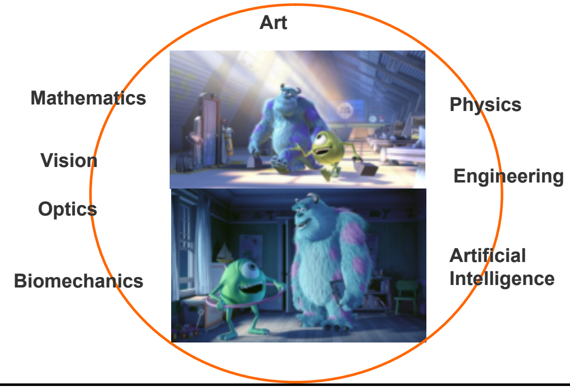
CS174A – Lecture 2

Linear Algebra:
The Algebra of Vectors and Matrices
(and Scalars)

Last Lecture: Computer Graphics

The Art and Science of creating imagery by computer



Why Study 3D Computer Graphics?

- CG has applications in many industries:
  - entertainment, scientific visualization, education
- Intellectually Challenging
  - Create and interact with realistic virtual world
  - Requires understanding of all aspects of physical world
  - New computing methods, displays, technologies
- Technically Challenging
  - Math of (perspective) projections, curves, surfaces
  - Physics of lighting and shading
  - 3D graphics software programming and hardware

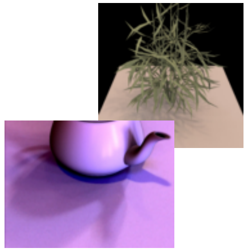
Course Goals

- **Systems:** Write complex 3D graphics programs (real-time scene in OpenGL, offline raytracer)
- **Theory:** Mathematical aspects and algorithms underlying modern 3D graphics systems
- This course is ***not*** about the specifics of 3D graphics programs and APIs like Maya, Alias, DirectX but about the concepts underlying them.

Course Outline

- 3D Graphics Pipeline

Modeling → Animation → Rendering

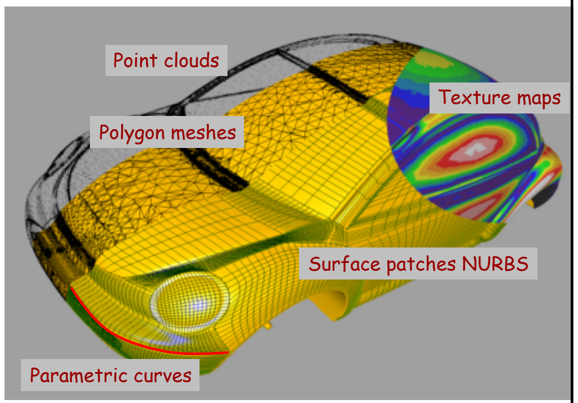


Sub-areas of CG

- Modeling
  - How do we model (mathematically represent) objects?
  - How do we construct models of specific objects?
- Animation
  - How do we represent the motions of objects?
  - How do we give animators control of this motion?
- Rendering
  - How do we simulate the real-world behavior of light?
  - How do we simulate the formation of images?
- Interaction
  - How do we enable humans and computers to interact?
  - How do we design human-computer interfaces?

Modeling

Representing objects geometrically on a computer



Modeling

Primitives
- 3D points
- 3D lines and curves
- surfaces (BREPs): polygons, patches
- volumetric representations
- image-based representations

Attributes
- Color, texture maps
- Lighting properties

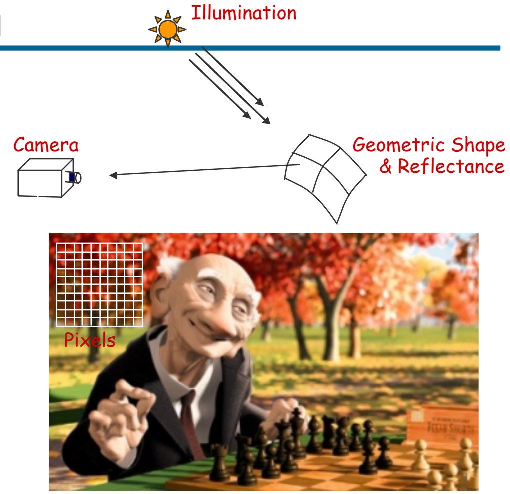
Geometric transformations

Animation

- Keyframe animation
- Motion capture
- Physics-based animation
- Autonomous motion planning

Rendering

Key elements



Linear Algebra Review: The Algebra of Vectors and Matrices (and Scalars)

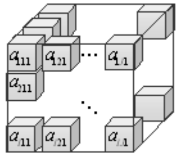
Motivation

- Many graphics concepts need basic math, such as linear algebra
  - Vectors (dot products, cross products,…)
  - Matrices (matrix-matrix, matrix-vector mult., …)
  - Operations like translation, or rotation of the points that form an object are most efficiently performed as a matrix-vector multiply
  - Thus, we will go out of our way to write everything as a matrix-vector multiply
- Chapters 2.4 (vectors) and 5.2 (matrices)
  - *Fundamentals of Computer Graphics* by Shirley and Marschner
  - Worthwhile to read all of chapters 2 and 5
- Should be refresher on very basic material for most of you

Lecture 2 - Outline

- Vectors,
  - Vector Basics
  - Vector Addition, Subtraction
  - Vector Multiplication
    - Dot product *(2.4.3)*
    - Cross product *(2.4.4)*
  - Constructing Coordinate Frames *(2.4.5,6)*
- Matrix *(5.2)*
  - Matrix Basics
  - Matrix Addition
  - Matrix Multiplication
  - Matrix Properties
  - Vector Multiplication in Matrix Form
- Points vs Vectors
- Lines and Planes

Notation

- Scalar a
- Vector $\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}$
- Matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1J} \\ a_{21} & & & \\ & & \ddots & \\ a_{I1} & a_{I2} & & a_{IJ} \end{bmatrix}$
- Tensor $\mathcal{A} =$ 

Vectors

N-tuple of scalar elements

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ \vdots \\ a_N \end{bmatrix} \qquad a_n \in \Re, \quad 1 \le n \le N$$

Vector:
Bold lower-case

Scalar:
Italic lower-case

- ***Length and direction. Absolute position is not important***
- ***Use to store offsets, displacements***
  - Note: Positions/locations are also represented by N-tuples, but strictly speaking, positions are not vectors and cannot be added.
    A location implicitly involves an origin, while an offset does not.

Column vs. Row Vectors

- vector: $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} \qquad \mathbf{a} = \underbrace{[\,a_1 \quad a_2 \quad \dots \quad a_n\,]^T}_{\text{Note: This is a column}}$
- row vector: $\mathbf{a}^T = [\,a_1 \quad a_2 \quad \dots \quad a_n\,]$

(denoted by a vector symbol with a transpose)

Vectors

N-tuple: $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n & \cdots & x_N \end{bmatrix}^T \quad x_n \in \Re$

Magnitude: $|\mathbf{x}| = \sqrt{x_1^2 \quad + \quad x_2^2 \quad \cdots \quad x_N^2}$

Unit vectors $\hat{\mathbf{x}}: \quad |\hat{\mathbf{x}}| = 1$

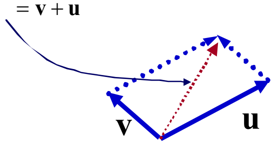
Normalizing a vector $\hat{\mathbf{x}} = \dfrac{\mathbf{x}}{|\mathbf{x}|}$

Vector-Vector Addition

- Parallelogram rule
  - tail to head, complete the triangle

geometric interpretation:
parallelogram rule

$\mathbf{u} + \mathbf{v} \; = \mathbf{v} + \mathbf{u}$



algebraic:
add coordinates

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

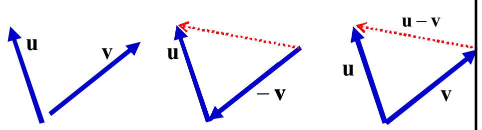
examples:

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \qquad \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} =$$

Vector-Vector Subtraction

- *Adding a negatively scaled vector*

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$$



$\mathbf{u} - \mathbf{v} =$
$\mathbf{u} + (-\mathbf{v})$

- *Example*

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \qquad \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} =$$

Operations with Vectors

- Addition $\quad \mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 & \cdots & x_n + y_n & \cdots & x_N + y_N \end{bmatrix}^T \quad 1 \le n \le N$
- Multiplication with scalar (scaling)

$$a\mathbf{x} = \begin{bmatrix} ax_1 & \cdots & ax_n & \cdots & ax_N \end{bmatrix}^T \quad a, x_n \in \Re$$

- Properties

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	commutative
$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	associative
$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}, \quad a \in \Re$	distributive
$\mathbf{u} - \mathbf{u} = 0$	inverse

Examples:

Linear Combination of Vectors

- Definition

A linear combination of the m vectors $\mathbf{v}_1,\ldots,\mathbf{v}_m$ is a vector of the form:

$\mathbf{w} = a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m$, $a_1,\ldots,a_m$ in R

Special Cases

- Linear combination
  - $\mathbf{w} = a_1\mathbf{v}_1 + \ldots a_m\mathbf{v}_m$, $a_1,\ldots,a_m$ in IR
- Affine combination:
  - A linear combination of vectors for which $a_1+\ldots+a_m = 1$
- Convex combination
  - An affine combination for which $a_i \geq 0$ for i = 1,…,m

Linear Independence

- Definition:
  - Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are called *linearly independent,* if $a_1\mathbf{v}_1+\ldots+a_m\mathbf{v}_m = 0$ *if and only if* $a_1 = a_2 = \ldots = a_m = 0$

Vector Multiplication

- ***Dot product***
- Cross product
- Orthonormal bases and coordinate frames

Dot Product

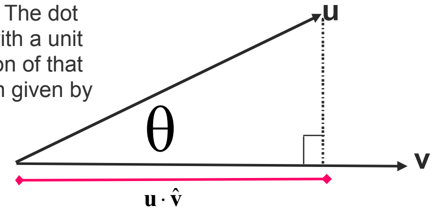
Motivation:

- Find angle between two vectors
  - e.g. cosine of angle between light source and surface for shading
- Finding projection of one vector on another
  - e.g. coordinates of a point in various coordinate frameworks -- object coordinate framework, world coordinate framework, camera coordinate framework, image coordinate framework

- Advantage: computed easily in cartesian components

Dot (Scalar) Product – Angle Computation

- Geometric definition: The dot product of a vector with a unit vector is the projection of that vector in the direction given by the unit vector.



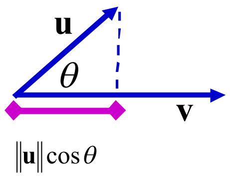
$$\mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}=|\mathbf{u}|\cos(\theta)$$

$$\theta=\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

$$\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos(\theta)$$

Dot Product Geometry – Projection (of u on v)

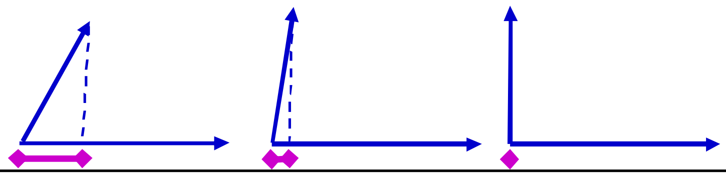
- The dot product of a vector with a unit vector is the projection of that vector in the direction given by the unit vector.



$$\mathbf{u}\cdot\frac{\mathbf{v}}{\|\mathbf{v}\|}=\|\mathbf{u}\|\cos\theta$$

$$\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

- as vectors become perpendicular, $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos\theta\to 0$



Dot (Scalar) Product

- Linear algebra definition of the dot product, aka inner product or the scalar product: $\mathbf{u}\bullet\mathbf{v}$

$$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\bullet\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=$$

$$\begin{bmatrix}6\\1\\2\end{bmatrix}\bullet\begin{bmatrix}1\\7\\3\end{bmatrix}=$$

Dot (Scalar) Product

- multiply: vector * vector = scalar $\quad \mathbf{u} \bullet \mathbf{v}$
- dot product, aka inner product

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1 * v_1) + (u_2 * v_2) + (u_3 * v_3)$$

$$\begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} = (6*1) + (1*7) + (2*3) = 6 + 7 + 6 = 19$$

Dot Product

Definition: $\mathbf{a}, \mathbf{b} \in \Re^N$

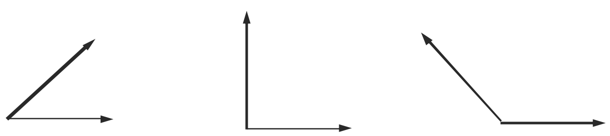
$$\mathbf{a} \cdot \mathbf{b} = \sum_{n=1}^{N} a_n b_n \quad = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

Properties

1. Symmetry: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. Linearity: $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
3. Homogeneity: $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
4. $|\mathbf{b}|^2 = \mathbf{b} \cdot \mathbf{b}$
5. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$

Dot Product and Perpendicularity

From Property 5. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$



$a \cdot b > 0$ $\qquad$ $a \cdot b = 0$ $\qquad$ $a \cdot b < 0$

Perpendicular Vectors

The dot product can be employed to determine if two vectors are perpendicular.

- Vectors **a** and **b** are perpendicular iff **a · b** = 0
- Also called orthogonal vectors

- It is easy to see that the standard unit vectors form an orthogonal basis:

i · j = 0, **j · k** = 0, **i · k** = 0

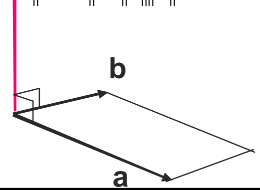
Vector Multiplication

- Dot product
- ***Cross product***
- Orthonormal bases and coordinate frames

Cross (vector) product

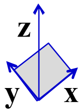
- Motivation: Useful in constructing coordinate systems (later)
  - Problem: Construct a orthonormal camera coordinate frame into which to transform world objects, given **a** (viewing direction), and a second vector **b** (up direction of camera)
- Cross product is a vector orthogonal to the two initial vectors
- Direction determined by *right-hand rule*

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$$



RHS vs. LHS Coordinate Systems

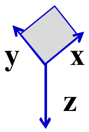
- right-handed coordinate system convention



right hand rule:
index finger x, second finger y;
right thumb points up

$$\mathbf{z} = \mathbf{x}\times\mathbf{y}$$

- left-handed coordinate system



left hand rule:
index finger x, second finger y;
left thumb points down

$$\mathbf{z} = \mathbf{x}\times\mathbf{y}$$

Cross Product

Defined only for 3D vectors and with respect to the standard unit vectors

Algebraic Definition:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

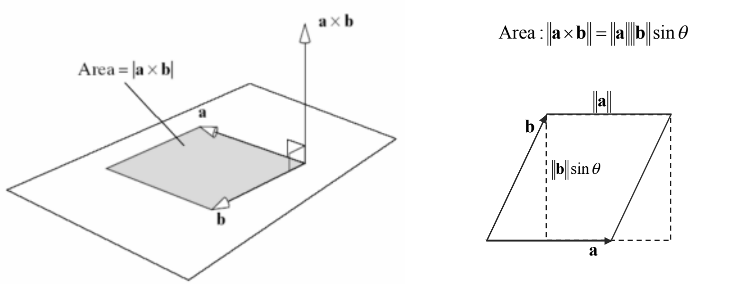
$$= (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

Cross Product

Geometric Definition: $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$

$\|\mathbf{a}\times\mathbf{b}\|$ parallelogram area

$\mathbf{a}\times\mathbf{b}$ perpendicular to parallelogram



Properties of the Cross Product

1. $\mathbf{i}\times\mathbf{j}=\mathbf{k}, \quad \mathbf{i}\times\mathbf{k}=-\mathbf{j}, \quad \mathbf{j}\times\mathbf{k}=\mathbf{i}$
2. Antisymmetry: $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$
3. Linearity: $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
4. Homogeneity: $(s\mathbf{a})\times\mathbf{b}=s(\mathbf{a}\times\mathbf{b})$
5. The cross product is normal to both vectors: $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=0$
6. $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin(\theta)$

Vector Multiplication

- Dot product
- Cross product
- ***Orthonormal bases and coordinate frames***

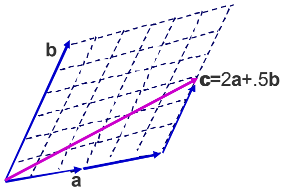
Orthonormal bases/coordinate frames

- Important for representing points, locations
- Often, many sets of coordinate systems
  - Global, local, world, model, parts of model (head, hands, …)
- Critical issue is transforming between these systems/bases
  - Topic of next 3 lectures

Basis Vectors

- Given any two vectors that are linearly independent (nonzero and nonparallel)
  - Can define any vector in the plane a a linear combination of these two vectors

$$\mathbf{c} = w_1\mathbf{a} + w_2\mathbf{b}$$



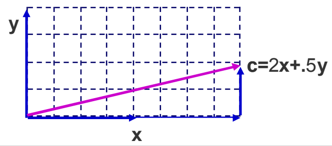
Orthonormal Basis Vectors

- If the basis vectors are orthonormal (perpendicular/orthogonal and unit length)

$$\mathbf{x} \bullet \mathbf{y} = 0$$

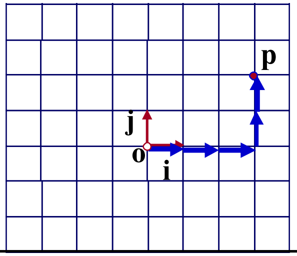
$$\|\mathbf{x}\| = \|\mathbf{y}\| = 1$$

  - we have Cartesian coordinate system
  - familiar Pythagorean definition of distance



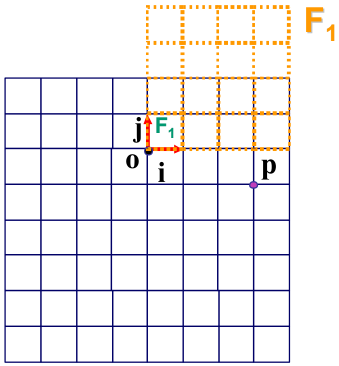
Basis Vectors and Origins

- Cordinate system: just basis vectors
  - can only specify offset: vectors
- Coordinate frame: basis vectors and origin
  - can specify location as well as offset: points & vectors



$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

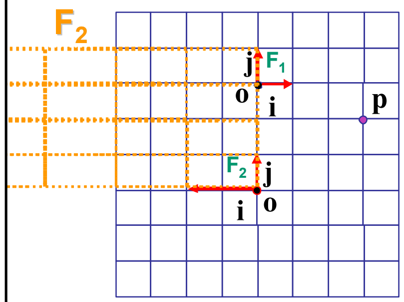
Working with Frames



$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

F_1 **p = (3,-1)**

Working with Frames

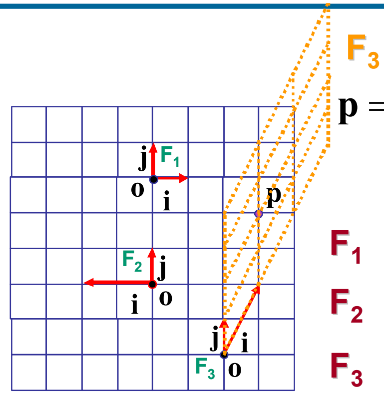


$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

F_1 p = (3,-1)

F_2 p = (-1.5,2)

Working with Frames



$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

F_1 p = (3,-1)

F_2 p = (-1.5,2)

F_3 p = (1,2)

Generators and Base Vectors

How many vectors are needed to generate a vector space?

- Given a vector space $\mathbf{R}^n$ we can prove that we need minimum n vectors to generate all vectors **v** in $\mathbf{R}^n$
- Any set of vectors that generate a vector space is called a ***generator set***
- A generator set with minimum size is called a ***basis*** for the given vector space

Standard Unit Vectors

For any vector space R^n:

$$\mathbf{i}_1 = (1,0,0,\ldots,0,0)^T$$
$$\mathbf{i}_2 = (0,1,0,\ldots,0,0)^T$$
$$\ldots$$
$$\mathbf{i}_n = (0,0,0,\ldots,0,1)^T$$

The elements of a vector v in R^n are the scalar coefficients of the linear combination of the basis vectors

Standard Unit Vectors

$$\mathbf{v} = (x_1, \ldots, x_n)^T, \quad x_i \in \Re$$

$$\begin{aligned}(x_1, x_2, \ldots, x_n)^T = \; & x_1(1,0,0,\ldots,0,0)^T \\ & + x_2(0,1,0,\ldots,0,0)^T \\ & \ldots \\ & + x_n(0,0,0,\ldots,0,1)^T\end{aligned}$$

Representation of Vectors Through Basis Vectors

Given a vector space R^n, a set of basis vectors **B** $\{\mathbf{b}_i$ in R^n, i=1,…n} and a vector **v** in R^n we can always find scalar coefficients such that:

$\mathbf{v} = a_1\mathbf{b}_1 + \ldots + a_n\mathbf{b}_n$

So, vector **v** expressed with respect to *B* is:

$\mathbf{v}_B = (a_1, \ldots, a_n)$

Outline

- Vectors,
  - Vector Basics
  - Vector Addition, Subtraction
  - Vector Multiplication
    - Dot product
    - Cross product
  - Coordinate Frames
- Matrix
  - Matrix Basics
  - Matrix Addition
  - Matrix Multiplication
  - Matrix Properties
  - Vector Multiplication in Matrix Form
- Points vs Vectors
- Lines and Planes

Matrices

- Can be used to transform points (vectors)
  - Translation, rotation, shear, scale (more detail next lecture)
- Section 5.2.1 and 5.2.2 of text
  - Instructive to read all of 5 but not that relevant to course

What is a Matrix

- It is a two-way array ($m \times n$ = m rows, n columns) that maps one vector space into another

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

- Can be used to transform points (vectors)
  - Translation, rotation, shear, scale (more detail next lecture)

Matrix-Matrix Addition

- add: matrix + matrix = matrix

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} n_{11}+m_{11} & n_{12}+m_{12} \\ n_{21}+m_{21} & n_{22}+m_{22} \end{bmatrix}$$

- example

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ 7 & 1 \end{bmatrix} =$$

Matrix Addition

Addition:

$$\mathbf{A}_{m\times n} + \mathbf{B}_{m\times n} = (a_{ij} + b_{ij})$$

Properties:

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
3. $f(\mathbf{A} + \mathbf{B}) = f\mathbf{A} + f\mathbf{B}$
4. Transpose: $\mathbf{A}^T = (a_{ij})^T = (a_{ji})$

Scalar-Matrix Multiplication

- multiply: scalar * matrix = matrix

$$a\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} a*m_{11} & a*m_{12} \\ a*m_{21} & a*m_{22} \end{bmatrix}$$

- example

$$3\begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} =$$

Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

Matrix-Matrix Multiplication

- row by column

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Matrix-Matrix Multiplication

- Number of columns in first must = rows in second

$$\begin{pmatrix} \boxed{1 \;\; 3} \\ 5 \;\; 2 \\ 0 \;\; 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix} = \begin{matrix} \boxed{9 \;\; 27 \;\; 33 \;\; 13} \\ 19 \;\; 44 \;\; 61 \;\; 26 \\ 8 \;\; 28 \;\; 32 \;\; 12 \end{matrix}$$

- Element (i,j) in product is the matrix product of row i of first matrix and column j of second matrix

$$\begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \mathbf{a}_3^T \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^T\mathbf{b}_1 & \mathbf{a}_1^T\mathbf{b}_2 & \mathbf{a}_1^T\mathbf{b}_3 & \mathbf{a}_1^T\mathbf{b}_4 \\ \mathbf{a}_2^T\mathbf{b}_1 & \mathbf{a}_2^T\mathbf{b}_2 & \mathbf{a}_2^T\mathbf{b}_3 & \mathbf{a}_2^T\mathbf{b}_4 \\ \mathbf{a}_3^T\mathbf{b}_1 & \mathbf{a}_3^T\mathbf{b}_2 & \mathbf{a}_3^T\mathbf{b}_3 & \mathbf{a}_3^T\mathbf{b}_4 \end{pmatrix}$$

Dot Product in Matrix Form

*A vector is a column matrix

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= \sum_{n=1}^{N} a_n b_n \\ &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= (a_1, a_2, a_3)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= \mathbf{a}^T\mathbf{b} \end{aligned}$$

Cross Product in Matrix Form

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Rewrite

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \mathbf{Ab}$$

A is the dual matrix of vector **a**

Summary: Multiplication with Matrices

Definition:

$$\mathbf{C}_{m\times r} = \mathbf{A}_{m\times n}\mathbf{B}_{n\times r}$$

$$(\mathbf{C}_{ij}) = (\sum_{k=1}^{n} a_{ik}b_{kj})$$

Properties:

1. $\mathbf{AB} \neq \mathbf{BA}$
2. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
3. $f(\mathbf{AB}) = (f\mathbf{A})\mathbf{B}$
4. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$,
   $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$
5. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
6. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\ \mathbf{A}^{-1}$

Outline

- Vectors,
  - Vector Basics
  - Vector Addition, Subtraction
  - Vector Multiplication
    - Dot product
    - Cross product
  - Constructing Coordinate Frames
- Matrix
  - Matrix Basics
  - Matrix Addition
  - Matrix Multiplication
  - Matrix Properties
  - Vector Multiplication in Matrix Form
- Points vs Vectors
- Lines and Planes

Homogeneous Coordinates

Vectors and Points are represented as column matrices

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{pmatrix} \qquad P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix}$$

Lines and Planes

In addition to vectors and points, lines and planes are fundamental geometric entities in computer graphics

- Recall how we represent them mathematically…

Lines

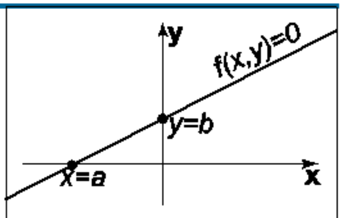
- **Explicit**
  - slope-intercept form

  $y = mx + b$

- **Implicit**

  $y - mx - b = 0$
  Ax + By + C = 0
  $f(x,y) = 0$

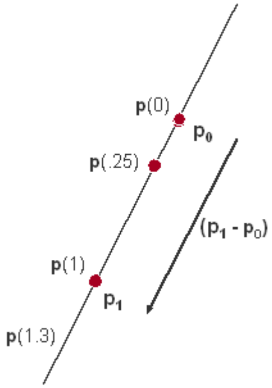
  - find where function is 0
  - plug in (x,y), check if
    - 0: on line
    - < 0: inside
    - > 0: outside



$f(x,y) = y - mx - b$
$m = -b/a$

2D Parametric Lines

- Start at point $\mathbf{p}_0$, go towards $\mathbf{p}_1$, according to parameter t
  - $\mathbf{p}(0) = \mathbf{p}_0$, $\mathbf{p}(1) = \mathbf{p}_1$
- $\mathbf{p}(t) = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0)$
- $\mathbf{p}(t) = \mathbf{o} + t(\mathbf{d})$
- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 + t(x_1 - x_0) \\ y_0 + t(y_1 - y_0) \end{bmatrix}$



Summary: Lines

Representations of a line (in 2D)

- Explicit - we have one dependent variable on the left-hand side of an equation, and all the independent variables and constants on the right-hand side of the equation

$$y = \frac{dy}{dx}(x - x_0) + y_0$$

- Implicit - An implicit equation is an algebraic line/curve formed by points that satisfy an equation. Implicits TEST their *(x,y)* input values and return true if the implicit relationship is satisfied. The general solution of implicit equations requires a search!

$$F(x,y) = (x - x_0)dy - (y - y_0)dx$$

$$\begin{array}{llll} \text{if} & F(x,y) = 0 & \text{then} & (x,y) \text{ is on line} \\ & F(x,y) > 0 & & (x,y) \text{ is below line} \\ & F(x,y) < 0 & & (x,y) \text{ is above line} \end{array}$$

- Parametric - also called generating functions, they generate coordinate pairs as output, given parameter values as input.

$$x(t) = x_0 + t(x_1 - x_0)$$
$$y(t) = y_0 + t(y_1 - y_0)$$
$$t \in [0, 1]$$

$$P(t) = P_0 + t(P_1 - P_0), \text{ or}$$
$$P(t) = (1 - t)P_0 + tP_1$$

Planes

Plane equations

Implicit $F(x,y,z) = Ax + By + Cz + D = 0 = \mathbf{N} \bullet P + D$

Points on Plane $F(x,y,z) = 0$

Explicit $z = -(A/C)x - (B/C)y - D/C, \; C \neq 0$

Parametric

$$Plane(s,t) = P_0 + s(P_1 - P_0) + t(P_2 - P_0)$$

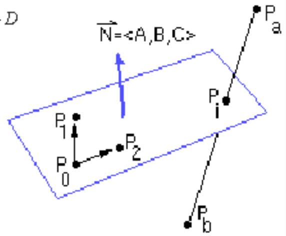
P_0, P_1, P_2 not collinear

or

$$Plane(s,t) = P_0 + sV_1 + tV_2 \text{ where } V_1, V_2 \text{ are basis vectors}$$

convex combination defines a triangle:

$$Plane(s,t) = (1 - s - t)P_0 + sP_1 + tP_2$$



Implicit Functions

- find where function is 0
  - plug in (x,y), check if
    - 0: on line
    - < 0: inside
    - > 0: outside
- analogy: terrain
  - sea level: f=0
  - altitude: function value
  - topo map: equal-value contours (level sets)