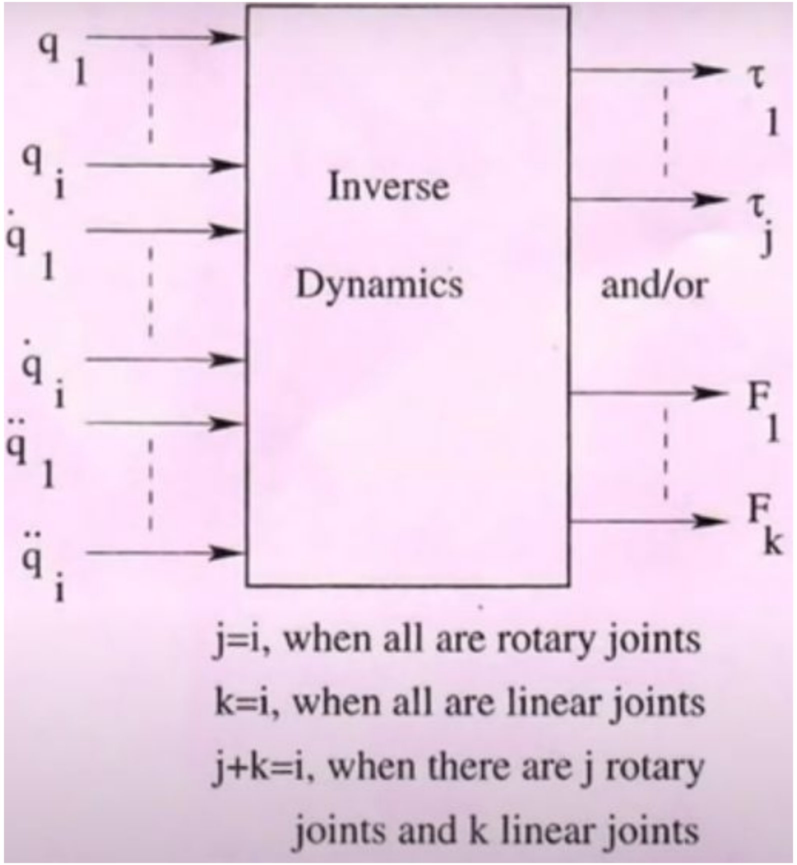
Robot Dynamics

Lecture-18

Dynamics

- The purpose of dynamics to determine the amount of force if it is linear joint or amount of torque if it is rotary joint.
- To carry out dynamic analysis the prerequisite is kinematics and trajectory planning.
- Expression for joint torque or joint force which is going to create moment.

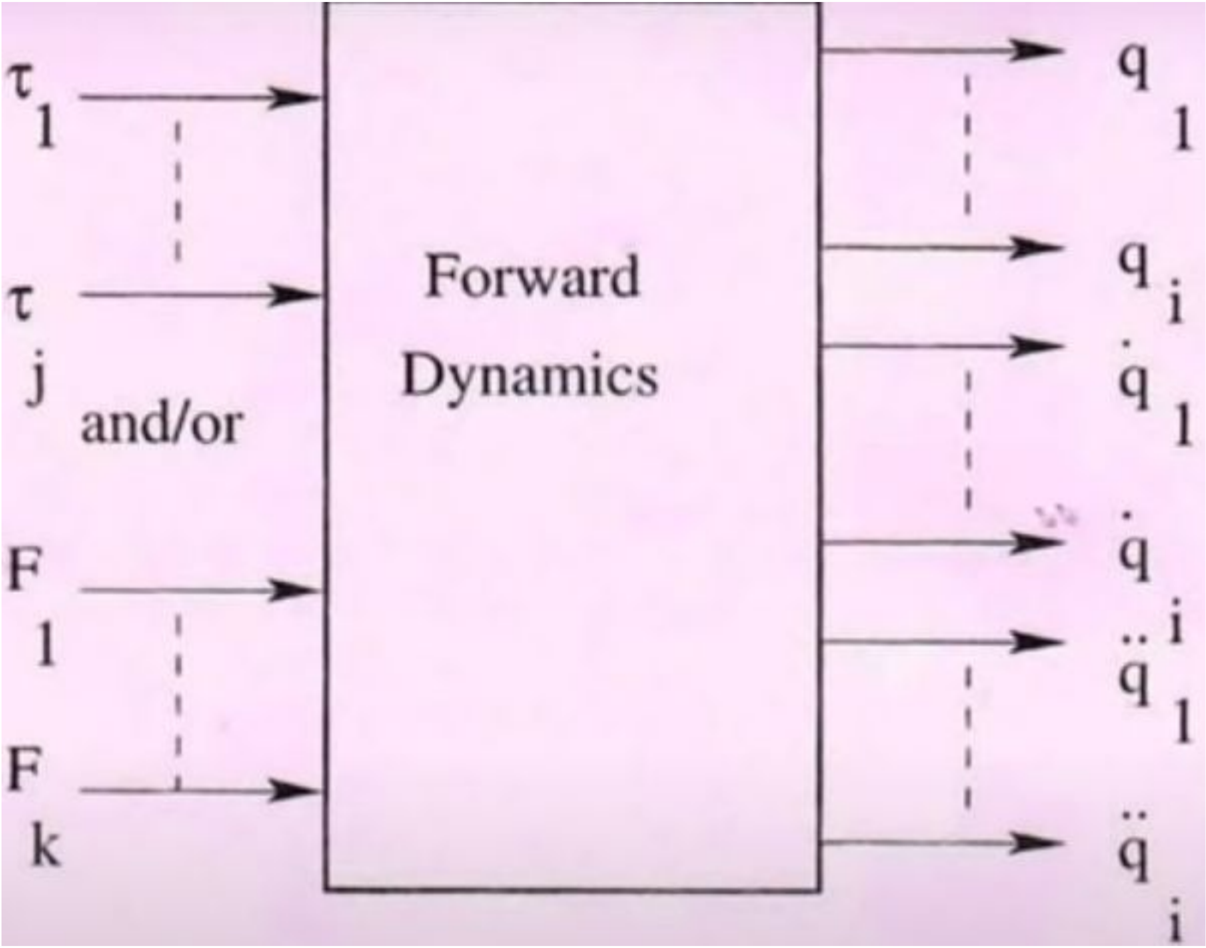


Inverse Dynamics

q=generalized coordinate

q= Θ for rotary joint

= d for linear joint

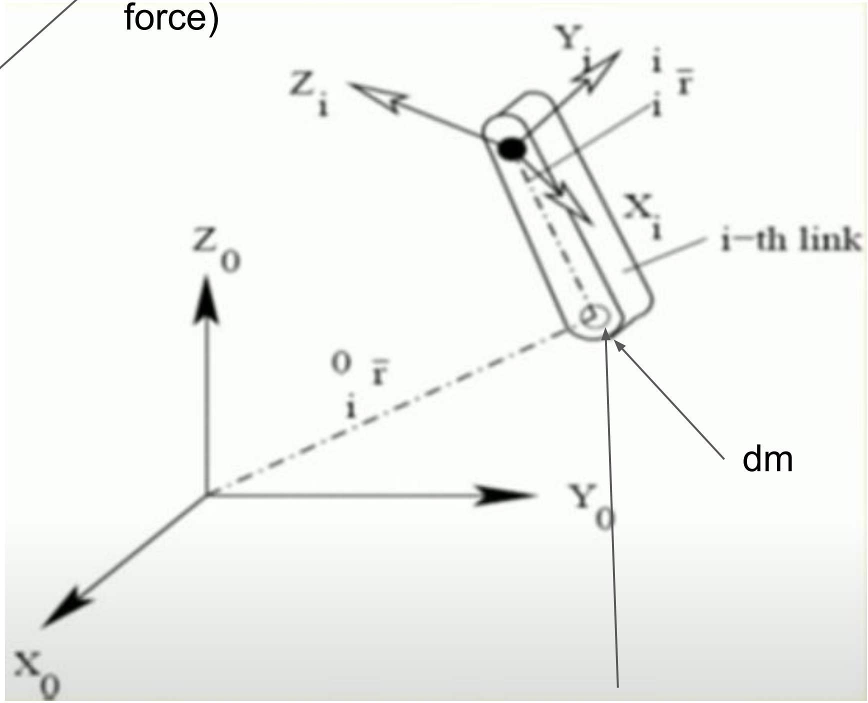


Forward Dynamics

Robot Dynamics

- To determine joint torques/forces
- Robotic joint torque consists of inertia, centrifugal and Coriolis and gravity terms

Rough sketch of robotic link (it is subjected to some amount of force, which is coriolis force)



dm

Depends on acceleration due to gravity

Inertia Term

Depends on mass distribution of the links and it is expressed in terms of moment of inertia tensor

You have to measure reaction force

The moment of Inertia:

- The moment of inertia, also known as rotational inertia, is a property of a rigid body that describes its resistance to rotational motion around a particular axis
- It is similar to mass in linear motion, in that it measures the body's resistance to changes in its rotational motion
- The moment of inertia depends on both the mass and the distribution of the mass of the body with respect to the axis of rotation
- The formula for calculating the moment of inertia of a rigid body is:

$$I = \int r^2 dm$$

Where I is the moment of inertia, r is the perpendicular distance from the axis of rotation to a small element of mass dm, and the integral is taken over the entire mass of the body.

Let,

${}^{i}_{i}\bar{r}$ =position of a fixed point lying on i-th rigid link expressed in its own coordinate system

$$= \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$$

The same point can be expressed in base coordinates system as follows:

$${}^{0}_{i}\bar{r} = {}^{0}_{i}T\,{}^{i}_{i}\bar{r}$$

where $${}^{0}_{i}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T \ldots {}^{i-1}_{i}T$$

Inertia Tensor of i-th link (moment of inertia)

$$J_i = \int {}^i_i\bar{r} \, {}^i_i\bar{r}^{T'} dm$$

$$= \begin{bmatrix} \int x_i^2 dm & \int x_i y_i dm & \int x_i z_i dm & \int x_i dm \\ \int x_i y_i dm & \int y_i^2 dm & \int y_i z_i dm & \int y_i dm \\ \int x_i z_i dm & \int y_i z_i dm & \int z_i^2 dm & \int z_i dm \\ \int x_i dm & \int y_i dm & \int z_i dm & \int dm \end{bmatrix}$$

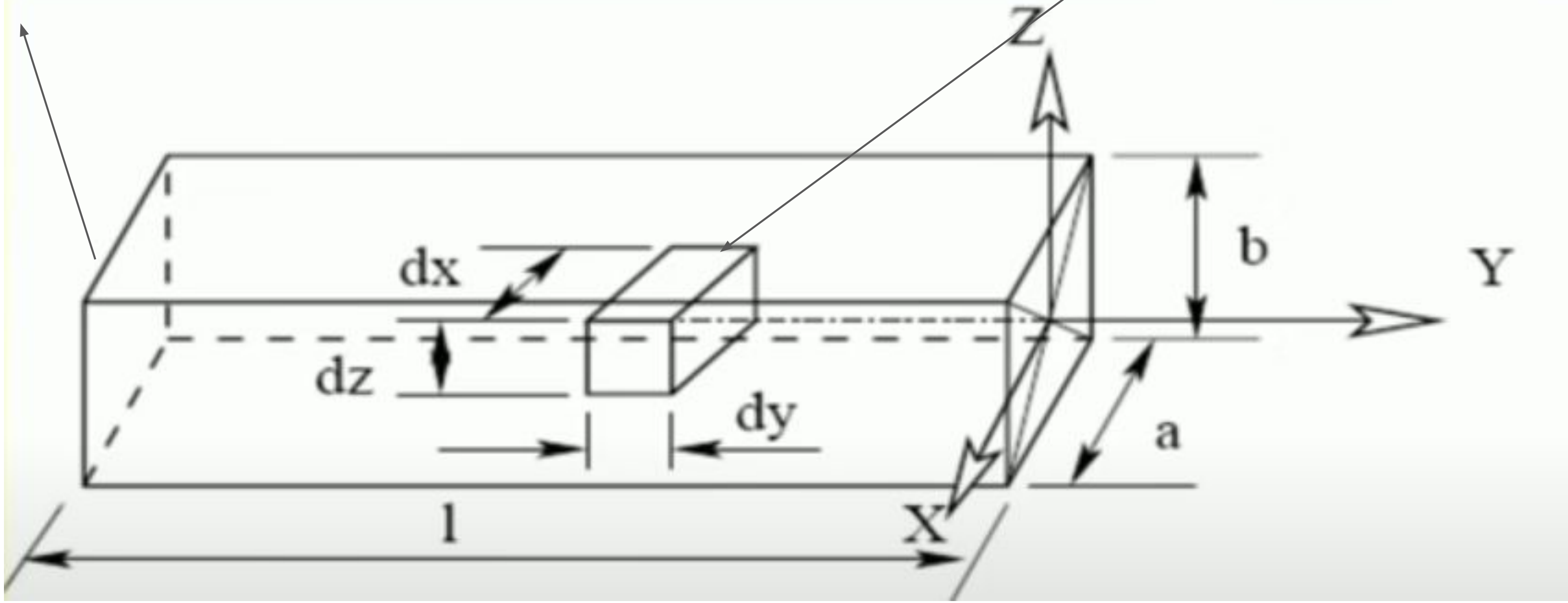
Various Cases

How to find out inertia tensor

Case 1: Link with Rectangular Cross-section

Differential mass

Motor is connected to this end



Moment of Inertia (Positive value)

Differential mass $dm = \rho dxdydz$ Density X Volume

Moment of inertia about X axis

$$I_{XX} = \int_{-b/2}^{b/2}\int_{-l}^{0}\int_{-a/2}^{a/2}(y^2 + z^2)\rho dxdydz$$

$$= m\left(\frac{l^2}{3} + \frac{b^2}{12}\right)$$

$m = \rho\, a\, b\, l$

$$I_{YY} = \int_{-b/2}^{b/2}\int_{-l}^{0}\int_{-a/2}^{a/2}(x^2 + z^2)\rho dxdydz$$

$$= m\left(\frac{a^2}{12} + \frac{b^2}{12}\right)$$

$$I_{ZZ} = \int_{-b/2}^{b/2} \int_{-l}^{0} \int_{-a/2}^{a/2} \left(x^2 + y^2\right) \rho dx dy dz$$

$$= m\left(\frac{l^2}{3} + \frac{a^2}{12}\right)$$

Product of Inertia (Positive/Negative/Zero)

$$I_{XY} = \int_{-b/2}^{b/2} \int_{-l}^{0} \int_{-a/2}^{a/2} xy \rho dx dy dz = 0$$

$$I_{YZ} = \int_{-b/2}^{b/2} \int_{-l}^{0} \int_{-a/2}^{a/2} yz \rho dx dy dz = 0$$

$$I_{ZX} = \int_{-b/2}^{b/2} \int_{-l}^{0} \int_{-a/2}^{a/2} zx \rho dx dy dz = 0$$

$$\int xdm = \int_{-b/2}^{b/2}\int_{-l}^{0}\int_{-a/2}^{a/2} x\rho dxdydz = 0$$

$$\int ydm = -m\frac{l}{2} = m\bar{y}_i$$

$$\int zdm = 0$$

Mass Center $= (\bar{x}_i, \bar{y}_i, \bar{z}_i)$

$(0, -l/2, 0)$

$$\int dm = m$$

Inertia tensor, J_i can be written as

$$J_i = \begin{bmatrix} \frac{-I_{XX}+I_{YY}+I_{ZZ}}{2} & I_{XY} & I_{ZX} & m_i\bar{x}_i \\ I_{XY} & \frac{I_{XX}-I_{YY}+I_{ZZ}}{2} & I_{YZ} & m_i\bar{y}_i \\ I_{ZX} & I_{YZ} & \frac{I_{XX}+I_{YY}-I_{ZZ}}{2} & m_i\bar{z}_i \\ m_i\bar{x}_i & m_i\bar{y}_i & m_i\bar{z}_i & m_i \end{bmatrix}$$

$$J_i = \begin{bmatrix} \frac{ma^2}{12} & 0 & 0 & 0 \\ 0 & \frac{ml^2}{3} & 0 & -\frac{ml}{2} \\ 0 & 0 & \frac{mb^2}{12} & 0 \\ 0 & -\frac{ml}{2} & 0 & m \end{bmatrix}$$

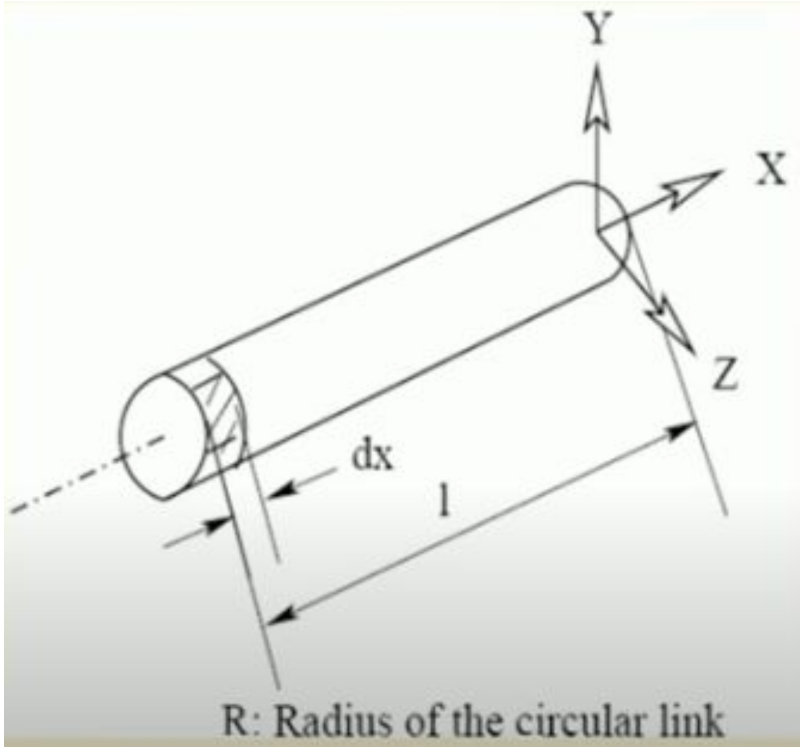
For a slender link,
($l \rangle\rangle a \; and \; l \rangle\rangle b$)

$$J_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{ml^2}{3} & 0 & -\frac{ml}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{ml}{2} & 0 & m \end{bmatrix}$$

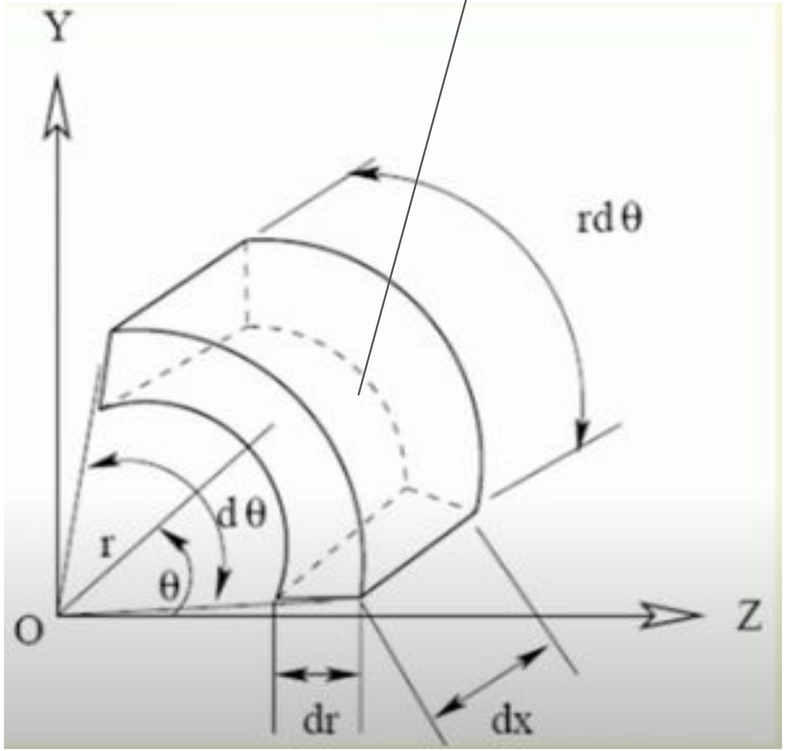
Robot Dynamics

Lecture-19

Case-2: Robotic link of circular cross-section



$$dm = \rho dv = \rho r d\theta dr dx$$



Let us consider a link of length l having circular cross-section of radius r

$$y = r\ \sin\theta$$

$$z = r\cos\theta$$

Volume of small element $dv = rd\theta drdx$

Mass of small element $dm = \rho dv,\ where\, \rho = density$

Moment of Inertia

$$I_{XX} = \int_V (y^2 + z^2)dm = \int_{-l}^{0}\int_{0}^{r}\int_{0}^{2\pi} r^2 \rho r d\theta dr dx = \frac{1}{2}mr^2$$

$$I_{YY} = \int_V (x^2 + z^2)\, dm = \int_{-l}^{0} \int_0^{r} \int_0^{2\pi} (x^2 + r^2 \cos^2 \theta)\, \rho r d\theta dr dx = \frac{ml^2}{3} + \frac{mr^2}{4}$$

$$I_{ZZ} = \int_V (x^2 + y^2)\, dm = \int_{-l}^{0} \int_0^{r} \int_0^{2\pi} (x^2 + r^2 \sin^2 \theta)\, \rho r d\theta dr dx = \frac{ml^2}{3} + \frac{mr^2}{4}$$

Product of Inertia

$$I_{XY} = \int_V xydm = \int_{-l}^{0}\int_0^r\int_0^{2\pi} xr\sin\theta\rho r d\theta dr dx = 0$$

Similarly $I_{YZ} = 0$ $I_{ZX} = 0$

$$\int_V x\,dm = \int_{-l}^{0}\int_0^r\int_0^{2\pi} x\rho r\,d\theta\,dr\,dx = -\frac{1}{2}ml$$

$$\int_V y\,dm = 0$$

$$\int_V z\,dm = 0$$

Mass Center= $(\bar{x}_i, \bar{y}_i, \bar{z}_i) = \left(-\frac{l}{2}, 0, 0\right)$

$$\int dm = m$$

$$J_i = \begin{bmatrix} \frac{ml^2}{3} & 0 & 0 & -\frac{ml}{2} \\ 0 & \frac{mr^2}{4} & 0 & 0 \\ 0 & 0 & \frac{mr^2}{4} & 0 \\ -\frac{ml}{2} & 0 & 0 & m \end{bmatrix}$$

For a slender link,
($l \rangle\rangle r$)

$$J_i = \begin{bmatrix} \frac{ma^2}{3} & 0 & 0 & -\frac{ml}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{ml}{2} & 0 & 0 & m \end{bmatrix}$$

Determination of Robotic Joint Torques

Lagrange-Euler Formulation

Mathematical expression for this term

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i$$

Where $i = 1, 2, \ldots, n$

$n = \text{No. of joints}$

$L = \text{Lagrangian function}$

$L = K(K.E) - P(P.E)$

$q_i = \text{Generalized coordinates}$

$q_i = \theta_i$ for a rotary joint

$q_i = d_i$ for a rotary joint

$\dot{q}_i = \text{first time (t) derovatove of } q_i$

τ_i : Generalized torque for a rotary joint

τ_i : Generalized torque for a linear joint

Let us consider i-th link of a serial manipulator

Position of a fixed point lying on this link

$$
{}^{i}_{i}\bar{r} = \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}
$$

The same point can be expressed in base coordinates system as follows:

$$
{}^{0}_{i}\bar{r} = {}^{0}_{i}T\,{}^{i}_{i}\bar{r}
$$

where $\quad {}^{0}_{i}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T \ldots {}^{i-1}_{i}T$

Determination of Kinetic Energy (K) of the Manipulator

Velocity of a particle of link i w.r.t. base coordinate system

$$\text{Kinetic Energy} = \frac{1}{2}mv^2$$

$$ {}_i^0\bar{V} = \frac{d}{dt}\left({}_i^0\bar{r}\right)$$

$$ {}_i^0\bar{V} = \frac{d}{dt}\left({}_i^0T\,{}_i^i\bar{r}\right) = {}_1^0\dot{T}\,{}_2^1T\ldots{}_i^{i-1}T\,{}_i^i\bar{r} + \ldots + {}_1^0T\,{}_2^1T\ldots{}_i^{i-1}\dot{T}\,{}_i^i\bar{r} + {}_1^0T\,{}_i^i\dot{\bar{r}}$$

$$= \left(\sum_{j=1}^{i}\frac{\partial\,{}_i^0T}{\partial q_j}\dot{q}_j\right){}_i^i\bar{r}, \text{ as } {}_i^i\dot{\bar{r}} = 0$$

Rigid link

$$ {}_i^0T = {}_1^0T\,{}_2^1T\ldots{}_i^{i-1}T$$

$$\frac{d}{dt}\,{}_i^0T = \frac{\partial}{\partial q_j}\,{}_i^0T\cdot\frac{dq}{dt}$$

$$\text{Let } \frac{\partial\,{}_i^0T}{\partial q_j} = U_{ij} \qquad \text{Therefore, } \quad {}_i^0\bar{V} = \left(\sum_{j=1}^{i}U_{ij}\dot{q}_j\right){}_i^i\bar{r}$$

$$\text{Note: } U_{ijk} = \frac{\partial U_{ij}}{\partial q_k}$$

Kinetic energy of the particle having differential mass dm

$$dK_i = \frac{1}{2}\left(\dot{x}_i{}^2 + \dot{y}_i^2 + \dot{z}_i^2\right)dm = \frac{1}{2}T_r\left({}^0_i\bar{V}\ {}^0_i\bar{V}^T\right)dm$$

where T_r : Trace of a matrix

$$dk_i = \frac{1}{2}T_r\left[\sum_{a=1}^{i} U_{ia}\dot{q}_a\ {}^i_i\bar{r}\left[\sum_{b=1}^{i} U_{ib}\dot{q}_b\ {}^i_i\bar{r}\right]^T\right]dm$$

$$= \frac{1}{2}T_r\left[\sum_{a=1}^{i}\sum_{b=1}^{i} U_{ia}\ {}^i_i\bar{r}\ {}^i_i\bar{r}^T U_{ib}^T\dot{q}_a\dot{q}_b\right]dm$$

$$= \frac{1}{2}T_r\left[\sum_{a=1}^{i}\sum_{b=1}^{i} U_{ia}\left({}^i_i\bar{r}\ dm\,{}^i_i\bar{r}^T\right)U_{ib}^T\dot{q}_a\dot{q}_b\right]$$

Kinetic energy of i-th link

$$K_i = \int dk_i = \frac{1}{2} T_r \left[\sum_{a=1}^{i} \sum_{b=1}^{i} U_{ia} \left(\int {}_i^i\bar{r}\, {}_i^i\bar{r}^T dm \right) U_{ib}^T \dot{q}_a \dot{q}_b \right]$$

$$= \frac{1}{2} T_r \left[\sum_{a=1}^{i} \sum_{b=1}^{i} U_{ia} J_i U_{ib}^T \dot{q}_a \dot{q}_b \right]$$

Where inertia tensor

$$J_i = \int {}_i^i\bar{r}\, {}_i^i\bar{r}^T dm$$

Total K.E. of the serial manipulator having n links

$$K = \sum_{i=1}^{n} k_i = \sum_{i=1}^{n} \frac{1}{2} T_r \left[\sum_{a=1}^{i} \sum_{b=1}^{i} U_{ia} J_i U_{ib}^T \dot{q}_a \dot{q}_b \right]$$

Total K.E. of the serial manipulator having n links

$$K=\sum_{i=1}^{n} k_i=\sum_{i=1}^{n}\frac{1}{2}T_r\left[\sum_{a=1}^{i}\sum_{b=1}^{i}U_{ia}J_iU_{ib}^{T}\dot{q}_a\dot{q}_b\right]$$

$$K=\frac{1}{2}\sum_{i=1}^{n}\sum_{a=1}^{i}\sum_{b=1}^{i}\left[T_r\left(U_{ia}J_iU_{ib}^{T}\right)\dot{q}_a\dot{q}_b\right]$$

Determination of Potential Energy of the manipulator

Potential energy of i-th link

$$P_i=-m_i\bar{g}\,{}^{0}_{i}\bar{r}=-m_i\bar{g}\left({}^{0}_{i}T\,{}^{i}_{i}\bar{r}\right)$$

where $$\bar{g}=(g_x,g_y,g_z,0)$$

Total potential energy of the manipulator

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} -m_i \bar{g} \left({}^0_i T \, {}^i_i \bar{r} \right)$$

Now $L = K - P$

$$L = \frac{1}{2} \sum_{i=1}^{n} \sum_{a=1}^{i} \sum_{b=1}^{i} \left[T_r \left(U_{ia} J_i U_{ib}^T \right) \dot{q}_a \dot{q}_b \right] + \sum_{i=1}^{n} m_i \bar{g} \left({}^0_i T \, {}^i_i \bar{r} \right)$$

Using Lagrange-Euler equation, we get

$$\tau_i = \sum_{c=1}^{n} D_{ic} \ddot{q}_c + \sum_{c=1}^{n} \sum_{d=1}^{n} h_{icd} \dot{q}_c \dot{q}_d + C_i$$

where $i = 1, 2, \ldots, n$

Inertia term

$$D_{ic} = \sum_{j=\max(i,c)}^{n} Tr\left(U_{jc} J_j U_{ji}^T\right) \qquad i, c = 1, 2, \ldots, n$$

Coriolis and centrifugal term

$$h_{icd} = \sum_{j=\max(i,c,d)}^{n} Tr\left(U_{jcd} J_j U_{ji}^T\right) \qquad i, c, d = 1, 2, \ldots, n$$

Gravity term

$$C_i = \sum_{i=1}^{n} \left(-m_i \bar{g}\, U_{ji}\ {}^{j}_{j}\bar{r}\right) \qquad i = 1, 2, \ldots, n$$

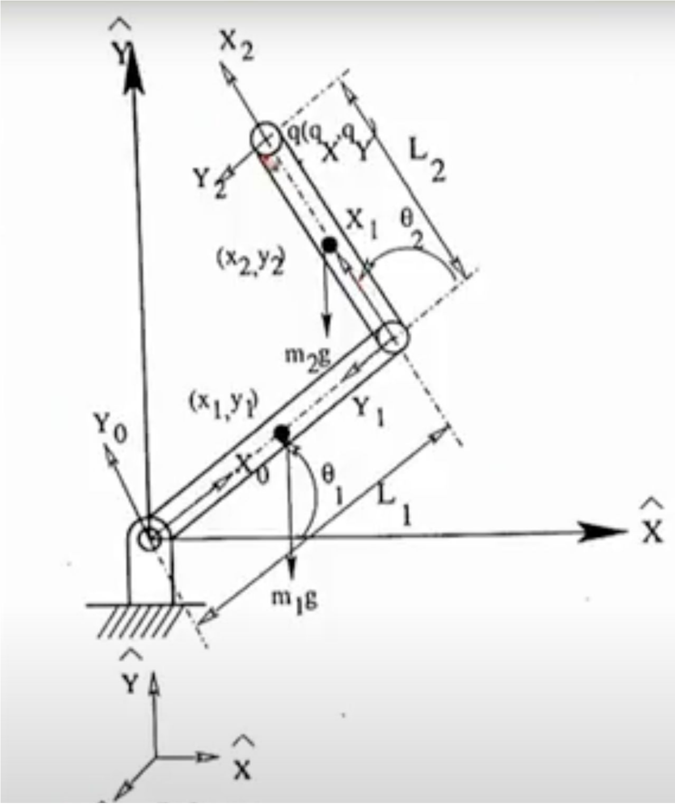
Robot Dynamics

Lecture-20

An Example

$$
{}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & L_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & L_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0_2T = \begin{bmatrix} c\theta_{12} & -s\theta_{12} & 0 & L_1 c\theta_1 + L_2 c\theta_{12} \\ s\theta_{12} & c\theta_{12} & 0 & L_1 s\theta_1 + L_2 s\theta_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Reference coordinate system

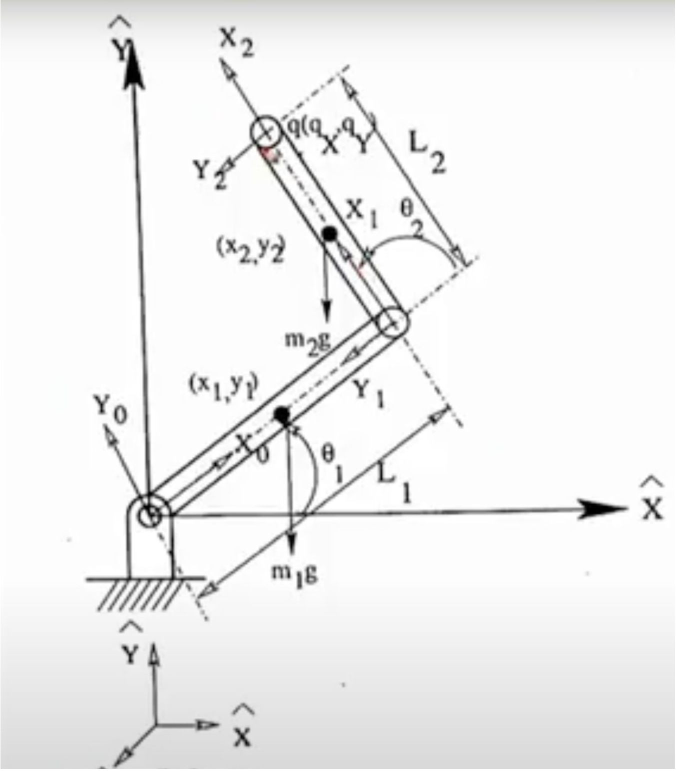
An Example

$${}^0_1T = Rot(Z, \theta_1)Trans(X, L_1)$$

$${}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & L_1c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & L_1s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_2T = \begin{bmatrix} c\theta_{12} & -s\theta_{12} & 0 & L_1c\theta_1 + L_2c\theta_{12} \\ s\theta_{12} & c\theta_{12} & 0 & L_1s\theta_1 + L_2s\theta_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_2T = Rot(Z, \theta_2)Trans(X, L_2)$$



D-H

Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	0	L_1
2	θ_2	0	0	L_2

$$\tau_1 = (D_{11}\ddot{\theta}_1 + D_{12}\ddot{\theta}_2) + h_{111}\dot{\theta}_1^{\,2} + h_{112}\dot{\theta}_1\dot{\theta}_2 + h_{121}\dot{\theta}_1\dot{\theta}_2 + h_{122}\dot{\theta}_2^{\,2} + C_1$$

$$\tau_2 = (D_{21}\ddot{\theta}_1 + D_{22}\ddot{\theta}_2) + h_{211}\dot{\theta}_1^{\,2} + h_{212}\dot{\theta}_1\dot{\theta}_2 + h_{221}\dot{\theta}_1\dot{\theta}_2 + h_{222}\dot{\theta}_2^{\,2} + C_2$$

$$\begin{aligned} U_{11} &= \frac{\partial_1^0 T}{\partial\theta_1} \\ &= \begin{bmatrix} -s\theta_1 & -c\theta_1 & 0 & -L_1 s\theta_1 \\ c\theta_1 & -s\theta_1 & 0 & L_1 c\theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
U_{21} &= \frac{\partial_2^0 T}{\partial\theta_1} \\
&= \begin{bmatrix} -s\theta_{12} & -c\theta_{12} & 0 & -L_1 s\theta_1 - L_2 s\theta_{12} \\ c\theta_{12} & -s\theta_{12} & 0 & L_1 c\theta_1 + L_2 c\theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
U_{22} &= \frac{\partial_2^0 T}{\partial\theta_2} \\
&= \begin{bmatrix} -s\theta_{12} & -c\theta_{12} & 0 & -L_2 s\theta_{12} \\ c\theta_{12} & -s\theta_{12} & 0 & L_2 c\theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
D_{11} &= Tr(U_{11}J_1U_{11}^{T'}) + Tr(U_{21}J_2U_{21}^{T'}) \\
&= (\frac{1}{3}m_1 + m_2)L_1^2 + \frac{1}{3}m_2L_2^2 + m_2L_1L_2c\theta_2 + \frac{1}{4}r^2(m_1 + m_2)
\end{aligned}
$$

$$
\begin{aligned}
D_{12} &= Tr(U_{22}J_2U_{21}^{T'}) \\
&= \frac{1}{3}m_2L_2^2 + \frac{1}{4}m_2r^2 + \frac{1}{2}m_2L_1L_2c\theta_2
\end{aligned}
$$

$$
\begin{aligned}
D_{22} &= Tr(U_{22}J_2U_{22}^{T'}) \\
&= \frac{1}{3}m_2L_2^2 + \frac{1}{4}m_2r^2
\end{aligned}
$$

$$
\begin{aligned}
D_{21} &= Tr(U_{21}J_2U_{22}^{T'}) \\
&= \frac{1}{3}m_2L_2^2 + \frac{1}{4}m_2r^2 + \frac{1}{2}m_2L_1L_2c\theta_2
\end{aligned}
$$

$$h_{111} = Tr(U_{111} J_1 U_{11}^{T'}) + Tr(U_{211} J_2 U_{21}^{T'}),$$

$$U_{111} = \frac{\partial U_{11}}{\partial \theta_1} = \begin{bmatrix} -c\theta_1 & s\theta_1 & 0 & -L_1 c\theta_1 \\ -s\theta_1 & -c\theta_1 & 0 & -L_1 s\theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$U_{211} = \frac{\partial U_{21}}{\partial \theta_1} = \begin{bmatrix} -c\theta_{12} & s\theta_{12} & 0 & -L_1 c\theta_1 - L_2 c\theta_{12} \\ -s\theta_{12} & -c\theta_{12} & 0 & -L_1 s\theta_1 - L_2 s\theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$h_{111} = 0.$$

$$h_{112} = Tr(U_{212} J_2 U_{21}^{T'}),$$

$$U_{212} = \frac{\partial U_{21}}{\partial \theta_2}$$

$$= \begin{bmatrix} -c\theta_{12} & s\theta_{12} & 0 & -L_2 c\theta_{12} \\ -s\theta_{12} & -c\theta_{12} & 0 & -L_2 s\theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$h_{112} = -\tfrac{1}{2} m_2 L_1 L_2 s\theta_2.$$

$$h_{121} = Tr(U_{221} J_2 U_{21}^{T'}),$$

$$U_{221} = \frac{\partial U_{22}}{\partial \theta_1}$$

$$= \begin{bmatrix} -c\theta_{12} & s\theta_{12} & 0 & -L_2 c\theta_{12} \\ -s\theta_{12} & -c\theta_{12} & 0 & -L_2 s\theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$h_{121} = -\frac{1}{2} m_2 L_1 L_2 s\theta_2$$

$$h_{122} = Tr(U_{222} J_2 U_{21}^{T'}),$$

$$\begin{aligned}
U_{222} &= \frac{\partial U_{22}}{\partial \theta_2} \\
&= \begin{bmatrix} -c\theta_{12} & s\theta_{12} & 0 & -L_2 c\theta_{12} \\ -s\theta_{12} & -c\theta_{12} & 0 & -L_2 s\theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$h_{122} = -\tfrac{1}{2} m_2 L_1 L_2 s\theta_2.$$

$$
\begin{aligned}
h_1 &= h_{111}\dot{\theta}_1^2 + h_{112}\dot{\theta}_1\dot{\theta}_2 + h_{121}\dot{\theta}_1\dot{\theta}_2 + h_{122}\dot{\theta}_2^2 \\
&= -m_2 L_1 L_2 s\theta_2 \dot{\theta}_1 \dot{\theta}_2 - \frac{1}{2} m_2 L_1 L_2 s\theta_2 \dot{\theta}_2{}^2
\end{aligned}
$$

$$
h_{211} = Tr(U_{211} J_2 U_{22}^{T'}),
$$

$$
h_{21\mathrm{i}} = \tfrac{1}{2} m_2 L_1 L_2 s\theta_2
$$

$$h_{212} = Tr(U_{212} J_2 U_{22}^{T'}),$$

$$U_{212} = \frac{\partial U_{21}}{\partial \theta_2}$$

$$= \begin{bmatrix} -c\theta_{12} & s\theta_{12} & 0 & -L_2 c\theta_{12} \\ -s\theta_{12} & -c\theta_{12} & 0 & -L_2 s\theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$h_{212} = 0$$

$$h_{221} = Tr(U_{221} J_2 U_{22}^{T'}),$$

$$\begin{aligned} U_{221} &= \frac{\partial U_{22}}{\partial \theta_1} \\ &= \begin{bmatrix} -c\theta_{12} & s\theta_{12} & 0 & -L_2 c\theta_{12} \\ -s\theta_{12} & -c\theta_{12} & 0 & -L_2 s\theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$h_{221} = 0$$

$$h_{222} = Tr(U_{222} J_2 U_{22}^{T'}),$$

$$U_{222} = \frac{\partial U_{22}}{\partial \theta_2}$$

$$= \begin{bmatrix} -c\theta_{12} & s\theta_{12} & 0 & -L_2 c\theta_{12} \\ -s\theta_{12} & -c\theta_{12} & 0 & -L_2 s\theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{h_{222} = 0}$$

$$
\begin{aligned}
h_2 &= h_{211}\dot{\theta}_1^2 + h_{212}\dot{\theta}_1\dot{\theta}_2 + h_{221}\dot{\theta}_1\dot{\theta}_2 + h_{222}\dot{\theta}_2^2 \\
&= \frac{1}{2}m_2 L_1 L_2 s\theta_2 \dot{\theta}_1{}^2
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= \sum_{j=1}^{2}(-m_j \bar{g} U_{j1} {}^j_j\bar{r}) \\
&= -m_1 \bar{g} U_{11} {}^1_1\bar{r} - m_2 \bar{g} U_{21} {}^2_2\bar{r}
\end{aligned}
$$

Substituting the values of $\bar{g} = (0 \ -g \ 0 \ 0)$, U_{11}, U_{21}, ${}^1_1\bar{r} = (-\frac{L_1}{2} \ 0 \ 0 \ 1)^{T'}$ and ${}^2_2\bar{r} = (-\frac{L_2}{2} \ 0 \ 0 \ 1)^{T'}$ in the above expression, we get

$$
C_1 = \frac{1}{2} m_1 g L_1 c\theta_1 + m_2 g L_1 c\theta_1 + \frac{1}{2} m_2 g L_2 c\theta_{12}
$$

$$C_2 = -m_2 \bar{g} U_{22} {}^2_2\bar{r}$$

Substituting the values of $\bar{g} = (0 \; -g \; 0 \; 0)$, U_{22}, ${}^2_2\bar{r} = (-\frac{L_2}{2} \; 0 \; 0 \; 1)^{T'}$ in the above expression, we get

$$C_2 = \frac{1}{2} m_2 g L_2 c\theta_{12}$$

$$
\begin{aligned}
\tau_1 \quad = \quad & ((\frac{1}{3}m_1 + m_2)L_1^2 + \frac{1}{3}m_2L_2^2 + m_2L_1L_2c\theta_2 + \frac{1}{4}r^2(m_1 + m_2))\ddot{\theta}_1 + \\
& (\frac{1}{3}m_2L_2^2 + \frac{1}{4}m_2r^2 + \frac{1}{2}m_2L_1L_2c\theta_2)\ddot{\theta}_2 - m_2L_1L_2s\theta_2\dot{\theta}_1\dot{\theta}_2 - \\
& \frac{1}{2}m_2L_1L_2s\theta_2\dot{\theta}_2^{\,2} + \frac{1}{2}m_1gL_1c\theta_1 + m_2gL_1c\theta_1 \\
& + \frac{1}{2}m_2gL_2c\theta_{12}
\end{aligned}
$$

$$
\begin{aligned}
\tau_2 \quad = \quad & ((\frac{1}{3}m_2L_2^2 + \frac{1}{4}m_2r^2 + \frac{1}{2}m_2L_1L_2c\theta_2)\ddot{\theta}_1 + (\frac{1}{3}m_2L_2^2 + \frac{1}{4}m_2r^2)\ddot{\theta}_2) + \\
& \frac{1}{2}m_2L_1L_2s\theta_2\dot{\theta}_1^{\,2} + \frac{1}{2}m_2gL_2c\theta_{12}
\end{aligned}
$$