Tutorial Sheet-2 **Autumn 2019**

MATHEMATICS-I(MA10001)

1. Determine the following limits using L'Hospital rule:

$a)\ \lim_{x\to\infty} (1-\frac{1}{2x})^{x+1}$ $\quad$ $b)\ \lim_{x\to 0^+} x^{\sin x}$

$c)\ \lim_{x\to 0} (\frac{\sin x}{x})^{\frac{1}{x}}$ $\quad$ $d)\ \lim_{x\to 0} (\frac{1}{2-2\cos x} - \frac{1}{x^2})$

$e)\ \lim_{x\to 0} |\sin x|^x$ $\quad$ $f)\ \lim_{x\to \frac{\pi}{2}} \frac{\log(\sin x)}{(\pi-x)^2}$

$g)\ \lim_{x\to 0} (\frac{2^x+3^x}{2})^{\frac{1}{x}}$ $\quad$ $h)\ \lim_{x\to\infty} \{x - \sqrt[n]{(x-1)(x-2)\dots(x-n)}\}$

2. a) Find the value of a for which the limit

$$\lim_{x\to 0} \frac{\sin(ax) - \sin x - x}{x^3}$$

is finite and evaluate the limit.

b) Find the values of a and b such that

$$\lim_{x\to 0} \frac{\cos(ax) - b}{2x^2} = -1.$$

3. Use Taylor's theorem to prove that

$a)\ x - \frac{x^2}{2} < \log(1+x) < x$ for $x > 0$.

$b)\ \cos x \geq 1 - \frac{x^2}{2}$ for $-\pi < x < \pi$.

$c)\ 1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$ for $x > 0$.

4. Let $c \in \mathbb{R}$ and a real function f be such that f'' is continuous on some neighbourhood of c. Prove that

$$\lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

5. Let $a \in \mathbb{R}$ and a real function f defined on some neighbourhood $N(a)$ of a such that f'' is continuous at a and $f''(a) \neq 0$. Prove that $\lim_{h\to 0} \theta = \frac{1}{2}$, where θ is given by $f(a+h) = f(a) + hf'(a+\theta h)$ $(0 < \theta < 1)$.

6. Each of the series in the following is the value of the Taylor series at $x = 0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

   $a)$ $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \ldots$

   $b)$ $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \ldots$

   $c)$ $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \ldots$

7. Using Taylor series expansion, evaluate

   $a)$ $\lim\limits_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ $b)$ $\lim\limits_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

   $c)$ $\lim\limits_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$ $d)$ $\frac{\cosh x - \cos x}{x \sin x}$

8. If f is continuous at x_0, and there are constants a_0 and a_1 such that

$$\lim_{x\to x_0} \frac{f(x) - a_0 - a_1(x - x_0)}{x - x_0},$$

   then prove that $a_0 = f(x_0)$, f is differentiable at x_0, and $f'(x_0) = a_1$.

9. Find the Maclaurin's infinite series expansion for

   $a)$ $e^x, \ x \in \mathbb{R}$
   $b)$ $\log(1+x), \ x \in (-1, 1]$
   $c)$ $e^x \cos x, \ x \in \mathbb{R}$

10. Obtain the fourth degree Taylor's polynomial approximation to $f(x) = e^{2x}$ about $x = 0$. Find the maximum error when $0 \leq x \leq 0.5$.

11. Using Taylor's series find the approximate value of $a)$ $\sqrt{1.5}$ and $b)$ $\cos 31^\circ$.

12. Obtain the Maclaurin's series expansion of $f(x) = \sin(m \sin^{-1} x)$, where m is a constant.

13. For the Taylor's polynomial approximation of degree less than or equal to n about the point $x = 0$ for the function e^x, determine the value of n such that the error satisfies $|R_n(x)| \leq 0.005$, when $-1 \leq x \leq 1$.

14. Can the function $f(x)$ defined by $f(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ be expanded by Maclaurin's theorem?

15. Find the Taylor's series expansion of $\sin^2 x$ upto five terms with Lagrange's form of remainder.

16. Find the Maclaurin's series expansion of $\tan^{-1} x$ upto four terms with Lagrange's form of remainder.