Logistic regression

Data Science in a Box

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Predicting categorical data

Spam filters

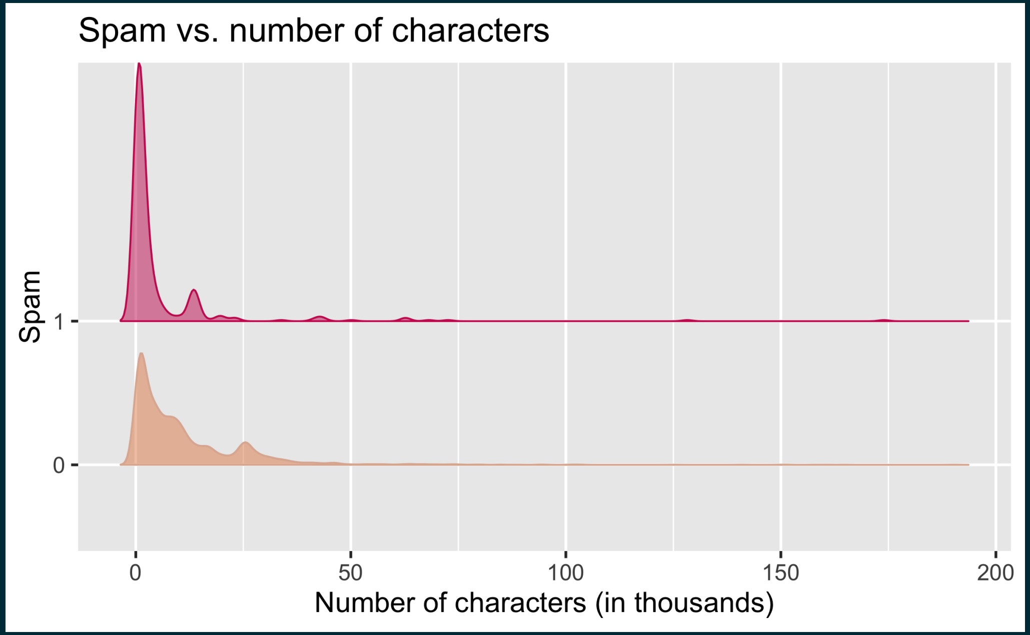
- Data from 3921 emails and 21 variables on them
- Outcome: whether the email is spam or not
- Predictors: number of characters, whether the email had "Re:" in the subject, time at which email was sent, number of times the word "inherit" shows up in the email, etc.

```r
library(openintro)
glimpse(email)
```

```
## Rows: 3,921
## Columns: 21
## $ spam         <fct> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, …
## $ to_multiple  <fct> 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, …
## $ from         <fct> 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, …
## $ cc           <int> 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, …
## $ sent_email   <fct> 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, …
## $ time         <dttm> 2012-01-01 08:16:41, 2012-01-01 09:03:59,…
## $ image        <dbl> 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, …
## $ attach       <dbl> 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, …
## $ dollar       <dbl> 0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, …
## $ winner       <fct> no, no, no, no, no, no, no, no, no, no, no…
## $ inherit      <dbl> 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, …
## $ viagra       <dbl> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, …
## $ password     <dbl> 0, 0, 0, 0, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, …
## $ num_char     <dbl> 11.370, 10.504, 7.773, 13.256, 1.231, 1.09…
## $ line_breaks  <int> 202, 202, 192, 255, 29, 25, 193, 237, 69, …
## $ format       <fct> 1, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, …
## $ re_subj      <fct> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, …
## $ exclaim_subj <dbl> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, …
## $ urgent_subj  <fct> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, …
## $ exclaim_mess <dbl> 0, 1, 6, 48, 1, 1, 1, 18, 1, 0, 2, 1, 0, 1…
## $ number       <fct> big, small, small, small, none, none, big,…
```

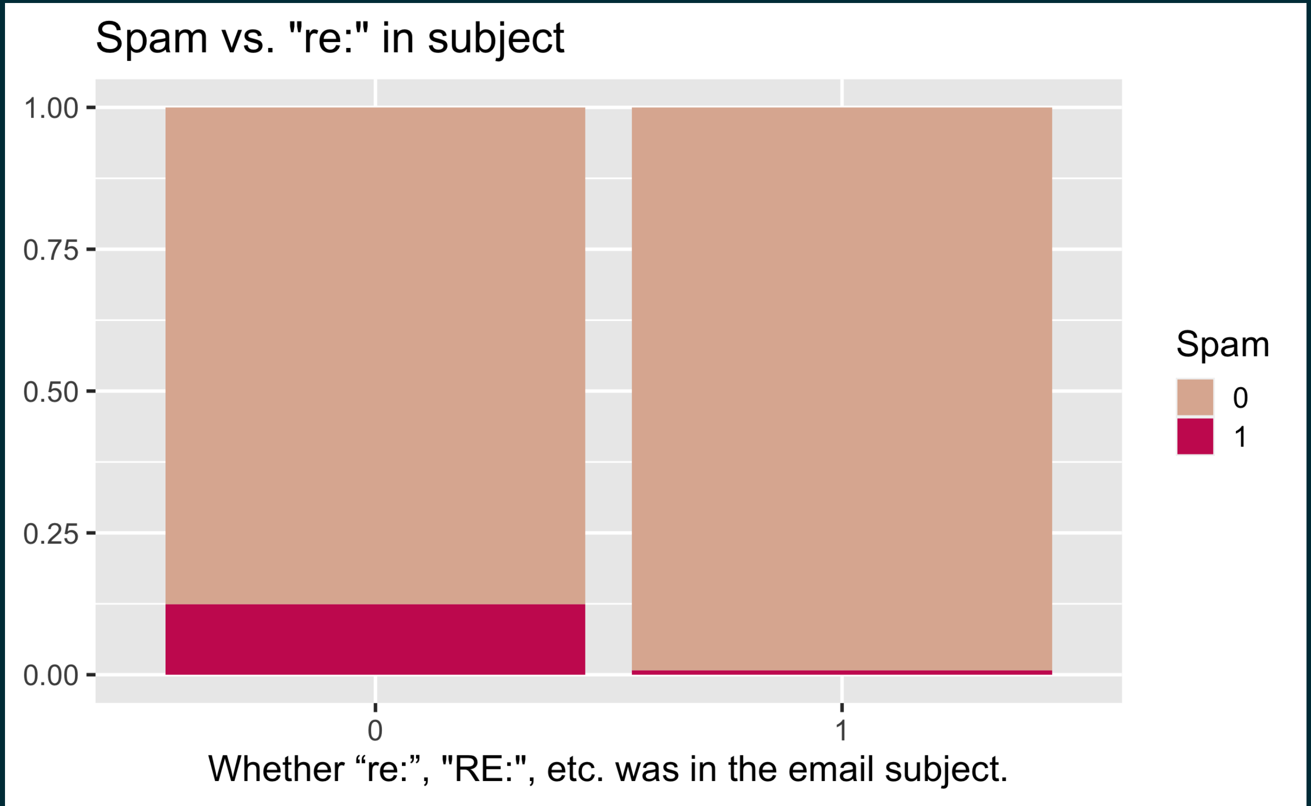
Would you expect longer or shorter emails to be spam?

```
## Warning: `guides(<scale> = FALSE)` is deprecated. Please use
## `guides(<scale> = "none")` instead.
```




```
## # A tibble: 2 × 2
##   spam  mean_num_char
##   <fct>         <dbl>
## 1 0             11.3
## 2 1              5.44
```

Would you expect emails that have subjects starting with "Re:", "RE:", "re:", or "rE:" to be spam or not?



Modelling spam

- Both number of characters and whether the message has "re:" in the subject might be related to whether the email is spam. How do we come up with a model that will let us explore this relationship?
- For simplicity, we'll focus on the number of characters (`num_char`) as predictor, but the model we describe can be expanded to take multiple predictors as well.

Modelling spam

This isn't something we can reasonably fit a linear model to -- we need something different!

```
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```

Framing the problem

- We can treat each outcome (spam and not) as successes and failures arising from separate Bernoulli trials
  - Bernoulli trial: a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted
- Each Bernoulli trial can have a separate probability of success

$$y_i \sim Bern(p)$$

- We can then use the predictor variables to model that probability of success, p_i
- We can't just use a linear model for p_i (since p_i must be between 0 and 1) but we can transform the linear model to have the appropriate range

Generalized linear models

- This is a very general way of addressing many problems in regression and the resulting models are called **generalized linear models (GLMs)**
- Logistic regression is just one example

Three characteristics of GLMs

All GLMs have the following three characteristics:

1. A probability distribution describing a generative model for the outcome variable
2. A linear model:

$$\eta = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k$$

3. A link function that relates the linear model to the parameter of the outcome distribution

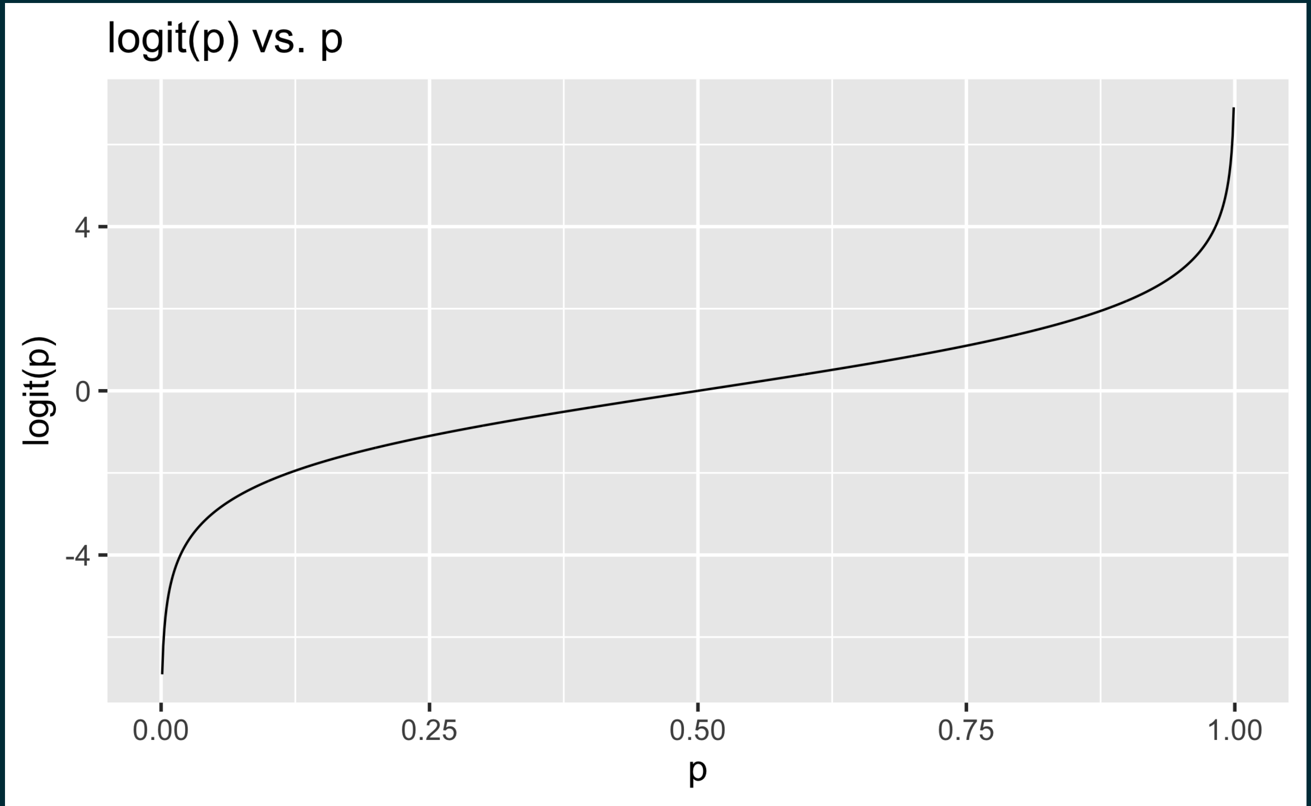
Logistic regression

Logistic regression

- Logistic regression is a GLM used to model a binary categorical outcome using numerical and categorical predictors
- To finish specifying the Logistic model we just need to define a reasonable link function that connects η_i to p_i: logit function
- **Logit function:** For $0 \le p \le 1$

$$logit(p) = \log\left(\frac{p}{1-p}\right)$$

Logit function, visualised



Properties of the logit

- The logit function takes a value between 0 and 1 and maps it to a value between $-\infty$ and ∞
- Inverse logit (logistic) function:

$$g^{-1}(x) = \frac{\exp(x)}{1+\exp(x)} = \frac{1}{1+\exp(-x)}$$

- The inverse logit function takes a value between $-\infty$ and ∞ and maps it to a value between 0 and 1
- This formulation is also useful for interpreting the model, since the logit can be interpreted as the log odds of a success -- more on this later

The logistic regression model

- Based on the three GLM criteria we have
  - $y_i \sim \text{Bern}(p_i)$
  - $\eta_i = \beta_0 + \beta_1 x_{1,i} + \cdots + \beta_n x_{n,i}$
  - $\text{logit}(p_i) = \eta_i$
- From which we get

$$p_i = \frac{\exp(\beta_0 + \beta_1 x_{1,i} + \cdots + \beta_k x_{k,i})}{1 + \exp(\beta_0 + \beta_1 x_{1,i} + \cdots + \beta_k x_{k,i})}$$

Modeling spam

In R we fit a GLM in the same way as a linear model except we

- specify the model with `logistic_reg()`
- use `"glm"` instead of `"lm"` as the engine
- define `family = "binomial"` for the link function to be used in the model

```
spam_fit <- logistic_reg() %>%
  set_engine("glm") %>%
  fit(spam ~ num_char, data = email, family = "binomial")

tidy(spam_fit)
```

```
## # A tibble: 2 × 5
##   term        estimate std.error statistic   p.value
##   <chr>          <dbl>     <dbl>     <dbl>     <dbl>
## 1 (Intercept)  -1.80     0.0716     -25.1  2.04e-139
## 2 num_char     -0.0621   0.00801     -7.75 9.50e- 15
```

Spam model

```
tidy(spam_fit)
```

```
## # A tibble: 2 × 5
##   term        estimate std.error statistic   p.value
##   <chr>          <dbl>     <dbl>     <dbl>     <dbl>
## 1 (Intercept)  -1.80     0.0716     -25.1  2.04e-139
## 2 num_char     -0.0621   0.00801     -7.75 9.50e- 15
```

$$\log\left(\frac{p}{1-p}\right) = -1.80 - 0.0621 \times \text{num\_char}$$

P(spam) for an email with 2000 characters

$$\log\left(\frac{p}{1-p}\right) = -1.80 - 0.0621 \times 2$$

$$\frac{p}{1-p} = \exp(-1.9242) = 0.15 \rightarrow p = 0.15 \times (1-p)$$

$$p = 0.15 - 0.15p \rightarrow 1.15p = 0.15$$

$$p = 0.15/1.15 = 0.13$$

> What is the probability that an email with 15000 characters is spam? What about an email with 40000 characters?

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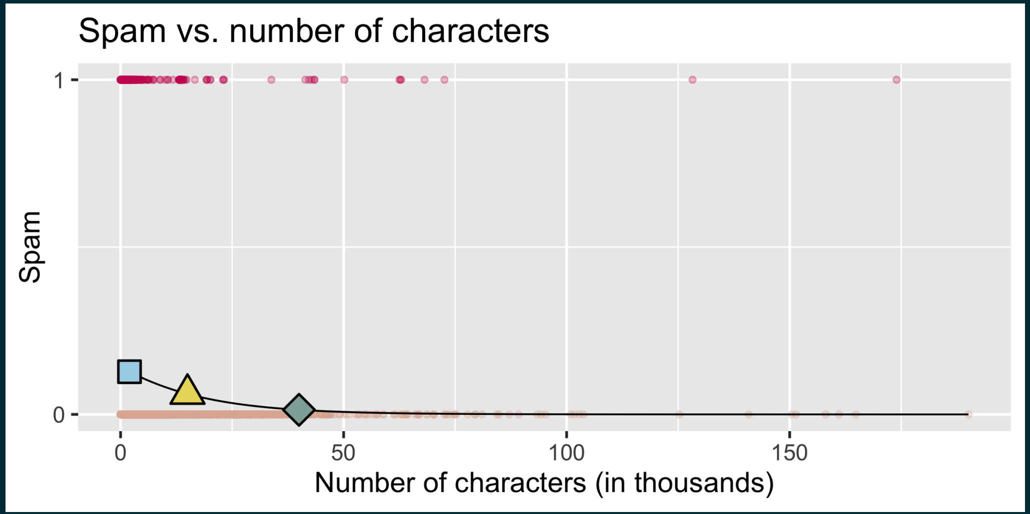
```



- 2K chars: P(spam) = 0.13
- 15K chars, P(spam) = 0.06
- 40K chars, P(spam) = 0.01

Would you prefer an email with 2000 characters to be labelled as spam or not? How about 40,000 characters?

```
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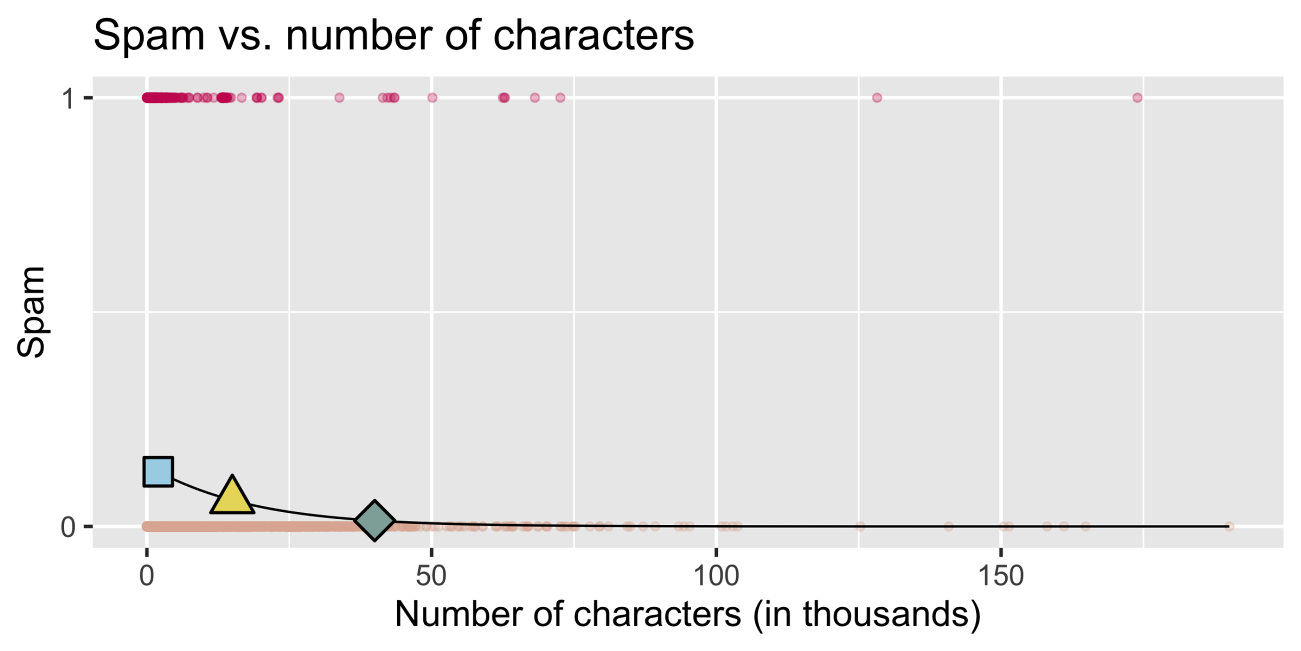
```



Sensitivity and specificity

False positive and negative

	Email is spam	Email is not spam
Email labelled spam	True positive	False positive (Type 1 error)
Email labelled not spam	False negative (Type 2 error)	True negative

- False negative rate = P(Labelled not spam | Email spam) = FN / (TP + FN)
- False positive rate = P(Labelled spam | Email not spam) = FP / (FP + TN)

Sensitivity and specificity

	Email is spam	Email is not spam
Email labelled spam	True positive	False positive (Type 1 error)
Email labelled not spam	False negative (Type 2 error)	True negative

- Sensitivity = P(Labelled spam | Email spam) = TP / (TP + FN)
  - Sensitivity = 1 − False negative rate
- Specificity = P(Labelled not spam | Email not spam) = TN / (FP + TN)
  - Specificity = 1 − False positive rate

If you were designing a spam filter, would you want sensitivity and specificity to be high or low? What are the trade-offs associated with each decision?