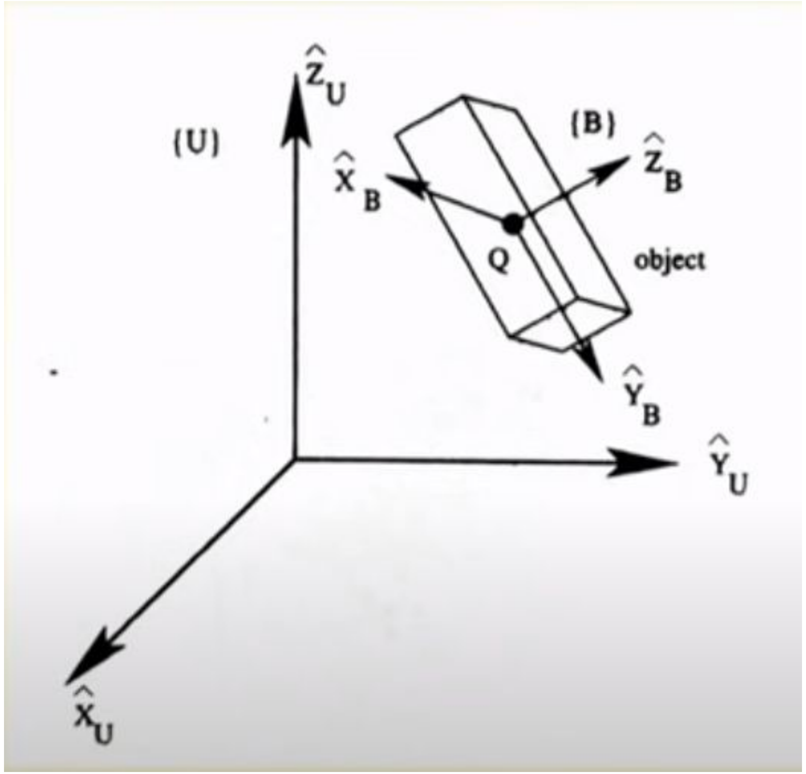
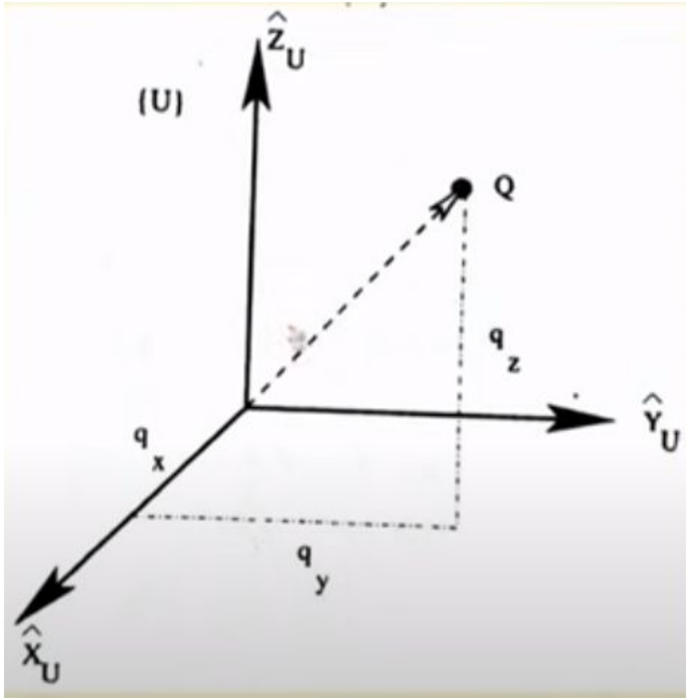
Robot Kinematics

Representation of an object in 3-Dspace

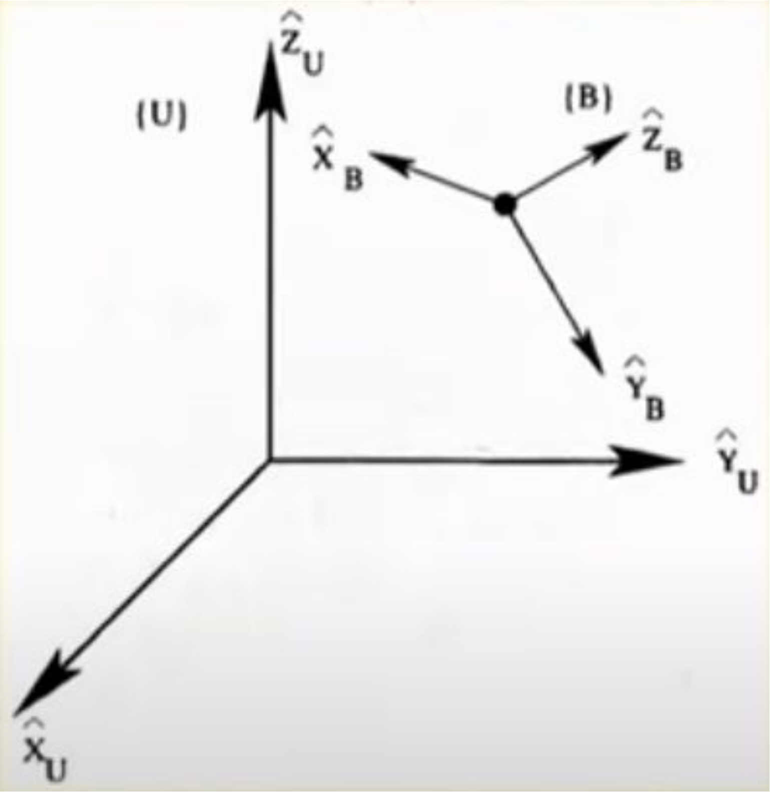


Representation of the Position



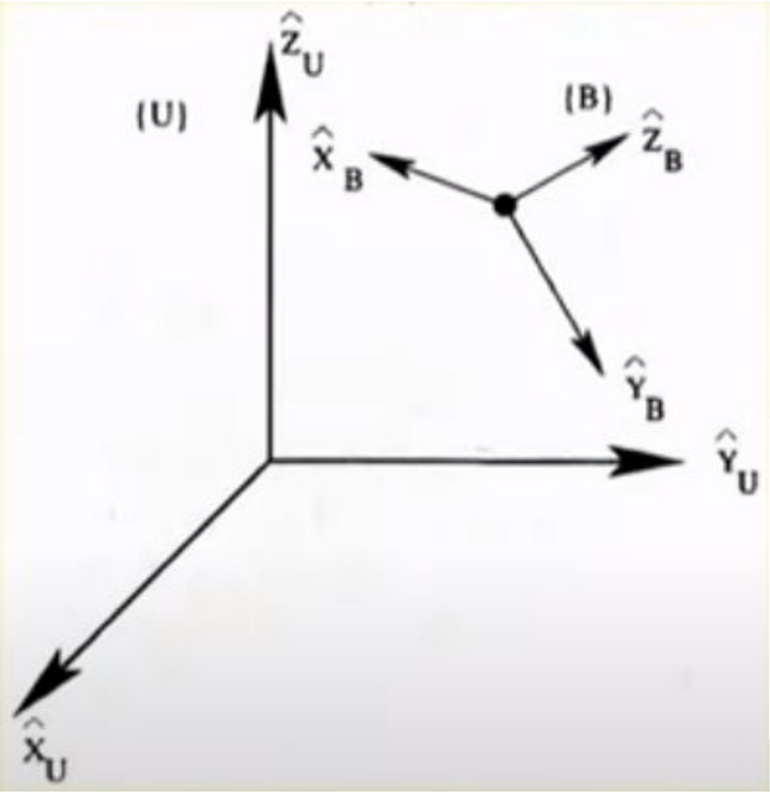
$$^U\bar{Q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix};$$ 3x1 matrix

Representation of the Orientation



$$^{U}R_{B} = \left[{}^{U}\bar{X}_{B} \quad {}^{U}\bar{Y}_{B} \quad {}^{U}\bar{Z}_{B} \right]_{3\cdot 3}$$

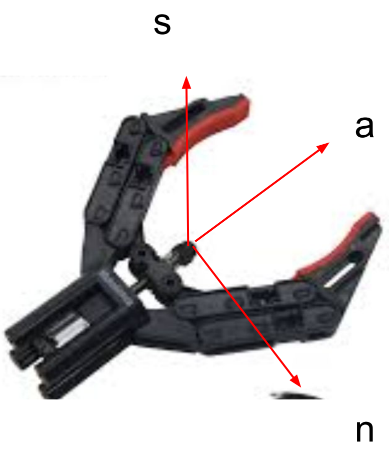
Representation of the Orientation



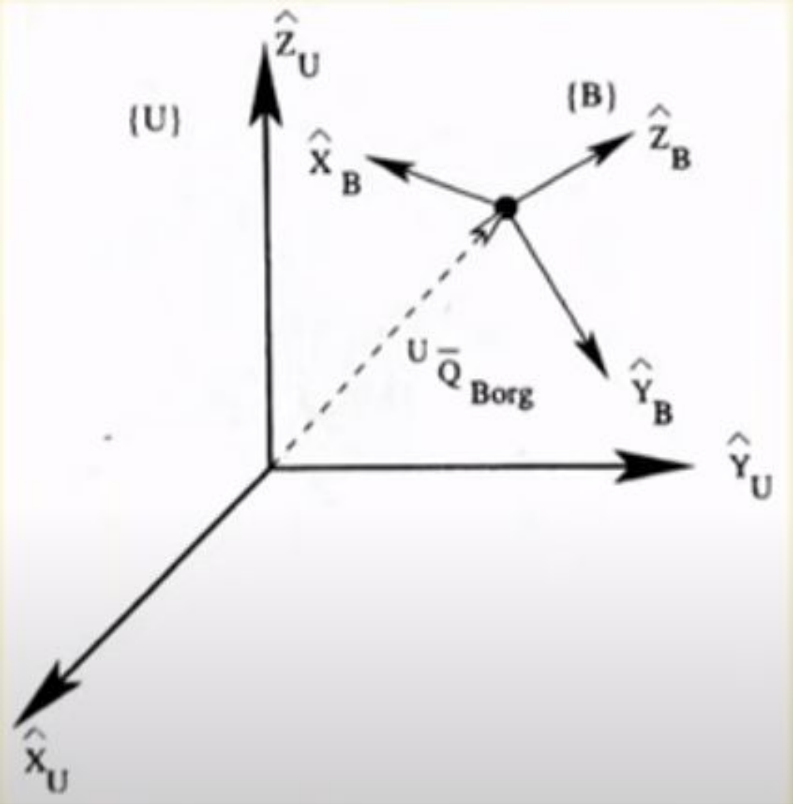
$$ {}^{U}_{B}R = \left[{}^{U}\bar{X}_B \quad {}^{U}\bar{Y}_B \quad {}^{U}\bar{Z}_B \right]_{3\cdot 3} $$

A set of 3 vector

$$\begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix}$$

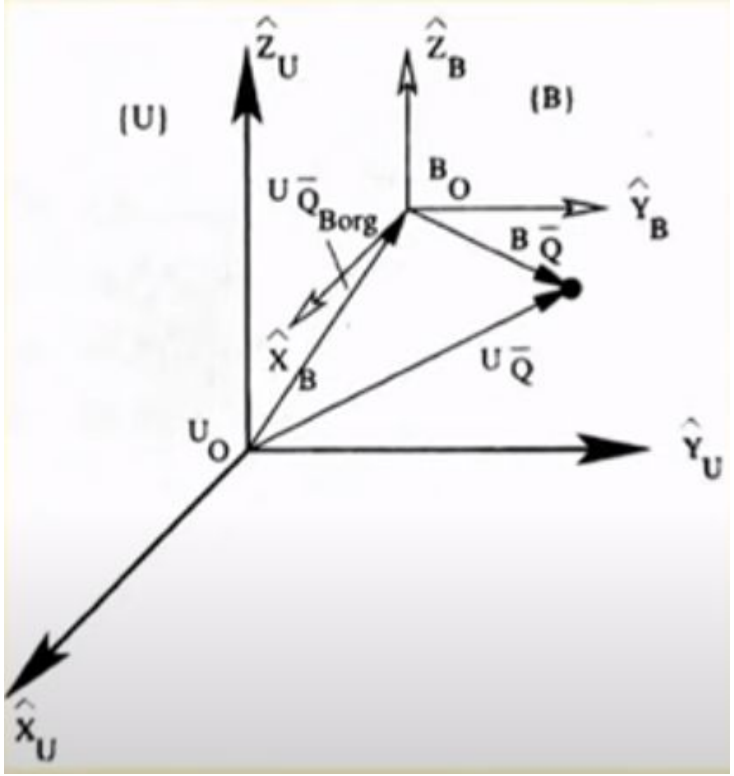


Frame Transformation



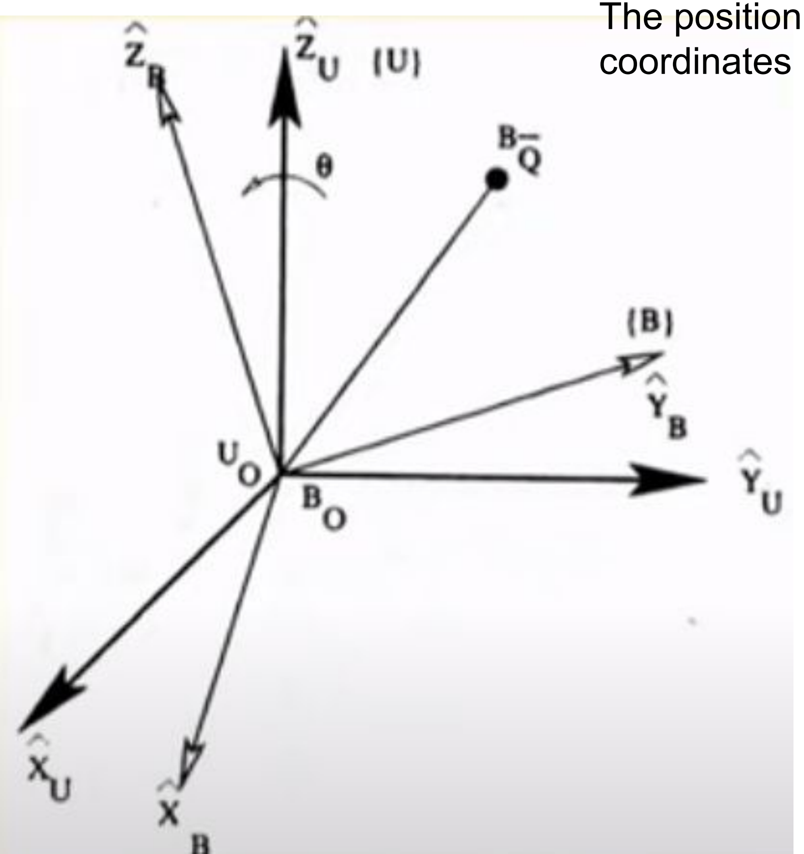
Frame: **A set of four vectors carrying position and orientation information**

Translation of a Frame



$$^U\bar{Q} = {^U\bar{Q}}_{Borg} + {^B\bar{Q}}$$

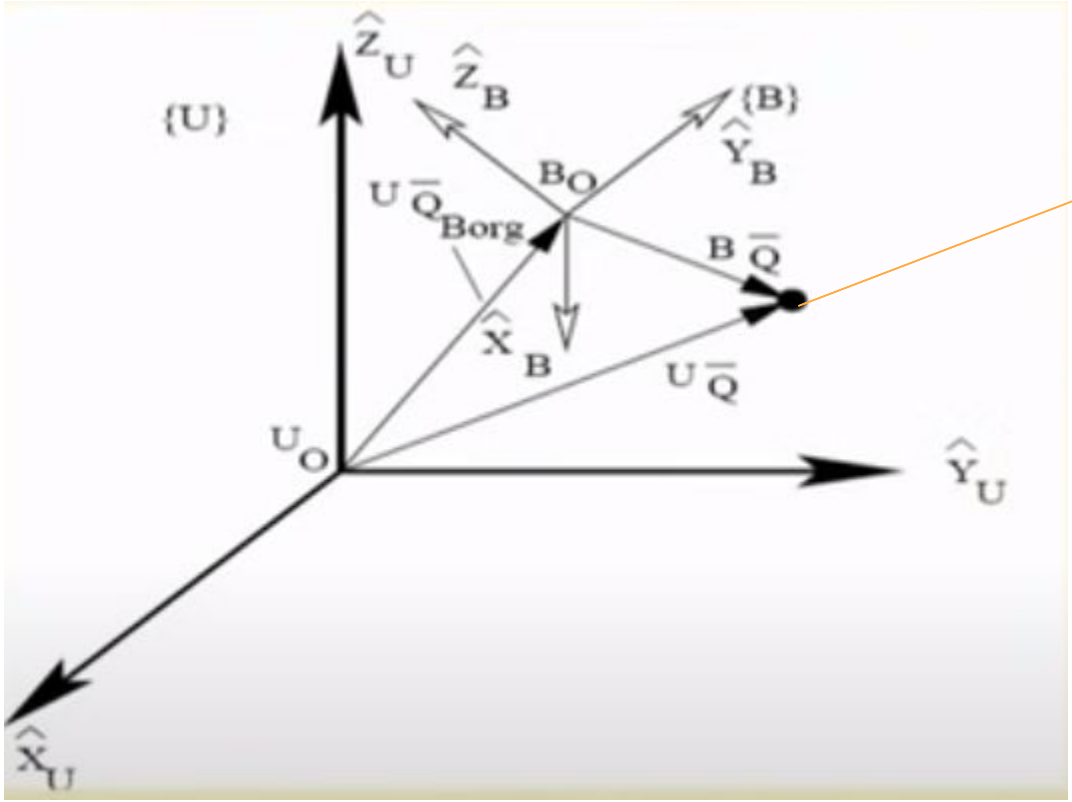
Rotation of a Frame



The position of point Q with respect to rotated body coordinates is known

$$^{U}\bar{Q} = {}^{U}_{B}R\; {}^{B}\bar{Q}$$

Translation and Rotation of a Frame



$${}^{U}\bar{Q} = {}^{U}_{B}R\ {}^{B}\bar{Q} + {}^{U}\bar{Q}_{Borg}$$

$${}^{U}\bar{Q} = {}^{U}_{B}R\ {}^{B}\bar{Q} + {}^{U}\bar{Q}_{Borg}$$

$${}^{U}\bar{Q} = {}^{U}_{B}T\ {}^{B}\bar{Q}$$

Where T: transformation(translation and rotation)

$$\Rightarrow \begin{bmatrix} {}^{U}\bar{Q}(3X1) \\ ----- \\ \end{bmatrix} = \begin{bmatrix} {}^{U}_{B}R(3X3) & || & {}^{U}_{B}\bar{Q}_{Borg}(3X1) \\ --- & || & --- \\ & || & \end{bmatrix} \begin{bmatrix} {}^{B}\bar{Q}(3X1) \\ ----- \\ \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} {}^{U}\bar{Q}(3X1) \\ ----- \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{U}_{B}R(3X3) & || & {}^{U}_{B}\bar{Q}_{Borg}(3X1) \\ --- & || & --- \\ 0 \quad 0 \quad 0 & || & 1 \end{bmatrix} \begin{bmatrix} {}^{B}\bar{Q}(3X1) \\ ----- \\ 1 \end{bmatrix}$$

Perspective transformation

Scaling factor

$${}^{U}\bar{Q} = {}^{A}_{B}T \; {}^{B}\bar{Q}$$

$$[T]^{-1} = \frac{adj\,T}{|T|}$$

Robot Kinematics

Lecture-10

Let [T]:**Homogeneous Transformation matrix**

$$[T] = \begin{bmatrix} {}^{U}_{B}R(3 \cdot 3) & || & {}^{U}\bar{Q}_{Borg}(3 \cdot 1) \\ ------ & || & --- \\ 0\ 0\ 0 & || & 1 \end{bmatrix}$$

Say

$$[T] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & q_x \\ r_{21} & r_{22} & r_{23} & q_y \\ r_{31} & r_{32} & r_{33} & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation operator

$Trans\left(\hat{X}, q\right)$: Translation of q units along x-director

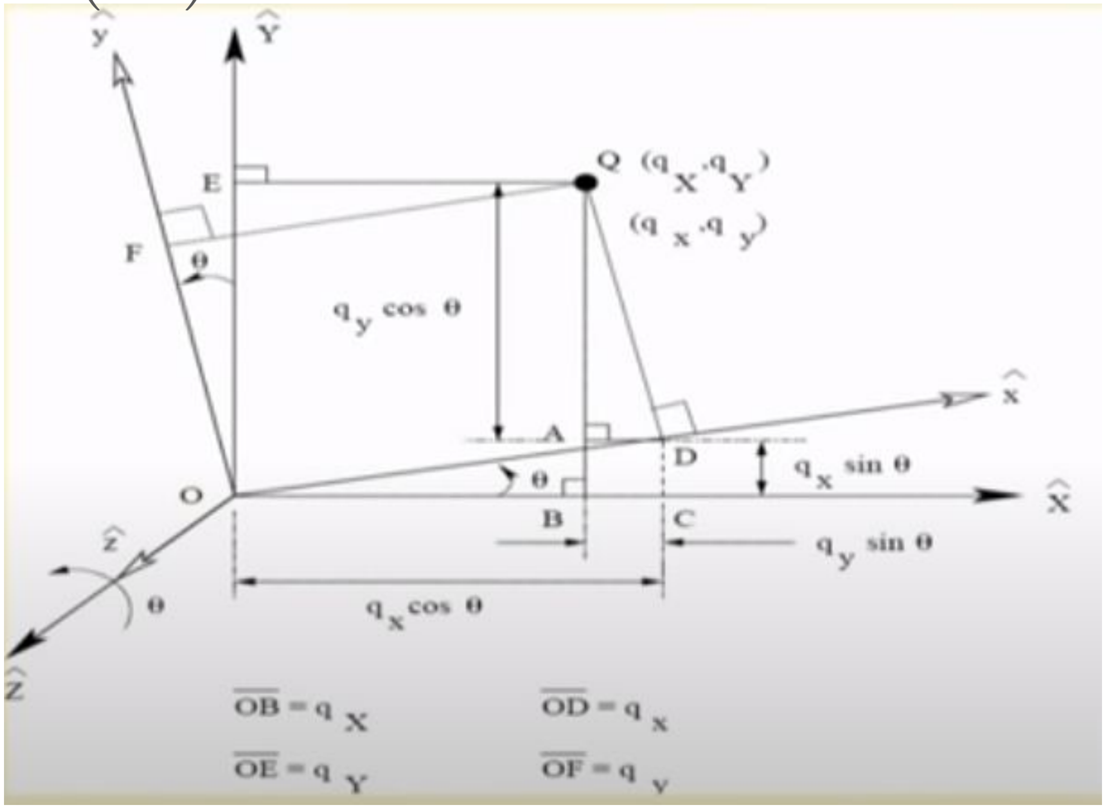
$$Trans\left(\hat{X}, q\right) = \begin{bmatrix} 1 & 0 & 0 & q \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: Trans operators are commutative in nature

$$Trans\left(\hat{X}, q_x\right) Trans\left(\hat{Y}, q_y\right) = Trans\left(\hat{Y}, q_y\right) Trans\left(\hat{X}, q_x\right)$$

Rotational Operator

$Rot\left(\hat{Z},\theta\right)$: Rotation about $\hat{Z}$ axis by an angle θ (anticlockwise sense)

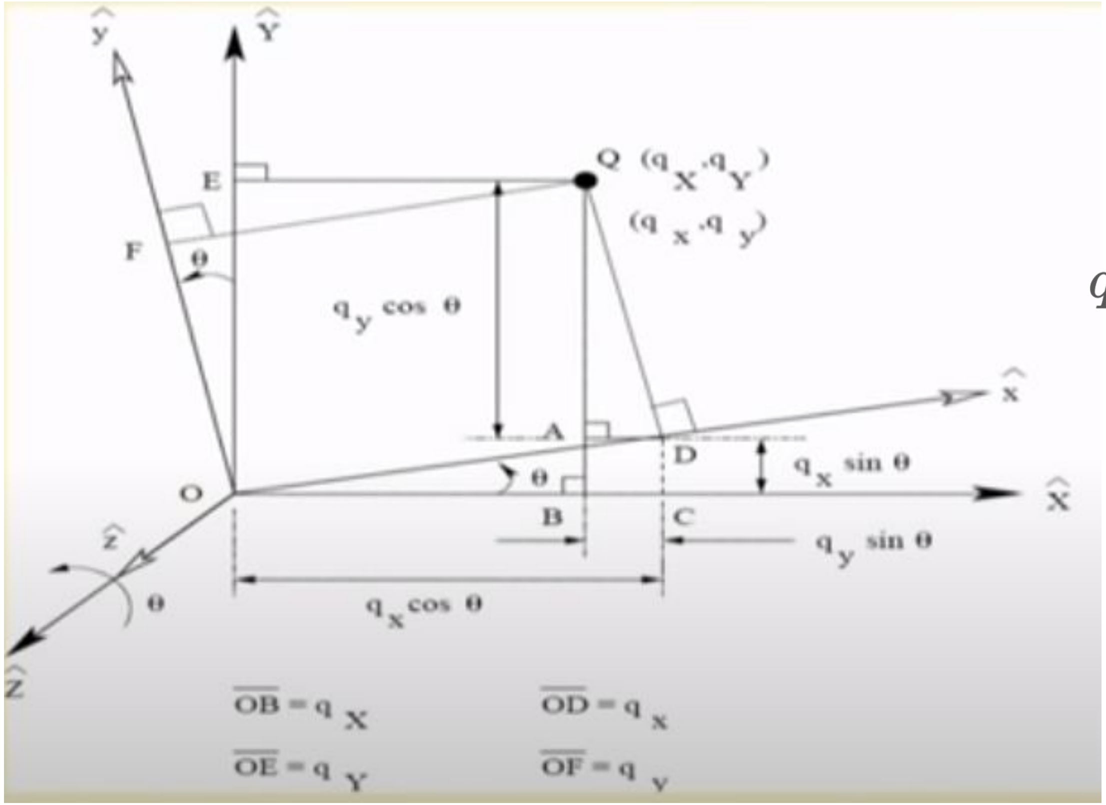


$$\overline{DC} = q_x \mathrm{Sin}\,\theta$$

$$\overline{AQ} = q_y \mathrm{Cos}\,\theta$$

$$\overline{OC} = q_x \mathrm{Cos}\,\theta$$

$$\overline{AD} = BC = q_y \mathrm{Sin}\theta$$



$$q_X = q_x \mathrm{Cos}\theta - q_y \mathrm{Sin}\theta + q_z X0$$

$$q_Y = q_x \mathrm{Sin}\theta + q_y \mathrm{Cos}\theta + q_z X0$$

$$q_Z = q_x X0 + q_y X0 + q_z X1$$

$$\begin{bmatrix} q_X \\ q_Y \\ q_Z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$$

$$\mathrm{Rot}(\hat{Z}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, we get

$$\mathrm{Rot}(\hat{X},\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$$

$$\mathrm{Rot}(\hat{Y},\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$

Properties of Rotation Matrix

- Each row/column of a rotation matrix is a unit vector.
- Inner(dot) product of each row of a rotation matrix with each other row becomes equal to 0. The same is true for each column also.
- Rotation matrices are not commutative in nature

$$Rot\left(\hat{X},\ \theta_1\right) Rot\left(\hat{Y},\theta_2\right) \neq Rot\left(\hat{Y},\theta_2\right) Rot\left(\hat{X},\theta_1\right)$$

- Inverse of a rotation matrix is nothing but its transpose

$$Rot^{-1}\left(\hat{X},\theta\right) = Rot^{T}\left(\hat{X},\theta\right)$$

$${}^{A}_{B}T = {}^{A}_{B}T^{-1}$$

A numerical Example

A frame {B} is rotated about $\hat{X}_U$ axis of the universal coordinate system by 45 degrees and translated along $\hat{X}_U$, $\hat{Y}_U$, and $\hat{Z}_U$ by 1, 2, and 3 units, respectively. Let the position of a point Q in {B} is given by $[3.0\ 2.0\ 1.0]^T$

Determine ${}^U\bar{Q}$?

A numerical Example

A frame {B} is rotated about $\hat{X}_U$ axis of the universal coordinate system by 45 degrees and translated along $\hat{X}_U$, $\hat{Y}_U$, and $\hat{Z}_U$ by 1, 2, and 3 units, respectively. Let the position of a point Q in {B} is given by $[3.0\ 2.0\ 1.0]^T$

Determine ${}^U\bar{Q}$?

Solution: $${}^UQ = {}^{\bar{U}}_B R \times {}^BQ$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \cos 45 & -\sin 45 & 2 \\ 0 & \sin 45 & \cos 45 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 2\cos 45 - \sin 45 + 2 \\ 2\sin 45 + \cos 45 + 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2.707 \\ 5.121 \\ 1 \end{bmatrix}$$

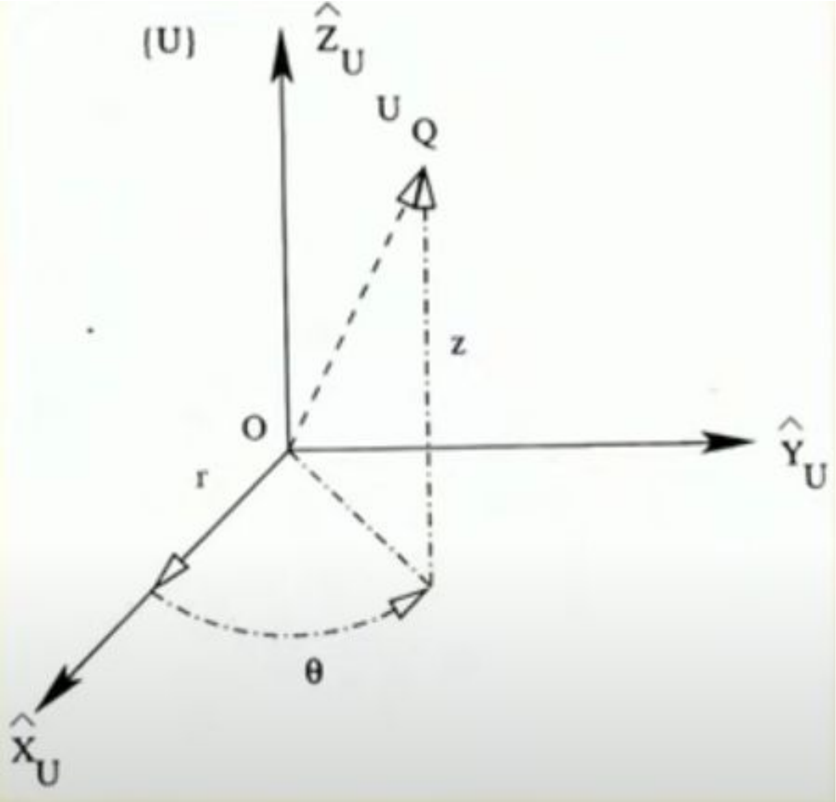
Composite Rotation Matrix

Composite rotation matrix representing a rotation of α angle about $\hat{Z}$, followed by a rotation of β angle about $\hat{Y}$ axis, followed by a rotation of γ angle about $\hat{X}$ axis.

$$Rot_{\text{composite}} = Rot\left(\hat{X}, \gamma\right) Rot\left(\hat{Y}, \beta\right) Rot\left(\hat{Z}, \alpha\right)$$

Representations of position in other than cartesian coordinate system

Cylindrical coordinate System



Steps:

1. Starting from the origin O, translate by r units along axis $\hat{X}_U$
2. Rotate in anticlockwise sense about $\hat{Z}_U$ axis by an angel θ
3. Translate along $\hat{Z}_U$ axis by Z units

$$[T]_{\text{composite}} = Trans\left(\hat{Z}_U, z\right) Rot\left(\hat{Z_U}, \theta\right) Trans\left(\hat{X}_U, r\right)$$

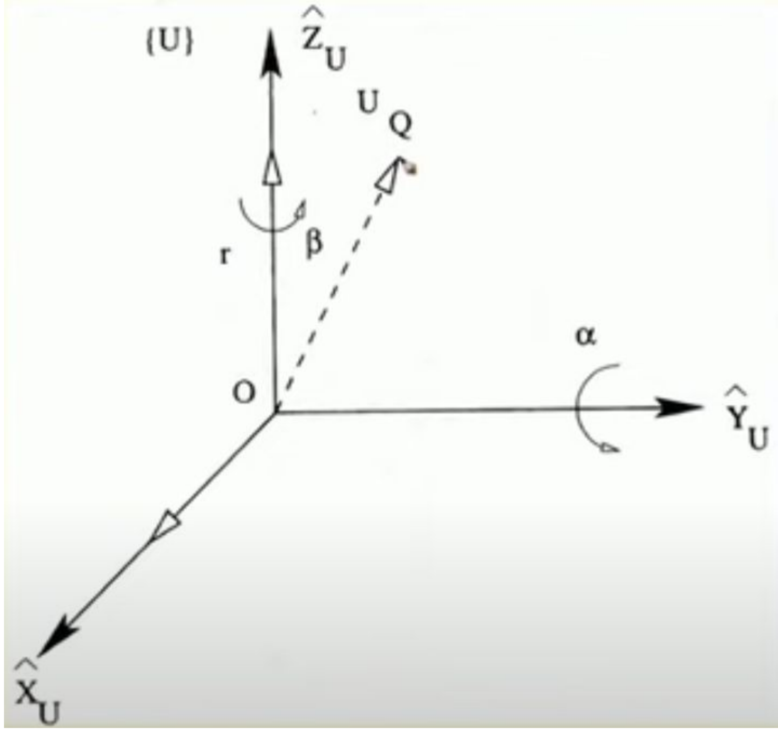
$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & r\cos\theta \\ \sin\,\theta & \cos\theta & 0 & r\,\sin\,\theta \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We get: $q_x = r\cos\theta$

$$q_y = r\sin\theta$$

$$q_z = z$$

Spherical coordinate System



Steps:

1. Starting from the origin O, translate by r units along axis $\hat{Z}_U$
2. Rotate in anticlockwise sense about $\hat{Y}_U$ axis by an angel α
3. Rotate in anticlockwise sense about $\hat{Z}_U$ axis by an angel β

$$[T]_{\text{composite}} = Rot\left(\hat{Z}_U, \beta\right) Rot\left(\hat{Y_U}, \alpha\right) Trans\left(\hat{Z}_U, r\right)$$

$$= \begin{bmatrix} \cos\alpha\cos\beta & -\sin\beta & \sin\alpha\ \cos\beta & r\ \sin\ \alpha\ \sin\beta \\ \cos\alpha\cos\beta & \cos\beta & \sin\alpha\ \cos\beta & r\ \sin\ \alpha\ \sin\beta \\ -\sin\ \alpha & 0 & \cos\alpha & r\ \cos\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We get: $q_x = r\ \sin\ \alpha\cos\beta$

$$q_y = r\sin\alpha\ \sin\beta$$

$$q_z = r\ \cos\alpha$$