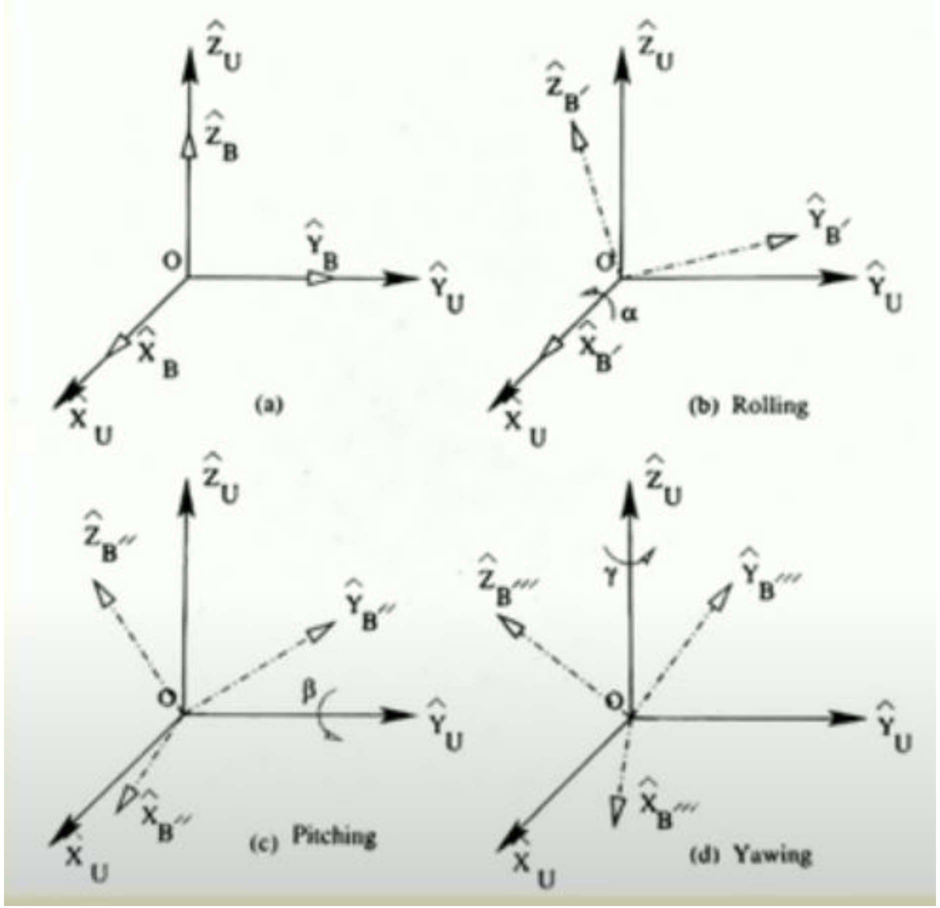
Robot Kinematics

Roll, Pitch and Yaw Angles



Steps:

1. Rotate $\{B\}$ about $\hat{X}_U$ by an angle α $\Longrightarrow$ Rolling
2. Rotate $\{B'\}$ about $\hat{Y}_U$ by an angle β $\Longrightarrow$ Pitching
3. Rotate $\{B''\}$ about $\hat{Z}_U$ by an angle γ $\Longrightarrow$ Yawing

$$
{}^{U}_{B}R_{\text{composite rpy}} = \overset{\text{3X3}}{ROT\left(\hat{Z}_U, \gamma\right)}\,\overset{\text{3X3}}{ROT\left(\hat{Y}_U, \beta\right)}\,\overset{\text{3X3}}{ROT\left(\hat{X}_U, \alpha\right)}
$$

$$
= \underset{\text{3X3}}{\begin{bmatrix} c\beta c\gamma & -c\alpha s\gamma + s\alpha s\beta c\gamma & s\alpha s\gamma + c\alpha s\beta c\gamma \\ c\beta s\gamma & c\alpha c\gamma + s\alpha s\beta s\gamma & -s\alpha c\gamma + c\alpha s\beta s\gamma \\ -s\beta & c\beta s\alpha & c\alpha s\beta \end{bmatrix}}
$$

We compare with

$$
{}^{U}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad \text{3X3}
$$

We get

$$\alpha = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right)$$

$$\beta = \tan^{-1}\left(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}\right)$$

$$\gamma = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right)$$

A numerical Example

The concept of roll, pitch and yaw angles has been used to represent the rotation of a frame $\{B\}$ with respect to the reference frame $\{U\}$, that is ${}^{U}_{B}R$. Let us suppose that the above rotation can also be expressed by a 3X3 rotation matrix as given below

$$
{}^{U}_{B}R = \begin{bmatrix} -0.250 & 0.433 & -0.866 \\ 0.433 & -0.750 & -0.500 \\ -0.866 & -0.500 & 0.000 \end{bmatrix}
$$

Determine the angles of rolling, pitching and yawing.

Solution:

Angle of rolling $\alpha = \tan^{-1} \frac{r_{32}}{r_{33}} = \tan^{-1} \frac{-.500}{.000} = 90^o$

Angle of pitching $\beta = \tan^{-1} \frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}$

$$= \tan^{-1} \frac{.866}{\sqrt{(-0.250)^2 + (0.433)^2}}$$

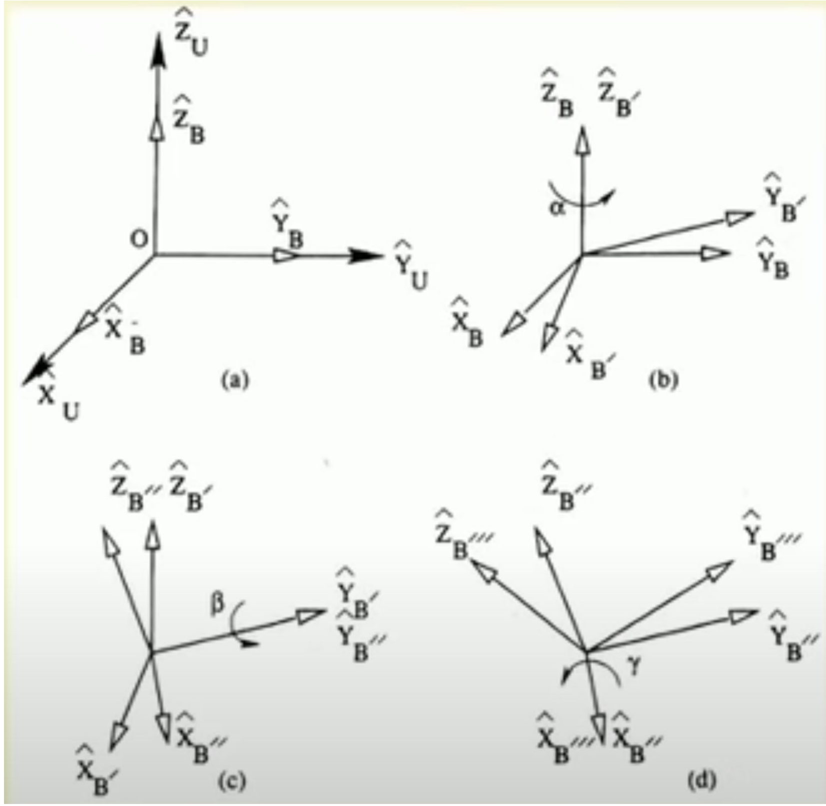
$$= 40.89^o$$

Solution:

Angle of yawing $$\gamma = \tan^{-1}\frac{-r_{21}}{r_{11}} = \tan^{-1}\frac{0.433}{-0.250} = -59.99 \approx -60^o$$

Using Euler angles

$${}^{U}_{B}R = {}^{B}_{U}R^{-1}$$



Steps:

1. Rotate $\{B\}$ about $\hat{Z}_B$ by an angle α in anti-clockwise sense
2. Rotate $\{B\}$ about $\hat{Y}_{B'}$ by an angle β in anti-clockwise sense
3. Rotate $\{B\}$ about $\hat{X}_{B''}$ by an angle γ in anti-clockwise sense

$$ {}^{B}_{U}R_{\mathrm{Eulerangles}} = ROT\left(\hat{X_{B''}}, -\gamma\right) ROT\left(\hat{Y_{B'}}, -\beta\right) ROT\left(\hat{Z}_{B}, -\alpha\right) $$

$$ {}^{U}_{B}R = \begin{bmatrix} c\alpha c\beta & s\beta s\gamma c\alpha - s\alpha c\gamma & s\beta c\gamma c\alpha + s\alpha s\gamma \\ s\alpha c\beta & s\beta s\gamma s\alpha + c\alpha c\gamma & s\beta c\gamma s\alpha - s\gamma c\alpha \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

We compare with

$$ {}^{U}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

We get

$$\alpha = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right)$$

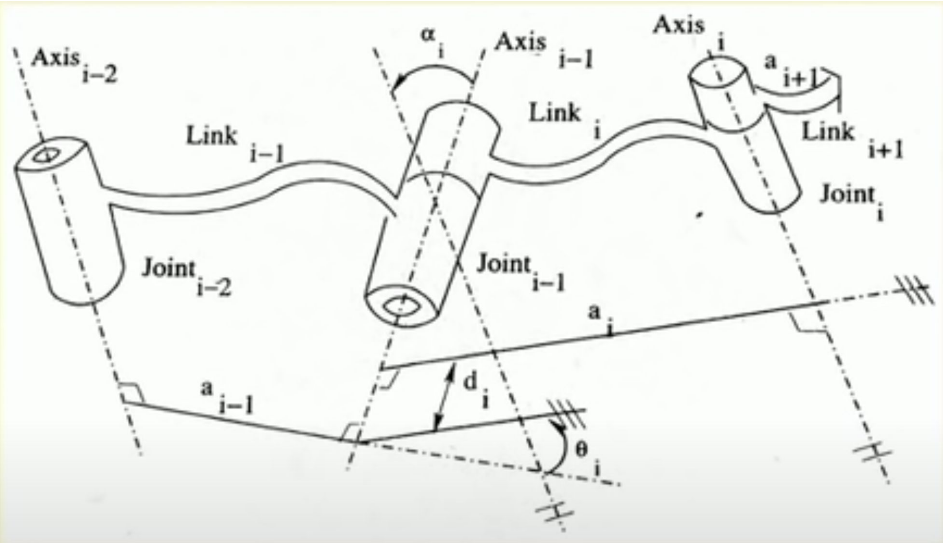
$$\beta = \tan^{-1}\left(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}\right)$$

$$\gamma = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right)$$

Lecture-12

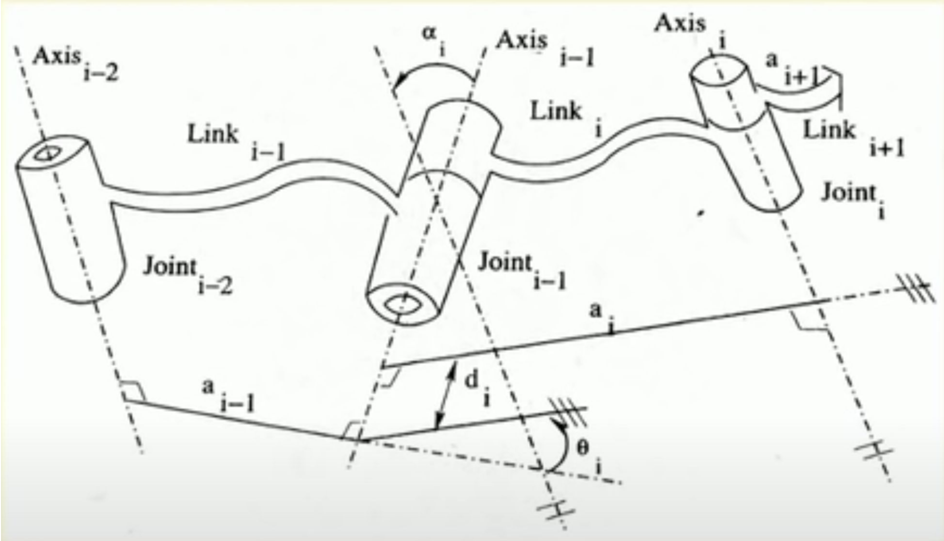
Denavit-Hartenberg Notations: Proposed in the year 1955

Link and Joint Parameters



- Length of $\mathrm{link}_i\,(a_i)$: it is the mutual perpendicular distance between Axis_{i-1} and Axis_i

- Angle of twist of $\mathrm{link}_i\,(a_i)$: it is the defined as the angle between Axis_{i-1} and Axis_i



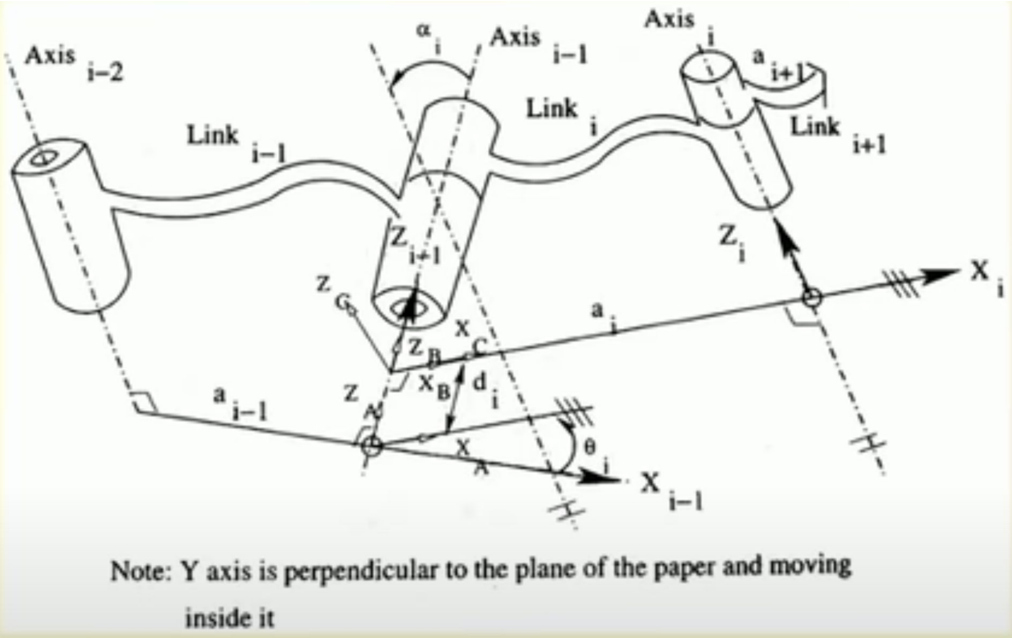
- Offset of $\mathrm{link}_i\,(d_i)$: it is the distance measured from a point where a_{i-1} intersects the Axis_{i-1} to the point where a_i intersects the Axis_{i-1} measured along the said axis

- Joint Angle (θ_i) : It is defined as the angle between the extension of a_{i-1} and a_i measured about the Axis_{i-1}

Notes:

- Revolute joint: θ_i is variable
- Prismatic Joint: d_i is variable

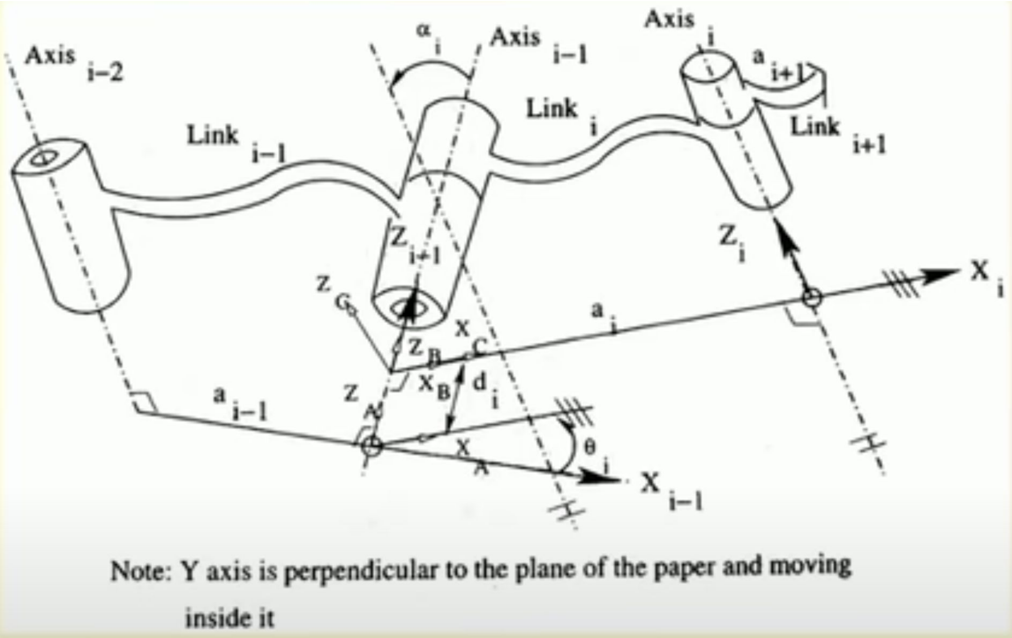
Rules for Coordinate Assignment



- Z_i is an axis about which the rotation is considered or along which the translation takes place

- If Z_{i-1} and Z_i axes are parallel to each other, X axis will be directed from Z_{i-1} to Z_i along their common normal

Rules for Coordinate Assignment



- If Z_{i-1} and Z_i axes intersect each other, X can be selected along either of two remaining directions
- If Z_{i-1} and Z_i axes act along a straight line, X axis can be selected anywhere in a plane perpendicular to them
- Y axis is decided as Y=ZxX

We have

$$ {}^{i-1}_{i}T = {}^{i-1}_{A}T\,{}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}_{i}T $$

$$ = Rot(Z, \theta_i) Trans(Z, d_i) Rot(X, \alpha_i) Trans(X, \alpha_i) $$

$$ = Screw_Z\, Screw_X $$

Robot Kinematics

Lecture-12

We have

$${}^{i-1}_{i}T = {}^{i-1}_{A}T{}^{A}_{B}T{}^{B}_{C}T{}^{C}_{i}T$$

$$= Rot(Z, \theta_i) Trans(Z, d_i) Rot(X, \alpha_i) Trans(X, \alpha_i)$$

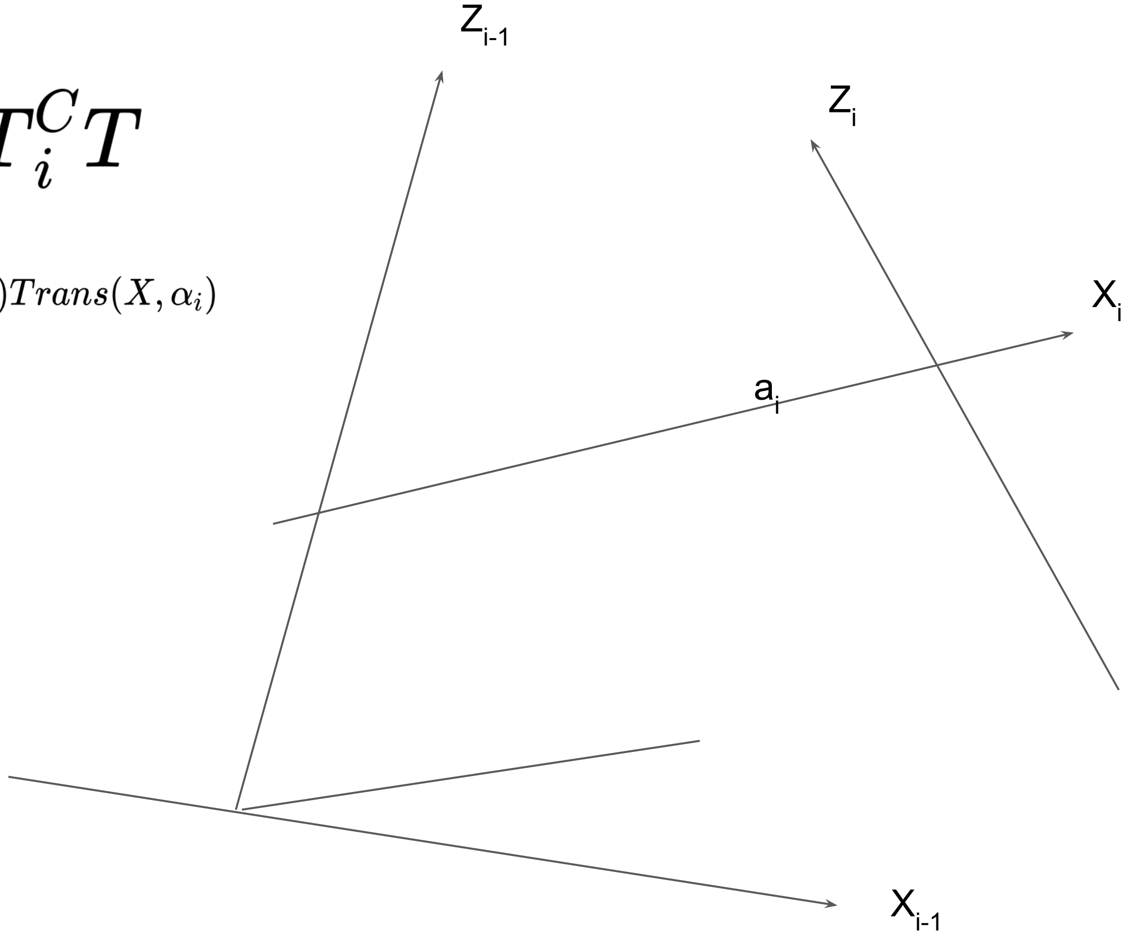
$$= Screw_Z \, Screw_X$$

We have

$$ {}^{i-1}_{i}T = {}^{i-1}_{A}T^{A}_{B}T^{B}_{C}T^{C}_{i}T $$

$$ = Rot(Z,\theta_i)Trans(Z,d_i)Rot(X,\alpha_i)Trans(X,\alpha_i) $$

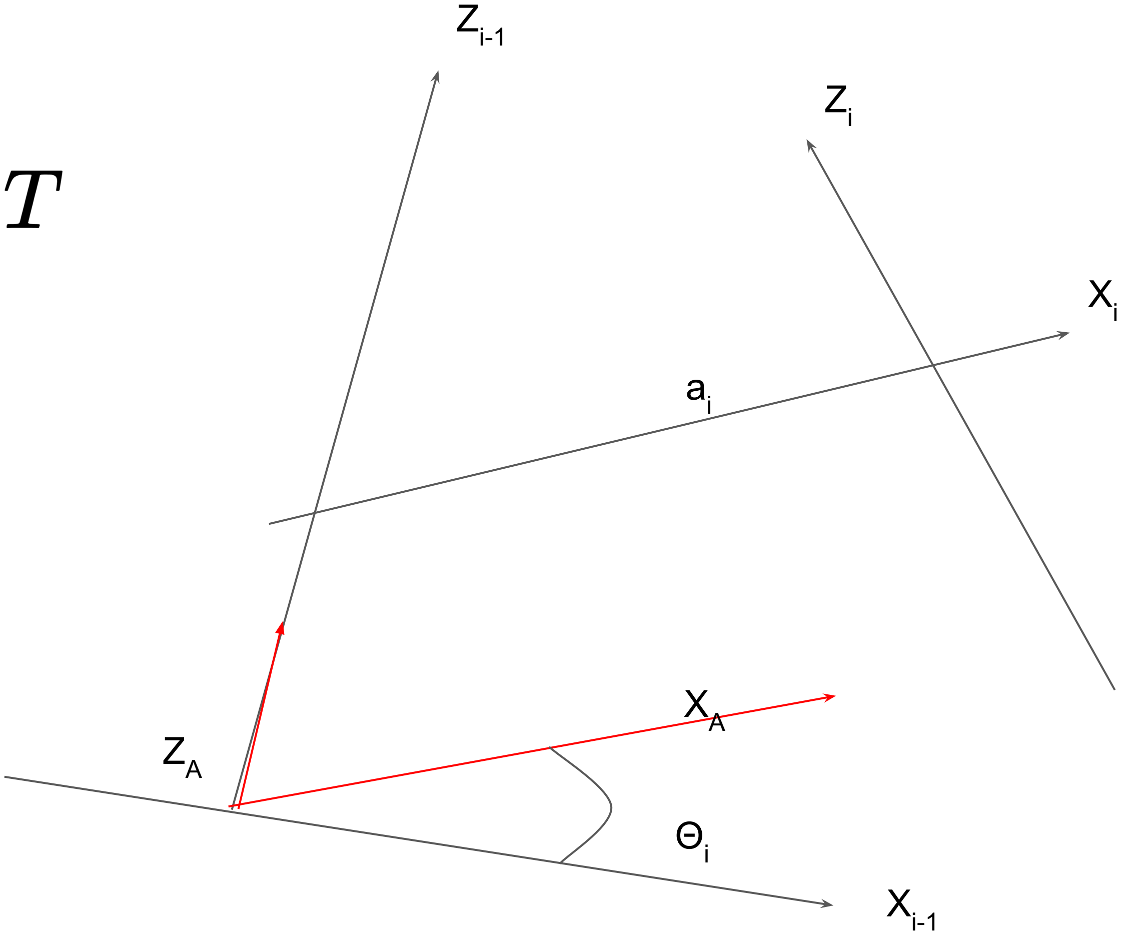
$$ = Screw_Z\, Screw_X $$



We have

$$^{i-1}_{i}T = {}^{i-1}_{A}T$$

$$= Rot(Z, \theta_i)$$

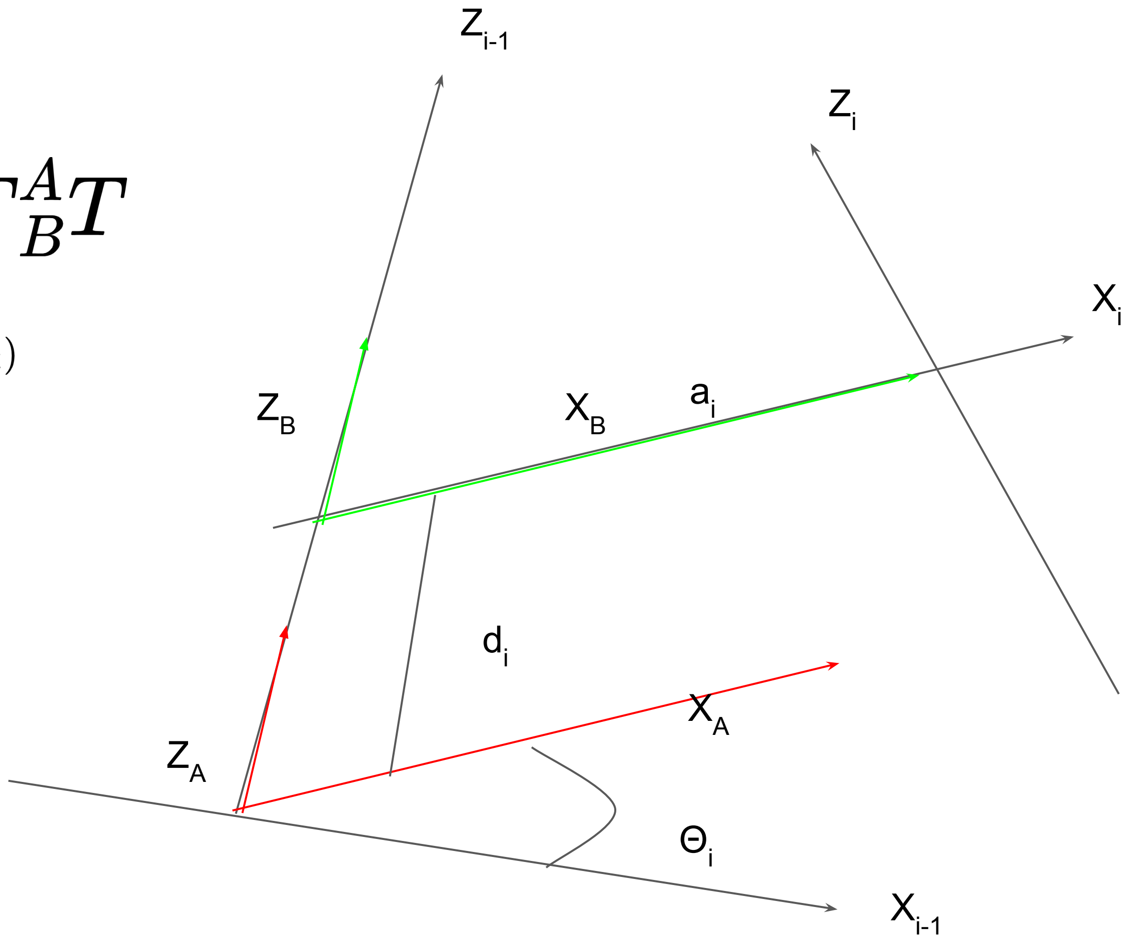


We have

$$ {}^{i-1}_{i}T = {}^{i-1}_{A}T\,{}^{A}_{B}T $$

$$ = Rot(Z, \theta_i)Trans(Z, d_i) $$

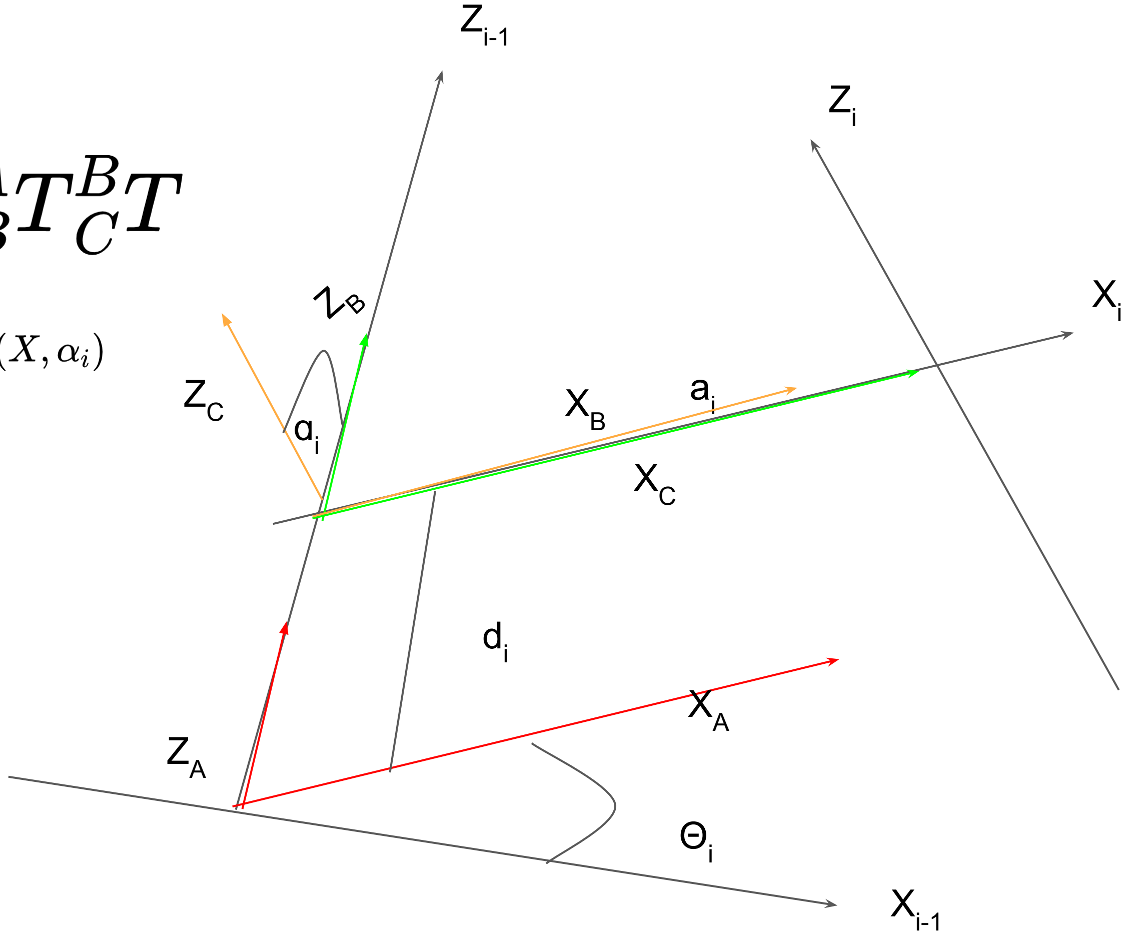
$$ = Screw_Z $$



We have

$$ {}^{i-1}_{i}T = {}^{i-1}_{A}T\,{}^{A}_{B}T\,{}^{B}_{C}T $$

$$ = Rot(Z,\theta_i)Trans(Z,d_i)Rot(X,\alpha_i) $$

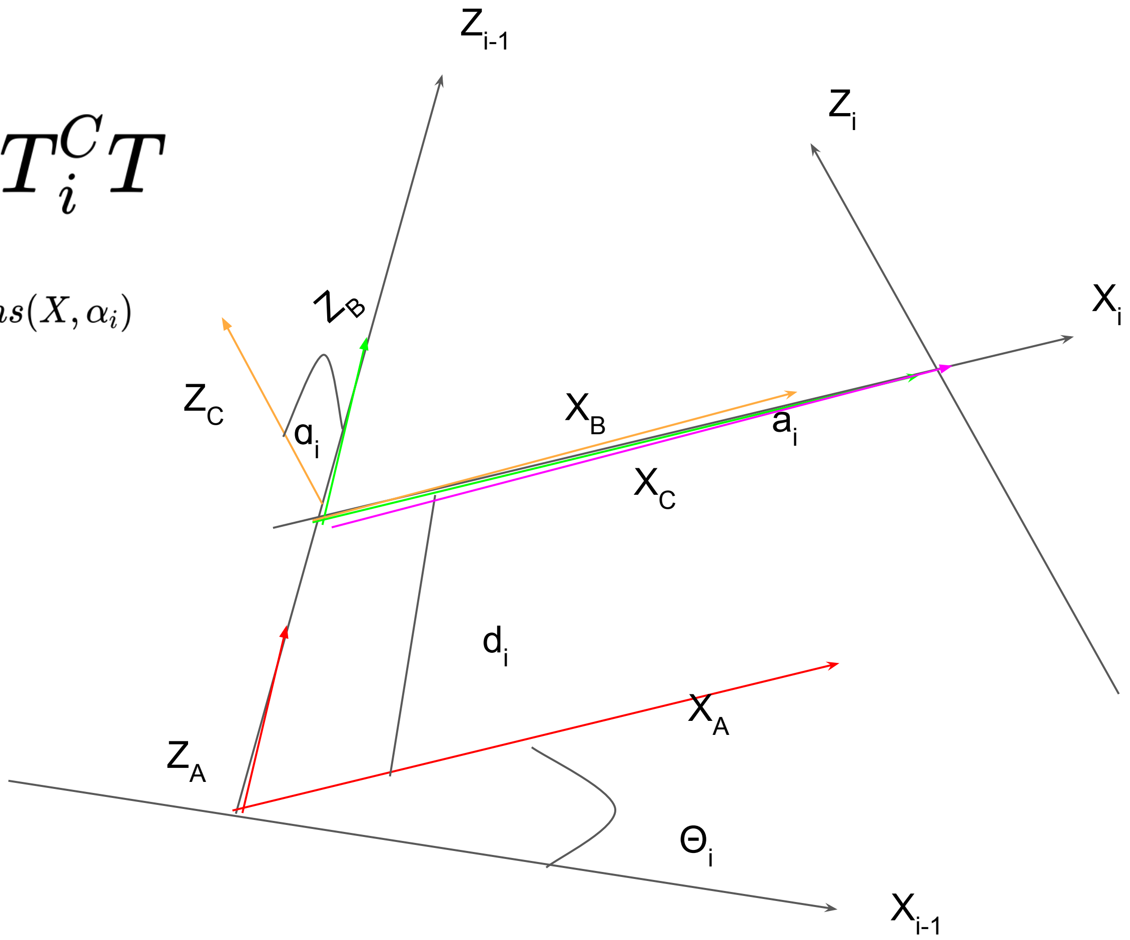


We have

$$ {}^{i-1}_{i}T = {}^{i-1}_{A}T {}^{A}_{B}T {}^{B}_{C}T {}^{C}_{i}T $$

$$ = Rot(Z,\theta_i)Trans(Z,d_i)Rot(X,\alpha_i)Trans(X,\alpha_i) $$

$$ = Screw_Z \, Screw_X $$



$$
{}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & \alpha_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & \alpha_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{i}_{i-1}T = \left[{}^{i-1}_{i}T\right]^{-1}
$$

$$
= \begin{bmatrix} c\theta_i & s\theta_i & 0 & -a_1 \\ -s\theta_i c\alpha_i & c\theta_i c\alpha_i & s\alpha_i & -d_i s\alpha_i \\ s\theta_i s\alpha_i & -c\theta_i s\alpha_i & c\alpha_i & -d_i c\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

We have

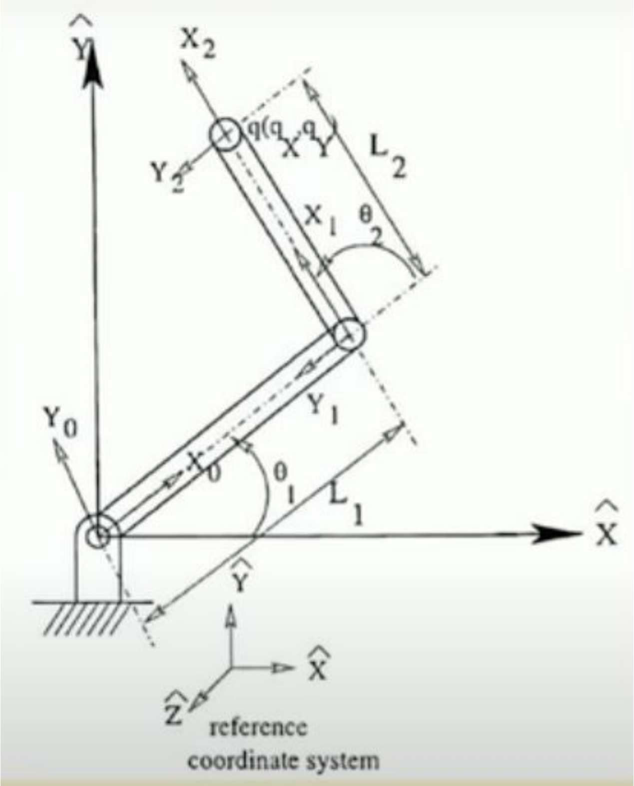
$$ {}^{i-1}_{i}T = {}^{i-1}_{A}T\,{}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}_{i}T $$

$$ = Rot(Z, \theta_i) Trans(Z, d_i) Rot(X, \alpha_i) Trans(X, \alpha_i) $$

$$ = Screw_Z\, Screw_X $$

Example 1

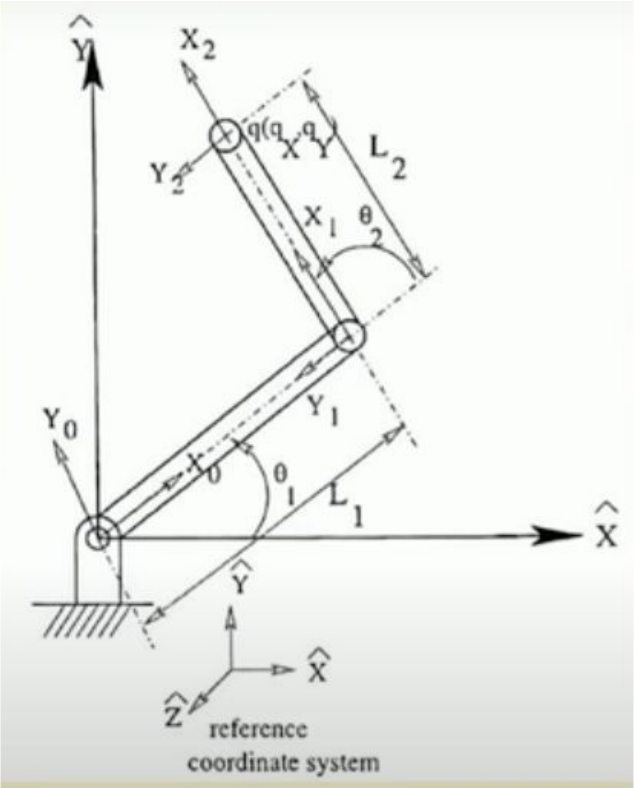
2 dof serial manipulator



Frame	θ_i	d_i	α_i	a_i
1				
2				

Example 1

2 dof serial manipulator



Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	0	L_1
2	θ_2	0	0	L_2

Forward Kinematics

$$ {}^{Base}_{2}T = {}^{Base}_{1}T\, {}^{1}_{2}T $$

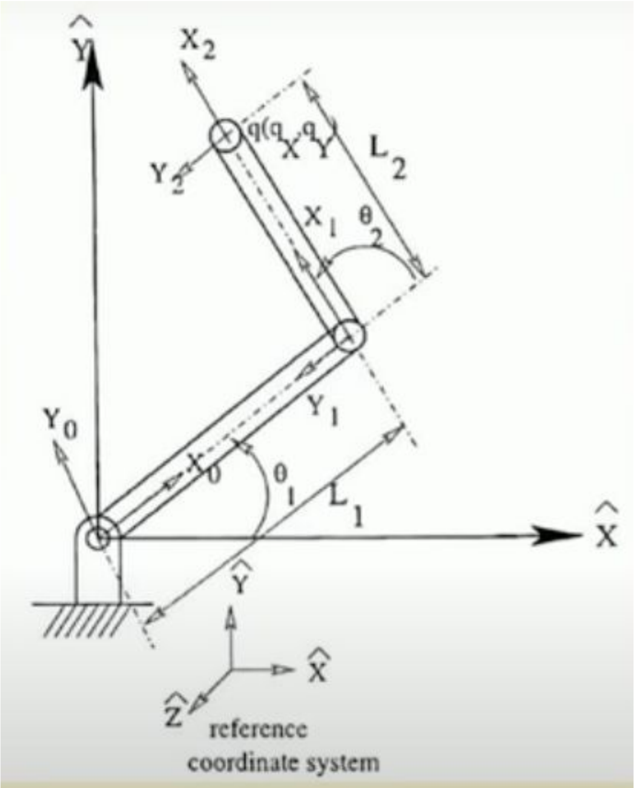
$$
\begin{aligned}
{}^{Base}_{1}T &= ROT(\hat{Z}, \theta_1) TRANS(\hat{X}, L_1) \\
&= \begin{bmatrix} c_1 & -s_1 & 0 & L_1 c_1 \\ s_1 & c_1 & 0 & L_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}_{2}T &= ROT(\hat{Z}, \theta_2)TRANS(\hat{X}, L_2) \\
&= \begin{bmatrix} c_2 & -s_2 & 0 & L_2c_2 \\ s_2 & c_2 & 0 & L_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{Base}_{2}T &= {}^{Base}_{1}T\,{}^{1}_{2}T \\
&= \begin{bmatrix} c_{12} & -s_{12} & 0 & L_1c_1 + L_2c_{12} \\ s_{12} & c_{12} & 0 & L_1s_1 + L_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Example 1

2 dof serial manipulator



Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	0	L_1
2	θ_2	0	0	L_2

$$q_X = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$

$$q_Y = L_1 sin\theta_1 + L_1 sin(\theta_1 + \theta_2)$$

Robot Kinematics

Lecture-13

Inverse Kinematics:

To determine θ_1 and θ_2

$$ {}^{Base}_{2}T = \begin{bmatrix} c\phi & -s\phi & 0 & q_x \\ s\phi & c\phi & 0 & q_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad = \text{Known} $$

$$\cos(\theta_1)$$

$$q_x = \underbrace{L_1 c_1}_{} + L_2 c_{12} \rightarrow \cos(\theta_1 + \theta_2)$$ Eq.-1

$$q_y = L_1 s_1 + L_2 s_{12} \rightarrow \sin(\theta_1 + \theta_2)$$ Eq.-2

By squaring and adding equation 1 and 2

$$q_x^2 + q_y^2 = L_1^2 + L_2^2 + 2L_1L_2c_{12}c_1 + 2L_1L_2s_{12}s_1,$$

$$c_2 = \frac{q_x^2+q_y^2-L_1^2-L_2^2}{2L_1L_2}$$

$$q_x^2 + q_y^2 - L_1^2 - L_2^2 = 2L_1L_2\cos\theta_2$$

$$\theta_2 = \arccos\left(\frac{q_x^2+q_y^2-L_1^2-L_2^2}{2L_1L_2}\right).$$

Since it is inverse, you will get two theta values

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

Eq. 1 we further rewrite

$$\begin{aligned} q_x &= L_1c_1 + L_2c_1c_2 - L_2s_1s_2 \\ &= c_1(L_1 + L_2c_2) - s_1(L_2s_2) \end{aligned}$$

Known

Known

$$L_1 + L_2C_2 = \rho\sin\Psi$$

$$L_2S_2 = \rho\cos\Psi$$

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

Eq. 1 we further rewrite

$$\begin{aligned} q_x &= L_1 c_1 + L_2 c_1 c_2 - L_2 s_1 s_2 \\ &= c_1(L_1 + L_2 c_2) - s_1(L_2 s_2) \end{aligned}$$

Known

Known

$$L_1 + L_2 C_2 = \rho \sin\Psi$$

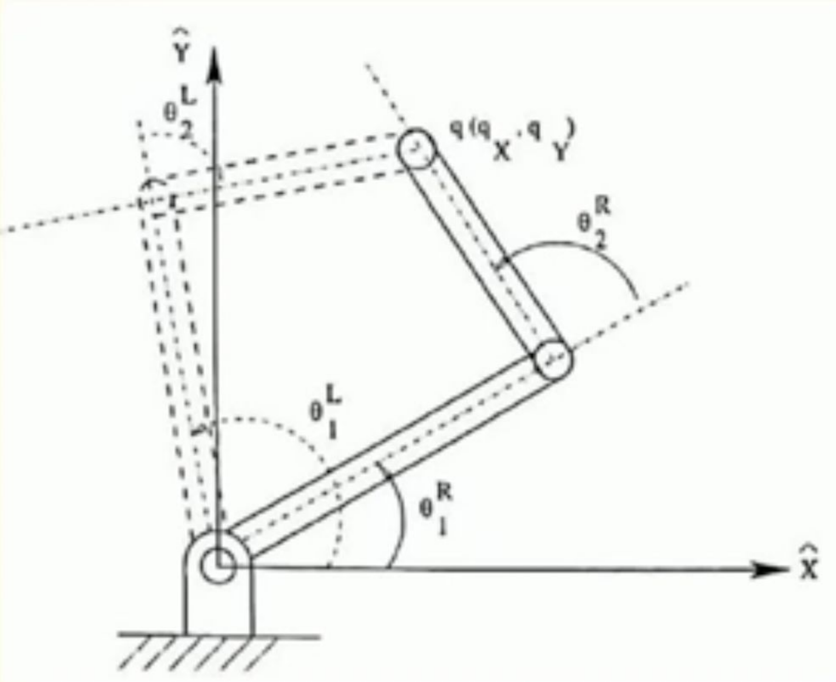
$$L_2 S_2 = \rho \cos\Psi$$

$$\rho = \sqrt{(L_1 + L_2 c_2)^2 + (L_2 s_2)^2}$$

$$\Psi = \tan^{-1} \frac{L_1 + L_2 c_2}{L_2 s_2}$$

$$q_x = c_1 \rho \sin\Psi - s_1 \rho \cos\Psi$$

$$q_x = \rho \sin(\Psi - \theta_1)$$



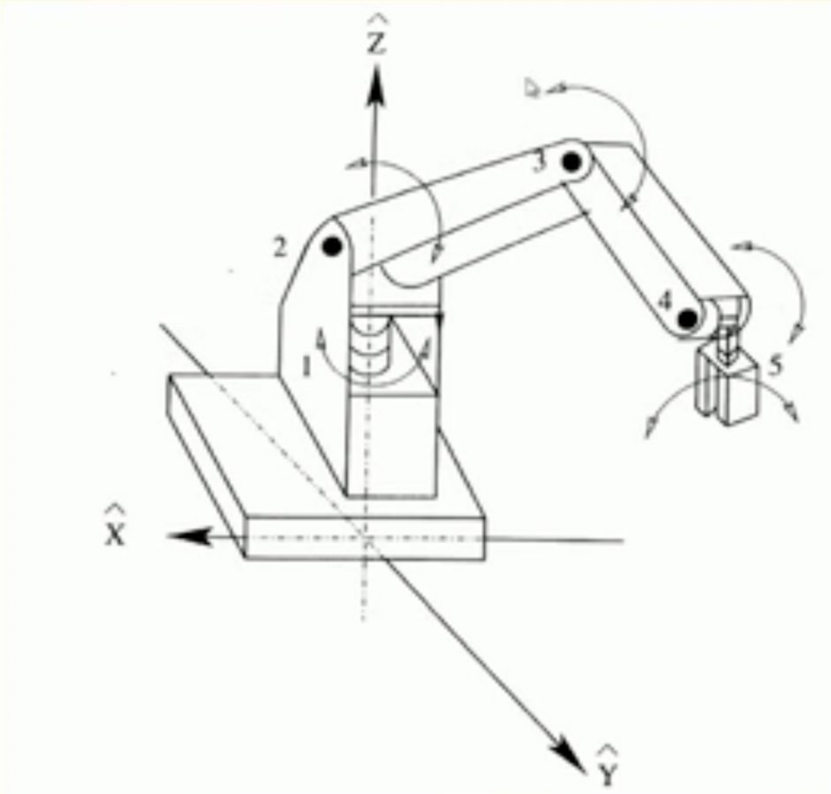
$$q_y \quad = \quad \rho cos(\psi - \theta_1)$$

$$\theta_1 = \psi - \arctan(\frac{q_x}{q_y})$$

Robot Kinematics

Lecture-14

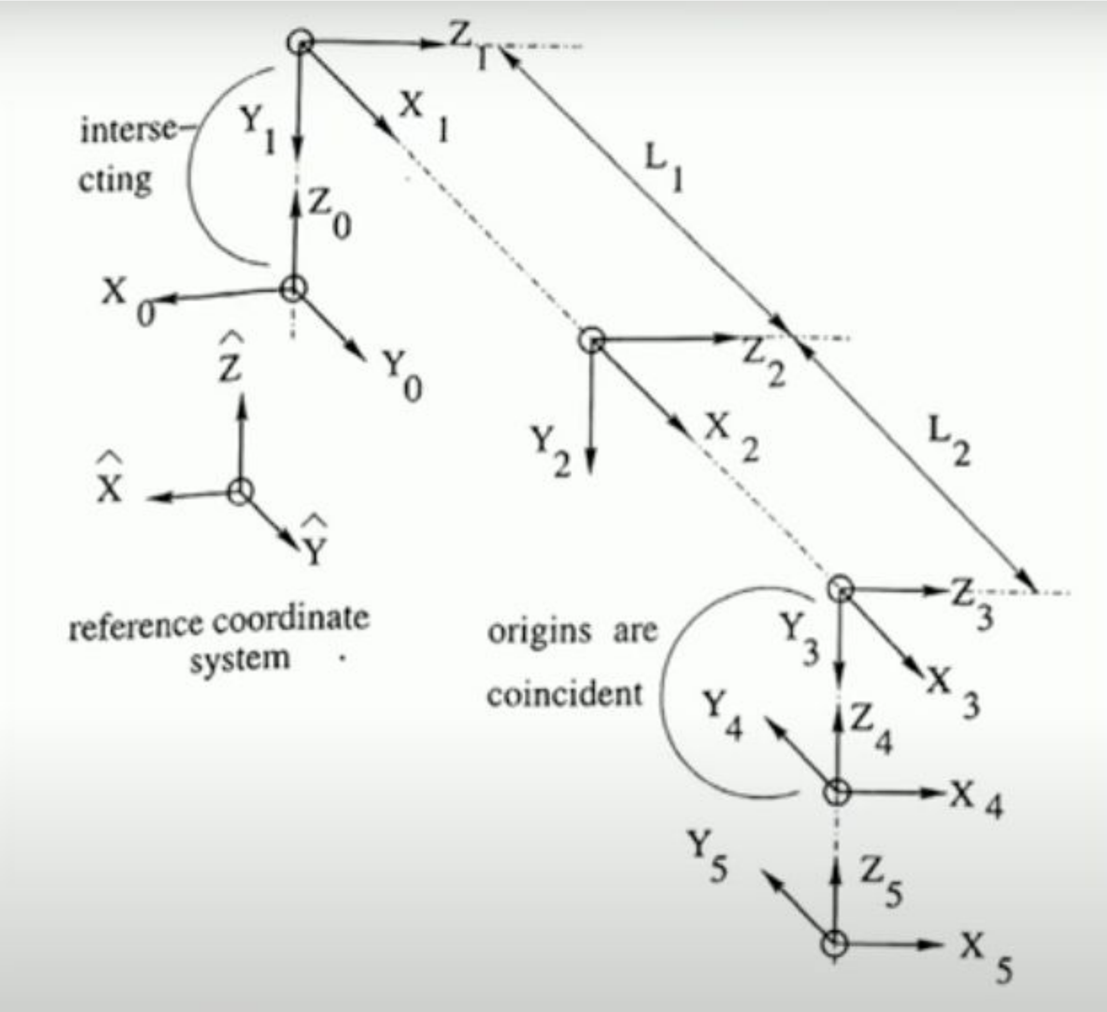
Example-2



Mini Mover (5dof)

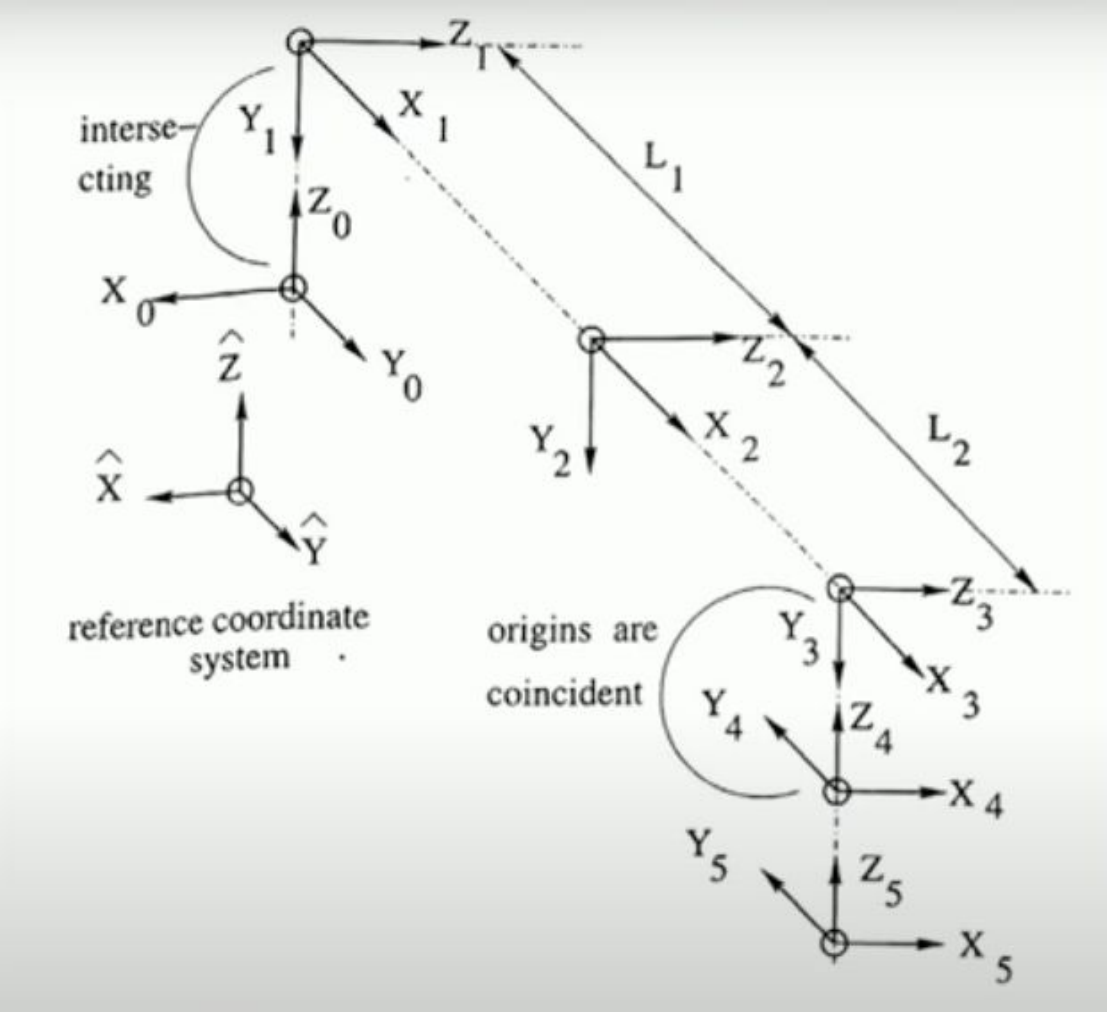
T-R-R-R-T

Kinematic diagram



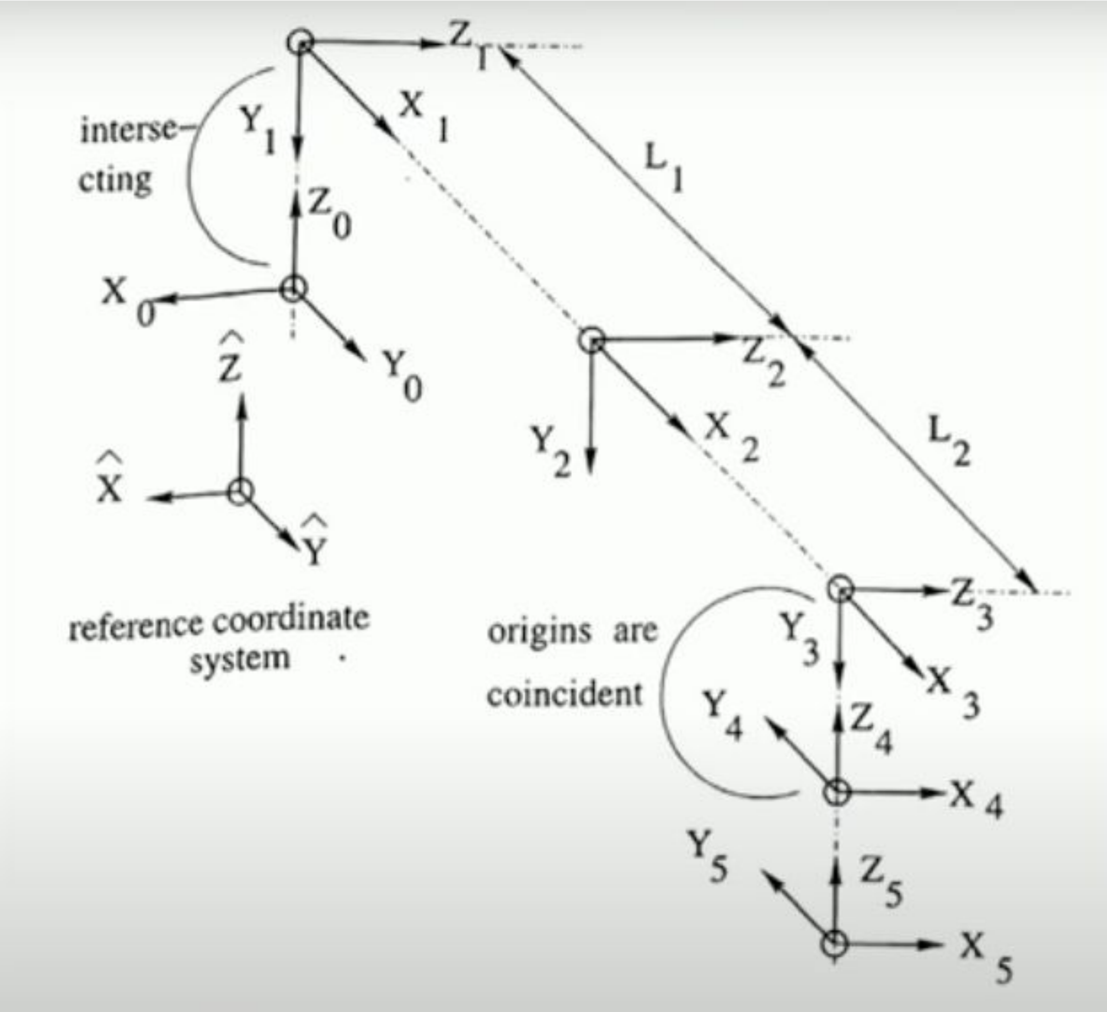
D-H parameter Table

Frame	θ_i	d_i	α_i	a_i
1				
2				
3				
4				
5				



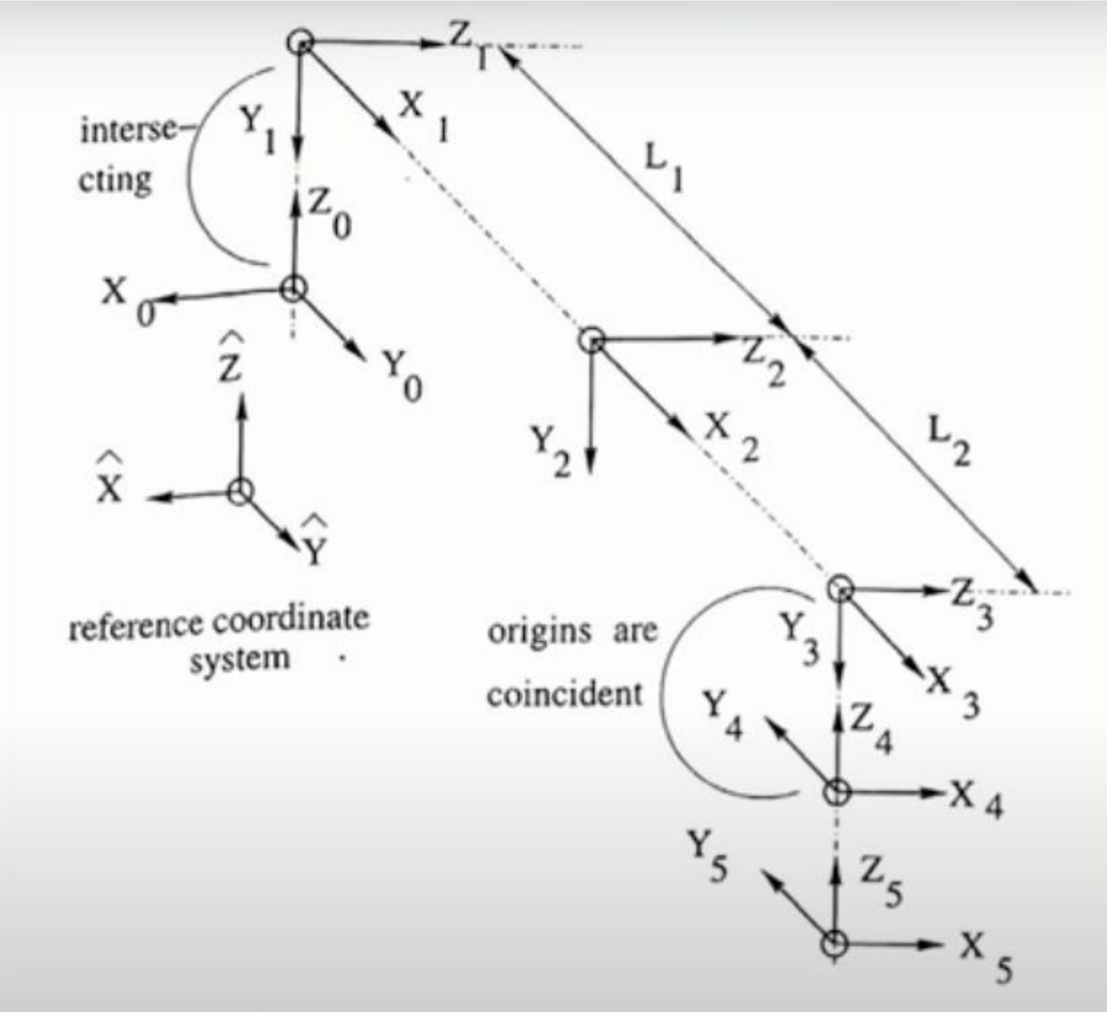
D-H parameter Table

Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	-90	0
2				
3				
4				
5				



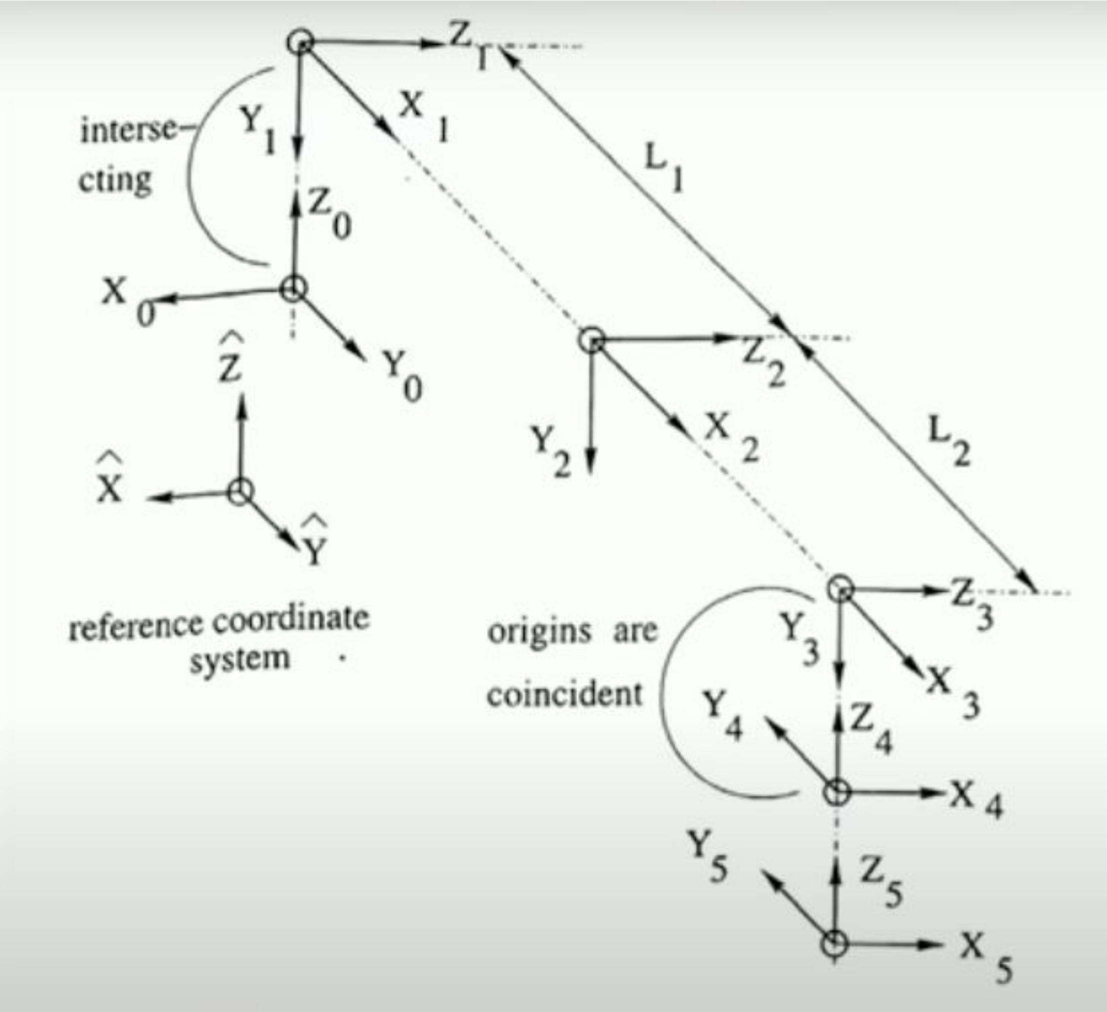
D-H parameter Table

Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	-90	0
2	θ_2	0	0	L_1
3				
4				
5				



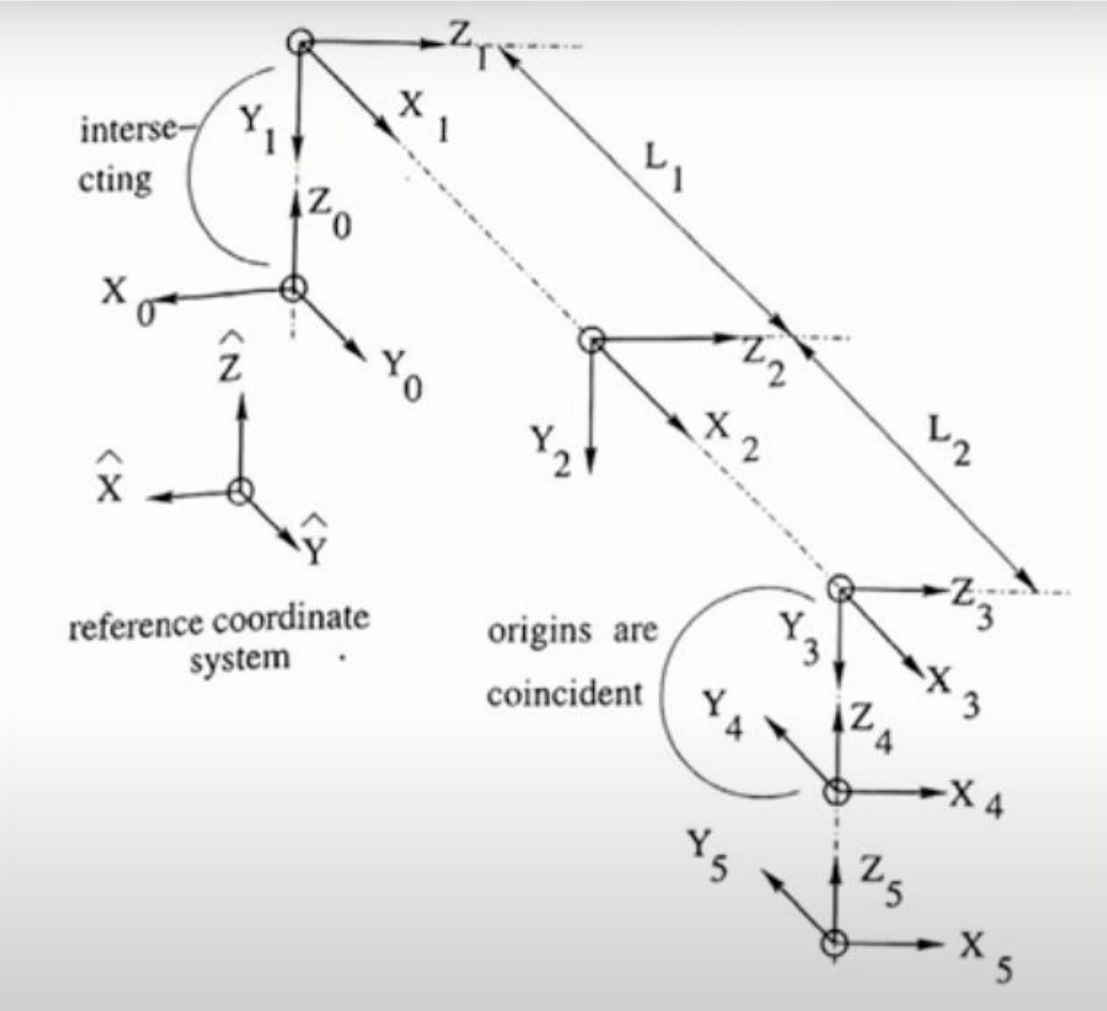
D-H parameter Table

Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	-90	0
2	θ_2	0	0	L_1
3	θ_3	0	0	L_2
4				
5				



D-H parameter Table

Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	-90	0
2	θ_2	0	0	L_1
3	θ_3	0	0	L_2
4	θ_4	0	90	0
5				



D-H parameter Table

Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	-90	0
2	θ_2	0	0	L_1
3	θ_3	0	0	L_2
4	θ_4	0	90	0
5	θ_5	0	0	0

Forward kinematics: To determine the position and orientation of end effector with respect to base coordinate system provided length of the link and joint angles are known

$$ {}^{0}_{5}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T $$

$$ \begin{aligned} {}^{0}_{1}T &= Rot(\hat{Z},\theta_1)Rot(\hat{X},-90) \\ &= \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} $$

Here, c_1 and s_1 denote $c\theta_1$ (or, $cos\theta_1$) and $s\theta_1$ (or, $sin\theta_1$), respectively.

$$
{}^{0}_{1}T^{-1} = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}T = Rot(\hat{Z}, \theta_2) Trans(\hat{X}, L_1)
$$

$$
= \begin{bmatrix} c_2 & -s_2 & 0 & L_1 c_2 \\ s_2 & c_2 & 0 & L_1 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
{}^{2}_{3}T &= Rot(\hat{Z},\theta_3)Trans(\hat{X},L_2) \\
&= \begin{bmatrix} c_3 & -s_3 & 0 & L_2c_3 \\ s_3 & c_3 & 0 & L_2s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{3}_{4}T &= Rot(\hat{Z},\theta_4)Rot(\hat{X},90) \\
&= \begin{bmatrix} c_4 & 0 & s_4 & 0 \\ s_4 & 0 & -c_4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$ {}^4_5T = Rot(\hat{Z}, \theta_5) $$

$$ = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$
\begin{aligned}
{}^0_5T &= {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T \\
&= \begin{bmatrix} v_{11} & v_{12} & v_{13} & p_x \\ v_{21} & v_{22} & v_{23} & p_y \\ v_{31} & v_{32} & v_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
v_{11} &= c_1 c_{234} c_5 - s_1 s_5 \\
v_{12} &= -c_1 c_{234} s_5 - s_1 c_5 \\
v_{13} &= c_1 s_{234} \\
v_{21} &= s_1 c_{234} c_5 + c_1 s_5 \\
v_{22} &= -s_1 c_{234} s_5 + c_1 c_5 \\
v_{23} &= s_1 s_{234} \\
v_{31} &= -s_{234} c_5 \\
v_{32} &= s_{234} s_5 \\
v_{33} &= c_{234} \\
p_x &= c_1(L_1 c_2 + L_2 c_{23}) \\
p_y &= s_1(L_1 c_2 + L_2 c_{23}) \\
p_z &= -L_1 s_2 - L_2 s_{23}
\end{aligned}
$$

Inverse Kinematics: to determine joint angles provided position and orientation of end effector with respect to base coordinate system

$$
{}^{0}_{5}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & q_x \\ r_{21} & r_{22} & r_{23} & q_y \\ r_{31} & r_{32} & r_{33} & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
{}^{0}_{5}T &= {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T \\
\Rightarrow {}^{0}_{1}T^{-1}({}^{0}_{5}T) &= {}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T
\end{aligned}
$$

$$\Rightarrow \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & q_x \\ r_{21} & r_{22} & r_{23} & q_y \\ r_{31} & r_{32} & r_{33} & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} c_{234}c_5 & -c_{234}s_5 & s_{234} & L_1c_2 + L_2c_{23} \\ s_{234}c_5 & -s_{234}s_5 & -c_{234} & L_1s_2 + L_2s_{23} \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} r_{11}c_1 + r_{21}s_1 & r_{12}c_1 + r_{22}s_1 & r_{13}c_1 + r_{23}s_1 & q_xc_1 + q_ys_1 \\ -r_{31} & -r_{32} & -r_{33} & -q_z \\ -r_{11}s_1 + r_{21}c_1 & -r_{12}s_1 + r_{22}c_1 & -r_{13}s_1 + r_{23}c_1 & -q_xs_1 + q_yc_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} c_{234}c_5 & -c_{234}s_5 & s_{234} & L_1c_2 + L_2c_{23} \\ s_{234}c_5 & -s_{234}s_5 & -c_{234} & L_1s_2 + L_2s_{23} \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$q_x c_1 + q_y s_1 = L_1 c_2 + L_2 c_{23}$$

$$-q_x s_1 + q_y c_1 = 0$$

$$q_z = -L_1 s_2 - L_2 s_{23}$$

$$s_{234} = r_{13} c_1 + r_{23} s_1$$

$$c_{234} = r_{33}$$

$$-r_{11} s_1 + r_{21} c_1 = s_5$$

$$-r_{12} s_1 + r_{22} c_1 = c_5$$

$$\begin{aligned} -q_x s_1 + q_y c_1 &= 0 \\ \Rightarrow \theta_1 &= \arctan\left(\frac{q_y}{q_x}\right) \end{aligned}$$

$$
\begin{aligned}
q_x^2 + q_y^2 + q_z^2 &= L_1^2 + L_2^2 + 2L_1L_2c_3 \\
\Rightarrow \theta_3 &= \arccos\left(\frac{q_x^2 + q_y^2 + q_z^2 - L_1^2 - L_2^2}{2L_1L_2}\right)
\end{aligned}
$$

$$
\begin{aligned}
L_1c_2 + L_2c_{23} &= q_xc_1 + q_ys_1 \\
\Rightarrow (L_1 + L_2c_3)c_2 - (L_2s_3)s_2 &= q_xc_1 + q_ys_1
\end{aligned}
$$

Let us assume $L_1 + L_2c_3 = \rho\sin\alpha$ and $L_2s_3 = \rho\cos\alpha$,
where $\rho \neq 0$ and $\rho = \sqrt{(L_1 + L_2c_3)^2 + (L_2s_3)^2}$; $\alpha = \arctan(\frac{L_1+L_2c_3}{L_2s_3})$.
Thus, the above expression can be written as follows:

$$\rho \sin \alpha c_2 - \rho \cos \alpha s_2 \quad = \quad q_x c_1 + q_y s_1$$

$$\rho \sin(\alpha - \theta_2) = q_x c_1 + q_y s_1$$

$$\rho \cos(\alpha - \theta_2) = -q_z$$

$$\tan(\alpha - \theta_2) = \frac{q_x c_1 + q_y s_1}{-q_z}$$

$$\Rightarrow \theta_2 = \alpha - \arctan\left(\frac{q_x c_1 + q_y s_1}{-q_z}\right)$$

$$\theta_2 + \theta_3 + \theta_4 = \arctan\left(\frac{r_{13} c_1 + r_{23} s_1}{r_{33}}\right)$$

$$\Rightarrow \theta_4 = \arctan\left(\frac{r_{13} c_1 + r_{23} s_1}{r_{33}}\right) - \theta_2 - \theta_3$$

$$\theta_5 = \arctan\left(\frac{-r_{11} s_1 + r_{21} c_1}{-r_{12} s_1 + r_{22} c_1}\right)$$