Problem Set - 6

SPRING 2020

MATHEMATICS-II (MA1002)(Integral Calculus)

1. Discuss the convergence of the following improper integral using definition:

   i) $\int_0^1 \frac{1}{1-x}dx$, ii) $\int_0^2 \frac{1}{\sqrt{x(2-x)}}dx$,

   iii)$\int_1^\infty \frac{1}{x\log x}dx$, iv)$\int_a^b \frac{1}{(x-a)^p}dx$, $p>0$,

   v)$\int_{-\infty}^\infty \frac{1}{1+x^2}dx$, vi)$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan x dx$.

2. Discuss the convergence of the following improper integral:

   i) $\int_0^1 \frac{x^{p-1}}{1-x}dx$, ii) $\int_0^1 x^{n-1}\log x dx$,

   iii)$\int_0^{\frac{\pi}{2}} \log(\sin x)dx$, iv)$\int_0^1 \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}}dx$,

   v)$\int_0^{\frac{\pi}{2}} \sin^{m-1}x\cos^{n-1}x dx$, vi)$\int_0^\infty \frac{x^{n-1}}{1+x}dx$,

   vii)$\int_0^\infty (\frac{1}{1+x}-\frac{1}{e^x})\frac{1}{x}dx$, viii) $\int_0^\infty \frac{\cos x}{\sqrt{x^3+x}}dx$,

   ix)$\int_0^{\frac{\pi}{2}} \frac{x^m}{\sin^n x}dx$, x)$\int_0^{\frac{\pi}{2}} \frac{1}{e^x-\cos x}dx$.

3. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if m and n both are positive.

4. A function f is defined on $[0,1]$ by $f(0)=0$, $f(x)=(-1)^{n+1}(n+1)$, for $\frac{1}{n+1}<x\leq\frac{1}{n}$, n=1,2,3,... Examine the convergence of the integral $\int_0^1 f(x)dx$.

5. Prove that the integral $\int_0^\infty \frac{\sin x}{x}dx$ is convergent but $\int_0^\infty |\frac{\sin x}{x}|dx$ is not convergent.

6. Prove that $\int_0^\infty \frac{\sin mx}{x^n}dx$ $(m>0)$ is convergent if $0<n<2$.

7. Show that the improper integral $\int_0^\infty \frac{1}{1+x^2\sin^2 x}dx$ is divergent.

8. Prove that $\Gamma(m)\Gamma(1-m)=\frac{\pi}{\sin m\pi}$, $0<m<1$(Using $\int_0^\infty \frac{x^{m-1}}{1+x}dx=\frac{\pi}{\sin m\pi}$).

9. Prove that $\int_0^\infty x^{m-1}e^{-x}dx$ is convergent if $m > 0$.

10. Prove that

i) $\int_0^{\frac{\pi}{2}} \cot^p x dx = \frac{\pi}{2}\sec\frac{p\pi}{2}$ and indicate the restriction on the values of p.

ii) $\int_0^1 \frac{1}{(1-x^3)^{\frac{1}{3}}}dx = \frac{2\pi}{3\sqrt{3}}$.

iii) $\int_0^1 x^{m-1}(\log\frac{1}{x})^{n-1}dx = \frac{\Gamma(n)}{m^n}$, if $m > 0,\ n > 0$.

iv) $(\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx)(\int_0^1 \frac{1}{\sqrt{1+x^4}}dx) = \frac{\pi}{4\sqrt{2}}$.

v) $\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx = (b-a)^{m+n-1}B(m,n)$, $m > 0, n > 0$.

11. Evaluate i) $\int_0^\infty \frac{b\sin ax - a\sin bx}{x^2}dx,\ 0 < b < a,$ ii) $\int_0^1 x^6(1-\sqrt{x})^8 dx$.

12. Prove that $\sqrt{\pi}\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2}),\ n > 0$.

13. If n be a positive integer, prove that

$$\Gamma(\frac{1}{n})\Gamma(\frac{2}{n})\Gamma(\frac{3}{n})\ldots\Gamma(\frac{n-1}{n}) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$$

(Use $\sin\frac{\pi}{n}\sin\frac{2\pi}{n}...\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$).