CV202, HW2

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Problem 1

The condition is that A should be an invertible matrix. Then $f^{-1}: x \to A^{-1}x$. ■

Problem 2

For f to be a linear map it must satisfy:

$$f(\delta x) = \delta f(x), \forall \delta \in \mathbb{R}$$

In our case, considering $f(x) = ax + b, b \neq 0$:

$$f(\delta x) = a\delta x + b$$

$$\delta f(x) = \delta(ax + b) = a\delta x + \delta b$$

If we want f to be linear, we must have:

$$b = \delta b, \forall \delta \in \mathbb{R}$$

This is only true if $b = 0$, which is not the case, therefore f is not a linear map. ■

Problem 3

Let's observe our f:

$$f(x) = Ax + b$$

For f to be invertible, we need to find $g : \mathbb{R}^m \to \mathbb{R}^n$ such that $\forall x$:

$$\forall x \in \mathbb{R}^n, y \in \mathbb{R}^m \quad : \quad g\big(f(x)\big) = x \; and \; f\big(g(y)\big) = y$$

Choosing $g(y) = A^{-1}y - A^{-1}b$:

$$g\big(f(x)\big) = g(Ax + b) = A^{-1}(Ax + b) - A^{-1}b = x$$

$$f\big(g(y)\big) = f(A^{-1}y - A^{-1}b) = A(A^{-1}y - A^{-1}b) + b = y$$

As we can see, this is the only possible inverse for b.

But its existence assumes two major conditions! A must be invertible (not only right/left invertible), which also means it must be a square matrix, meaning $m = n$.

So in conclusion f can be invertible only on the conditions that $m = n$ and A is invertible. ■

Problem 4

$g: x \to A_2 x + b_2$
$f: x \to A_1 x + b_1$

$h = g \; o \; f = g\big(f(x)\big) = A_2(A_1 x + b_1) + b_2 = A_2 A_1 x + A_2 b_1 + b_2$

Define $A_3 = A_2 A_1,\ A_3 \in \mathbb{R}^{kxn}$, and $b_3 = A_2 b_1 + b_2,\ b_3 \in \mathbb{R}^k$.

Therefore h is affine $h: x \to A_3 x + b_3$. ∎

Problem 5

We have $f: \mathbb{R}^n \to \mathbb{R}^n\ and\ g: \mathbb{R}^n \to \mathbb{R}^n$ invertible affine maps.

This means:

$$f(x) = A_f x + b_f,\ \ g(x) = A_g x + b_g$$

Such that A_f, A_g are invertible (we've shown that in problem 3).

Now by definition h is an affine map (as we've shown before), of the following structure:

$$h(x) = g\big(f(x)\big) = g\big(A_f x + b_f\big) = A_g\big(A_f x + b_f\big) + b_g = A_g A_f x + A_g b_f + b_g$$

And knowing that A_f^{-1}, A_g^{-1} exist, we can see that $A_g A_f$ is invertible (by $A_f^{-1} A_g^{-1}$), therefore we can conclude from what we've proven in problem 3 that h is invertible affine map. ∎

Problem 6

We will show that G1, G2 and G3 definitions are satisfied:

$GL(n) = \{Q | Q \in R^{nxn}, \det Q\ \neq 0\}$

<u>*G1:*</u> $I_{nxn}\ \in R^{nxn}\ and\ \det(I_{nxn}) \neq 0$

<u>*G2:*</u> $A, B \in GL(n)$

$$A_{nxn} B_{nxn}\ \in R^{nxn}$$

$$\det(A) \neq 0 \text{ and } \det(B) \neq 0$$

$$\det(AB) = \det(A)\det(B) \neq 0$$

<u>*G3:*</u> *define* $A \in GL(n)$

$$\det(A) \neq 0, \qquad therefore\ A\ is\ invertible$$

$$A^{-1} \in R^{nxn}\ and\ \det(A^{-1}) \neq 0$$

Problem 7

Define:

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$A, B\ \in R^{nxn}, \det(A) \neq 0\ and\ \det(B) \neq 0, therefore\ A, B\ \in GL(2), but:$

$$AB = \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = BA$$

We found 2 matrices that belongs to GL(2) so that the Abelian definition is not satisfied, therefore GL(n) is not Abelian.

Problem 8

We will show that G1, G2 and G3 definitions are satisfied:

$GL(n)_+ = \{Q|Q \in R^{nxn}, \det Q > 0\}$

<u>G1:</u> $I_{nxn} \in R^{nxn}$, so $\det(I_{nxn}) = 1 > 0$

<u>G2:</u> $A, B \in GL(n)_+$

$$A_{nxn}B_{nxn} \in R^{nxn}$$

$$\det(A) > 0 \text{ and } \det(B) > 0$$

$$\det(AB) = \det(A)\det(B) > 0$$

<u>G3:</u> $A \in GL(n)$

$$\det(A) > 0, also \det(A) \neq 0, \quad therefore\ A\ is\ invertible$$

$$\det(AA^{-1}) = \det(A)\det(A^{-1}) = 1 \quad => \det(A^{-1}) = \frac{1}{\det(A)}$$

$$A^{-1} \in R^{nxn}\ and \det(A^{-1}) = \frac{1}{\det(A)} > 0$$

Problem 9

Lets take a look at G2 definition:

$$A, B \in GL_-(n)$$

$$A_{nxn}B_{nxn} \in R^{nxn}$$

$$\det(A) < 0 \text{ and } \det(B) < 0$$

$$\det(AB) = \det(A)\det(B) > 0$$

$AB\ does\ not\ belong\ to\ GL_-(n)\ , therefore\ GL(n)_-\ is\ not\ a\ matrix\ group$

Problem 10

$A\ and\ B\ are\ symetric\ matrices\ A = A^T, B = B^T$

$(AB)^T = B^T A^T = BA$

$AB = (AB)^T = BA, satisfied\ only\ when$:

- $A = B$
- $A = cI\ or\ B = cI$
- $A\ and\ B\ diagnal\ matrices$

$Those\ terms\ usually\ are\ not\ satisfied\ so\ \boldsymbol{usually}\ AB \neq (AB)^T$

Problem 11

We will show that G1, G2 and G3 definitions are satisfied:

$US(n) = \{Q|Q \in SI_{nxn}, S \in R_{>0}\}$

G1: $1I_{nxn} \in R^{nxn}$ and $\mathrm{S} = 1 > 0$ so $I_{n\times n} \in US(n)$

G2: $A, B \in US(n)$

$$A_{nxn}B_{nxn} \in R^{nxn}$$

$$A = S_1 I, B = S_2 I$$

$$S_1, S_2 > 0$$

$$AB = S_1 I S_2 I = S_1 S_2 I$$

$$S_1 S_2 > 0$$

$$AB \in US(n)$$

G3: $A \in GL(n)$

define $\mathrm{A} = cI, A^{-1} = \frac{1}{c} I$ $, so$ $A^{-1} exists$

$$c' = \frac{1}{c} > 0 \Rightarrow A^{-1} = c' I$$

$therefore$ $A^{-1} \in US(n)$

Problem 12

$Define$ $A, B \in US(n)$

$A = S_1 I, B = S_2 I$

$AB = S_1 I S_2 I =^{S_1 and\ S_2\ are\ scalars} S_2 I S_1 I = BA$

Problem 13

We can see that the matrix 0_{nxn} satisfies the following:

- for all $X \in R^{nxn}$ $, X + 0_{nxn} = X$
- for all $X \in R^{nxn}$ $, X \cdot 0_{nxn} = 0_{nxn}$

Therefore the matrix 0_{nxn} is the zero element.

Problem 14

A linear subspace of a linear space must contain the 0 element.

We know that if A belongs to a matrix group there exists A^{-1}.

However the $0_{n\times n}$ is clearly non-invertible, therefore can't be in any matrix group.

Therefore any matrix group cannot be a linear subspace of $\mathbb{R}^{n\times n}$.

Problem 15

We will show that G1, G2 and G3 definitions are satisfied:

G1: $I_{n+1xn+1} = \begin{bmatrix} I_{nxn} & 0_{1xn} \\ 0 & 1 \end{bmatrix}, A = I_{nxn}, b = 0_{1xn}.$

$$A = I_{nxn} \quad => \quad \det(A) \neq 0$$

$$\Longrightarrow I_{(n+1)\times(n+1)} \in (the\ affine\ group)$$

<u>G2:</u> $let\ \ matrices\ M, N \in affine\ group$

$Define: M = \begin{bmatrix} A & b_m \\ 0 & 1 \end{bmatrix}, N = \begin{bmatrix} B & b_n \\ 0 & 1 \end{bmatrix}$

$$MN = \begin{bmatrix} AB & Ab_n + b_m \\ 0 & 1 \end{bmatrix}$$

We know $\det(A)$, $\det(B) \neq 0$ as M, N are in the affine group.

So: $\det(AB) = \det(A)\det(B) \neq 0$

$also\ MN\ is\ of\ the\ form\ of\ affine\ group, AB\ \in R^{nxn}, Ab_n + b_m \in \mathbb{R}^n, MN\ \in R^{n+1\ x\ n+1}$

<u>G3:</u> $let\ M \in affine\ group$

Define: $M = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$

$we\ know\ that\ there\ exist\ A^{-1}\ such\ that\ AA^{-1} = A^{-1}A = I, because\ \det(A) \neq 0\ (invertible).$

$lets\ compose\ a\ new\ matrix\ N = \begin{bmatrix} A^{-1} & -A^{-1}b \\ 0 & 1 \end{bmatrix}$

$\mathrm{MN} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A^{-1} & -A^{-1}b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} AA^{-1} & A \cdot -A^{-1}b + b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I_{nxn} & 0 \\ 0 & 1 \end{bmatrix} = I_{n+1\ x\ n+1}$

$\det(A^{-1}) \neq 0, obviously$

$\mathrm{N} = M^{-1} \in affine\ group$

Problem 16

We will show that G1, G2 and G3 definitions are satisfied:

<u>G1:</u> $I_{n+1xn+1} = \begin{bmatrix} I_{nxn} & 0_{1xn} \\ 0 & 1 \end{bmatrix}, A = I_{nxn}, b = 0_{1xn}.$

$$A = I_{nxn} \quad => \quad \det(A) = \det(I_{n+1\ x\ n+1}) > 0$$

<u>G2:</u> $let\ M, N \in identity\ component\ of\ affine\ group$

$M = \begin{bmatrix} A_m & b_m \\ 0 & 1 \end{bmatrix}, N = \begin{bmatrix} A_n & b_n \\ 0 & 1 \end{bmatrix}$

$$MN = \begin{bmatrix} A_m A_n & A_m b_n + b_m \\ 0 & 1 \end{bmatrix}$$

We know $\det(A)$, $\det(B) > 0$ as M, N are in our group.

So: $\det(AB) = \det(A)\det(B) > 0$

$also\ MN\ is\ of\ the\ form\ of\ identity\ component\ of\ affine\ group, AB\ \in R^{nxn}, A_m b_n + b_m \in \mathbb{R}^n, MN\ \in R^{n+1\ x\ n+1}$

So MN is in our group.

G3: $let\ M$ be a matrix in our group.

Define: $M = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$

$we\ know\ that\ there\ exists\ A^{-1}\ such\ that\ AA^{-1} = A^{-1}A = I, because\ \det(A) > 0\ (invertible).$

$lets\ compose\ a\ new\ matrix\ N = \begin{bmatrix} A^{-1} & -A^{-1}b \\ 0 & 1 \end{bmatrix}$

$\mathrm{MN} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A^{-1} & -A^{-1}b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} AA^{-1} & A \cdot -A^{-1}b + b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I_{nxn} & 0 \\ 0 & 1 \end{bmatrix} = I_{n+1\ x\ n+1}$

We know $\det(\mathrm{A}) > 0, \det(I) > 0$, and $\det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$ which means $\det(A^{-1}) > 0$ also.

So we can conclude:

$\mathrm{N} = M^{-1} \in the\ identity\ component\ of\ the\ affine\ group$

Problem 17

Lets look at G2:

$M, N \in matrix\ group\ of\ the\ form\ \begin{bmatrix} A_m & b_m \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} A_n & b_n \\ 0 & 1 \end{bmatrix}, \det(A) < 0$

$$M_{n+1\ x\ n+1}N_{n+1\ x\ n+1} \in R^{n+1\ x\ n+1}$$

$$\det(A_m) < 0 \text{ and } \det(A_n) < 0$$

$$\det(A_m A_n) = \det(A_m)\det(A_n) > 0$$

But $MN = \begin{bmatrix} A_m B_n & \cdots \\ 0 & 1 \end{bmatrix}$ which means MN is not in our group, therefore the group is not a matrix group.

Problem 18

The answer is no. There is no continuous curve c, from the unit interval to the affine group, such that $c(0) = f$ and $c(1) = g$.

We will show that by observing any possible continuous curve $c(t), t \in [0,1]$, such that:

$$c(t)(x) = A(t)x + b(t)$$

$$A(0) = A_0, A(1) = A_1$$

$$b(0) = b_0, b(1) = b_1$$

We know that:

$$\det\big(A(0)\big) = \det(A_0) > 0$$

$$\det\big(A(1)\big) = \det(A_1) < 0$$

c is a continuous curve, so for any i, j:

$$\big(A(t)\big)_{i,j} = c_{i,j}(t)$$

$$c_{i,j} : [0,1] \to \mathbb{R}$$

And $c_{i,j}$ is continuous!

Now, as we know from the Leibniz formula, a determinant of a matrix is a polynomial expression of the matrix entries. Every entry in our $A(t)$ is continuous over $[0,1]$, therefore we can say $\det(A(t))$ is a continuous curve as well!

Now we know it begins in $\det(A_0) > 0$, and ends at $\det(A_1) < 0$, therefore exists t_0 such that $\det\big(A(t_0)\big) = 0$!

So we can see that we do not have the case $\det\big(A(t)\big) \neq 0, \forall t \in [0,1]$, showing no continuous curve exists between the two affine maps. ■

Problem 19

Part (i):

We need to find the least square estimator for $\boldsymbol{\theta}$.

We want to find $\boldsymbol{\theta}$ such that for all i: $\boldsymbol{u}_i \approx \boldsymbol{P} \cdot \begin{bmatrix} \boldsymbol{X}_i \\ 1 \end{bmatrix} + \boldsymbol{\epsilon}_i$.

So our problem is described by:

$$f(\boldsymbol{\theta}) = \sum_{i=1}^{N} \left\| \boldsymbol{P} \begin{bmatrix} \boldsymbol{X}_i \\ 1 \end{bmatrix} + \boldsymbol{\epsilon}_i - \boldsymbol{u}_i \right\|^2 \; and \; \boldsymbol{\theta_{LS}} = argmin_{\boldsymbol{\theta}}\big(f(\boldsymbol{\theta})\big)$$

Now let's observe $f(\boldsymbol{\theta})$:

$$f(\boldsymbol{\theta}) = \sum_{i=1}^{N} \left\| \begin{bmatrix} \theta_1 X_i + \theta_2 Y_i + a \\ \theta_3 X_i + bY_i + \theta_4 \end{bmatrix} + \boldsymbol{\epsilon}_i - \boldsymbol{u}_i \right\|^2 = \sum_{i=1}^{N} \left\| \begin{bmatrix} \theta_1 X_i + \theta_2 Y_i \\ \theta_3 X_i + \theta_4 \end{bmatrix} + \begin{bmatrix} a \\ bY_i \end{bmatrix} + \boldsymbol{\epsilon}_i - \boldsymbol{u}_i \right\|^2 =$$

$$= \sum_{i=1}^{N} \left\| \begin{bmatrix} X_i & Y_i & 0 & 0 \\ 0 & 0 & X_i & 1 \end{bmatrix} \cdot \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} + \begin{bmatrix} a \\ bY_i \end{bmatrix} + \boldsymbol{\epsilon}_i - \boldsymbol{u}_i \right\|^2$$

Now let:

$$\boldsymbol{H}_i = \begin{bmatrix} X_i & Y_i & 0 & 0 \\ 0 & 0 & X_i & 1 \end{bmatrix}, \; \boldsymbol{y}_i = \boldsymbol{u}_i - \begin{bmatrix} a \\ bY_i \end{bmatrix} - \boldsymbol{\epsilon}_i = \begin{bmatrix} u_i - a - \boldsymbol{\epsilon}_i(1) \\ v_i - bY_i - \boldsymbol{\epsilon}_i(2) \end{bmatrix}$$

And we get:

$$f(\boldsymbol{\theta}) = \sum_{i=1}^{N} \|\boldsymbol{H}_i \cdot \boldsymbol{\theta} - \boldsymbol{y}_i\|^2$$

Now:

$$\boldsymbol{H} = [\boldsymbol{H}_1^T \quad \cdots \quad \boldsymbol{H}_N^T]^T = \left[\begin{bmatrix} X_1 & 0 \\ Y_1 & 0 \\ 0 & X_1 \\ 0 & 1 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} X_N & 0 \\ Y_N & 0 \\ 0 & X_N \\ 0 & 1 \end{bmatrix}\right]^T = \begin{bmatrix} X_1 & Y_1 & 0 & 0 \\ 0 & 0 & X_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ X_N & Y_N & 0 & 0 \\ 0 & 0 & X_N & 1 \end{bmatrix}$$

$$\boldsymbol{y} = [\boldsymbol{y}_1^T \quad \cdots \quad \boldsymbol{y}_N^T]^T = \left[\begin{bmatrix} u_1 - a - \boldsymbol{\epsilon}_1(1) \\ v_1 - bY_1 - \boldsymbol{\epsilon}_1(2) \end{bmatrix}^T \quad \cdots \quad \begin{bmatrix} u_N - a - \boldsymbol{\epsilon}_N(1) \\ v_N - bY_N - \boldsymbol{\epsilon}_N(2) \end{bmatrix}^T\right]^T = \begin{bmatrix} u_1 - a - \boldsymbol{\epsilon}_1(1) \\ v_1 - bY_1 - \boldsymbol{\epsilon}_1(2) \\ \vdots \\ u_N - a - \boldsymbol{\epsilon}_N(1) \\ v_N - bY_N - \boldsymbol{\epsilon}_N(2) \end{bmatrix}$$

And as we've learned, the minimizer satisfies:

$$\boldsymbol{\theta_{LS}} = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{y} =$$

$$= \boxed{\begin{bmatrix} \sum X_i^2 & \sum X_iY_i & 0 & 0 \\ \sum X_iY_i & \sum Y_i^2 & 0 & 0 \\ 0 & 0 & \sum X_i^2 & \sum X_i \\ 0 & 0 & \sum X_i & N \end{bmatrix}^{-1} \cdot \begin{bmatrix} X_1 & 0 & \cdots & X_N & 0 \\ Y_1 & 0 & \cdots & Y_N & 0 \\ 0 & X_1 & \cdots & 0 & X_N \\ 0 & 1 & \cdots & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 - a - \boldsymbol{\epsilon}_1(1) \\ v_1 - bY_1 - \boldsymbol{\epsilon}_1(2) \\ \vdots \\ u_N - a - \boldsymbol{\epsilon}_N(1) \\ v_N - bY_N - \boldsymbol{\epsilon}_N(2) \end{bmatrix}} \blacksquare$$

Part (ii):

For this part we will note that we are looking for (let p be our probability density function):

$$g(\boldsymbol{\theta}) = \prod_{i=1}^{N} \mathrm{p}(\boldsymbol{H}_i \cdot \boldsymbol{\theta} = \boldsymbol{y_i}) = \prod_{i=1}^{N} \mathrm{p}\left(\boldsymbol{H}_i \cdot \boldsymbol{\theta} = \boldsymbol{u}_i - \begin{bmatrix} a \\ bY_i \end{bmatrix} - \boldsymbol{\epsilon}_i\right) = \prod_{i=1}^{N} \mathrm{p}\left(\boldsymbol{\epsilon}_i = -\boldsymbol{H}_i \cdot \boldsymbol{\theta} + \boldsymbol{u}_i - \begin{bmatrix} a \\ bY_i \end{bmatrix}\right) =$$

$$= \prod_{i=1}^{N} \mathrm{p}\left(\boldsymbol{\epsilon}_i = \begin{bmatrix} -X_i \cdot \theta_1 - Y_i \cdot \theta_2 + u_i - a \\ -X_i \cdot \theta_3 - \theta_4 + v_i - bY_i \end{bmatrix}\right)$$

We know the probability density function of ϵ_i, so:

$$= \prod_{i=1}^{N} \frac{1}{2\pi\sigma^2} \cdot \exp\left(-\frac{\begin{bmatrix} -X_i \cdot \theta_1 - Y_i \cdot \theta_2 + u_i - a \\ -X_i \cdot \theta_3 - \theta_4 + v_i - bY_i \end{bmatrix}^T \cdot \begin{bmatrix} -X_i \cdot \theta_1 - Y_i \cdot \theta_2 + u_i - a \\ -X_i \cdot \theta_3 - \theta_4 + v_i - bY_i \end{bmatrix}}{2\sigma^2}\right) =$$

$$\Rightarrow \boxed{g(\boldsymbol{\theta}) = \left(\frac{1}{2\pi\sigma^2}\right)^N \cdot \prod_{i=1}^{N} \exp\left(-\frac{h_i(\theta_1, \theta_2, \theta_3, \theta_4)}{2\sigma^2}\right)}$$

Where:

$$\begin{aligned} h_i(\theta_1, \theta_2, \theta_3, \theta_4) \\ &= X_i^2\theta_1^2 + 2X_iY_i\theta_1\theta_2 + (a - u_i)X_i\theta_1 + Y_i^2\theta_2^2 + (a - u_i)Y_i\theta_2 + (a - u_i)^2 + X_i^2\theta_3^2 \\ &+ 2X_i\theta_3\theta_4 + (bY_i - v_i)X_i\theta_3 + \theta_4^2 + (bY_i - v_i)\theta_4 + (bY_i - v_i)^2 \quad \blacksquare \end{aligned}$$

Part (iii):

The maximum likelihood estimator for $\boldsymbol{\theta}$ will be achieved by getting the maximum of the **log likelihood function** (as it is a monotonic non-decreasing function):

$$\log\big(g(\boldsymbol{\theta})\big) = \mathrm{N} \cdot \log\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{N} h_i(\theta_1, \theta_2, \theta_3, \theta_4)$$

To get the maximum $\boldsymbol{\theta}^* = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$, we will look for a gradient of $\mathbf{0}$.

$$\frac{\partial(\log(g))}{\partial\theta_1} = -\frac{1}{2\sigma^2} \cdot \sum_{i=1}^{N} 2X_i^2\theta_1 + 2X_iY_i\theta_2 + (a - u_i)X_i = 0$$

$$\frac{\partial(\log(g))}{\partial\theta_2} = -\frac{1}{2\sigma^2} \cdot \sum_{i=1}^{N} 2Y_i^2\theta_2 + 2X_iY_i\theta_1 + (a - u_i)Y_i = 0$$

$$\frac{\partial(\log(g))}{\partial\theta_3} = -\frac{1}{2\sigma^2} \cdot \sum_{i=1}^{N} 2X_i^2\theta_3 + 2X_i\theta_4 + (bY_i - v_i)X_i = 0$$

$$\frac{\partial(\log(g))}{\partial\theta_4} = -\frac{1}{2\sigma^2} \cdot \sum_{i=1}^{N} 2\theta_4 + 2X_i\theta_3 + (bY_i - v_i) = 0$$

We will start with θ_1, θ_2:

$$\begin{cases} \left(\sum_{i=1}^{N} X_i^2\right)\theta_1 + \left(\sum_{i=1}^{N} X_iY_i\right)\theta_2 = \sum_{i=1}^{N} \frac{(u_i - a)X_i}{2} \\ \left(\sum_{i=1}^{N} X_iY_i\right)\theta_1 + \left(\sum_{i=1}^{N} Y_i^2\right)\theta_2 = \sum_{i=1}^{N} \frac{(u_i - a)Y_i}{2} \end{cases} \Rightarrow \begin{bmatrix} \sum_{i=1}^{N} X_i^2 & \sum_{i=1}^{N} X_iY_i \\ \sum_{i=1}^{N} X_iY_i & \sum_{i=1}^{N} Y_i^2 \end{bmatrix} \cdot \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} \frac{(u_i - a)X_i}{2} \\ \sum_{i=1}^{N} \frac{(u_i - a)Y_i}{2} \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \frac{1}{\left(\sum_{i=1}^{N} X_i^2\right) \cdot \left(\sum_{i=1}^{N} Y_i^2\right) - \left(\sum_{i=1}^{N} X_iY_i\right)^2} \begin{bmatrix} \sum_{i=1}^{N} Y_i^2 & -\sum_{i=1}^{N} X_iY_i \\ -\sum_{i=1}^{N} X_iY_i & \sum_{i=1}^{N} X_i^2 \end{bmatrix} \cdot \begin{bmatrix} \sum_{i=1}^{N} \frac{(u_i - a)X_i}{2} \\ \sum_{i=1}^{N} \frac{(u_i - a)Y_i}{2} \end{bmatrix}}$$

And now for θ_3, θ_4:

$$\begin{cases} \left(\sum_{i=1}^{N} X_i^2\right)\theta_3 + \left(\sum_{i=1}^{N} X_i\right)\theta_4 = \sum_{i=1}^{N} \frac{(v_i - bY_i)X_i}{2} \\ \left(\sum_{i=1}^{N} X_i\right)\theta_3 + N \cdot \theta_4 = \sum_{i=1}^{N} \frac{v_i - bY_i}{2} \end{cases} \Rightarrow \begin{bmatrix} \sum_{i=1}^{N} X_i^2 & \sum_{i=1}^{N} X_i \\ \sum_{i=1}^{N} X_i & N \end{bmatrix} \cdot \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} \frac{(v_i - bY_i)X_i}{2} \\ \sum_{i=1}^{N} \frac{v_i - bY_i}{2} \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{bmatrix}\theta_3\\\theta_4\end{bmatrix} = \frac{1}{N\cdot\sum_{i=1}^N X_i^2 - \left(\sum_{i=1}^N X_i\right)^2}\cdot\begin{bmatrix} N & -\sum_{i=1}^N X_i \\ -\sum_{i=1}^N X_i & \sum_{i=1}^N X_i^2\end{bmatrix}\cdot\begin{bmatrix}\sum_{i=1}^N \frac{(v_i - bY_i)X_i}{2} \\ \sum_{i=1}^N \frac{v_i - bY_i}{2}\end{bmatrix}} \quad \blacksquare$$

Problem 20

We need to find:

$$\arg\min_{\boldsymbol{\theta}\in\mathbb{R}^k} \|\boldsymbol{H\theta} - \boldsymbol{y}\|_{l_2}^2 + \lambda\|\boldsymbol{\theta}\|_{l_2}^2$$

Let's observe:

$$\|\boldsymbol{H\theta}-\boldsymbol{y}\|_{l_2}^2 + \lambda\|\boldsymbol{\theta}\|_{l_2}^2 = \|\boldsymbol{H\theta}-\boldsymbol{y}\|_{l_2}^2 + \left\|\sqrt{\lambda}\cdot\boldsymbol{\theta}\right\|_{l_2}^2 = \sum_{i=1}^N\left((\boldsymbol{H\theta}-\boldsymbol{y})(i)\right)^2 + \sum_{i=1}^k\left(\left(\sqrt{\lambda}\cdot\boldsymbol{\theta}\right)(i)\right)^2 =$$

$$= \left\|\begin{bmatrix}\boldsymbol{H\theta}-\boldsymbol{y}\\\sqrt{\lambda}\cdot\boldsymbol{\theta}\end{bmatrix}\right\|_{l_2}^2 = \left\|\begin{bmatrix}\boldsymbol{H\theta}\\\sqrt{\lambda}\cdot\boldsymbol{\theta}\end{bmatrix} - \begin{bmatrix}\boldsymbol{y}\\\boldsymbol{0}_{k\times1}\end{bmatrix}\right\|_{l_2}^2 = \left\|\begin{bmatrix}\boldsymbol{H}\\\sqrt{\lambda}\cdot\boldsymbol{I}_{k\times k}\end{bmatrix}\boldsymbol{\theta} - \begin{bmatrix}\boldsymbol{y}\\\boldsymbol{0}_{k\times1}\end{bmatrix}\right\|_{l_2}^2$$

And this is of LS form, so we know our $\boldsymbol{\theta}$ solution is:

$$\boldsymbol{\theta} = \left(\begin{bmatrix}\boldsymbol{H}\\\sqrt{\lambda}\cdot\boldsymbol{I}_{k\times k}\end{bmatrix}^T\begin{bmatrix}\boldsymbol{H}\\\sqrt{\lambda}\cdot\boldsymbol{I}_{k\times k}\end{bmatrix}\right)^{-1}\begin{bmatrix}\boldsymbol{H}\\\sqrt{\lambda}\cdot\boldsymbol{I}_{k\times k}\end{bmatrix}^T\begin{bmatrix}\boldsymbol{y}\\\boldsymbol{0}_{k\times1}\end{bmatrix} =$$

$$\Rightarrow \boxed{\boldsymbol{\theta} = (\boldsymbol{H}^T\boldsymbol{H} + \lambda\cdot\boldsymbol{I}_{k\times k})^{-1}\boldsymbol{H}^T\boldsymbol{y}} \quad \blacksquare$$

Problem 21

So we need to find:

$$\arg\min_{\hat{x}\in span(v_1,\dots,v_k)} \|x - \hat{x}\|_{l_2}$$

But $\hat{x}\in span(v_1,\dots,v_k)$ means:

$$\hat{x} = \textstyle\sum_{j=1}^k \theta_j v_j = \boldsymbol{V}\cdot\boldsymbol{\theta} \text{ for some } \boldsymbol{\theta} = \begin{bmatrix}\theta_1\\\vdots\\\theta_k\end{bmatrix}\in\mathbb{R}^k$$

So we can rewrite our problem:

$$\arg\min_{\theta\in\mathbb{R}^k}\|\boldsymbol{x} - \boldsymbol{V}\cdot\boldsymbol{\theta}\|_{l_2}$$

As when we find the right $\boldsymbol{\theta}$, we can get our $\hat{x}$.

Now, as we know x^2 is a monotonic non-decreasing function, so we can rewrite again:

$$\arg\min_{\theta\in\mathbb{R}^k}\|\boldsymbol{x} - \boldsymbol{V}\cdot\boldsymbol{\theta}\|_{l_2} = \arg\min_{\theta\in\mathbb{R}^k}\|\boldsymbol{x} - \boldsymbol{V}\cdot\boldsymbol{\theta}\|_{l_2}^2 = \arg\min_{\theta\in\mathbb{R}^k}\|\boldsymbol{V\theta} - \boldsymbol{x}\|_{l_2}^2$$

And now we can it's a LS problem. For $N = 1$:

$$\boldsymbol{H} = \boldsymbol{H}_1 = \boldsymbol{V} \in \mathbb{R}^{d\times k}$$

$$\boldsymbol{y} = \boldsymbol{y}_1 = \boldsymbol{x} \in \mathbb{R}^{d\times 1}$$

$$\boldsymbol{\theta} \in \mathbb{R}^{k\times 1}$$

$$\Longrightarrow \boldsymbol{r} = \boldsymbol{H\theta} - \boldsymbol{y} = \boldsymbol{V\theta} - \boldsymbol{x}$$

$$\Longrightarrow \boldsymbol{\theta_{LS}} = \arg\min_{\boldsymbol{\theta}\in\mathbb{R}^k} \|\boldsymbol{r}\|_{l_2}^2 = \arg\min_{\theta\in\mathbb{R}^k} \|\boldsymbol{V\theta} - \boldsymbol{x}\|_{l_2}^2$$

So now we can use our known solution:

$$\boldsymbol{\theta_{LS}} = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{y} = (\boldsymbol{V}^T\boldsymbol{V})^{-1}\boldsymbol{V}^T\boldsymbol{x} = \boldsymbol{V}^T\boldsymbol{x}$$

$$\Longrightarrow \boxed{\boldsymbol{\theta} = \boldsymbol{V}^T\boldsymbol{x}}$$

$$\Longrightarrow \boxed{\widehat{\boldsymbol{x}} = \boldsymbol{V}\cdot\boldsymbol{\theta} = \boldsymbol{V}\cdot\boldsymbol{V}^T\cdot\boldsymbol{x}} \quad \blacksquare$$

Problem 22

Definition: $(a * b)(i,j) = \sum_{k,l} a(i-k, j-l)\, b(k,l)$

$$\begin{aligned}
\big((a*b)*c\big)(i,j) &= \sum\nolimits_{m,n} (a*b)(i-m, j-n)\, c(m,n) \\
&= \sum\nolimits_{m,n} \left(\sum\nolimits_{k,l} a(i-m-k, j-n-l) b(k,l)\right) c(m,n) =_{t:=k+m, s:=l+n} \\
&= \sum\nolimits_{m,n} \left(\sum\nolimits_{t,s} a(i-t, j-s) b(t-m, s-n)\right) c(m,n) \\
&= \sum\nolimits_{m,n} \left(\sum\nolimits_{t,s} a(i-t, j-s) b(t-m, s-n) c(m,n)\right) \\
&= \sum\nolimits_{t,s} \left(\sum\nolimits_{m,n} a(i-t, j-s) b(t-m, s-n) c(m,n)\right) \\
&= \sum\nolimits_{t,s} a(i-t, j-s) \cdot \sum\nolimits_{m,n} b(t-m, s-n) c(m,n) \\
&= \sum\nolimits_{t,s} a(i-t, j-s) \cdot (b*c)(t,s) = a*(b*c)\,(i,j) \blacksquare
\end{aligned}$$

Problem 23

First of all, we have seen in definition 9 that $(I * h)(i,j) = (h * I)(i,j)$ for all h.

So all we need to show is that $(h * \delta)(\mathrm{i},\mathrm{j}) = h(i,j)$ for all i, j.

Intuition: if at (0,0) the value of δ is 1 and otherwise its 0, it means that it preserve only the pixel that the convolution is affecting, without any influence by other pixels (they are multiplied by 0).

$$(\mathrm{h} * \delta)(\mathrm{i},\mathrm{j}) = \sum\nolimits_{k,l} h(i-k, j-l)\, \delta(k,l)$$

$$\delta(k,l) = \begin{cases} 1, & k=0, l=0 \\ 0, & otherwise \end{cases}$$

$$(\mathrm{h} * \delta)(\mathrm{i},\mathrm{j}) = 0 + 0 + \cdots\ h(i-0, j-0)\delta(0,0) = h(i,j), for\ all\ i, j.$$

Therefore $\mathrm{h} * \delta = h$ ■

Problem 24

First of all let's be clear about our "column vector" versions of x and y:

$$(1)\ \ \boldsymbol{x}(N \cdot (i-1) + j) = x(i,j)$$

$$(2)\ \ \boldsymbol{y}(N \cdot (i-1) + j) = y(i,j)$$

If we want to do filter $(3x3)$ on $x(i,j)$ the definition of it is:

$$y(i,j) = \sum_{k=-1}^{1}\sum_{l=-1}^{1} x(i-k, j-l)h(k,l) =_{(1)} \sum_{k=-1}^{1}\sum_{l=-1}^{1} \boldsymbol{x}(N(i-k-1) + j - l)h(k,l) =$$

$$= \sum_{k=-1}^{1}\sum_{l=-1}^{1} \boldsymbol{x}(N(i-1) + j - Nk - l)h(k,l)$$

But we know $\boldsymbol{y}(N \cdot (i-1) + j) = y(i,j)$, so:

$$\boldsymbol{y}(N \cdot (i-1) + j) = \sum_{k=-1}^{1}\sum_{l=-1}^{1} \boldsymbol{x}(N(i-1) + j - Nk - l)h(k,l)$$

Now let's re-purpose our indexes, marking $i \coloneqq N(i-1) + j$. Recall we are using the zero-boundary assumption.

$$(*)\ \ \boldsymbol{y}(i) = \sum_{k=-1}^{1}\sum_{l=-1}^{1} \boldsymbol{x}(i - Nk - l)h(k,l)$$

Now let's be clear about our goal. We need to define $\boldsymbol{H}(i,j)$ for all $1 \le i, j \le MN$ such that:

$$\boldsymbol{y} = \boldsymbol{H} \cdot \boldsymbol{x}$$

$\boldsymbol{y}$ and $\boldsymbol{x}$ are both column vectors, so according to our goal, lets see what each value (which is a row) of $\boldsymbol{y}$ should be:

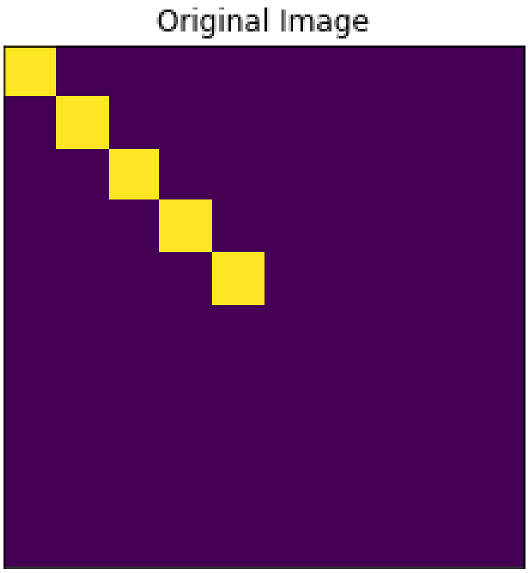
$$\Rightarrow 1 \le i \le MN : \boldsymbol{y}(i) = \sum_{j=1}^{MN} H(i,j) \cdot \boldsymbol{x}(j)$$

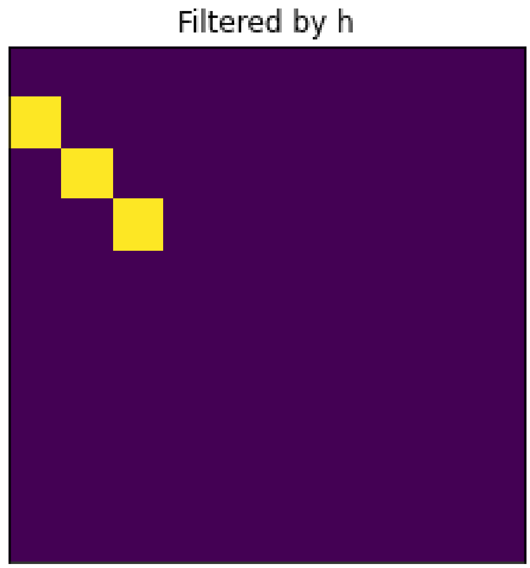
Now using this with our $(*)$, we can safely define $H(i,j)$:

$$\boxed{H(i,j) = \begin{cases} h(k,l) & j = i - Nk - l, \quad -1 \le k,l \le 1 \\ 0 & else \end{cases}} \quad ■$$

Problem 25

The effect of convolving an image with this given h, is moving the entire image 2 pixels to the left, and 1 pixel upwards. This is a very small change, barely noticeable in sizeable images, so we can see this with this very small image example:





Problem 26

Part (i)

$$1 = \det(I_{n\times n}) = \det(YY^T) = \det Y \cdot \det Y^T = (\det Y)^2$$

$$\Longrightarrow \boxed{\det Y = \pm 1} \ \blacksquare$$

Part (ii)

From $YY^T = I_{n\times n}$, we get $Y^{-1} = Y^T$ by definition of the inverse matrix. Therefore directly we can see that:

$$I_{n\times n} = YY^{-1} = Y^{-1}Y = Y^TY = Y^T(Y^T)^T$$

Proving that Y^T is an orthogonal square matrix. $\blacksquare$

Part (iii)

For any i, directly from part (ii), by definition of matrix multiplication we get (as y_i is a <u>column</u> of Y):

$$1 = (I_{n\times n})_{i,i} = (Y^TY)_{i,i} = y_i^T y_i$$

$$\Longrightarrow \boxed{y_i^T y_i = 1} \ \blacksquare$$

Part (iv)

For any i, by definition:

$$\|y_i\|_{l_2} = \sum_{j=1}^{n} (y_i)_j^2 = y_i^T y_i$$

$$\Longrightarrow \boxed{\|y_i\|_{l_2} = 1} \ \blacksquare$$

Part (v)

Similarly to part (iii), for any i:

$$0 = (I_{n\times n})_{i,j} = (Y^T Y)_{i,j} = y_i^T y_j$$

$$\Longrightarrow \boxed{y_i^T y_j = 0}$$

This means the angle between y_i and y_j is 90°. ■

Part (vi)

We will prove this by definition. We mark this group of all $n \times n$ orthogonal matrices G.

1. $I_{n\times n} \in G$
   This is clear because $I_{n\times n} \cdot (I_{n\times n})^T = I_{n\times n} \cdot I_{n\times n} = I_{n\times n}$, meaning it's orthogonal so in G.
2. We need to show that if $A, B \in G$, so is AB.
   Let's observe, remembering both A, B are orthogonal matrices:
   $$(AB) \cdot (AB)^T = A \cdot B \cdot B^T \cdot A^T = A \cdot I_{n\times n} \cdot A^T = A \cdot A^T = I_{n\times n}$$
   $\Longrightarrow AB$ is an orthogonal matrix so in G.
3. We need to show that if $A \in G$, A^{-1} exists and it's in G.
   A is orthogonal so $AA^T = I_{n\times n}$ meaning A^{-1} exists and $A^{-1} = A^T$.
   We've proven before that A^T is also orthogonal, meaning it's in G, so A^{-1} is in G as we wanted.

And that shows by definition that our group is a matrix group. ■

Problem 27

K is a $n \times n$ separable filter. Is it invertible? Well a matrix is invertible if and only if it has a full rank. So we need to see whether $rank(K) = n$.

K is separable so:

$$K = USV^T$$

Such that S is a diagonal matrix with values $s_1, \dots, s_n$, and **only on of them** is a non-zero!

This directly means $rank(S) = 1$!

And from fact 3 we know:

$$rank(K) = rank(USV^T) \leq \min\big(rank(US), rank(V^T)\big) \leq \min\big(rank(U), rank(S), rank(V^T)\big) \leq 1$$

So we got $rank(K) \leq 1$ (it's actually 1 otherwise it's a 0 matrix). So unless $n = 1$ which is a very trivial case, K is not invertible. ■

Problem 28

Bilateral filtering is not a linear operation, because as we can see in equation (28), the filter operation is dependent on the f_r in a non-linear way.

Any change to the input image will not result in a proportional change in the output.
First of all f_r is not necessarily linear, and even if it is: It's input is in absolute value which is not linear.
In addition, the $\frac{1}{c}$ component which has f_r in it, is not linear in any way.

Problem 29

$$\nabla^2 G(x,y,\sigma) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(x,y,\sigma)$$

$$G(x,y,\sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

First derivative:

$$\frac{\partial\, G(x,y,\sigma)}{\partial x} = -\frac{x}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$\frac{\partial\, G(x,y,\sigma)}{\partial y} = -\frac{y}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Second derivative:

$$\frac{\partial^2\, G(x,y,\sigma)}{\partial^2 x} = \frac{(-1 + \frac{x^2}{\sigma^2})}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$\frac{\partial^2\, G(x,y,\sigma)}{\partial^2 y} = \frac{(-1 + \frac{y^2}{\sigma^2})}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$\begin{aligned}\nabla^2 G(x,y,\sigma) &= \left(\frac{\partial^2 G(x,y,\sigma)}{\partial x^2} + \frac{\partial^2 G(x,y,\sigma)}{\partial y^2}\right) = \frac{(-1 + \frac{x^2}{\sigma^2})}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}} + \frac{(-1 + \frac{y^2}{\sigma^2})}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}} \\ &= \left(\frac{\left(-1 + \frac{x^2}{\sigma^2}\right)}{\sigma^2} + \frac{\left(-1 + \frac{y^2}{\sigma^2}\right)}{\sigma^2}\right) \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} = \left(\frac{x^2}{\sigma^4} + \frac{y^2}{\sigma^4} - \frac{2}{\sigma^2}\right) G(x,y,\sigma) \\ &= \left(\frac{x^2+y^2}{\sigma^4} - \frac{2}{\sigma^2}\right) G(x,y,\sigma)\end{aligned}$$

Problem 30

First derivatives:

$$\frac{\partial I}{\partial x} = \frac{\partial I}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial I}{\partial y'} \cdot \frac{\partial y'}{\partial x} = \frac{\partial I}{\partial x'} \cdot \cos\theta + \frac{\partial I}{\partial y'} \cdot \sin\theta$$

$$\frac{\partial I}{\partial y} = \frac{\partial I}{\partial x'} \cdot \frac{\partial x'}{\partial y} + \frac{\partial I}{\partial y'} \cdot \frac{\partial y'}{\partial y} = \frac{\partial I}{\partial x'} \cdot (-\sin\theta) + \frac{\partial I}{\partial y'} \cdot \cos\theta$$

Second derivatives:

$$
\begin{aligned}
\frac{\partial^2 I}{\partial^2 x} = \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial x'} \cdot \cos\theta + \frac{\partial I}{\partial y'} \cdot \sin\theta\right) = \left(\frac{\partial}{\partial x}\frac{\partial I}{\partial x'}\right) \cdot \cos\theta + \left(\frac{\partial}{\partial x}\frac{\partial I}{\partial y'}\right) \cdot \sin\theta \\
&= \left(\frac{\partial}{\partial x'}\frac{\partial I}{\partial x}\right) \cdot \cos\theta + \left(\frac{\partial}{\partial y'}\frac{\partial I}{\partial x}\right) \cdot \sin\theta \\
&= \frac{\partial}{\partial x'}\left(\frac{\partial I}{\partial x'} \cdot \cos\theta + \frac{\partial I}{\partial y'} \cdot \sin\theta\right) \cdot \cos\theta + \frac{\partial}{\partial y'}\left(\frac{\partial I}{\partial x'} \cdot \cos\theta + \frac{\partial I}{\partial y'} \cdot \sin\theta\right) \cdot \sin\theta \\
&= \frac{\partial^2 I}{\partial^2 x'} \cdot \cos^2\theta + 2 \cdot \frac{\partial^2 I}{\partial x' \partial y'} \cdot \sin\theta\cos\theta + \frac{\partial^2 I}{\partial^2 y'}\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 I}{\partial^2 y} = \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial y}\right) &= \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial x'} \cdot (-\sin\theta) + \frac{\partial I}{\partial y'} \cdot \cos\theta\right) = \left(\frac{\partial}{\partial y}\frac{\partial I}{\partial x'}\right) \cdot (-\sin\theta) + \left(\frac{\partial}{\partial y}\frac{\partial I}{\partial y'}\right) \cdot \cos\theta \\
&= \left(\frac{\partial}{\partial x'}\frac{\partial I}{\partial y}\right) \cdot (-\sin\theta) + \left(\frac{\partial}{\partial y'}\frac{\partial I}{\partial y}\right) \cdot \cos\theta \\
&= \frac{\partial}{\partial x'}\left(\frac{\partial I}{\partial x'} \cdot (-\sin\theta) + \frac{\partial I}{\partial y'} \cdot \cos\theta\right) \cdot (-\sin\theta) + \frac{\partial}{\partial y'}\left(\frac{\partial I}{\partial x'} \cdot (-\sin\theta) + \frac{\partial I}{\partial y'} \cdot \cos\theta\right) \\
&\cdot \cos\theta = \frac{\partial^2 I}{\partial^2 x'} \cdot \sin^2\theta - 2 \cdot \frac{\partial^2 I}{\partial x' \partial y'} \cdot \sin\theta\cos\theta + \frac{\partial^2 I}{\partial^2 y'}\cos^2\theta
\end{aligned}
$$

Now finally we can see:

$$
\begin{aligned}
\frac{\partial^2 I}{\partial^2 x} + \frac{\partial^2 I}{\partial^2 y} &= \frac{\partial^2 I}{\partial^2 x'} \cdot \cos^2\theta + \cancel{2 \cdot \frac{\partial^2 I}{\partial x' \partial y'} \cdot \sin\theta\cos\theta} + \frac{\partial^2 I}{\partial^2 y'}\sin^2\theta + \frac{\partial^2 I}{\partial^2 x'} \cdot \sin^2\theta - \cancel{2 \cdot \frac{\partial^2 I}{\partial x' \partial y'}} \\
&\cancel{\cdot \sin\theta\cos\theta} + \frac{\partial^2 I}{\partial^2 y'}\cos^2\theta = \frac{\partial^2 I}{\partial^2 x'} \cdot (\cos^2\theta + \sin^2\theta) + \frac{\partial^2 I}{\partial^2 y'}(\sin^2\theta + \cos^2\theta) \\
&= \frac{\partial^2 I}{\partial^2 x'} + \frac{\partial^2 I}{\partial^2 y'} \quad \blacksquare
\end{aligned}
$$