Chapter 5 - Convergence

Q1: $X_1, X_2, \dots, X_n$ IID, $E(X_1)=\mu$ $V(X_1)=\sigma^2$

$\bar{X}_n$ sample mean S_n sample var

a) show $E(S_n^2)=\sigma^2$

$$E(S_n^2)=\frac{1}{n-1}E\Big(\sum x_i^2+\sum\bar{x}^2-2\underbrace{\sum X_i\bar{X}}_{n\bar{x}^2}\Big)$$

$$=\frac{n}{n-1}\left(E(x_i^2)-E(\bar{x}^2)\right)$$

$$E(x_i^2)=\sigma^2+\mu^2 \qquad E(\bar{x}^2)=\frac{\sigma^2}{n}+\mu^2$$

$$\Rightarrow E(S_n^2)=\frac{n}{n-1}\left(\sigma^2+\mu^2-\frac{\sigma^2}{n}-\mu^2\right)=\sigma^2$$

b) show $S_n^2 \xrightarrow{P} \sigma^2$

hints: - show $S_n^2 = c_n n^{-1}\sum_{i=1}^{n} X_i^2 - d_n \bar{X}_n^2$

$c_n \to 1$, $d_n \to 1$

- apply LLN to $n^{-1}\sum X_i^2$ and to $\bar{X}_n$

- use theorem $\begin{cases} X_n \rightsquigarrow X \\ Y_n \rightsquigarrow c \end{cases} \Rightarrow X_n Y_n \rightsquigarrow cX$

$$S_n^2 = \frac{1}{n-1}\sum (X_i-\bar{X})^2 = \underbrace{\frac{n}{n-1}}_{c_n}\frac{1}{n}\sum X_i^2 - \underbrace{\frac{n}{n-1}}_{d_n}\bar{X}^2$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 1$$

LLN: $\bar{X}_n \xrightarrow{P} \mu \Rightarrow \bar{X}_n^2 \xrightarrow{P} \mu^2$

$\Rightarrow d_n \bar{X}_n^2 \xrightarrow{P} \mu^2$

$\bar{Y}_n = \frac{1}{n}\sum X_i^2 \xrightarrow{P} E(X_i^2) \Rightarrow Y_n \xrightarrow{P} \sigma^2+\mu^2$

$\Rightarrow c_n Y_n \xrightarrow{P} \sigma^2+\mu^2$

$S_n^2 = c_n Y_n - d_n \bar{X}_n^2$

$\Rightarrow S_n^2 \xrightarrow{P} \sigma^2+\mu^2-\mu^2$

$\Rightarrow \boxed{S_n^2 \xrightarrow{P} \sigma^2}$

Q2: $X_1, X_2, \ldots$ seq of r.v.

show $X_n \xrightarrow{qm} b \iff \begin{cases} \lim_{n\to\infty} E(X_n) = b \\ \lim_{n\to\infty} V(X_n) = 0 \end{cases}$

Case 1:

$$X_n \xrightarrow{qm} b \Rightarrow X_n \xrightarrow{P} b \Rightarrow X_n \rightsquigarrow b$$

$$\Rightarrow \lim_{n\to\infty} E(X_n) = b \quad \& \quad \lim_{n\to\infty} Var(X_n) = 0$$

Case 2:

$$\lim_{n\to\infty} E(X_n - b)^2 = \lim \left[E(X_n^2) - 2b\,E(X_n) + b^2 \right]$$

$$= \lim \left[V(X_n) + E(X_n)^2 - 2b\,E(X_n) + b^2 \right]$$

$$= 0 + b^2 - 2b^2 + b^2 = 0$$

$$\Rightarrow X_n \xrightarrow{qm} b$$

Q3: $X_1, \dots, X_n$ IID, $\mu = E(X_1)$, $V(X_1)$ finite

Show $\bar{X}_n \xrightarrow{qm} \mu$

$$E(\bar{X}_n - \mu)^2 = E(\bar{X}_n^2 + \mu^2 - 2\mu\bar{X})$$

$$= E(\bar{X}_n^2) - \mu^2 = \frac{\sigma^2}{n} + \cancel{\mu^2} - \cancel{\mu^2}$$

$$\Rightarrow E(\bar{X}_n - \mu)^2 \to 0 \Rightarrow \bar{X}_n \xrightarrow{qm} \mu$$

Q4: $X_1, \dots, X_n$ r.v. such that

$P(X_n = 1/n) = 1 - 1/n^2 \qquad P(X_n = n) = 1/n^2$

Conv. in prob? Conv in qm?

$X_n \xrightarrow{P} 0$? $\qquad P(|X_n| > \varepsilon) = P(X_n = n) = 1/n^2 \to 0$

$\Rightarrow$ Converges in prob.

$X_n \xrightarrow{qm} 0$? $\qquad E(X_n^2) = \frac{1}{n^2}\left(1 - \frac{1}{n^2}\right) + n^2 \frac{1}{n^2} \to 1$

$\Rightarrow$ does not conv. in qm

Q5: $X_1, \dots, X_n \sim$ Bernoulli(p)

prove $\frac{1}{n} \sum_{i=1}^{n} X_i^2 \xrightarrow{P} p$

prove $\frac{1}{n} \sum X_i^2 \xrightarrow{qm} p$

only need to show qm, if qm then it conv. in prob. as well

$Y = \frac{1}{n} \sum X_i^2 - p$

prove that $E(Y^2) \to 0$ equals to $\frac{1}{n} \sum X_i^2 \xrightarrow{qm} p$

$$E(y^2) = V(y) + (Ey)^2$$

$$Ey = \frac{1}{n} n E(X_i^2) - P = P - P = 0$$

$$V(y) = \frac{n}{n^2} V(X_i^2) = \frac{P(1-P)}{n}$$

$$\Rightarrow E(y^2) = P(1-P)/n \Rightarrow E(y^2) \xrightarrow[n \to \infty]{} 0$$

Q6: $\mu = 68 \quad \sigma = 2.6 \quad n = 100$

$$P(\bar{X}_n > 68) = ?$$

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \Rightarrow P(\bar{X} > \mu) = P(Z > 0) = 0.5$$

$$Z \sim N(0,1)$$

Q7: $\lambda_n = 1/n \quad n = 1, 2, \dots \quad X_n \sim \text{Poisson}(\lambda_n)$

a) Show $X_n \xrightarrow{P} 0$

b) $Y_n = n X_n$, show $Y_n \xrightarrow{P} 0$

a) $E(X_n) = \lambda_n \quad V(X_n) = \lambda_n$

$$P(|X_n - 0| > \varepsilon) \leq \frac{V(X_n)}{\varepsilon^2} \quad \text{chebyshev}$$

$$\Rightarrow P(|X_n| > \varepsilon) \leq 1/n\varepsilon^2 \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\Rightarrow X_n \xrightarrow{P} 0$$

b) $f(x) = e^{-\lambda} \dfrac{\lambda^x}{x!} \quad x = 0, 1, \cdots, \infty$

$Y_n = nX_n = 0, n, 2n, \cdots, \infty$

$$\Rightarrow P(|Y_n| > \varepsilon) = 1 - P(Y_n = 0)$$

$$P(Y_n = 0) = P(X_n = 0) = e^{-\lambda_n} = e^{-1/n} \xrightarrow[n \to \infty]{} 1$$

$$\Rightarrow P(|Y_n| > \varepsilon) \rightarrow 0 \Rightarrow Y_n \xrightarrow{P} 0$$

Q8: $X_i \sim \text{Poisson}(\lambda = 1)$

$Y = \sum_{i=1}^{n} X_i \quad , \quad n = 100$

$P(Y < 90) = ?$

$\mu = 1$, $\sigma^2 = 1$, $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \rightsquigarrow Z \sim N(0,1)$

$y = n\bar{X} \Rightarrow P(y < 90) = P(\bar{X} < 0.9)$

$= P\left(\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \dfrac{0.9 - 1}{0.1}\right) = P(Z < -1)$

$= \Phi(-1) = 0.1587$

Q9: $P(X = 1) = P(X = -1) = 0.5$

$$X_n = \begin{cases} X & \text{with prob } 1 - 1/n \\ e^n & \text{with prob } 1/n \end{cases}$$

$X_n \underset{?}{\xrightarrow{P}} X$ $\quad X_n \underset{?}{\xrightarrow{qm}} X$ $\quad X_n \underset{?}{\rightsquigarrow} X$

$P(|X_n - X| > \varepsilon) = P(X_n = e^n) = 1/n \to 0$

$\Rightarrow \quad X_n \xrightarrow{P} X$

$\Rightarrow \quad X_n \rightsquigarrow X$

$$E[(X_n - X)^2] = 0 \times (1 - \tfrac{1}{n})$$

$$+ \tfrac{1}{n}\left(\tfrac{1}{2}(e^n+1)^2 + \tfrac{1}{2}(e^n-1)^2\right)$$

$$= \tfrac{1}{2n}\left(e^{2n} + 1 + \cancel{2e^n} + e^{2n} + 1 - \cancel{2e^n}\right)$$

$$= \frac{e^{2n}+1}{n} \to \infty$$

$\Rightarrow$ does not conv. in qm.

Q10: $Z \sim N(0,1)$ $t > 0$ $k > 0$

Show $P(|Z| > t) \leq \dfrac{E|Z|^k}{t^k}$

Compare to Mill's inequality

$$E|Z|^k = \int_{-\infty}^{\infty} |z|^k f(z)\,dz = \int_{-\infty}^{-t} \cdots + \int_{-t}^{t} \cdots + \int_{t}^{\infty} \geq \int_{-\infty}^{-t} + \int_{t}^{\infty}$$

$$\Rightarrow E|Z|^k \geq \int_{-\infty}^{-t} |z|^k f(z)\,dz + \int_{t}^{\infty} |z|^k f(z)\,dz$$

$$\geq t^k \left(\int_{-\infty}^{-t} f(z)\,dz + \int_{t}^{\infty} f(z)\,dz \right) = t^k P(|Z| > t)$$

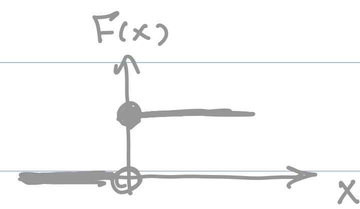
$$\Rightarrow E|Z|^k \geq t^k P(|Z| > t)$$

$$\Rightarrow P(|Z|>t) \leqslant \frac{E|Z|^k}{t^k}$$

Mill's: $$P(|Z|>t) \leqslant \sqrt{\frac{2}{\pi}}\, \frac{e^{-t^2/2}}{t}$$

Q11: $X_n \sim N(0, 1/n)$

$$X \sim F(x) = \begin{cases} 0 & x<0 \\ 1 & x \geqslant 0 \end{cases}$$



$P(X=0)=1$

$$X_n \xrightarrow[?]{P} X \qquad X_n \underset{?}{\rightsquigarrow} X$$

$$P(|X_n|>\varepsilon) \leqslant \frac{\sigma^2}{\varepsilon^2} = \frac{1}{n\varepsilon^2} \to 0$$

$$\Rightarrow X_n \xrightarrow{P} 0 \Rightarrow X_n \rightsquigarrow 0$$

Q12: $X_1, \dots, X_n$ r.v. positive integers

Show $X_n \rightsquigarrow X \Leftrightarrow \lim_{n\to\infty} P(X_n=k)=P(X=k)$ $\forall k$

part1: $\lim_{n\to\infty} P(X_n = k) = P(X=k) \quad \forall k$

$\Rightarrow \lim_{n\to\infty} F_n(t) = F(t) \Rightarrow X_n \rightsquigarrow X$

part2: $X_n \rightsquigarrow X$

$\Rightarrow \lim_{n\to\infty} F_n(t) = F(t) \quad \forall t$ which F in continous

$\Rightarrow$ Since $X_1, \dots, X_n$ are integer r.v.

$\Rightarrow \lim_{n\to\infty} P(X_n = k) = P(X=k) \quad \forall k$

Q13: $Z_1, \dots, Z_n$ IID r.v. density f

$P(Z_i > 0) = 1 \qquad \lambda = \lim_{x \downarrow 0} f(x) > 0$

$X_n = n \min\{Z_1, \dots, Z_n\}$

Show $X_n \rightsquigarrow Z$

where Z exp. dist. with mean $1/\lambda$

$$f(z) = Exp(\tfrac{1}{\lambda}) = \lambda e^{-\lambda z}$$

$$F(z) = \int_0^z \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^z = 1 - e^{-\lambda z}$$

now must prove $\lim\limits_{n\to\infty} F_n(x) = 1 - e^{-\lambda x}$?

$$P(X_n \leq x) = 1 - P(X_n > x)$$

$$= 1 - P(\min\{Z_1, \dots, Z_n\} > x/n)$$

$$= 1 - P(Z_1 > x/n)\,P(Z_2 > x/n) \cdots P(Z_n > x/n)$$

$$= 1 - P(Z_1 > x/n)^n$$

$$\underset{n\to\infty}{P}(Z_1 > x/n) = \underset{n\to\infty}{1 - P}(Z_1 < x/n) = 1 - \int_0^{x/n} \lambda\,dx = 1 - \lambda \frac{x}{n}$$

$$\Rightarrow \underset{n\to\infty}{P}(X_n \leq x) = \lim_{n\to\infty} 1 - \underbrace{\left(1 - \lambda \frac{x}{n}\right)^n}_{e^{-\lambda x} \text{ as } n\to\infty}$$

$$\Rightarrow \lim_{n\to\infty} F_n(x) = 1 - e^{-\lambda x}$$

Q14: $X_1, \dots, X_n \sim \text{Unif}(0,1)$, $Y_n = \bar{X}_n^2$

find limiting dist. of Y_n

$\bar{X}_n \approx N(\mu, \frac{\sigma^2}{n})$

using delta method $g(x) = x^2$

$\Rightarrow g(\bar{X}_n) = \bar{X}_n^2 \approx N\left(g(\mu), (g'(\mu))^2 \frac{\sigma^2}{n}\right)$

$Y_n \approx N(\mu^2, 4\mu^2\sigma^2/n)$

Q15: $\begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix} \dots \begin{pmatrix} X_{1n} \\ X_{2n} \end{pmatrix}$ IID random vector

$\mu = (\mu_1, \mu_2)$

$\bar{X}_1 = \frac{1}{n} \sum X_{1i}$ $\quad$ $\text{Var}\ \Sigma$

$\bar{X}_2 = \frac{1}{n} \sum X_{2i}$

$Y_n = \bar{X}_1 / \bar{X}_2$ $\quad$ $Y_n \rightsquigarrow ?$

$Y_n = g(X)$, $\sqrt{n}\left(g(x) - g(\mu)\right) \rightsquigarrow N(0, \nabla_\mu^T \Sigma \nabla_\mu)$

$g(S_1, S_2) = \frac{S_1}{S_2} \Rightarrow \nabla g(S) = \begin{bmatrix} \partial g/\partial S_1 \\ \partial g/\partial S_2 \end{bmatrix} = \begin{bmatrix} 1/S_2 \\ -S_1/S_2^2 \end{bmatrix}$

$$\Rightarrow \nabla_\mu = \begin{bmatrix} \mu_2^{-1} \\ -\mu_1 \mu_2^{-2} \end{bmatrix}$$

$$\sqrt{n}\left(Y_n - \frac{\mu_1}{\mu_2}\right) \rightsquigarrow N\left(0, \nabla_\mu^T \Sigma \nabla_\mu\right)$$

Q16: give example $\begin{cases} X_n \rightsquigarrow X \\ Y_n \rightsquigarrow Y \end{cases} \not\Rightarrow X_n + Y_n \rightsquigarrow X + Y$

let $X_n \rightsquigarrow N(0,1)$

$Y_n = -X_n \rightsquigarrow N(0,1)$

$$\Rightarrow X_n + Y_n = 0 \neq N(0,1) + N(0,1)$$