The Goemans-Williamson Rounding Alg. for MAXCUT

Just coming up with a ~~feasible~~ ~~sol.~~ approximation to a NP-HARD problem is in practice insufficient as we generally require also a feasible sol. Today, we see a method to generate a feasible ~~rounding~~ sol. to MAXCUT that achieves a multiplicative guarantee $\rho \approx 0.879 - \varepsilon$.

Let $G = (V, E)$ be a graph with $n = |V|, m = |E|$. Then we can define MAXCUT as

$$\mathrm{MAXCUT}(G) = \begin{cases} \max & \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2} \\ \text{s.t.} & x_i \in \{\pm 1\} \qquad \forall\, i \in V. \end{cases}$$

We can consider the SDP relaxation as

$$\mathrm{SDP}(G) = \begin{cases} \sup & \sum_{(i,j)\in E} \frac{1 - x_i \cdot x_j}{2} \\ \text{s.t.} & \|x_i\| = 1 \qquad \forall\, i \in V. \end{cases}$$

Let's quickly reverify that $\mathrm{SDP}(G) \geq \mathrm{MAXCUT}(G)$.

<u>Proof</u>. Map each vector $x_i \longmapsto$ a vector $(\pm 1, 0, \dots, 0) \in \mathbb{R}^n$ and $x_i x_j \longmapsto x_i \cdot x_j$. Then easy to see that $\mathrm{SDP}(G) \geq \mathrm{MAXCUT}(G)$.

Let's write the SDP in the canonical form. Make it easier to complete the pf. of the correctness of the rounding alg.

Define a matrix $X = (X_{ij})$ where $X_{ij} = x_i \cdot x_j$. The matrix X is PSD as it can be expressed as

$$X = \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix} \cdot \begin{pmatrix} — & x_1 & — \\ — & x_2 & — \\ & \vdots & \\ — & x_n & — \end{pmatrix} = \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix} \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix}^T$$

The constraint $\|x_i\| = 1$ is the same as $X_{ii} = 1$ which is equivalent to $X \bullet E_{ii} = 1$. As we are maximizing, we can simplify to $X \bullet E_{ii} \le 1$.

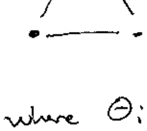
Define $A = (A_{ij})$ by $A_{ij} = \frac{1}{2} \mathbb{1}_{\{(i,j) \in E\}}$.

$$\max \sum_{(i,j) \in E} \frac{1 - x_i x_j}{2} = \frac{m}{2} + \max \sum_{i,j} (-A_{ij}\, x_i \cdot x_j)$$

$$= \frac{m}{2} + \max(-A \bullet X)$$

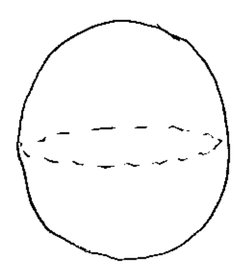
$$SDP(G) = \begin{cases} \sup & (-A) \bullet X \\ \text{s.t.} & E_{ii} \bullet X \le 1 \qquad i = 1, \dots, n \\ & X \succeq 0. \end{cases}$$

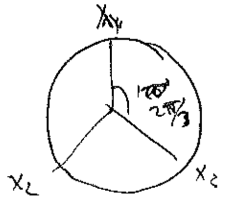
Let's take a second now to explore how the SDP can be visualized.

Let $G = K_3 =$  . Obviously, the $MAXCUT(K_3) = 2$. Consider the vector definition of SDP. For vectors $x_1, x_2, x_3 \in \mathbb{R}^3$ with norm 1, $x_i \cdot x_j = \cos(\theta_{ij})$ where θ_{ij} = the angle between them.

Goal is then to maximize $-\cos(\theta_{ij})$ or minimize $\cos(\theta_{ij})$.

i.e. all as close to 180° apart.



Choose, 

. Then $\cos(\theta_{ij}) = -\frac{1}{2}$. So,

$$SDP(G) \geq 3\left(\frac{1-(-\frac{1}{2})}{2}\right) = \frac{9}{4}.$$

So $\frac{SDP(K_3)}{MAXCUT(K_3)} \geq \frac{\frac{9}{4}}{2} = \frac{9}{8}$. i.e. we shouldn't hope for a multiplicative guarantee too strong.

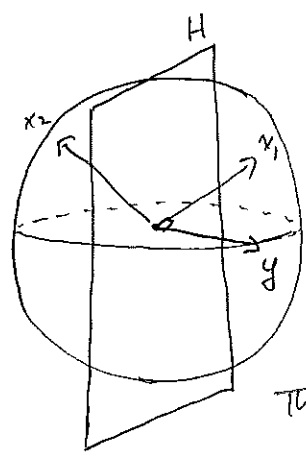
The Rounding Procedure:

Starting from an optimal sol. of (x_i), select a uniformly random $y \in \mathbb{R}^n$ and apply the rounding procedure

$$x_i \longmapsto \operatorname{sign}(x_i \cdot y) := \varepsilon_i.$$

Unlike other problems that are NP-hard, any assignment of ± 1 to the vertices is a valid assignment. But, we just now need to ensure that the cut produced is close to optimal.

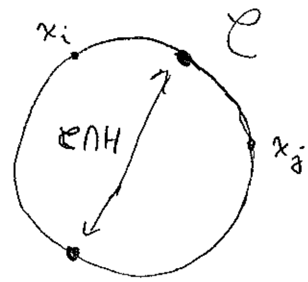
<u>Prop</u>. For any i,j, $\Pr_y(\varepsilon_i \neq \varepsilon_j) = \frac{\arccos(x_i \cdot x_j)}{\pi}$.

<u>Pf</u>. 

Let H be the half plane $\perp$ to y.

Splits the unit ball S^{n-1} into two halves, where $x \cdot y > 0$ in one and $x \cdot y < 0$ in the other.

Consider $\mathcal{C}$ circle containing both x_i and x_j on S^{n-1}.

Then 

Then $H \cap \mathcal{C}$ is two antipodal pts.

If one of the antipodal pts. is on the arc $\widehat{x_i x_j}$, then they are in different halves.

That occurs with probability $\frac{\Theta}{\pi}$ where Θ is the angle between them.

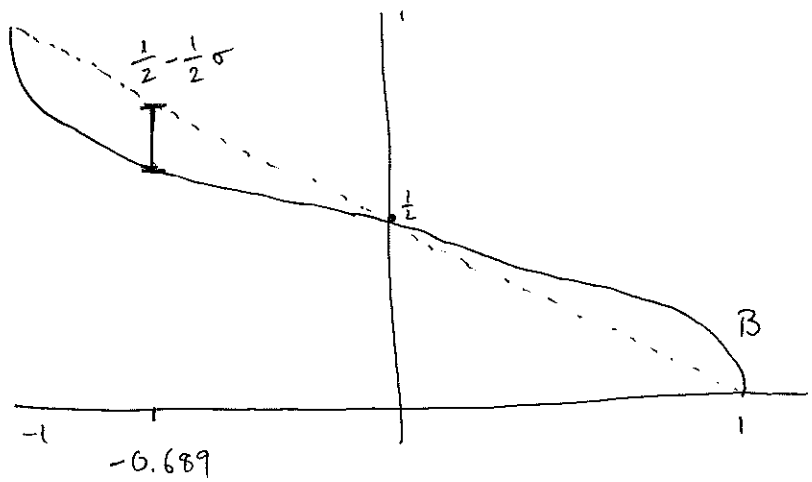
As $x_i \cdot x_j = \|x_i\| \|x_j\| \cos\Theta \implies \Theta = \arccos(x_i \cdot x_j)$.

$$\text{So } \Pr_y(\varepsilon_i \neq \varepsilon_j) = \frac{\arccos(x_i \cdot x_j)}{\pi}.$$

Let $B : [-1,1] \longrightarrow [0,1]$ be the fn. $B(\sigma) = \frac{\arccos(\sigma)}{\pi}$.

$$\mathbb{E}(GW(G)) = \sum_{i,j} A_{ij} \frac{\arccos(x_i \cdot x_j)}{\pi} \quad \Big| \quad SDP(G) = \sum_{i,j} A_{ij}\left(\frac{1}{2} - \frac{1}{2} x_i \cdot x_j\right)$$

How much worse is the rounding than the SDP?



We can verify the largest gap occurs at approximately -0.689, and the ratio is approx. 0.879.

Therefore, we know that

$$GW(G) \geq 0.879\ SDP(G).$$

Recall that $SDP(G) \geq MAXCUT(G)$. And as GW defines a cut, then

$MAXCUT(G) \geq GW(G)$. $\implies$

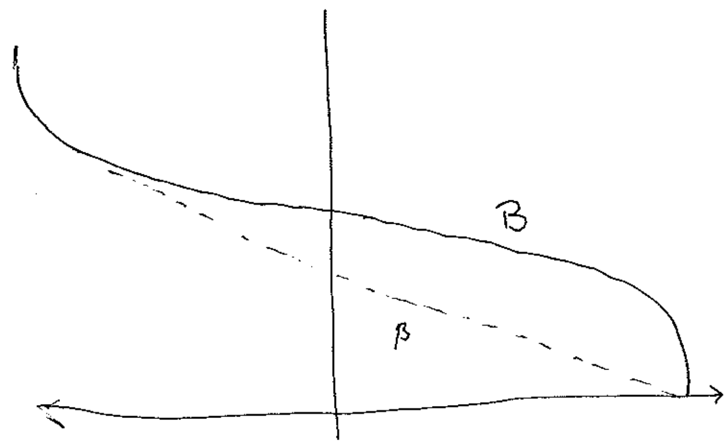
$$0.879\ SDP(G) \leq GW(G) \leq MAXCUT(G) \leq SDP(G).$$

So, GW also yields a 0.879 approx. of MAXCUT.

* The $0.879 - \varepsilon$ approx. result comes from the inability to calculate a $y \in \mathbb{R}^n$ perfectly randomly. The best strategy there is to choose $y \sim [\mathcal{N}(0,1)]^n$.

But, we're not done yet. The argument here was made for the worst case, but if the SDP is close to optimal, then we can actually make stronger arguments about the GW rounding.

Consider the largest convex-envelope of B, define it β. i.e. we need to choose the pt. c_1 s.t. from c_1 the tangent curve is always below the graph of B. To solve, find the c_1 s.t. the ~~tangent~~ tangent from c_1 hits $(1,0)$.



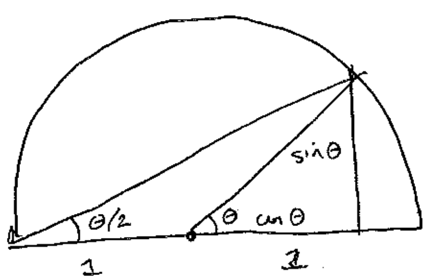
Solve: $0 = B(1) = B(c_1) + B'(c_1)(1 - c_1)$

$$0 = \frac{\arccos(c_1)}{\pi} - \frac{(1-c_1)}{\pi\sqrt{1-c_1^2}}.$$

i.e. $$\arccos(c_1) = \frac{1-c_1}{\sqrt{1-c_1^2}}.$$

Assume $c_1 = \cos\theta$ for some $\theta \in (0,\pi)$. Therefore,

$$\theta = \frac{1-\cos\theta}{\sin\theta} = \frac{(\sin\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} = \frac{\sin\theta}{1+\cos\theta} := \tan\frac{\theta}{2}.$$



so $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$.

So, $c_1 = \cos\theta$ and if we define $c_2 = \frac{2}{\pi\sin\theta}$ where $\theta = \tan\frac{\theta}{2}$. Then,

$$\beta(\sigma) = \begin{cases} B(\sigma) & \text{if } \sigma < c_1 \\ B(c_1) - \frac{c_2}{2}(\sigma - c_1) & \text{o.w.} \end{cases}$$

$c_1 \sim -0.689$

$c_2 \sim 0.879.$

Prop. If $SDP(G) \geq \delta m \Rightarrow \mathbb{E}(GW(G)) \geq \beta(1-2\delta)m$.

Pf. $SDP(G) \geq \delta m$ equiv. to

$$\frac{m}{2} - \sum_{(i,j)\in E} \frac{x_i \cdot x_j}{2} \geq \delta m \quad \text{where } (x_i) \text{ is optimal. SDP sol.}$$

$$\Rightarrow \sum_{(i,j)\in E} x_i \cdot x_j \leq (1-2\delta)m.$$

We have $Pr_y(\varepsilon_i \neq \varepsilon_j) = B(x_i \cdot x_j) \geq \beta(x_i \cdot x_j)$. As β is convex,
$\sum \beta(y_i) \geq \beta\left(\sum y_i\right)$. And as β is a negative fn $\beta(y) \geq \beta(x)$ for $y < x$.

$$\Rightarrow \mathbb{E}(GW(G)) = \sum_{(i,j)\in E} Pr_y(\varepsilon_i \neq \varepsilon_2)$$
$$\geq \sum_{(i,j)\in E} \beta(x_i \cdot x_j)$$
$$\geq \beta\left(\sum_{(i,j)\in E} x_i \cdot x_j\right) \quad \text{convexity}$$
$$\geq \beta((1-2\delta)m) \quad \text{negativity}$$
$$\geq \beta(1-2\delta)m \quad \text{convexity}$$

Now lets assume $SDP(G) \geq (1-\varepsilon)m$ for small $\varepsilon > 0$. Then, we will see that
$$\mathbb{E}(GW(G)) \geq \left(1 - \frac{2}{\pi}\sqrt{\varepsilon} - o(\sqrt{\varepsilon})\right)m.$$

Pf. By previous, $\mathbb{E}(GW(G)) \geq \beta(1-2\delta)m = \beta(-1+2\varepsilon)m$ where $1-\varepsilon = \delta$.

Then for $0 < \varepsilon \leq \frac{1+c_1}{2}$, $\beta(-1+2\varepsilon) = B(-1+2\varepsilon) = \dfrac{\arccos(-1+2\varepsilon)}{\pi}$.

Now, consider angle γ s.t. $\cos(\gamma) = -1 + 2\varepsilon$. Then $\beta(-1+2\varepsilon) = \frac{\gamma}{\pi}$.

Or, $\cos(\gamma - \pi) = \cos(\pi - \gamma) = 1 - 2\varepsilon$. Note $\gamma \approx \pi$ so for $\varphi := \pi - \gamma$.

Small angle approximation, $1 - 2\varepsilon = 1 - \frac{\varphi^2}{2} \pm O(\varphi^4)$.

A little math and, $\varphi = 2\sqrt{\varepsilon} + o(\sqrt{\varepsilon})$.

So, $\gamma = \pi - 2\sqrt{\varepsilon} - o(\sqrt{\varepsilon})$. Thus,

$$\mathbb{E}(GW(G)) \geq \frac{\pi - 2\sqrt{\varepsilon} - o(\sqrt{\varepsilon})}{\pi} m = \left(1 - \frac{2}{\pi}\sqrt{\varepsilon} - o(\sqrt{\varepsilon})\right) m.$$