Answer sheet - 12

Mathematics-I(MA10001) Autumn 2019

1. a) Ans: $z = 0$(simple pole).
   b) Ans: $z = 0$(pole of order 3), $z = n\pi$(simple pole) where n is a non-zero integer.
   c) Ans: $z = -1$(simple pole), $z = 1$(pole of order 2).
   d) Ans: $z = 0$(pole of order 3).
   e) Ans: $z = 0$(removable).
   f) Ans: $z = 0$(simple pole), $z = \sqrt{2n\pi i}$(simple pole) where n is a non-zero integer.
   g) Ans: $z = 0$(essential).
   h) Ans: $z = 0$(simple pole), $z = \pm i$(simple pole).
   i) Ans: $z = 0$(pole of order 2).

2. a) Ans: Res(f, $\pm i$)$=\pm\dfrac{3i}{4}$.
   b) Ans: Res(f, 1)$=-e$, Res(f, 0)$=e-1$.
   c) Ans: Res(f, 0)=0.
   d) Ans: Res(f, 0)=0.

3. a) Ans: $\dfrac{1}{(z-i)(z-2)} = \dfrac{1}{i-2}\displaystyle\sum_{n=0}^{n=\infty}[(\frac{1}{i})^{n-1} + \frac{1}{2^{n+1}}]z^n$. [Hint: $\dfrac{1}{z-i} = \dfrac{i}{1-\frac{z}{i}}$ and $\dfrac{1}{z-2} = -\dfrac{1}{2}\dfrac{1}{(1-\frac{z}{2})}$]
   b) Ans: $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{n=\infty}\frac{(z-2i)^n}{(1-2i)^{n+1}}, \mid z-2i \mid< \sqrt{5}$. [Hint: $\dfrac{1}{1-z} = \dfrac{1}{1-2i-(z-2i)} = \dfrac{1}{(1-2i)}\dfrac{1}{(1-\frac{z-2i}{1-2i})}$]

4. Ans: $\dfrac{z^2-2z+3}{z-2} = (z-1) + 1 + \displaystyle\sum_{n=1}^{n=\infty}\frac{3}{(z-1)^n}$.

5. a) Ans: $(z-3)\sin\dfrac{1}{z+2} = 1 - \dfrac{5}{z+2} - \dfrac{1}{6(z+2)^2} + \dfrac{5}{6(z+2)^3} + \ldots\ldots\ldots\ldots$, $z \neq 2$.[Hint: Put $u = z+2$ and expand $\sin\dfrac{1}{u}$ in Laurent series.]
   b) Ans: $\dfrac{e^{2z}}{(z-1)^3} = \dfrac{e^2}{(z-1)^3} + \dfrac{2e^2}{(z-1)^2} + \dfrac{2e^2}{z-1} + \dfrac{4e^2}{3} + \dfrac{2e^2}{3}(z-1) + \ldots\ldots\ldots\ldots$, for $z \neq 1$[Hint: put $u = z-1$, $\dfrac{e^{2z}}{(z-1)^3} = \dfrac{e^2}{u^3}e^{2u}$]

6. Ans: $z^2e^{\frac{1}{z}} = z^2 + z + \displaystyle\sum_{n=0}^{n=\infty}\frac{1}{(n+2)!}z^{-n}$.[Hint: Expand $e^{1/z}$ in Taylor series about $z = 0$]

7. a) Ans: $\dfrac{1}{z(1-z)(2-z)} = \dfrac{1}{2z} + \displaystyle\sum_{k=0}^{k=\infty}(1-2^{-(k+2)})z^k$, $0 <\mid z < 1$.[Hint: $f(z) = \dfrac{1}{z}(\dfrac{1}{1-z} - \dfrac{1}{2-z})$ and $\dfrac{1}{2-z} = \dfrac{1}{2}\dfrac{1}{(1-\frac{z}{2})}$]
   b) Ans: $\dfrac{1}{z(1-z)(2-z)} = -\displaystyle\sum_{n=-\infty}^{n=-2}z^n - \sum_{n=-1}^{n=\infty}2^{-(n+2)}z^n$, $1 <\mid z \mid< 2$.[Hint: $f(z) = \dfrac{1}{z}(\dfrac{1}{1-z} - \dfrac{1}{2-z})$ and $\dfrac{1}{1-z} = -\dfrac{1}{z}\dfrac{1}{1-\frac{1}{z}}$ and $\dfrac{1}{2-z} = \dfrac{1}{2}\dfrac{1}{(1-\frac{z}{2})}$]
   c) Ans: $\dfrac{1}{z(1-z)(2-z)} = \displaystyle\sum_{k=-\infty}^{k=-3}(2^{-(k+2)}-1)z^k$, $2 <\mid z \mid$. [Hint: Use (a) and (b)].

8. Ans: $\dfrac{e^z}{(z-1)^2} = \displaystyle\sum_{n=-2}^{n=\infty} \dfrac{e}{(n+2)^n}(z-1)^n$, $0 <\mid z \mid< \infty$.[Hint: Expand e^z in Taylor series about $z = 1$]

9. a) Ans: $10\pi i$.[Hint: Use Cauchy-Residue theorem]
   b) Ans: $-\dfrac{i\pi}{3}$.[Hint: Use Cauchy-Residue theorem]
   c) Ans: $\dfrac{\pi i}{60}$.[Hint: Use Cauchy-Residue theorem]
   d) Ans: $-\dfrac{\pi i}{2e}$.[Hint: Use Cauchy-Residue theorem]

10. Ans: $-4\pi + 12\pi i$.[Hint: Use Cauchy-Residue theorem]

11. a) Ans: $4\pi i$.[Hint: Use Cauchy-Residue theorem]
    b) Ans: Integration value$= 2\pi i$, if $n = 1$; Integration value$= 0$, if $n \neq 1$.[Hint: Put $z - a = re^{i\theta}, 0 \leq \theta \leq 2\pi$]

12. Ans: $2\pi i$.[Hint: Use Cauchy-Residue theorem]