CS 174A Homework 3 Solutions

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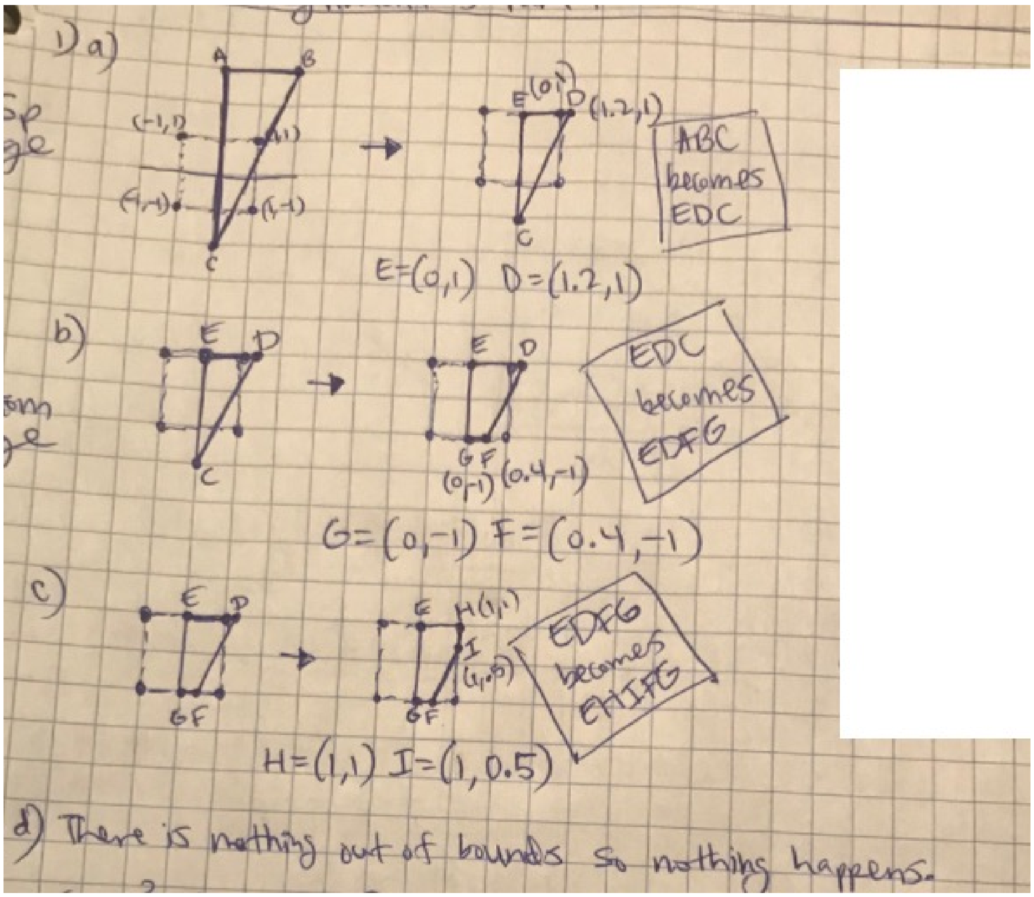
1. See image below:



2. (a)

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \text{ or } ||\mathbf{p} - \mathbf{c}||^2 = r^2$$

(b) Use the explicit representation of the circle from lecture:

for x = 0 to r
    y = sqrt(r^2 - x^2)
    SetPixel($x + c_x$, round($y + c_y$))
    SetPixel($x - c_x$, round($y + c_y$))
    SetPixel($x + c_x$, round($y - c_y$))

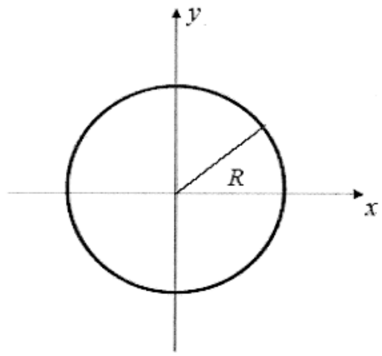
$$\text{SetPixel}(x - c_x, \text{round}(y - c_y))$$

(c) See images on next 2 pages:

Assume the circle is $x^2+y^2=R^2$

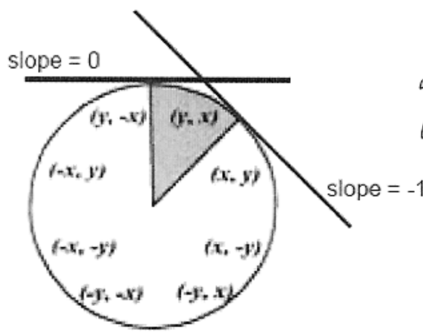
For the function $d=F(x,y)=x^2+y^2-R^2$

we have $\begin{cases} d=0 : (x,y) \text{ is on the circle} \\ d>0 : (x,y) \text{ is outside} \\ d<0 : (x,y) \text{ is inside} \end{cases}$



(1)

Since the symmetry of a circle, we only need to compute 1/8 of the whole circle (Figure 2), starting from (0, R) to the point where $x \geq y$



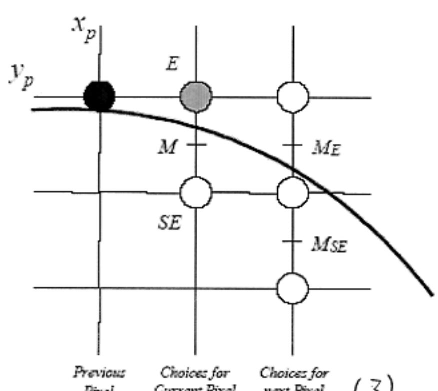
(2)

Similar to Bresenham algorithm, we only need to consider the E and SE directions for next step, depending on which pixel is closer to the circle. That is, whether the midpoint of E and SE is inside or outside the circle (figure 3)

The computation of d_{old}, d_{new} and $d_{initial}$ is:

$$d_{old} = F(x_p+1, y_p-\tfrac{1}{2}) = (x_p+1)^2+(y_p-\tfrac{1}{2})^2-R^2$$

$$d_{new} = \begin{cases} next = E: F(x_p+2, y_p-\tfrac{1}{2}) \\ \qquad = d_{old}+2x_p+3 \\ next = SE: F(x_p+2, y_p-\tfrac{3}{2}) \\ \qquad = d_{old}+2x_p-2y_p+5 \end{cases}$$



(3)

$$d_{initial} = F(x_0+1, y_0-\tfrac{1}{2}) = 5/4 - R$$

Reference: http://www.cs.sfu.ca/CourseCentral/361/hyounesy/Lectures/09-raster.pdf

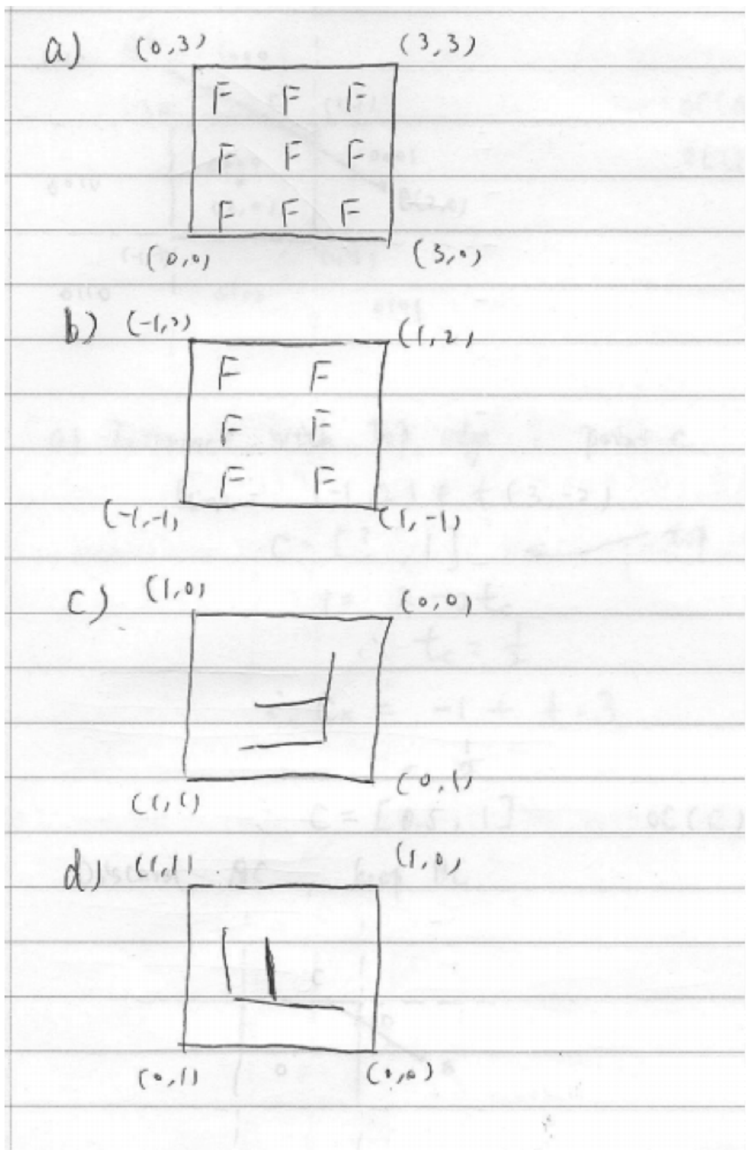
What we need is integer only algorithm. ~~So~~ But $d_{initial}$ is not an ~~it~~ integer! There are two methods to solve this. One is multiply 4 for each expression of d, the other is just using $1-R$ for $d_{initial}$ since ~~the sign of~~

$$\begin{cases} d \leq 0 \Leftrightarrow d - \frac{1}{4} \leq 0 \\ d > 0 \Leftrightarrow d - \frac{1}{4} > 0 \end{cases}$$

The pseudo code for drawing a circle is

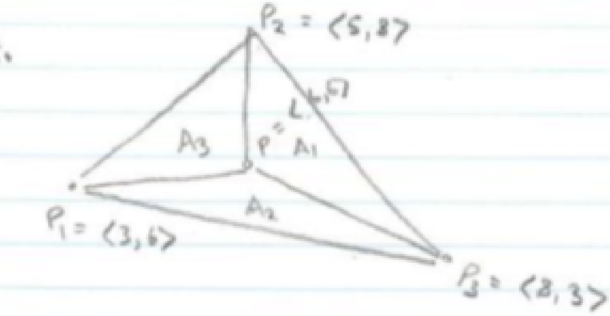
```
x = 0 ; y = R ; d = d_initial
draw 8 way point (x, y)
while (x < y) {
    if (d < 0) {
        E direction
    } else {
        SE direction
    }
    draw 8 way point (x, y)
}
```

3. See image below:



4. See image below:

4.



$\vec{PP_1} = P_1 - P = \langle -3, 0\rangle$

$\vec{PP_2} = P_2 - P = \langle -1, 2\rangle$

$\vec{PP_3} = P_3 - P = \langle 2, -3\rangle$

$A_1 = \frac{1}{2}\|\vec{PP_3} \times \vec{PP_2}\| = \frac{1}{2}\|\langle 0,0,1\rangle\| = \frac{1}{2}$

$A_2 = \frac{1}{2}\|\vec{PP_1} \times \vec{PP_3}\| = \frac{1}{2}\|\langle 0,0,9\rangle\| = 4.5$

$A_3 = \frac{1}{2}\|\vec{PP_2} \times \vec{PP_1}\| = \frac{1}{2}\|\langle 0,0,6\rangle\| = 3$

$A = A_1 + A_2 + A_3 = 8$

$$\boxed{\begin{aligned} \alpha &= \frac{A_1}{A} = \frac{1}{16} = .0625 \\ \beta &= \frac{A_2}{A} = \frac{9}{16} = .5629 \\ \gamma &= \frac{A_3}{A} = \frac{3}{8} = .375 \end{aligned}}$$

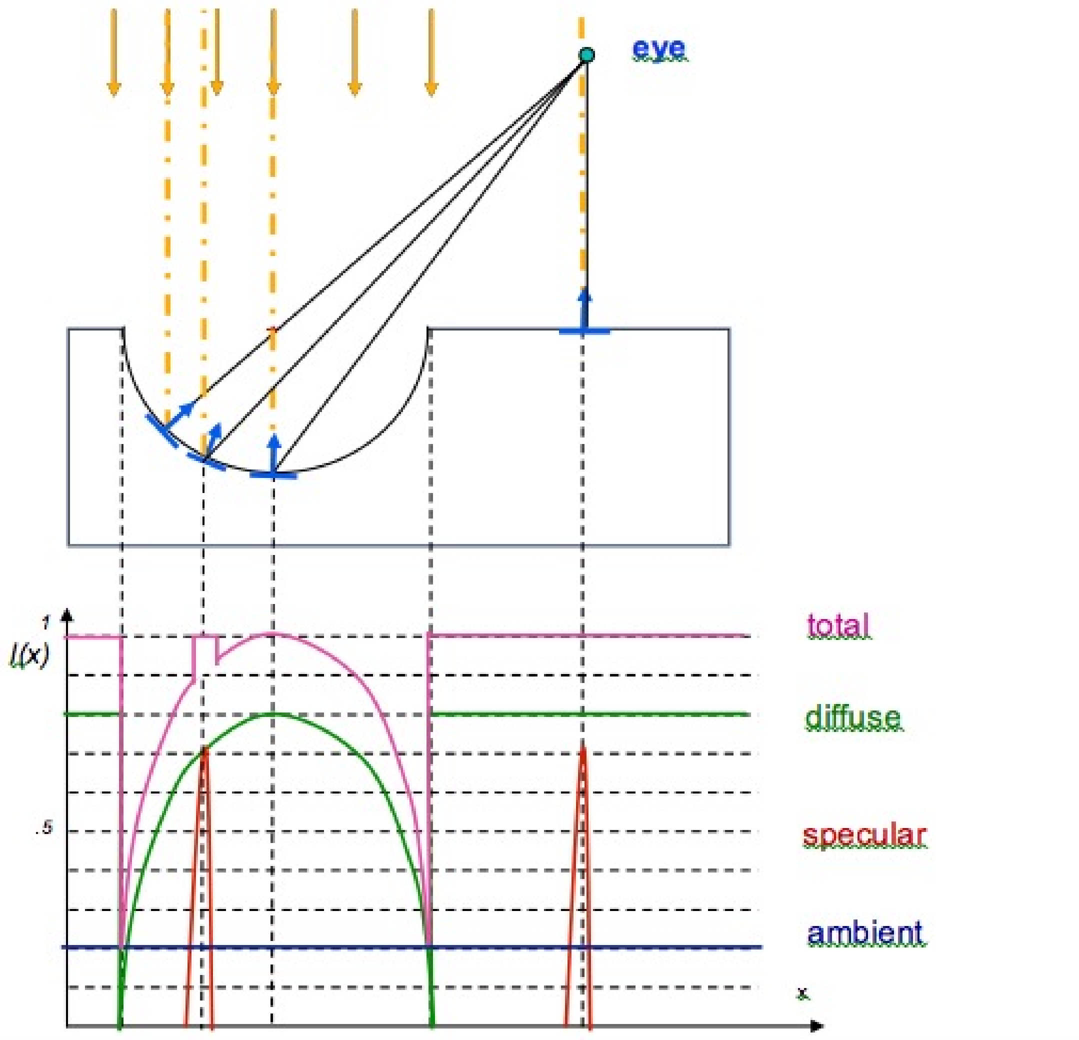
$\langle r\ g\ b\rangle$ @ $P = \alpha\langle .3\ .2\ .9\rangle$ or $\alpha\langle .3\ .9\ .2\rangle$ +

$\beta\langle .5\ .5\ .9\rangle$ +

$\gamma\langle .7\ .2\ .7\rangle$

$$\boxed{\begin{aligned} &= \langle .5625\ .3638\ .8250\rangle \text{ or} \\ &\langle .5625\ .4125\ .7812\rangle \end{aligned}}$$

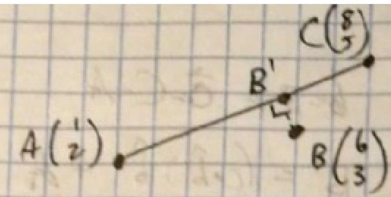
5. (a) See image below:



(b) They are done in the View Coordinate System (VCS). We cannot do it before since we need to know the eye location, and cannot do it after perspective since the space gets distorted.

6. (a) See images below:

6.



$\vec{c} = C - A = \binom{7}{3}$

$\vec{b} = B - A = \binom{5}{1}$ $\quad B' = A + \left(\vec{b}\circ\frac{\vec{c}}{|\vec{c}|}\right)\overset{\vec{c}/|\vec{c}|}{\hat{c}} = A + \left(\vec{b}\circ\frac{\vec{c}}{|\vec{c}|^2}\right)\vec{c}$

$$B' = \binom{1}{2} + \left(\binom{5}{1}\circ\frac{\binom{7}{3}}{49+9}\right)\binom{7}{3} = \binom{1}{2} + \frac{38}{58}\binom{7}{3} = \cancel{\binom{133/29}{57/29} = B'}$$

$$B' = \binom{162/29}{115/29} \qquad B' - A = \binom{133/29}{57/29} \to \text{wt } \frac{19}{29}\binom{7}{3}$$

$$A = \binom{29/29}{58/29} \qquad C - B' = \binom{70/29}{30/29} \to \text{wt } \frac{10}{29}\binom{7}{3}$$

$$C = \binom{232/29}{145/29}$$

normal at A is $\binom{-\sqrt{2}/2}{-\sqrt{2}/2}$ weighted by $C-B'$ segment weight $\frac{10}{29}$

normal at C is $\binom{1}{0}$ weighted by $B'-A$ segment weight $\frac{19}{29}$

So the normal at B is $\frac{10}{29}\begin{pmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} + \frac{19}{29}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} (19-5\sqrt{2})/29 \\ -5\sqrt{2}/29 \end{pmatrix}$

normalize:

$$\left\|\begin{pmatrix} (19-5\sqrt{2})/29 \\ -5\sqrt{2}/29 \end{pmatrix}\right\| = \sqrt{(19-5\sqrt{2})^2 + (-5\sqrt{2})^2}\Big/29 =$$

$$\sqrt{19^2 - 2\cdot 19\cdot 5\sqrt{2} + 50 + 50}\Big/29$$

$$= \sqrt{461 - 190\sqrt{2}}\Big/29$$

normalized normal at B

$$\begin{pmatrix} (19-5\sqrt{2})/29 \\ -5\sqrt{2}/29 \end{pmatrix}\Big/\left(\sqrt{461-190\sqrt{2}}/29\right) = \begin{pmatrix} (19-5\sqrt{2})/\sqrt{461-190\sqrt{2}} \\ -5\sqrt{2}/\sqrt{461-190\sqrt{2}} \end{pmatrix} \approx \begin{pmatrix} 0.86022 \\ -0.50991 \end{pmatrix}$$

(b) See images below:

b.)

$$H = \frac{L+V}{|L+V|} \qquad I = I_d k_d (N \cdot L) + I_s k_s (H \cdot N)^n + I_a k_a$$

$$k_a I_a = \begin{pmatrix} 0.01 \\ 0.02 \\ 0.01 \end{pmatrix} \qquad k_d I_L = \begin{pmatrix} 0.3 \\ 0.8 \\ 0.81 \end{pmatrix}$$

$$k_s I_L = \begin{pmatrix} 1.0 \\ 1.0 \\ 0.9 \end{pmatrix}$$

Point B:

$$L = (8,1,0) - (6,3,0) = (2,-2,0) = \overbrace{\left(\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, 0\right)}^{\text{normalized}}$$

$$V = (8,-1,0) - (6,3,0) = (2,-4,0) = \left(\tfrac{1}{\sqrt{5}}, \tfrac{-2}{\sqrt{5}}, 0\right)$$

Halfway Vector

$$H = \frac{L+V}{|L+V|} = \frac{(1.154, -1.602, 0)}{\sqrt{1.154^2 + 1.602^2}} = \begin{pmatrix} 0.584 \\ -0.811 \\ 0 \end{pmatrix}$$

$$I_{diffuse} = k_d I_L (N \cdot L)$$

$$= \begin{pmatrix} 0.3 \\ 0.8 \\ 0.81 \end{pmatrix} \begin{pmatrix} 0.860 \\ -0.510 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \\ 0 \end{pmatrix}$$

$$= 0.969 \begin{pmatrix} 0.3 \\ 0.8 \\ 0.81 \end{pmatrix} = \begin{pmatrix} 0.291 \\ 0.776 \\ 0.785 \end{pmatrix}$$

- $I_{specular} = k_s I_L (H \cdot N)^n$

$$= \begin{pmatrix} 1.0 \\ 1.0 \\ 0.9 \end{pmatrix} \left[\begin{pmatrix} 0.584 \\ -0.811 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0.860 \\ -0.510 \\ 0 \end{pmatrix} \right]^{20}$$

$$= 0.172 \begin{pmatrix} 1.0 \\ 1.0 \\ 0.9 \end{pmatrix} = \begin{pmatrix} 0.172 \\ 0.172 \\ 0.155 \end{pmatrix}$$

- $I_{ambient} = k_a I_a = \begin{pmatrix} 0.01 \\ 0.02 \\ 0.01 \end{pmatrix}$

$$I_{total} = I_{diff} + I_{spec} + I_{amb} = \begin{pmatrix} 0.473 \\ 0.968 \\ 0.950 \end{pmatrix}$$

Point C:

$$L = (8,1,0) - (8,5,0) = (0,-4,0) = \overbrace{(0,-1,0)}^{\text{normalized}}$$

$$V = (8,-1,0) - (8,5,0) = (0,-6,0) = (0,-1,0)$$

$$H = \frac{L+V}{|L+V|} = \frac{(0,-2,0)}{\sqrt{4}} = (0,-1,0)$$

- $I_{diff} = K_d I_L (N \cdot L)$

$$= \begin{pmatrix} 0.3 \\ 0.8 \\ 0.81 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- $I_{spec} = K_s I_L (H \cdot N)^n$

$$= \begin{pmatrix} 1.0 \\ 1.0 \\ 0.9 \end{pmatrix} \left[\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]^{20} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- $I_{amb} = K_a I_a = \begin{pmatrix} 0.01 \\ 0.02 \\ 0.01 \end{pmatrix}$

$$I_{total} = I_{diff} + I_{spec} + I_{amb} = \begin{pmatrix} 0.01 \\ 0.02 \\ 0.01 \end{pmatrix}$$

Point D:

Need to find normal at D by interpolating between B and C. Since the weights are equal (D is the midpoint of BC) we take the average of N_B & N_C

$$N_D = \frac{N_B + N_C}{2}$$

$$= \frac{\begin{pmatrix} 0.860 \\ -0.510 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{2} = \frac{1}{2}\begin{pmatrix} 1.860 \\ -0.510 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.930 \\ -0.255 \\ 0 \end{pmatrix}$$

$$\boxed{N_{D\,\text{normalized}} = \begin{pmatrix} 0.964 \\ -0.264 \\ 0 \end{pmatrix}}$$

Point D:

$$\overbrace{\qquad\qquad}^{\text{normalized}}$$

$$L = (8,1,0) - (7,4,0) = (1,-3,0) = \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}, 0\right)$$

$$V = (8,-1,0) - (7,4,0) = (1,-5,0) = \left(\frac{1}{\sqrt{26}}, -\frac{5}{\sqrt{26}}, 0\right)$$

$$H = \frac{L+V}{|L+V|} = \frac{(0.512, -1.929, 0)}{\sqrt{0.512^2 + 1.929^2}} = \begin{pmatrix} 0.257 \\ -0.967 \\ 0 \end{pmatrix}$$

- $I_{diffuse} = K_d I_L (N \cdot L)$

$$= \begin{pmatrix} 0.3 \\ 0.8 \\ 0.81 \end{pmatrix} \begin{pmatrix} 0.964 \\ -0.264 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \\ 0 \end{pmatrix}$$

$$= 0.555 \begin{pmatrix} 0.3 \\ 0.8 \\ 0.81 \end{pmatrix} = \begin{pmatrix} 0.167 \\ 0.444 \\ 0.450 \end{pmatrix}$$

- $I_{spec} = K_s I_L (H \cdot N)^n$

$$= \begin{pmatrix} 1.0 \\ 1.0 \\ 0.9 \end{pmatrix} \left[\begin{pmatrix} 0.257 \\ -0.967 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0.964 \\ -0.264 \\ 0 \end{pmatrix} \right]^{20}$$

$$= 1.076 \times 10^{-6} \begin{pmatrix} 1.0 \\ 1.0 \\ 0.9 \end{pmatrix} = \begin{pmatrix} 1.076 \times 10^{-6} \\ 1.076 \times 10^{-6} \\ 9.688 \times 10^{-7} \end{pmatrix}$$

- $I_{amb} = K_a I_a = \begin{pmatrix} 0.01 \\ 0.02 \\ 0.01 \end{pmatrix}$

$$I_{total} = I_{diff} + I_{spec} + I_{amb} = \begin{pmatrix} 0.177 \\ 0.464 \\ 0.460 \end{pmatrix}$$

6.) cont...

Flat Shading:

For the flat shading model, the illumination at B & C is the same. The illumination at point D can either equal that of B or C.

Point D: $I_D = I_B$

$I_{diff} = (0.291, 0.776, 0.785)$

$I_{specular} = (0.172, 0.172, 0.155)$

$I_{ambient} = (0.01, 0.02, 0.01)$

$I_{total} = (0.473, 0.968, 0.950)$

(c) See image below:

c.) Gouraud Shading Model

Again, the illumination at points B & C is the same as it was originally. The illumination at point D is the average of those at B & C because it is equidistant from B & C. (equal weights when interpolating)

Point D:

$I_{diff_D} = (I_{diff_B} + I_{diff_C})/2 = (0.146, 0.388, 0.393)$

$I_{spec_D} = (I_{spec_B} + I_{spec_C})/2 = (0.086, 0.086, 0.078)$

$I_{amb_D} = (0.01, 0.02, 0.01)$

$I_{total_D} = (0.242, 0.494, 0.481)$

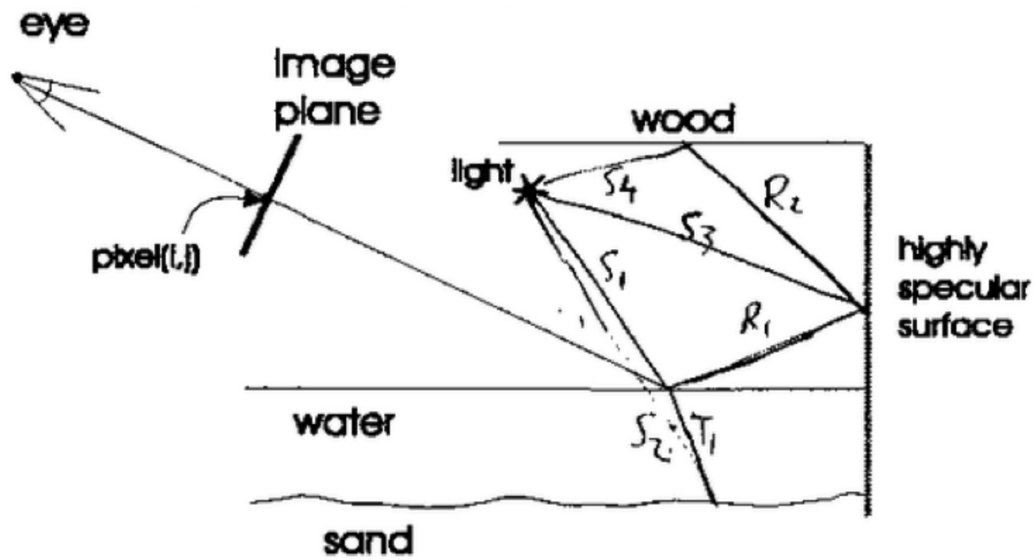
(d) See image below:

d.) <u>Phong Shading Model</u>

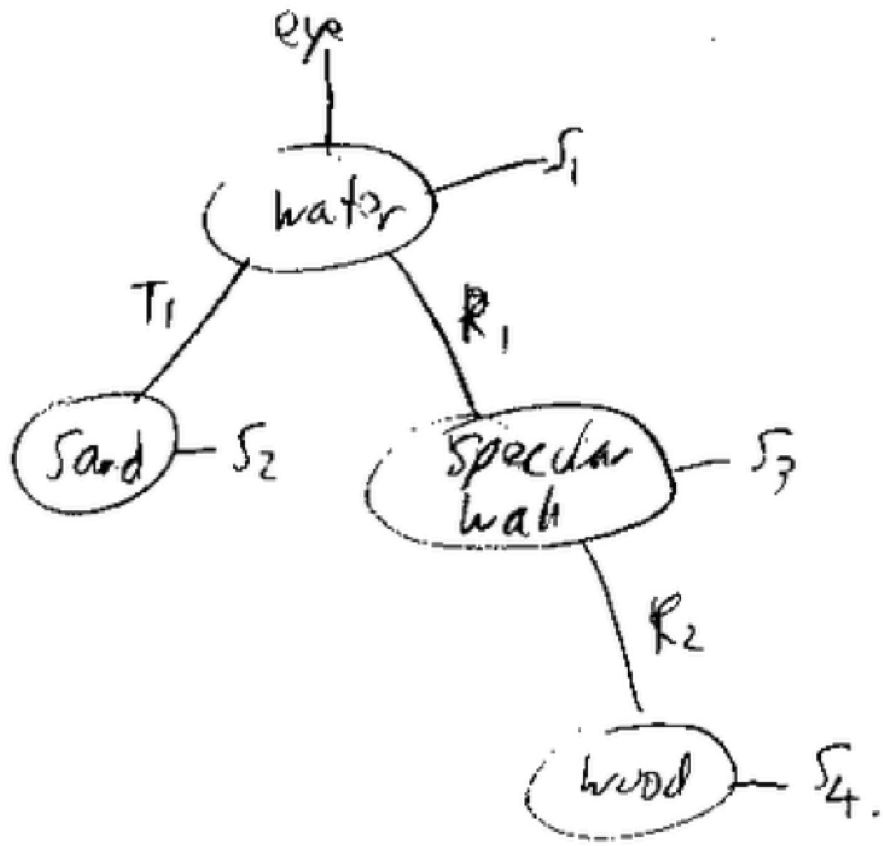
- Again, the illumination at points B & C are the same as what they were originally (calculated in part b).
- In order to calculate the illumination at point D, must find the normal at D by interpolating from B & C. This was done in part B using the half-vector.

$$I_{total_D} = \begin{pmatrix} 0.177 \\ 0.464 \\ 0.460 \end{pmatrix}$$

7. (a) See image below:

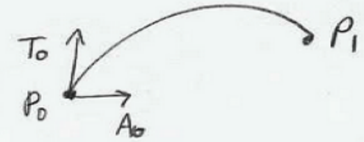


(b) See image below:



8. See image below:

$$p(t) = [t^3 \quad t^2 \quad t \quad 1]\, A$$
$$p'(t) = [3t^2 \quad 2t \quad 1 \quad 0]\, A$$
$$p''(t) = [6t \quad 2 \quad 0 \quad 0]\, A$$



$$P_{0x} = [0 \quad 0 \quad 0 \quad 1]\, A_x$$
$$T_{0x} = [0 \quad 0 \quad 1 \quad 0]\, A_x$$
$$A_{0x} = [0 \quad 2 \quad 0 \quad 0]\, A_x$$
$$P_{1x} = [1 \quad 1 \quad 1 \quad 1]\, A_x$$

$$\Rightarrow \begin{bmatrix} P_{0x} \\ T_{0x} \\ A_{0x} \\ P_{1x} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} A_x$$

constraint matrix

$$G_x = C\, A_x$$
$$A_x = \underbrace{C^{-1}}_{\text{basis matrix}}\, G_x$$

$$x(t) = T \cdot A_x$$
$$= T \cdot C^{-1} \cdot G_x$$

basis matrix is:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

Your answer may differ if your geometry vector, G, uses the position, velocity, and acceleration constraints in a different order.