Time Complexity Analysis of Extracting k Largest Numbers in a List Using Heaps

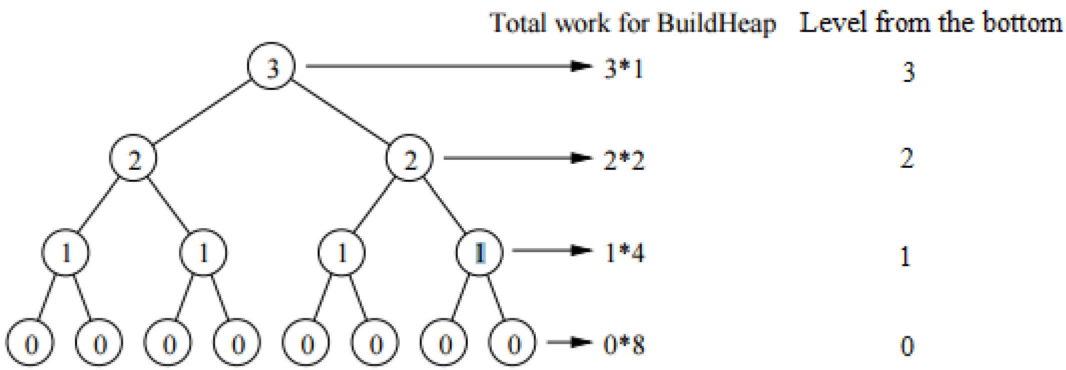
By Niruhan Viswarupan

A. Theoretical Evaluation of the Time Complexity of ExtractMaxK(NumberList, k)

1. Time Complexity Analysis of HeapBuildMaxHeap(A)

Let the number of elements in the input array = n

Assumption: $\exists h \in Z^+$ s.t $n = 2^{h+1} - 1$ i.e the heap is complete



Consider the case where we are at a node in the j^{th} level from the bottom of the heap. Assume that the children of the node are max heaps. The worst case time complexity to max-heapify the node is equal to the number of levels in the subtree rooted at the node if the current key has to sift down all the way to the bottom. That is equal to j.

Number of nodes at the j^{th} level = 2^{h-j}

Total time complexity to max heapify all the nodes at j^{th} level = $j2^{h-j}$

Counting from the bottom to the top of the heap,

Total time taken, $T(n) = \sum_{j=0}^{h} j2^{h-j}$

Now consider,

$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ $|x|<1$

Differentiate both sides w.r.t x,

$\sum_{j=0}^{\infty} jx^{j-1} = \frac{1}{(1-x)^2}$

Multiply by x both sides,

$\sum_{j=0}^{\infty} jx^{j} = \frac{x}{(1-x)^2}$

Substitute $x = \frac{1}{2}$

$\sum_{j=0}^{\infty} \frac{j}{2^j} = 2$

Now,

$T(n) = \sum_{j=0}^{h} j2^{h-j} = 2^h \sum_{j=0}^{h} \frac{j}{2^j} < 2^h \sum_{j=0}^{\infty} \frac{j}{2^j} = 2^h * 2 = 2^{h+1}$ = n+1 $\forall h \in Z^+$

T(n) < n+1

∴ T(n) ∈ O(n)

Also clearly T(n) ∈ Ω(n) because the algorithm has to access all the elements on the input list at least once.

∴T(n) ∈ θ(n)

2. Cost Modelling of HeapExtractMax(A)

Algorithm:

```
def HeapExtractMax(A):
    last_index = len(A)-1
    (A[1], A[last_index]) = (A[last_index], A[1])
    max_val = A.pop()
    HeapMaxHeapify(A, 1)
    return max_val
```

Line	Cost	Times
1	C1	1
2	C2	1
3	C3	1
4	C4	1
5	θ(logn)	1
6	C6	1

Time complexity of HeapExtractMax(A), T1(n) = C1*1 + C2*1 + C3*1 + C4*1 + θ(logn)*1 + C6*1

∴T1(n) ∈ θ(logn)

3. Cost Modelling of ExtractMaxK(NumberList, k)

Algorithm:

```
def ExtractMaxK (NumberList, k):
    output = []
    a = HeapBuildMaxHeap(NumberList)
    for i in range(k):
        output.append(HeapExtractMax(NumberList))
    return output
```

Line	Cost	Times
1	D1	1
2	D2	1
3	θ(n)	1
4	D4	k+1
5	D5+ θ(logn)	k
6	D6	1

Time complexity of ExtractMaxK(NumberList, k),
T2(n, k) = D1*1 + D2*1 + θ(n)*1 + D4*(k+1) + (D5+ θ(logn))*k + D6*1
Since n>>k
T2(n, k) ∈ θ(n)

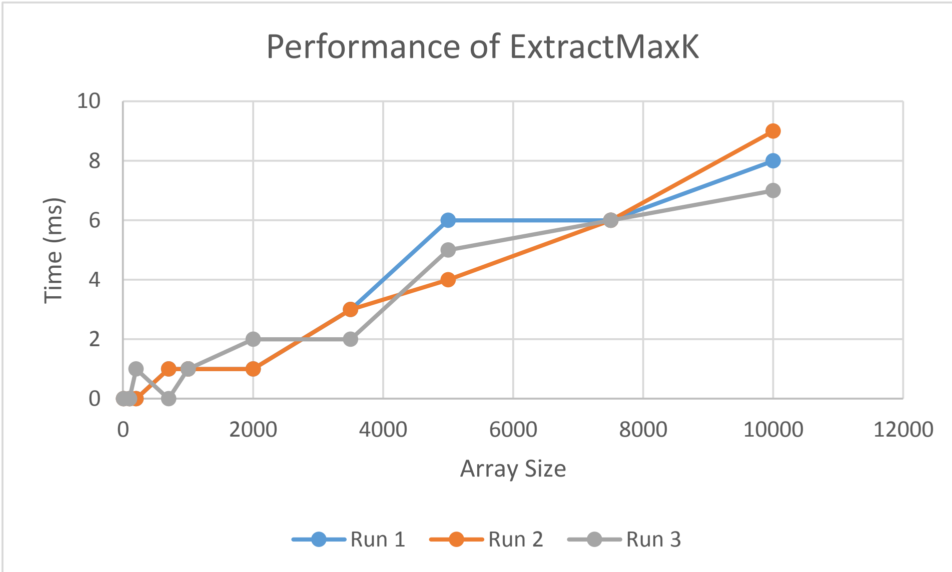
Note:
Here we assume that n>>k. At every iteration of the extract max from the number list the length of the list reduces by one. This happens k times. Since n>>k we neglect the reduction in length of the list during the k iterations.

B. Time Complexity Analysis of ExtractMaxK(NumberList, k) Code Written Using Input

Lists of random integers of the given array sizes were generated and times were measured.
Test Code:

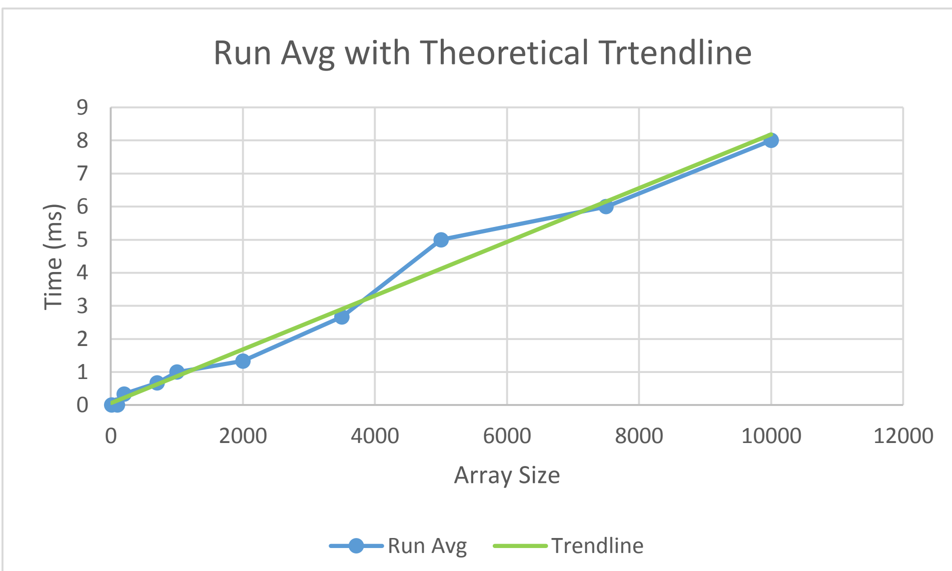
```
def RandListGen(n):
    return [random.randint(0, n) for i in range(n)]

test = [(RandListGen(10), 3), (RandListGen(100), 5), (RandListGen(200), 7), (RandListGen(700), 10),
(RandListGen(1000), 10),
        (RandListGen(2000), 15), (RandListGen(3500), 20), (RandListGen(5000), 30),
(RandListGen(7500), 35), (RandListGen(10000), 40)]

for element in test:
    start = time.time()
    ExtractMaxK(element[0], element[1])
    end = time.time();
    print (end-start)*1000
```

Array Size	**Time(ms)**			
	Run 1	Run 2	Run 3	Run Avg
10	0	0	0	0
100	0	0	0	0
200	0	0	0.999927521	0.333309174
700	1.000165939	1.000165939	0	0.666777293
1000	0.999927521	0.999927521	0.999927521	0.999927521
2000	0.999927521	0.999927521	2.00009346	1.333316167
3500	3.000020981	3.000020981	1.999855042	2.666632334
5000	6.000041962	3.999948502	5.000114441	5.000034968
7500	6.000041962	6.000041962	6.000041962	6.000041962
10000	7.999897003	9.000062943	6.999969482	7.999976476



One can see that the time taken for the algorithm increases somewhat linearly with the size of the input. So the time complexity of the ExtractMaxK algorithm can be stated as θ(n).

C. Comparison between Theoretical and Practical Observations of the Time Complexity



It is clear that the run averages follow closely the theoretical prediction of linear time complexity. The possible reasons for deviations are:

1. The distribution of numbers generated by the randint function. This will affect the time taken to build max heaps.
2. The assumptions made during the theoretical derivations
3. Effect of other processes running on the CPU. The time.time() function returns the wall clock time and not the CPU time spent on the process. So other processes running on the CPU could have taken place during the measurement.