considering the time modulated homogenous medium as

$$\varepsilon(x,t) = n_0^2 + \Delta\varepsilon\ m(t-t_0)\varepsilon_m(x),$$

with the properties given in the eq. (2) of the paper.
I didn't understand what it means by saying - the modulation essentially turns on a periodic grating perpendicular to the direction of propagation of the pulse"
Solving for the eq. (3) with the ansatz given in eq. (4) of the paper we get the following

$$\partial_{xx}E = (2ik_0\partial_x E^+ - k_0^2 E^+)e^{ik_0x-i\omega_0 t} - (2ik_0\partial_x E^- + k_0^2 E^-)e^{-ik_0x-i\omega_0 t}$$

$$\begin{aligned} & \partial_t(\varepsilon(x,t)E(x,t)) = \left[(\Delta\varepsilon\partial_t m\ \varepsilon_m(x))(E^+e^{ik_0x-i\omega_0 t} + E^-e^{-ik_0x-i\omega_0 t})\right] \\ + \quad & \left[n_0^2 + \Delta\varepsilon\ m(t-t_0)\varepsilon_m(x)\right]\left[(\partial_t E^+ - i\omega_0 E^+)e^{ik_0x-i\omega_0 t} + (\partial_t E^- - i\omega_0 E^-)e^{-ik_0x-i\omega_0 t}\right] \end{aligned}$$

Now

$$\begin{aligned} & \partial_{tt}(\varepsilon(x,t)E(x,t)) = \left[(\Delta\varepsilon\partial_t m\ \varepsilon_m(x))\left[(\partial_t E^+ - i\omega_0 E^+)e^{ik_0x-i\omega_0 t} + (\partial_t E^- - i\omega_0 E^-)e^{-ik_0x-i\omega_0 t}\right]\right] \\ + \quad & \left[n_0^2 + \Delta\varepsilon\ m\varepsilon_m(x)\right]\left[(-2i\omega_0\partial_t E^+ - \omega_0^2 E^+)e^{ik_0x-i\omega_0 t} + (-2i\omega_0\partial_t E^- - \omega_0^2 E^-)e^{-ik_0x-i\omega_0 t}\right] \\ + \quad & \left[(\Delta\varepsilon\partial_t m\ \varepsilon_m(x))\left[(\partial_t E^+ - i\omega_0 E^+)e^{ik_0x-i\omega_0 t} + (\partial_t E^- - i\omega_0 E^-)e^{-ik_0x-i\omega_0 t}\right]\right]. \end{aligned}$$

In writing this above equations we have neglected the terms containing $\partial_{tt}m,\ \partial_{tt}E^{\pm},\ \partial_{xx}E^{\pm}$. Now equating $\partial_{xx}E = \frac{1}{c^2}\partial_{tt}(\varepsilon(x,t)E(x,t))$ and rearranging the terms by dropping $e^{-i\omega_0 t}$ (as it occurs throughout the expression) gives us eq. (5) (paper).

Since $\varepsilon_m(x)$ is a periodic function we can write Fourier series as given in eq. (6) (with DC term in the series set to zero as average of ε_m is zero over one period) and neglecting the higher order terms from $|j| > 1$ we get $\varepsilon_m = \varepsilon_m^{(1)}e^{\frac{2\pi x}{d}} + \varepsilon_m^{*\,(-1)}e^{\frac{-2\pi x}{d}}$. Assuming that ε_m is a real function we have $\varepsilon_m^{(1)} = \varepsilon_m^{*\,(-1)}$ (in general $\varepsilon_m^{(1)}$ can be complex)and this results in

$$\varepsilon_m = \varepsilon_m^{(1)}[e^{i2k^{(g)}x} + e^{-i2k^{(g)}x}] = 2\cos(2k^{(g)}x)$$

Now substituting this relation in eq. (5) (paper) we get

$$\begin{aligned} & 2ik_0\partial_x E^+ + 2ik_0\frac{n_0}{c}\partial_t E^+ = \left[2ik_0\partial_x E^- - 2ik_0\frac{n_0}{c}\partial_t E^-\right]e^{-2ik_0x} \\ + \quad & \frac{\Delta\varepsilon\varepsilon_m^{(1)}}{c^2}e^{i2k^{(g)}x}\left[2\partial_t m(\partial_t E^+ - i\omega_0 E^+) - m(2i\omega_0\partial_t E^+ + \omega_0^2 E^+)\right] \\ + \quad & \frac{\Delta\varepsilon\varepsilon_m^{(1)}}{c^2}e^{-i2k^{(g)}x}\left[2\partial_t m(\partial_t E^+ - i\omega_0 E^+) - m(2i\omega_0\partial_t E^+ + \omega_0^2 E^+)\right] \\ + \quad & \frac{\Delta\varepsilon\varepsilon_m^{(1)}}{c^2}e^{-i2\delta kx}\left[\partial_t m(\partial_t E^- - i\omega_0 E^-) - m(2i\omega_0\partial_t E^- + \omega_0^2 E^-)\right] \\ + \quad & \frac{\Delta\varepsilon\varepsilon_m^{(1)}}{c^2}e^{-i2(k^{(g)}+k_0)x}\left[\partial_t m(\partial_t E^- - i\omega_0 E^-) - m(2i\omega_0\partial_t E^- + \omega_0^2 E^-)\right] \end{aligned}$$

For the sake of brevity we now redefine the above equation in the following manner

$$A = Be^{-i2k_0x} + \begin{pmatrix} Ce^{-i2\delta kx}e^{i2k_0x} \\ \text{or} \\ Ce^{i2(k^{(g)}+k_0)x}e^{-i2k_0x} \end{pmatrix} + \begin{pmatrix} Ce^{i2\delta k_0x}e^{-i2\delta k_0x} \\ \text{or} \\ Ce^{-i2(k^{(g)}+k_0)x}e^{i2\delta kx} \end{pmatrix} + De^{-i2\delta kx} + De^{-i2(k_g+k_0)x}$$

where $C = C_1 + C_2 + C_3 + C_4,\ D = D_1 + D_2 + D_3 + D_4,\ A = A_1 + A_2,\ B = B_1 + B_2$ are the respective terms in the above equation, for example $C_1 = \frac{\Delta\varepsilon\varepsilon_m^{(1)}}{c^2}2\partial_t m\partial_t E^+, C_2 = -i\frac{\Delta\varepsilon\varepsilon_m^{(1)}}{c^2}2(\partial_t m)\omega_0 E^+$ and so on. Now expanding the relation in terms of A, B, C, D we have

$$\begin{aligned} A &= Be^{-i2k_0x} + (C_1 + C_3)e^{-i2\delta kx}e^{i2k_0x} + (C_2 + C_4)e^{i2(k^{(g)}+k_0)x}e^{-i2k_0x} \\ &+ (C_1 + C_4)e^{i2\delta kx}e^{-i2k_0x} + (C_2 + C_3)e^{-i2(k^{(g)}+k_0)x}e^{i2k_0x} \\ &+ (D_1 + D_2)e^{i2k^{(g)}x}e^{-i2k_0x} + (D_2 + D_3)e^{-i2k_0x}e^{i2k^{(g)}x} \\ &+ (D_2 + D_4)e^{-i2k^{(g)x}}e^{-i2k_0x} + (D_1 + D_3)e^{i2\delta kx}e^{-i4k_0x}. \end{aligned}$$

Multiplying the above relation with e^{i2k_0x} and adding both of them will yield us an expression and now collecting terms with same exponential factors from the resulting expression leads us to the following

$$\begin{aligned} e^{i2k_0x} \quad \rightarrow \quad A &= (C_1 + C_3)e^{-i2\delta kx} + (C_2 + C_3)e^{-i2(k^{(g)}+k_0)x} + (D_1 + D_4)e^{-i2\delta kx} \\ &+ (C_2 + C_4)e^{i2k^{(g)x}} + (D_1 + D_4)e^{-i2(k^{(g)}+k_0)x} + (D_2 + D_3)e^{i2(k^{(g)}+k_0)x} \\ &+ (C_1 + C_3)e^{-i2(\delta k-2k_0)x} + (C_2 + C_3)e^{i2k^{(g)}x} \\ e^{-i2k_0x} \quad \rightarrow \quad 0 &= B + (C_2 + C_4)e^{i2(k^{(g)}+k_0)x} + (C_1 + C_4)e^{i2\delta kx} \\ &+ (D_1 + D_4)e^{-i2k^{(g)}x} + (D_1 + D_3)e^{i2\delta kx} + (D_2 + D_3)e^{i2(k^{(g)}+k_0)x} \\ &+ (D_2 + D_4)e^{-i2(k^{(g)}+k_0)x} + (D_1 + D_3)e^{i2(\delta k-2k_0)x} + (C_1 + C_4)e^{-i2k^{(g)x}} \end{aligned}$$

Now neglecting the fast oscillating terms (all terms other than $e^{\pm i2\delta kx}$ simplifies the above relation into two conditions given by

$$\begin{aligned} A &= (C_1 + C_3)e^{-i2\delta kx} + (D_1 + D_4)e^{-i2\delta kx} \\ -B &= (D_1 + D_3)e^{i2\delta kx} + (C_1 + C_4)e^{-i2\delta kx} \end{aligned}$$

Now replacing ABCD's we get the relations given in eq.(8) and(9), respectively, **but with the extra factor of -1 in RHS of both the equations (I think it is probably a typo in the paper)**. If now one assumes that $\varepsilon_m(x)$ is an even function then we have $\varepsilon_m^{*(1)}(x) = \varepsilon_m^{(1)}(x)$. After all the simplifications we directly get eq. (10) and eq. (11). Proceeding further to eq. (10)-(11) and transforming each equation in the moving frame as

$$x^{f,b} = x \mp vt; \quad t^{f,b} = t, \tag{1}$$

and converting the derivaties in terms of new variable gives us

$$\partial_t = \partial_{t^f} - v\partial_{x^f}, \quad \partial_{x^f} = \partial_x$$
$$\partial_t = \partial_{t^b} + v\partial_{x^b}, \quad \partial_{x^b} = \partial_x$$

Now eq.(10)(paper) transforms as

$$[\partial_{x^f} + \frac{n_0}{c}(\partial_{t^f} - v\partial_{x^f})]E^+(x^f, t^f) = 0$$
$$\frac{1}{v}\partial_{t^f} E^+(x^f, t^f) = 0$$
$$E^+(x^f, t^f) \equiv E^+(x^f, t) = E^+(x - vt, 0) = E^+(x - vt)$$

Similarly for E^- we have

$$[\partial_{x^b} - \frac{n_0}{c}(\partial_{t^b} + v\partial_{x^b})]E^-(x^b, t^b) = -i\kappa^* e^{i2\delta k(x^b - vt^b)m(t^b - t_0)E^+(x^b - 2vt^b)}$$
$$\partial_{t^b} E^-(x^b, t^b) = iv\kappa^* e^{i2\delta k x^b} e^{-i2\delta k t^b} m(t^b - t_0)E^+(x^b - 2vt^b),$$
$$E^-(x^b, t) = iv\kappa^* e^{i2\delta k x^b} \int_{-\infty}^{t} e^{-i2\delta k t^b} m(t^b - t_0)E^+(x^b - 2vt^b)dt^b.$$

This is the convolution integral given in eq. (14) (paper). Now for a given gaussian pulse input field and $m(t-t_0)$ eq. (15) (paper), the convolution integral takes the form (with $\delta k = 0 = t_0$)

$$\begin{aligned} E^-(x^b, t) &= iv\kappa^* \int_{-\infty}^{t} e^{-\frac{(t^b)^2}{T_m^2}} e^{-\frac{(x^b)^2 + 4v^2(t^b)^2 - 4vt^b x^b}{v^2 T_p^2}} dt^b \\ &= iv\kappa^* e^{-\frac{(x^b)^2}{v^2 T_p^2}} \int_{-\infty}^{t} e^{-(\frac{1}{T_m^2} + \frac{4}{T_p^2})(t^b)^2} e^{\frac{4vx^b}{v^2 T_p^2}} dt^b. \end{aligned}$$

Let

$$a = \frac{1}{T_m^2} + \frac{4}{T_p^2}$$
$$b = -\frac{4vx^b}{v^2 T_p^2}$$

and expressing the integral as

$$\begin{aligned} I &= \int_{-\infty}^{t} e^{-(ay^2 - by)} dy \\ I &= e^{(\frac{b^2}{4a})} \int_{-\infty}^{t} e^{-a(y - \frac{b}{2a})^2} dy \end{aligned}$$

Replacing for $a,\ b$ in I gives us E^- as

$$E^-(x^b, t) = iv\kappa^* e^{-\frac{(x^b)^2}{v^2(T_p^2 + 4T_m^2)}} \int_{-\infty}^{t} e^{-a(t^b - \frac{b}{2a})^2} dt^b$$

Now t is in the frame of x^b so the complete reflected pulse will evolve in the remote future as $t \to \infty$

$$\begin{aligned} E^-(x+vt) &= iv\kappa^* e^{-\frac{(x+vt)^2}{v^2(T_p^2+4T_m^2)}} \int_{-\infty}^{\infty} e^{-a(t^b-\frac{b}{2a})^2} dt^b \\ E^-(x+vt) &= iv\kappa^* \sqrt{\pi} e^{-\frac{(x+vt)^2}{v^2(T_p^2+4T_m^2)}} \sqrt{\frac{T_m^2 T_p^2}{4T_m^2+T_p^2}} \end{aligned}$$

This exactly matches with relation given in eq.(16) (paper)