Observing energy growth in Pseudo-Integrable Systems using smoothly oscillating dynamical Billiards

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Abstract

The motion of a mass-less particle in two types of Billiard tables was studied numerically. The first table was rectangular with a vertically oscillating bar and the second table was Stadium shaped. An ensemble of 100 systems was considered for each case. The simulations were run for 500,000 collisions, and changes in ensemble average energy with time were observed for both the Billiards tables. The particles in the slitted-rectangle experienced an exponential energy growth, and the particles in the Stadium Billiards experienced a series of energy gains and losses; the gains being higher than the losses, the energy gain is positive.

1 Introduction

Dynamical Billiards are simulations of a particle moving inside an enclosure and colliding elastically with its rigid walls. These simulations are used for a variety of systems in Physics like Classical, Statistical [10] and Quantum [30] mechanics [13]. Static Billiards have no moving parts for the particle to collide with. These billiards cannot change the energy of the particle. Billiards with oscillating parts are also called breathing Billiards.

1.1 Static Billiards

Static Billiards have stationary boundaries. They can be broadly classified according to the shapes of their boundaries. The boundary of a billiards may consist of Dispersing(convex inward), Focusing(convex outward) or Neutral(Flat) components.

1.1.1 Dispersing Billiards

These systems of billiards have chaotic properties. Dispersing boundaries cause the divergence of trajectories. Repeated reflections with the boundaries cause the trajectories to separate exponentially. The Sinai Billiard has a circular disc inside it, which provides the dispersing surface. The Sinai Billiards is used to model the Boltzmann gas consisting of elastically colliding hard balls in a box [10]. The Sinai Billiard also shows chaotic properties like ergodicity.

1.1.2 Focusing Billiards

Billiards with focusing boundaries can give both regular and chaotic dynamics depending on the exact shape of the Billiard table. Elliptical or Circular shaped Billiard tables are examples of focusing Billiards which give rise to integrable billiards. These types of billiard tables may be solved analytically by a coordinate transformation into the action-angle variables to give conserved quantities. Integrable systems have an invariant manifold which is diffeomorphic to a torus in their phase space. [6] Stadium shaped Billiard tables are not integrable because the path lengths of trajectories after converging are long enough to diverge after passing through a focal point. This is called the mechanism of defocusing. [1]

1.1.3 Polygonal Billiards

These type of systems can show a variety of different properties like being integrable(Rectangle and Equilateral Triangle), almost integrable [20] and pseudointegrable systems(square torus billiards) [26]. We are particularly interested in rational polygons. These types of billiards are particularly interesting because such systems can show chaotic properties which arise from the splitting of trajectories at the vertices of the polygon and not from divergence of nearby initial conditions. Pseudointegrable systems do not form an invariant torus in their phase space despite the presence of conserved quantities. [26] show that the square torus billiards forms a 5-handled sphere in its phase space.

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1.1.4 Other types of Billiards

There are several different kinds Billiards like the Mushroom Billiard which has coexistence of regular and chaotic dynamics. This arises due to presence of integrable trajectories that never leave the semi-circular part of the table [2].

1.2 Fermi Acceleration and Fermi-Ulam Model

Now, lets turn our attention to Billiards with moving parts or Breathing Billiards. Our study of Breathing Billiards is centered around Fermi Acceleration. It was first discovered by Enrico Fermi [8] in the acceleration of cosmic wave particle due to magnetic fields [24]. Ulam was the first to model this phenomena with a particle bouncing back and forth two walls, out of which one is oscillating slowly in comparison to the particle velocity [27]. The Fermi-Ulam model has been studied in detail. However, Fermi acceleration was only observed for non-periodic oscillations of the wall which have a stochastic component in its motion [14] [3]. Periodic oscillations can give energy growth upon in introduction of a potentials [22] [7] or relativistic correction [23].

1.3 Breathing Billiards and LRA conjecture

It was then found that 2D breathing Billiards show Fermi Acceleration for slow, periodic oscillations of the boundary [28] [11]. Fermi Acceleration was mostly found in Billiarrds which are Chaotic when frozen [18] [19]. No unbounded energy growth was observed in systems which are Integrable when frozen [21] [15]. . Similar results were obtained for the oscillating stadium-like [17] anad annular billiard systems [25].This was hypothesized by Loskutov et al. as 'A chaotic dynamics for the static billiard is a sufficient condition for the presence of Fermi acceleration in the billiard with time-dependent boundary'. This is also known as the LRA conjecture. This was proven by V. Gelfreich et al in 2002 [31]. More recently, Fermi Acceleration was found in breathing elliptically driven billiards by Lenz. et. al., which violates the LRA conjecture [9].

1.4 Energy growth in Breathing Billiards

Given unbounded energy growth, we are also interested in the rate of this growth. Smoothly oscillating chaotic billiards have shown quadratic in time energy growth [4] [12]. Exponential energy growth was observed in pseudointegrable Billiards. It was also proven that pseudointegribility is a sufficient condition for unbounded energy [16] [29]. Even Integrable systems show slow energy growth due to Arnold diffusion [5].

In this work, we compare a pseudointegrable and a chaotic dynamical Billiard. For the pseudointegrable billiard, we choose the Slitted rectangle Billiard table and for the chaotic one, we choose the breathing Stadium Billiard table. Our obejective is to replicate the results reported in [16] and [29]

2 Methodology

2.1 Slitted rectangle Billiard

This system consists of a particle moving inside a rectangular enclosure and elastically colliding with its boundaries. There is a horizontal slit in the center of the rectangle, which oscillates vertically with a fixed frequency ω.

The particle starts with an initial velocity $u_0 = 4\lambda\omega/\sqrt{5}$ and $v_0 = 41u_0$. Values for the other parameters in Figure 1 are $\lambda = 1$, $L = 2$, $h = 1$ and $s = 0.1$. The slit oscillates with $f(t) = \cos\frac{2\pi t}{70}$. The state variables of the system were recorded at every collision. The total energy of an ensemble of 100 systems over 500,000 collisions was plotted w.r.t. time. The initial positions of the particles were randomly sampled from the uniform distribution.

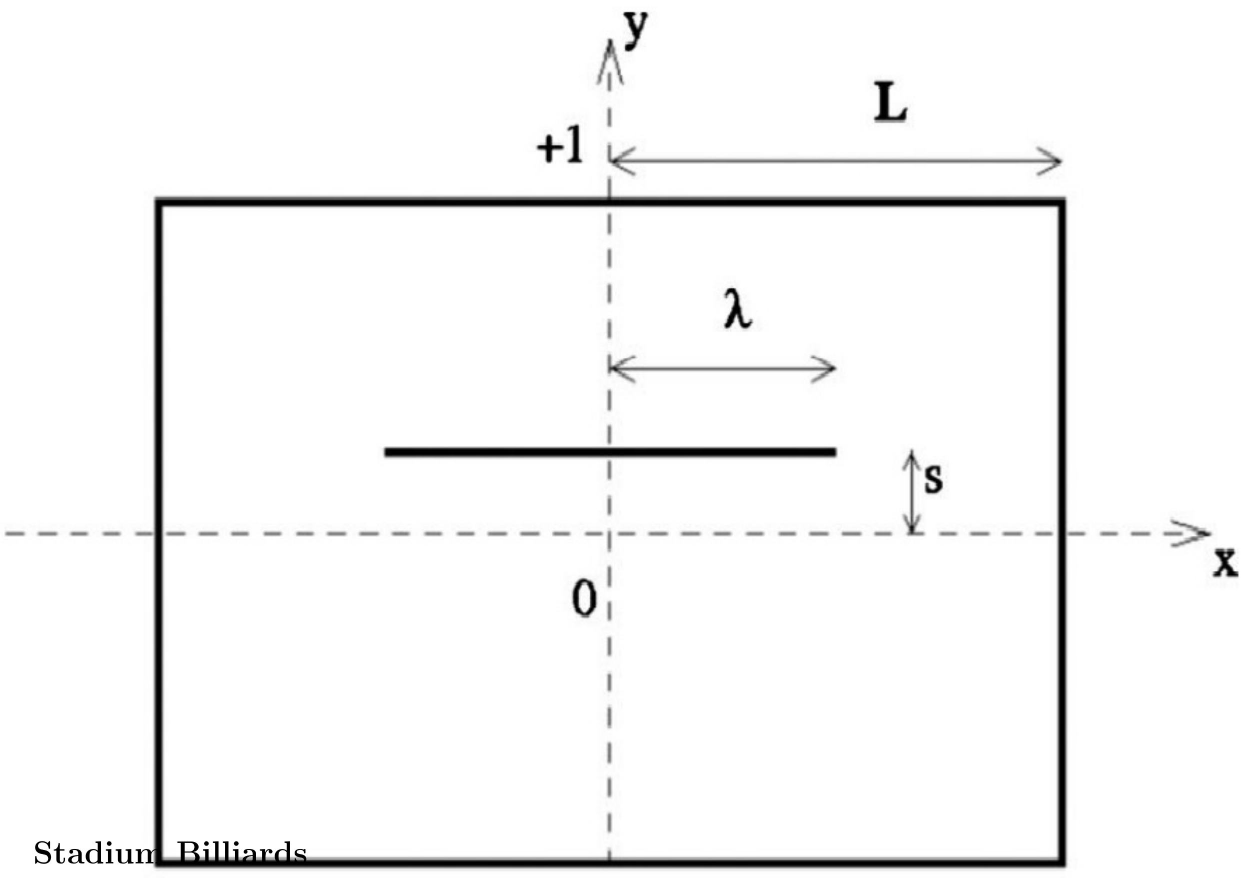

Figure 1: Geometry of the slitted rectangle Billiards

2.2 Stadium Billiards

This system consists of a particle moving inside a stadium shape (Figure 2) enclosure and elastically colliding with its boundaries. The bottom wall oscillates vertically with a fixed frequency ω

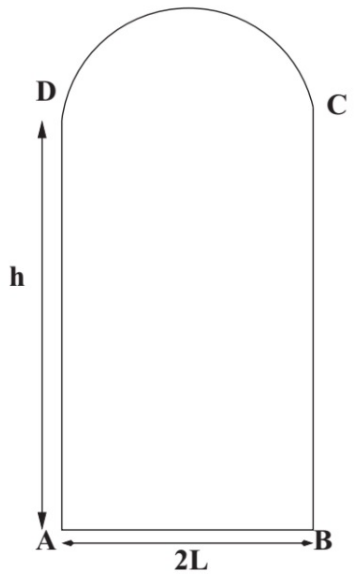

Figure 2: Geometry of the slitted rectangle Billiards

The particle starts with an initial velocity $u_0 = 4\lambda\omega/\sqrt{5}$ and $v_0 = 41u_0$. Values for the other parameters in Figure 2 are $L = 0.5$, $h = 4$ and $s = 0.1$. The slit oscillates with $f(t) = \cos\frac{2\pi t}{70}$. The state variables of the system were recorded at every collision. The total energy of an ensemble of 100 systems over 500,000 collisions was plotted w.r.t. time. The initial positions of the particles were randomly sampled from the uniform distribution.

3 Results and Discussions

3.1 Slitted rectangle Billiard

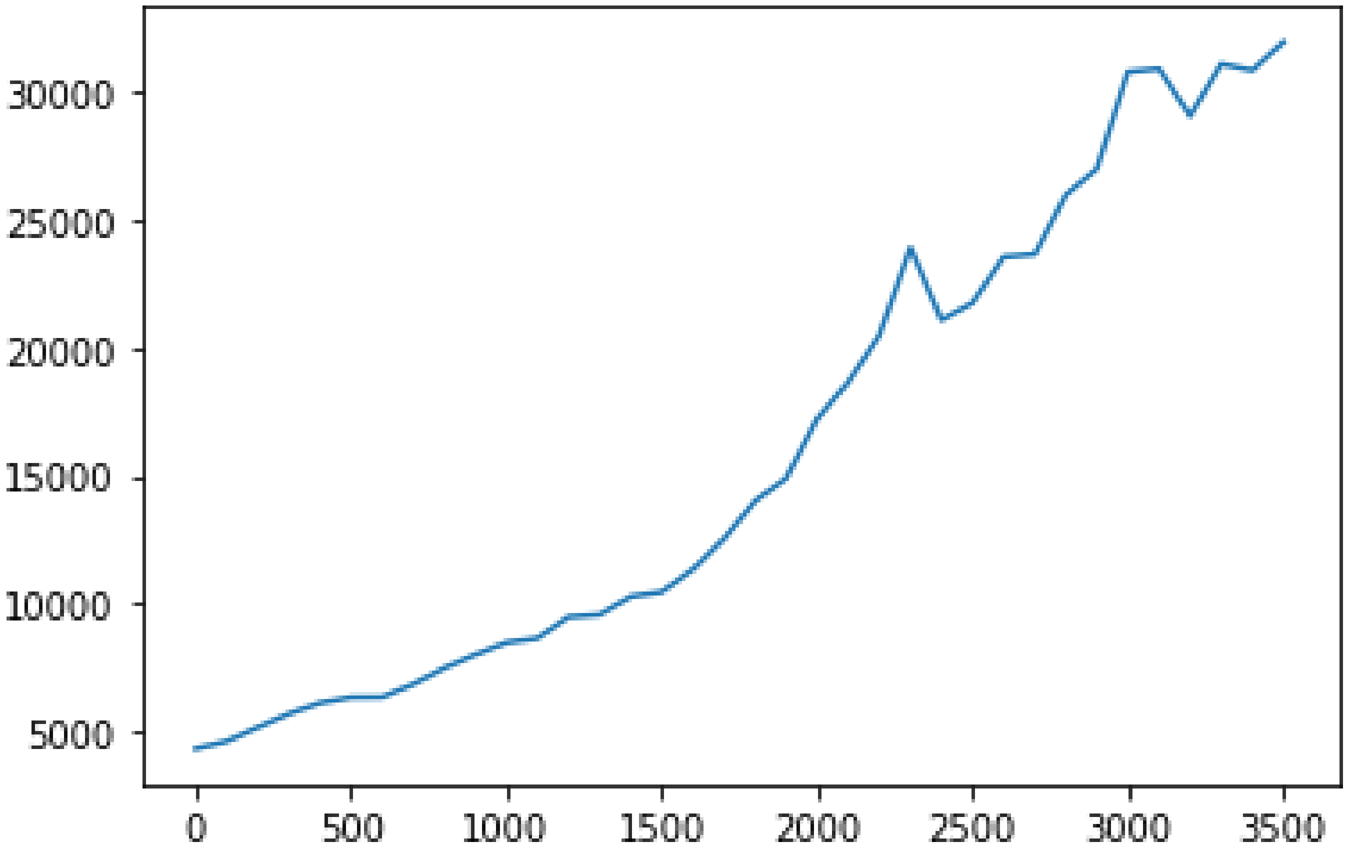

Figure 3: Total Energy of the slitted rectangle ensemble changing with time

Figure 3 shows the energy growth for an ensemble of 100 systems over 500,000 collisions. We can observe an exponential energy growth over time.

3.2 Stadium Billiards

Figure 4 shows the energy growth for an ensemble of 100 systems over 500,000 collisions. We can see growth in energy but cannot find any trend followed by the ensemble energies.

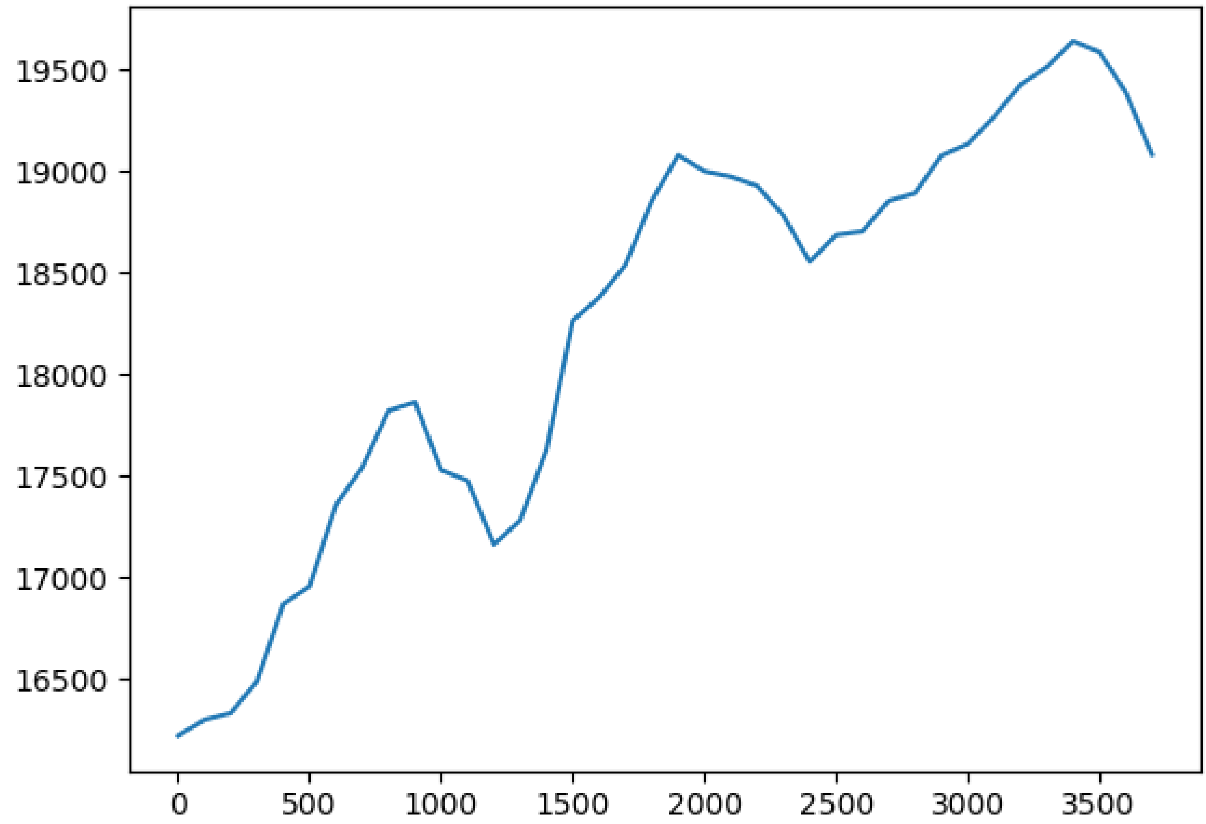

Figure 4: Total Energy of the stadium ensemble changing with time

We observe almost exponential energy growth in the slitted rectangular Billiards. For the Stadium Billiards, we observe a series of energy gains and losses. The net energy gain is positive.

4 Future plan

Following is a brief road map for further work in this project.

- Analyze the data obtained to fit a curve and find growth rates for these ensembles.
- Obtain results for the Trapezium Billiards ensemble for comparison with the Stadium Billiards.
- Vary other parameters like the amplitude and frequency of the slit oscillations, the shape of the enclosure.
- Introducing a small mass for the particle and making the collisions inelastic.
- Read the paper ”Robust Exponential Acceleration in Time-Dependent Billiards”

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