navier-Stokes equations, such as the Stokes equation. Almost all processes of fluid mechanics require considerations related to the Navier-Stokes equations. Navier-Stokes equation is non-linear, characterizing flow fluctuations. However, in case of laminar flow, i.e. when fluctuations are negligible, this linearized form of the Navier-Stokes equation is the Stokes equation.

Discontinuous Galerkin method (DGM) has found traction as numerical method for elliptic problems **pereire reference** as well as hyperbolic problems **Book on compressible flow reference**. This is due to its several advantages over Finite Element Method (FEM) and Finite Volume Method (FVM). In fact, DG method is considered as combination of FEM and FVM. DGM uses polynomial approximation of suitable degree providing higher accuracy as well as allows discontinuity at the interface, by the concept of numerical flux, allowing greater flexibility. This fact makes DGM naturally attractive to problems such as shock capturing due to presence of steep gradients or discontinuities. Additionally, since the Dirichlet conditions are applied as boundary penalty, it avoids necessity to work with subspace of FEM. Several variants of DGM exist based on computational advantages such as sparsity pattern or extension of computational stencil, complexity of numerical implementation etc.

Geometric parametrization has emerged as important application of Parametric Partial Differential Equations (PPDEs) and as alternative to shape optimization. The concept of geometric parametrization allows to transfer operator evaluated on one domain to another domain efficiently. For linear equations, this means exploiting affine parameter dependence as will be shown in later section. Model Order Reduction (MOR) on the other hand allows reducing the size of the system to be solved and working with the smaller system containing only dominant components and discarding the non-dominant components. It is pertinent to mention that identifying "dominant" components is critical to the success of model order reduction strategy. Optimization of engineering components using geometric parametrization combined with MOR for PPDEs has given quite useful results in the fields such as mechanical, naval and aeronautic designs. Also, the faster computations obtained by MOR has helped in many query context, real time computation and quick transfer of computational results to industrial problems.

In the present work, we first introduce Discontinuous Galerkin Interior Penalty Method (DG-IPM). We subsequently introduce notion of parametrization characterizing geometry of the domain under consideration, exploit affine parameter dependence and its application in the context of offline-online decomposition. We then apply Proper Orthogonal Decomposition (POD) for constructing reduced basis space and apply Galerkin projection to project the system of equations on the space constructed by POD. Finally we present a test problem to demonstrate the introduced method and present numerical result.

2 Discontinuous Galerkin formulation

Let $\Omega \subset \mathbb{R}^d$ be open bounded domain. The boundary of Ω, $\partial\Omega$ is divided into Neumann boundary Γ_N and Dirichlet boundary Γ_D i.e. $\partial\Omega = \Gamma_N \cup \Gamma_D$. The domain Ω is divide into N_{su} number of mutually non overlapping subdomains such that, $\Omega = \bigcup_{i=1}^{N_{su}} \Omega_i$, $\Omega_i \cap \Omega_j = \emptyset$, for $i \neq j$. Each subdomain is divided into nel number of triangular elements τ_k such that $\Omega = \bigcup_{k=1}^{nel} \tau_k$. The triangulation $\mathcal{T}$ is the set of all triangular elements i.e. $\mathcal{T} = \{\tau_k\}_{k=1}^{nel}$. The internal boundary $\Gamma = \{\partial\tau_k\}_{k=1}^{nel} \backslash \partial\Omega$. We represent $\overrightarrow{n}$ as the outward pointing normal to an edge of element.

The Stokes's equation in strong form can be stated as,

$$-\nu\Delta\overrightarrow{u} + \nabla p = \overrightarrow{f} \,, \text{ in } \Omega \,, \tag{1}$$

$$\nabla \cdot \overrightarrow{u} = 0 \,, \tag{2}$$

$$\overrightarrow{u} = \overrightarrow{u}_D \,, \text{ on } \Gamma_D \,, \tag{3}$$

$$-p\overrightarrow{n} + \nu\overrightarrow{n} \cdot \nabla\overrightarrow{u} = \overrightarrow{t} \,, \text{ on } \Gamma_N \,. \tag{4}$$

The vector variable velocity $\overrightarrow{u}$ and scalar pressure p are the unknowns. ν is the material property known as kinematic viscosity. Vector $\overrightarrow{f}$ is external force term or source term. $\overrightarrow{u}_D$ is the Dirichlet velocity and $\overrightarrow{t}$ is the Neumann value.

Before introducing weak form let us introduce broken Sobolev spaces for variables. The space for velocity is

$$\mathbb{V} = \{\overrightarrow{\phi} \in (L^2(\mathcal{T}))^d | \quad \overrightarrow{\phi} \in (P^D(\tau_k))^d \quad \forall \quad \tau_k \in \mathcal{T}\} \,. \tag{5}$$

The space for pressure is

$$\mathbb{Q} = \{\psi \in (L^2(\mathcal{T}))| \quad \psi \in (P^{D-1}(\tau_k)) \quad \forall \quad \tau_k \in \mathcal{T}\} \,. \tag{6}$$

Here, $P^D(\tau_k)$ denotes space of polynomials of degree at most D over τ_k.

In order to approximate the numerical flux we need the concept of Jump and Average operator. 1. For scalar quantity p the jump operator is defined as,

$$\begin{aligned} [p\overrightarrow{n}] &= p^+\overrightarrow{n}^+ + p^-\overrightarrow{n}^- \quad \text{on} \quad \Gamma, \\ [p\overrightarrow{n}] &= p\overrightarrow{n} \quad \text{on} \quad \Gamma_D. \end{aligned} \tag{7}$$

2. For vector quantity $\overrightarrow{u}$ the jump operator is defined as,

$$\begin{aligned} [\overrightarrow{n} \cdot \overrightarrow{u}] &= \overrightarrow{n}^+ \cdot \overrightarrow{u}^+ + \overrightarrow{n}^- \cdot \overrightarrow{u}^- \quad \text{on} \quad \Gamma, \\ [\overrightarrow{n} \cdot \overrightarrow{u}] &= \overrightarrow{n} \cdot \overrightarrow{u} \quad \text{on} \quad \Gamma_D. \end{aligned} \tag{8}$$

3. The average operator is defined as,

$$\left\{\overrightarrow{u}\right\} = \frac{\overrightarrow{u}^{+} + \overrightarrow{u}^{-}}{2}. \tag{9}$$

The weak form of Stokes equation is as follow,

$$a_{IP}(\overrightarrow{u}, \overrightarrow{\phi}) + b(\overrightarrow{\phi}, p) + (\{p\}, [\overrightarrow{n} \cdot \overrightarrow{\phi}])_{\Gamma \cup \Gamma_D} = l_{IP}(\overrightarrow{\phi}) \ . \tag{10}$$

$$\begin{aligned} a_{IP}(\overrightarrow{u}, \overrightarrow{\phi}) = (\nabla \overrightarrow{u}, \nabla \overrightarrow{\phi}) + C_{11}([\overrightarrow{u}], [\overrightarrow{\phi}])_{\Gamma \cup \Gamma_D} \\ -\nu(\nabla \overrightarrow{u}, [\overrightarrow{n} \otimes \overrightarrow{\phi}])_{\Gamma \cup \Gamma_D} - \nu([\overrightarrow{n} \otimes \overrightarrow{u}], \nabla \overrightarrow{\phi})_{\Gamma \cup \Gamma_D}. \end{aligned} \tag{11}$$

The penalty paramter $C_{11} > 0$ is an empirical constant to be kept large enough to maintain coercivity of bilinear form.

$$b(\phi, \psi) = -\int_{\mathcal{T}} \psi \nabla \cdot \overrightarrow{\phi} \ , \tag{12}$$

$$l_{IP}(\overrightarrow{\phi}) = (\overrightarrow{f}, \overrightarrow{\phi}) + (\overrightarrow{t}, \overrightarrow{\phi})_{\Gamma_N} + C_{11}(\overrightarrow{u}_D, \overrightarrow{\phi})_{\Gamma_D} - (\overrightarrow{n} \otimes \overrightarrow{u}_D, \nu \nabla \overrightarrow{\phi})_{\Gamma_D} \ . \tag{13}$$

Discrete form of equations can be written in Matrix form as,

$$\underbrace{\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}}_{\text{Stiffness matrix}} \underbrace{\begin{pmatrix} U \\ P \end{pmatrix}}_{\text{Solution vector}} = \underbrace{\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}}_{\text{Right hand side (Known)}} . \tag{14}$$

Here,

$$\begin{aligned} A_{ij} = \sum_{k=1}^{d} (\frac{\partial \phi_i}{\partial x_k}, \frac{\partial \phi_j}{\partial x_k}) + \sum_{k=1}^{d} C_{11}([\phi_i n_k], [\phi_j n_k])_{\Gamma \cup \Gamma_D} \\ - \sum_{k=1}^{d} \nu([\phi_i n_k], \{\frac{\partial \phi_j}{\partial x_k}\})_{\Gamma \cup \Gamma_D} - \sum_{k=1}^{d} \nu(\{\frac{\partial \phi_i}{\partial x_k}\}, [\phi_j n_k])_{\Gamma \cup \Gamma_D} \ . \end{aligned} \tag{15}$$

$$B_{ij} = -\int_{\mathcal{T}} \frac{\partial \phi_i}{\partial x_i} \psi_j + (\{\psi_j\}, [n \cdot \phi_i])_{\Gamma \cup \Gamma_D} \ . \tag{16}$$

3 Geometric parametrization

In the context of geometric parametrization, domain Ω is characterized by set of parameters μ belonging to parameter space $\mathbb{P}$. A domain, called reference domain $\hat{\Omega}$, whose configuration is defined by known parameter set $\bar{\mu} \in \mathbb{P}$ is selected. In other words, the configuration of $\hat{\Omega}$ is completely known. We denote the Dirichlet boundary on reference domain and Neumann boundary on reference domain as $\hat{\Gamma}_D$ and $\hat{\Gamma}_N$ respectively.

A mapping F and inverse mapping T is defined,

$$\begin{aligned} F : \hat{\Omega} \times \mathbb{P} \to \Omega \,, \\ T : \Omega \times \mathbb{P} \to \hat{\Omega} \,, \\ T = F^{-1} \ . \end{aligned} \tag{17}$$

In the case of affine transformation, F is of the form,

$$x = F(\hat{x}, \mu) = G_F(\mu)\hat{x} + c_F(\mu) \ ; \forall x \in \Omega \,,\ \hat{x} \in \hat{\Omega} \ . \tag{18}$$

The inverse map T is expressed in the form,

$$\hat{x} = T(x, \mu) = G_T(\mu)x + c_T(\mu) \ ; \forall x \in \Omega \,,\ \hat{x} \in \hat{\Omega} \,, \tag{19}$$

$$T = F^{-1} \,, \tag{20}$$

$$G_T = G_F^{-1} \,, \tag{21}$$

$$c_T = -G_T c_F \ . \tag{22}$$

The unit normal $\overrightarrow{n}$ on parametrized domain is transformed from $\hat{\overrightarrow{n}}$ by,

$$\overrightarrow{n} = G_T^T \hat{\overrightarrow{n}} \ ,\ \overrightarrow{n} \in \mathbb{M} \,,\ \hat{\overrightarrow{n}} \in \hat{\mathbb{M}} \ . \tag{23}$$

We emphasize the fact that the map T (or F) is dependent only on parameter μ.

4 Reduced basis method

4.1 Snapshot proper orthogonal decomposition

We present now snapshot proper orthogonal decomposition method. Here, "snapshot" means solution calculated by discontinuous Galerkin method. We calculate solution based on $\mu_n, n \in \{1, ..., n_s\}$ i.e. n_s snapshots are generated. We also introduce inner product matrices $M_u \in \mathbb{R}^{u_{ndofs} \times u_{ndofs}}$ and $M_p \in \mathbb{R}^{p_{ndofs} \times p_{ndofs}}$, formed by inner product of basis function with respect to suitable function space $\mathbb{W}$.

$$M_u =< \phi, \phi >_{\mathbb{W}} \ . \tag{24}$$

$$M_p =< \psi, \psi >_{\mathbb{W}} \ . \tag{25}$$

We also introduce matrices storing velocity snapshots S_u and storing pressure snapshots S_p. We discuss the method only for velocity snapshots. The method is similar for pressure snapshots. We note the size of matrices, useful for matrix operations presented hereafter.

$$S_u \in \mathbb{R}^{u_{ndofs} \times n_s} \ ,$$
$$S_p \in \mathbb{R}^{p_{ndofs} \times n_s} \ ,$$
$$M_u \in \mathbb{R}^{u_{ndofs} \times u_{ndofs}} \ ,$$
$$M_p \in \mathbb{R}^{p_{ndofs} \times p_{ndofs}} \ .$$

4.2 Spectral decomposition of snapshots

We denote the dimension of reduced basis as N and assert that $N < n_s$. We now perform the spectral decomposition of $S_u^T M_u S_u$,

$$S_u^T M_u S_u = V \Theta V^T . \tag{26}$$

The columns of V are eigenvectors and Θ has eigenvalues θ such that,

$$\Theta_{ij} = \theta_{ij} \delta_{ij} . \tag{27}$$

We also note that $\theta_{ij} > 0$ and $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{n_s}$ i.e. the eigenvalues are in sorted order. We form the reduced basis by linear combination of the snapshot vector,

$$B_{velocity} = S_u A, \quad A \in \mathbb{R}^{n_s \times N} . \tag{28}$$

Here,$B_{velocity}$ is defined such that, if $\phi \in \mathbb{R}^{n \times d_u}$ $B_{velocity} \in \mathbb{R}^{n \times N}$, the reduced basis for velocity $\psi_u \in \mathbb{R}^{d_u \times N}$ is formed by,

$$\psi_u = \phi^T B . \tag{29}$$

Considering orthonormality of reduced basis ψ_u with respect to inner product in function space $\mathbb{W}$,

$$< \psi_u, \psi_u >_{\mathbb{W}} = B_u^T M_u B_u = I \ . \tag{30}$$

Considering above orthonormality, we express matrix A as,

$$A = V \Theta^{-\frac{1}{2}} R, \quad R \in \mathbb{R}^{n_s \times N}, \quad R^T R = I . \tag{31}$$

where, I is identity matrix of suitable size.
We set now R as,

$$R = [I_{N \times N} ; 0_{(n_s \times N)}] \quad \text{and accordingly} \quad B = S V \Theta^{-\frac{1}{2}} R . \tag{32}$$

4.3 Galerkin reduced basis formulation

We now present the reduced bilinear form as,

$$a(u_N, \phi_N; \mu) + b(p_N, \phi_N; \mu) = f_1(\phi_N, \mu), \tag{33}$$

$$b(u_N, \psi_N; \mu) = f_2(\psi_N, \mu). \tag{34}$$

In discrete form, we form reduced equation as,

$$\underbrace{\begin{pmatrix} B_{velocity}^T A(\mu) B_{velocity} & B_{velocity}^T B(\mu) B_{pressure} \\ B_{pressure}^T B(\mu)^T B_{velocity} & 0 \end{pmatrix}}_{\tilde{K}} \underbrace{\begin{pmatrix} U_N \\ P_N \end{pmatrix}}_{\zeta} = \underbrace{\begin{pmatrix} B_{velocity}^T F_1(\mu) \\ B_{pressure}^T F_2(\mu) \end{pmatrix}}_{\tilde{F}}, \tag{35}$$

and accordingly we solve following variational form for reduced degrees of freedom ζ,

$$\tilde{K}\zeta = \tilde{F}, \quad \zeta_u \in \mathbb{R}^N, \tag{36}$$

and calculate reduced solution u_N as,

$$u_N = \psi_u \zeta_u, \quad u_N \in \mathbb{R}^{d_u}. \tag{37}$$

5 Numerical example

We perform the POD-Galerkin method as mentioned in section 4.1 and section 4.3. The boundary $x = 0$ is Dirichlet boundary with inflow velocity at point $(0, y)$ as $u = (y(1-y), 0)$. The boundary $x = 1$ is a Neumann boundary with zero Neumann value i.e. $t = (0, 0)$. Other boundaries are Dirichlet boundary with no slip condition. The source term is $f = (0, 0)$.

We consider inner-product matrix as (u_{rb}, p_{rb} is reduced basis solution and u_h, p_h is DG solution) i.e.

$$M_u = \int_\Omega \phi_i \cdot \phi_j + \sum_{k=1}^{nel} \int_{\tau_k} \nabla\phi_l : \nabla\phi_m \,,$$

$$i, j = 1, \ldots, u_{ndofs} \,,\; l, m = 1, \ldots, u_{ndofs\_per\_element}$$

$$M_p = \int_\Omega \psi_i \psi_j \,,\; i, j = 1, \ldots, p_{ndofs} \,.$$

The domain is shown in Figure **??**. The geometric parameters were coordinates of tip of the obstacle.

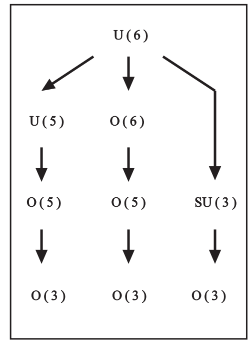

Fig. 1 If the width of the figure is less than 7.8 cm use the `sidecapion` command to flush the caption on the left side of the page. If the figure is positioned at the top of the page, align the sidecaption with the top of the figure – to achieve this you simply need to use the optional argument `[t]` with the `sidecaption` command

$$(x, y) = (\mu_1, \mu_2)\ .$$
$$(x_{ref}, y_{ref}) = (0.5, 0.3)\ .$$

1. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
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- Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development, cf. Table 1.
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Table 1 Please write your table caption here

Classes	Subclass	Length	Action Mechanism
Translation	mRNA[a]	22 (19–25)	Translation repression, mRNA cleavage
Translation	mRNA cleavage	21	mRNA cleavage
Translation	mRNA	21–22	mRNA cleavage
Translation	mRNA	24–26	Histone and DNA Modification

[a] Table foot note (with superscript)

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Type 1 That addresses central themes pertainng to migration, health, and disease. In Sect. **??**, Wilson discusses the role of human migration in infectious disease distributions and patterns.

Type 2 That addresses central themes pertainng to migration, health, and disease. In Sect. **??**, Wilson discusses the role of human migration in infectious disease distributions and patterns.

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Theorem 1 *Theorem text goes here.*

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Proof Proof text goes here. □

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Theorem 2 *Theorem text goes here.*

Definition 2 Definition text goes here.

Proof Proof text goes here. □

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Appendix

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$$a \times b = c \tag{38}$$

References

References

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