Solutions of Tutorial Sheet-7

1. (i) Given set is $\mathbb{R}^3$

The given operations are

$$(x,y,z)+(x',y',z') = (x+x', y+y', z+z') \quad \text{--- (1)}$$

for $(x,y,z),(x',y',z') \in \mathbb{R}^3$

and $K(x,y,z) = (Kx, y, z)$ for $K \in \mathbb{R}$, $(x,y,z) \in \mathbb{R}^3$.

The given addition (1) satisfies all the properties of vector addition.

Now, let $k_1, k_2 \in \mathbb{R}$ and $(x,y,z) \in \mathbb{R}^3$.

Then
$$(k_1+k_2)(x,y,z) = ((k_1+k_2)x, y, z)$$
$$= (k_1x+k_2x, y, z)$$
$$\neq (k_1x, y, z) + (k_2x, y, z)$$

$\therefore (k_1+k_2)(x,y,z) \neq k_1(x,y,z) + k_2(x,y,z)$.

Therefor $\mathbb{R}^3$ is NOT a vector space under these operations.

Counter example: Let $k_1 = 2$, $k_2 = 1$, $(x,y,z) = (-1,3,2)$

$(k_1+k_2)(x,y,z) = (2+1)(-1,3,2) = (-3, 3, 2)$

$k_1(x,y,z) = 2(-1,3,2) = (-2,3,2)$

$k_2(x,y,z) = 1(-1,3,2) = (-1,3,2)$

$\therefore k_1(x,y,z) + k_2(x,y,z) = (-2,3,2) + (-1,3,2)$
$= (-3, 6, 4)$.

(ii) Given set is $V = \{(x_1,x_2) \in \mathbb{R}^2 ; x_1+x_2 = 1, 0 \le x_1 \le 1, 0 \le x_2 \le 1\}$ with the operations

$$(x_1,x_2)+(y_1,y_2) = \left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}\right) \text{ for } (x_1,x_2),(y_1,y_2) \in \mathbb{R}^2.$$

and $r(x_1,x_2) = (rx_1, rx_2)$, $r \in \mathbb{R}$.

Given addition is closed in V.

Let $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in V$

$$((x_1,x_2)+(y_1,y_2)) + (z_1,z_2) = \left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}\right) + (z_1,z_2)$$

$$\therefore \left((x_1,x_2)+(y_1,y_2)\right)+(z_1,z_2) = \left(\frac{\frac{x_1+y_1}{2}+z_1}{2}, \frac{\frac{x_2+y_2}{2}+z_2}{2} \right)$$

$$= \left(\frac{x_1+y_1+2z_1}{4}, \frac{x_2+y_2+2z_2}{4} \right)$$

Again $(x_1,x_2) + \left((y_1,y_2)+(z_1,z_2)\right)$

$$= (x_1,x_2) + \left(\frac{y_1+z_1}{2}, \frac{y_2+z_2}{2} \right)$$

$$= \left(\frac{x_1 + \frac{y_1+z_1}{2}}{2}, \frac{x_2 + \frac{y_2+z_2}{2}}{2} \right)$$

$$= \left(\frac{2x_1+y_1+z_1}{4}, \frac{2x_2+y_2+z_2}{4} \right)$$

clearly, $\left((x_1,x_2)+(y_1,y_2)\right)+(z_1,z_2) \neq (x_1,x_2)+\left((y_1,y_2)+(z_1,z_2)\right)$ in general. Therefore '+' is not associative.

$\therefore$ V is NOT a vector space under these operations.

Counter example: let $(x_1,x_2) = (1,3)$, $(y_1,y_2) = (4,2)$
$(z_1,z_2) = (7,5)$

$$\therefore \left((x_1,x_2)+(y_1,y_2)\right)+(z_1,z_2) = \left((1,3)+(4,2)\right)+(7,5)$$

$$= \left(\frac{5}{2}, \frac{5}{2}\right) + (7,5)$$

$$= \left(\frac{\frac{5}{2}+7}{2}, \frac{\frac{5}{2}+5}{2} \right)$$

$$= \left(\frac{19}{4}, \frac{15}{4} \right)$$

$$(x_1,x_2)+\left((y_1,y_2)+(z_1,z_2)\right) = (1,3) + \left((4,2)+(7,5)\right)$$

$$= (1,3) + \left(\frac{11}{2}, \frac{7}{2}\right)$$

$$= \left(\frac{1+\frac{11}{2}}{2}, \frac{3+\frac{7}{2}}{2} \right)$$

$$= \left(\frac{13}{4}, \frac{13}{4} \right)$$

1. (iii) Given set is $\mathbb{R}_+$, with the operations

$$x + x' = x x' \quad \text{for } x, x' \in \mathbb{R}_+$$

and $$kx = x^k \quad \text{for } k \in \mathbb{R},\ x \in \mathbb{R}_+.$$

Now 1. $x + x' \in \mathbb{R}_+$

2. $(x + x') + x'' = x x' x'' = x + (x' + x'')$ for $x, x', x'' \in \mathbb{R}_+$

3. $x + 1 = x \cdot 1 = x = 1 \cdot x = 1 + x \quad \forall x \in \mathbb{R}_+$

(1 is the additive identity.)

4. $x + \frac{1}{x} = x \cdot \frac{1}{x} = 1 = \frac{1}{x} \cdot x = \frac{1}{x} + x$

($\frac{1}{x}$ is the additive inverse of x)

5. $x + x' = x x' = x' x = x' + x$

6. $kx = x^k \in \mathbb{R}_+$,

7. $(k_1 + k_2) x = x^{k_1 + k_2} = x^{k_1} x^{k_2}$

$= (k_1 x) + (k_2 x)$, for $k_1, k_2 \in \mathbb{R}$.

8. $k(x + x') = (x + x')^k$

$= (x x')^k$

$= x^k x'^k = (kx) + (kx')$, for $x, x' \in \mathbb{R}_+$

9. $1 \cdot x = x^1 = x$, for $x \in \mathbb{R}_+$

$\mathbb{R}_+$ with the given addition and scalar multiplication satisfies all the properties of a vector space.

(iv) Given set is $S = \left\{ \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} ;\ a, b \in \mathbb{R} \right\}$ with usual addition and scalar multiplication of matrices.

Let $\begin{pmatrix} a_1 & 1 \\ 1 & b_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & 1 \\ 1 & b_2 \end{pmatrix}$ are in S.

$$\begin{pmatrix} a_1 & 1 \\ 1 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 1 \\ 1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & 2 \\ 2 & b_1 + b_2 \end{pmatrix}$$

$$\notin S.$$

therefore S is NOT closed under matrix addition.
S is NOT a vector space.

(V) Given set is

$$V = \{ f \in C(\mathbb{R}) : \exists\, p \in \mathbb{N},\ f(x+p) = f(x),\ \forall x \in \mathbb{R} \}$$

with the operations

$$(f_1+f_2)(x) = f_1(x) + f_2(x), \quad \forall x \in \mathbb{R}.$$

and $$(r f_1)(x) = r f_1(x), \quad \forall x \in \mathbb{R},\ r \in \mathbb{R}.$$

Let $f_1(x), f_2(x) \in V$. Then $\exists\, p_1, p_2 \in \mathbb{N}$

such that $f_1(x+p_1) = f(x)$
$f_2(x+p_2) = f(x) \quad \forall x \in \mathbb{R}.$

Let $p' = \text{l.c.m}(p_1, p_2)$. Then $p' = k_1 p_1$ and $p' = k_2 p_2$ for some positive integers k_1, k_2.

Now ~~f(x)~~ $(f_1+f_2)(x+p')$

$= f_1(x+p') + f_2(x+p')$

$= f_1(x+k_1p_1) + f_2(x+k_2p_2)$ [Since p_1, p_2 are periods of f_1 and f_2 respectively. $\therefore$ k_1p_1 & k_2p_2 are also periods of f_1 & f_2 respectively]

$= f_1(x) + f_2(x) = (f_1+f_2)(x)$

$\therefore\ f_1 + f_2 \in V.$

For $r \in \mathbb{R}$, $(rf)(x+p'') = r f(x+p'')$, p'' is the period of $f \in V$.

$= r f(x)$

$= (rf)(x)$

$\therefore\ rf \in V$

All other properties of vector space are satisfied as $f_1, f_2 \in V \Rightarrow f_1, f_2 \in C(\mathbb{R})$, and $C(\mathbb{R})$ is a vector space. Therefore V is a vector space under these operations.

2.

(i) Given subset of $\mathbb{R}^3$ is

$$S = \{(a,b,c) \in \mathbb{R}^3 : b = a+c\}$$
$$= \{(a, a+c, c) : a, c \in \mathbb{R}\}$$

Let $(a_1, a_1+c_1, c_1), (a_2, a_2+c_2, c_2) \in S$

and $\alpha, \beta \in \mathbb{R}$

Then $\alpha(a_1, a_1+c_1, \cancel{c_2}) + \beta(a_2, a_2+c_2, c_2)$

$$= (\alpha a_1 + \beta a_2, \alpha a_1 + \beta a_2 + \alpha c_1 + \beta c_2, \alpha c_1 + \beta c_2)$$

$\in S$ as $\alpha a_1 + \beta a_2 \in \mathbb{R}$ & $\alpha c_1 + \beta c_2 \in \mathbb{R}$.

S is a subspace of $\mathbb{R}^3$.

(ii) Given subset of $M_{n \times n}$ is

$$S = \{A \in M_{n\times n} : A = A^T\}$$

Let $A, B \in S$ and $\alpha, \beta \in \mathbb{R}$

Then $(\alpha A + \beta B)^T = \alpha A^T + \beta B^T = \alpha A + \beta B$

[As $A = A^T, B = B^T$]

Therefore $\alpha A + \beta B \in S$.

S is a subspace of $M_{n\times n}$.

(iii) Given subset of $M_{2\times 2}$ is

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2} : a+d = 0 \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

Let $\begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix} \in S$

and $\alpha, \beta \in \mathbb{R}$. Then.

$$\alpha \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} + \beta \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & -(\alpha a_1 + \beta a_2) \end{pmatrix} \in S.$$ Therefore S is a subspace of $M_{2\times 2}$

(iv) Given subset of $M_{n\times n}$ is

$$S = \{A \in M_{n\times n} : \det A = 0\}$$

Let $A_1, A_2 \in S$. Then $\det A_1 = 0$ & $\det A_2 = 0$

But this does not necessarily imply $\det(A_1+A_2) = 0$

For example let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_{2\times 2}$

Both have $\det A_1 = 0$, $\det A_2 = 0$, but

$A_1 + A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\det(A_1+A_2) = 1 \neq 0$.

Hence $A_1 + A_2 \notin S$. S is NOT a subspace of $M_{n\times n}$.

(V) Given subset of $\mathbb{R}^3$ is

$$S = \{(a,b,c) \in \mathbb{R}^3 : ab = 0\}$$

Let $(a_1, 0, c_1) \in S$ with $a_1 \neq 0$ (~~let~~)

$(0, b_2, c_2) \in S$ with $b_2 \neq 0$ (~~let~~)

Then $(a_1, 0, c_1) + (0, b_2, c_2) = (a_1, b_2, c_1 + c_2)$

$\notin S$ as $a_1 b_2 \neq 0$

$\therefore$ S is NOT a subspace of $\mathbb{R}^3$.

(VI) Given subset of $\mathbb{R}^3$ is

$$W = \{(a,b,c) \in \mathbb{R}^3 ; a^3 = b^3\}$$

Since the Equation $a^3 = b^3$ implies $a = b$ in $\mathbb{R}$

Therefore $W = \{(a,b,c) \in \mathbb{R}^3 ; a = b\}$

$= \{(a,a,c) : a, c \in \mathbb{R}\}$

Let $(a_1, a_1, c_1), (a_2, a_2, c_2) \in W$, $\alpha, \beta \in \mathbb{R}$

Then $\alpha(a_1, a_1, c_1) + \beta(a_2, a_2, c_2)$

$= (\alpha a_1 + \beta a_2, \alpha a_1 + \beta a_2, \alpha c_1 + \beta c_2) \in W$

$\therefore$ W is a subspace of $\mathbb{R}^3$.

But if we think the subset of $\mathbb{C}^3$

$$W = \{(a,b,c) \in \mathbb{C}^3 : a^3 = b^3\}$$
$$= \{(a,b,c) : a = b,\ a = \omega b,\ a = \omega^2 b \text{ where } \omega^3 = 1, a,b,c \in \mathbb{C}\}$$

Now $(2,2,0) \in W$, $(1,\omega,0) \in W$

But $(2,2,0) + (1,\omega,0) = (3, 2+\omega, 0)$

$\notin W$ since $3^3 = 27$

$\therefore$ W is NOT a subspace of $\mathbb{C}^3$. $(2+\omega)^3 = 9+8\omega+4\omega^2$

3. $S = \{f \in C[0,1] : \int_0^1 f(x)\,dx = b, \text{ for some fixed } b \in \mathbb{R}\}$

If S is a subspace of $C[0,1]$, then

$0(x) \in S$. where $0(x) = 0 \ \forall x \in [0,1]$

Then $\int_0^1 0(x)\,dx = 0 = b$

Conversely, if $b = 0$, then $S = \{f \in C[0,1] : \int_0^1 f(x)\,dx = 0\}$

Let $f_1(x), f_2(x) \in S$ and $\alpha, \beta \in \mathbb{R}$.

$g(x) = \alpha f_1(x) + \beta f_2(x)$

$$\int_0^1 g(x)\,dx = \int_0^1 [\alpha f_1(x) + \beta f_2(x)]\,dx$$
$$= \alpha \int_0^1 f_1(x)\,dx + \beta \int_0^1 f_2(x)\,dx$$
$$= 0$$

$\left[\because \int_0^1 f_1(x) = 0 = \int_0^1 f_2(x)\,dx\right]$

$\therefore$ $g(x) = \alpha f_1(x) + \beta f_2(x) \in S$

$\therefore$ S is a subspace of $C[0,1]$.

4. The given sub set of $C[-4,4]$ is

$$S = \{ f(x) \in C[-4,4] \; ; \; f'(-1) = 3f(2) \}$$

Let $f_1, f_2 \in S$. Then $f_1'(-1) = 3f_1(2)$

$$f_2'(-1) = 3f_2(2)$$

for $\alpha, \beta \in \mathbb{R}$, set

$$g(x) = \alpha f_1(x) + \beta f_2(x)$$

$$\therefore g'(-1) = \alpha f_1'(-1) + \beta f_2'(-1)$$

$$= \alpha \cdot 3 \cdot f_1(2) + \beta \, 3 \cdot f_2(2)$$

$$= 3\left[\alpha f_1(2) + \beta f_2(2)\right]$$

$$= 3\, g(2)$$

$\therefore \quad g(x) = \alpha f_1(x) + \beta f_2(x) \in S$.

S is a subspace of $C[-4,4]$.

5. (a) If $E = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ is a linear combination of $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

Then there exists scalars α, β, γ such that

$$\underline{E} = \alpha A + \beta B + \gamma C$$

or $$\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \alpha \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha + \beta + \gamma & \alpha + \beta - \gamma \\ -\beta & -\alpha \end{pmatrix}$$

Comparing both sides,

$$\begin{aligned} \alpha + \beta + \gamma &= 3 \\ \alpha + \beta - \gamma &= -1 \\ -\beta &= 1 \\ -\alpha &= -2 \end{aligned} \quad \Longrightarrow \quad \alpha = 2, \beta = -1, \gamma = 2$$

$$\therefore \quad E = 2A - B + 2C.$$

5. (b) If $p = 2+2x+3x^2$ is a linear combination of $p_1 = 2+x+4x^2$, $p_2 = 1-x+3x^2$, $p_3 = 3+2x+5x^2$, then there exists scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$p = \alpha p_1 + \beta p_2 + \gamma p_3$$

or $2+2x+3x^2 = \alpha(2+x+4x^2) + \beta(1-x+3x^2) + \gamma(3+2x+5x^2)$

$$= (2\alpha+\beta+3\gamma) + (\alpha-\beta+2\gamma)x + (4\alpha+3\beta+5\gamma)x^2$$

Comparing both sides.

$$2\alpha+\beta+3\gamma = 2$$
$$\alpha-\beta+2\gamma = 2$$
$$4\alpha+3\beta+5\gamma = 3$$

Solution of this system is $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}, \gamma = \frac{1}{2}$

$\therefore \quad p = \frac{1}{2}p_1 - \frac{1}{2}p_2 + \frac{1}{2}p_3.$

(c) $u = (1,-1,3), \quad v = (2,4,0)$

(i) If $(3,3,3)$ is a linear combination of u and v then, $\exists$ scalars $\alpha, \beta \in \mathbb{R}$ such that

$$(3,3,3) = \alpha(1,-1,3) + \beta(2,4,0)$$
$$= (\alpha+2\beta, -\alpha+4\beta, 3\alpha)$$

Comparing, $\quad \alpha+2\beta = 3$
$\quad -\alpha+4\beta = 3 \quad \Rightarrow \quad \alpha = 1, \beta = 1$
$\quad 3\alpha = 3$

$\therefore \quad (3,3,3) = 1 \cdot u + 1 \cdot v.$

(C) (ii) If $(4,2,6)$ is a linear combination of u and v.

Then $\exists$ scalars α, β such that-

$$(4,2,6) = \alpha(1,-1,3) + \beta(2,4,0)$$
$$= (\alpha+2\beta, -\alpha+4\beta, 3\alpha)$$

Comparing, $\alpha+2\beta = 4$, $-\alpha+4\beta=2$, $3\alpha = 6$ $\Rightarrow$ $\alpha = 2$, $\beta = 1$.

$\therefore$ $(4,2,6) = 2.u + 1.v$.

(iii) If $(1,5,6)$ is a linear Combination of u and v

we get- $\alpha+2\beta = 1$, $-\alpha+4\beta = 5$, $3\alpha = 6$ has no solution

$\therefore$ $(1,5,6)$ can not be written in the linear Combination of u and v.

(iv) If $(0,0,0)$ is a linear Combination of u and v

we get $\alpha+2\beta = 0$, $-\alpha+4\beta = 0$, $3\alpha = 0$ $\Rightarrow$ $\alpha = 0$, $\beta = 0$

$\therefore$ $(0,0,0) = 0.u + 0.v$.

6. $u_1 = (1,2,1)$, $u_2 = (3,1,5)$, $u_3 = (3,-4,7)$.

Span$\{u_1, u_2\}$ = Span$\{u_1, u_2, u_3\}$ holds if

$u_3 = \alpha u_1 + \beta u_2$ for some $\alpha, \beta \in \mathbb{R}$.

or $(3,-4,7) = \alpha(1,2,1) + \beta(3,1,5)$
$$= (\alpha+3\beta, 2\alpha+\beta, \alpha+5\beta)$$

Comparing, $\alpha+3\beta = 3$, $2\alpha+\beta = -4$, $\alpha+5\beta = 7$ $\Rightarrow$ $\alpha = -3$, $\beta = 2$

$\therefore$ $u_3 = -3u_1 + 2u_2$

and Span$\{u_1, u_2\}$ = Span$\{u_1, u_2, u_3\}$

7. (a) $S = \{v_1, v_2, v_3, v_4\}$ spans V.

Let $W = \{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$

Now if Span W = Span S = V holds.

then we can get scalars $\alpha, \beta, \gamma, \delta$ such that

$$\alpha(v_1 - v_2) + \beta(v_2 - v_3) + \gamma(v_3 - v_4) + \delta v_4$$

$$= a v_1 + b v_2 + c v_3 + d v_4$$

Comparing, [For any $a, b, c, d \in F$]

$$\alpha = a$$
$$-\alpha + \beta = b$$
$$-\beta + \gamma = c$$
$$-\gamma + \delta = d$$

$\Rightarrow$ $\alpha = a, \beta = a+b, \gamma = a+b+c,$
$\delta = a+b+c+d.$

$\therefore$ Span W = Span S = V.

(b) $S = \{u_1, u_2, u_3\}$, $T = \{u_1, u_1+u_2, u_1+u_2+u_3\}$

and $U = \{u_1+u_2, u_2+u_3, u_3+u_4\}$ are sets of vectors in $\mathbb{R}^4$.

If span S = span T holds, then we can get scalars α, β, γ such that

$$\alpha(u_1) + \beta(u_1+u_2) + \gamma(u_1+u_2+u_3)$$

$$= a u_1 + b u_2 + c u_3$$

[For any $a, b, c \in F$]

Comparing,

$$\alpha + \beta + \gamma = a$$
$$\beta + \gamma = b$$
$$\gamma = c$$

$\Rightarrow$ $\alpha = a - b, \beta = b - c, \gamma = c$

$\therefore$ Span S = Span T.

Similarly if Span S = Span U then $\exists\ \alpha, \beta, \gamma \in F$ such that

$$\alpha(u_1+u_2)+\beta(u_2+u_3)+\gamma(u_3+u_1) = au_1+bu_2+cu_3$$

Comparing, $\alpha+\gamma = a$

$\alpha+\beta = b$

$\beta+\gamma = c$

$\therefore\quad \alpha = \dfrac{a+b-c}{2},\ \beta = \dfrac{b+c-a}{2},\ \gamma = \dfrac{a+c-b}{2}$

$\therefore$ Span S = Span U = Span T.

8. (a) $S = \{(4,-4,8,0),(2,2,4,0),(6,0,0,2),(6,3,-3,0)\}$

To check the linear dependency of S in $\mathbb{R}^4$.

~~Sol~~ Let us consider the relation

$$c_1(4,-4,8,0)+c_2(2,2,4,0)+c_3(6,0,0,2)+c_4(6,3,-3,0) = 0$$

$$\Rightarrow (4c_1+2c_2+6c_3+6c_4,\ -4c_1+2c_2+3c_4,\ 8c_1+4c_2-3c_4,\ 2c_3) = (0,0,0,0)$$

$4c_1+2c_2+6c_3+6c_4 = 0$

$-4c_1+2c_2+3c_4 = 0$

$8c_1+4c_2-3c_4 = 0$

$2c_3 = 0$

$$\Rightarrow \begin{pmatrix} 4 & 2 & 6 & 6 \\ -4 & 2 & 0 & 3 \\ 8 & 4 & 0 & -3 \\ 0 & 0 & 2 & 0 \end{pmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{vmatrix} 4 & 2 & 6 & 6 \\ -4 & 2 & 0 & 3 \\ 8 & 4 & 0 & -3 \\ 0 & 0 & 2 & 0 \end{vmatrix} = -480 \neq 0$$

$\therefore \; c_1 = c_2 = c_3 = c_4 = 0$

S is linearly independent.

(b) $S = \{2, 4\sin^2 x, \cos^2 x\}$ in $C[-\pi, \pi]$

To check the linear dependency of S let us consider the relation

$$c_1 \cdot 2 + c_2 \, 4\sin^2 x + c_3 \cos^2 x = 0$$

Differentiating w.r.t 'x'

$$c_1 \cdot 0 + c_2 \, 4\sin 2x + c_3(-\sin 2x) = 0$$

Again Differentiating w.r.t 'x'

$$c_1 \cdot 0 + c_2 \cdot 8\cos 2x + c_3(-2\cos 2x) = 0$$

In matrix form.

$$\begin{pmatrix} 2 & 4\sin^2 x & \cos^2 x \\ 0 & 4\sin 2x & -\sin 2x \\ 0 & 8\cos 2x & -2\cos 2x \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

As $\begin{vmatrix} 2 & 4\sin^2 x & \cos^2 x \\ 0 & 4\sin 2x & -\sin 2x \\ 0 & 8\cos 2x & -2\cos 2x \end{vmatrix}$

$$= 2(-8\sin 2x\cos 2x + 8\sin 2x\cos 2x)$$
$$= 0$$

$\exists$ non zero $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \cdot 2 + c_2 \, 4\sin^2 x + c_3 \cos^2 x = 0$$

$\therefore$ S is linearly dependent.

(c) $S = \{t^3-5t^2-2t+3,\ t^3-4t^2-3t+4,\ 2t^3-7t^2-7t+9\}$

To check the linear dependency of S in $\mathbb{P}_3$.

Let us consider the relation.

$$c_1(t^3-5t^2-2t+3)+c_2(t^3-4t^2-3t+4)$$
$$+c_3(2t^3-7t^2-7t+9) = 0$$

Comparing,

$c_1+c_2+2c_3 = 0$ —— (i)

$-5c_1-4c_2-7c_3 = 0$ —— (ii)

$-2c_1-3c_2-7c_3 = 0$ —— (iii)

$3c_1+4c_2+9c_3 = 0$ —— (iv)

Using (ii)+5(i) $\quad c_2+3c_3 = 0$ —(v)

(iii)+2(i) $\quad -c_2-3c_3 = 0$ —(vi)

(iv)−3(i) $\quad c_2+3c_3 = 0$ —(vii)

(v), (vi) & (vii) are same equations.

∴ The system reduces to

$$c_1+c_2+2c_3 = 0$$
$$c_2+3c_3 = 0$$

let $c_3 = k$, $\quad c_1 = c_3 = k \quad$ let $k \neq 0$ & $k \in \mathbb{R}$

$c_2 = -3k$.

Then $k(t^3-5t^2-2t+3)-3k(t^3-4t^2-3t+4)$
$$+k(2t^3-7t^2-7t+9) = 0$$

For any non zero value of k this relation holds

∴ S is a linearly dependent.

8. (d)

$f_1(t) = t \; , \; t \in [-1,1]$

$f_2(t) = \begin{cases} -t, & t \in [-1,0] \\ t, & t \in [0,1] . \end{cases}$

$\{f_1, f_2\}$ is linearly dependent ~~iff~~ in $[-1,1]$ if

$$f_1(t) = K f_2(t) \quad \forall t \in [-1,1] . \quad (1)$$

In $[-1,0]$, $f_1(t) = t$

$= -(-t)$

$= -f_2(t) \quad \forall t \in [-1,0]$

$\{f_1, f_2\}$ is linearly dependent in $[-1,0]$

In $[0,1]$ $\quad f_1(t) = t = f_2(t) \;\; \forall t \in [0,1]$

$\{f_1, f_2\}$ is linearly dependent in $[0,1]$

In $[-1,1]$. There does not exist any fixed k in $[-1,1]$ such that (1) holds.

$\therefore$ $\{f_1, f_2\}$ is linearly independent in $[-1,1]$.

(e) when $\{1+i, 1-i\} \in \mathbb{C}(\mathbb{R})$.

To check the linear dependency of this set

Let us consider the relation

$c_1(1+i) + c_2(1-i) = 0$ where $c_1, c_2 \in \mathbb{R}$.

$\Rightarrow \begin{array}{l} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{array} \Rightarrow c_1 = c_2 = 0$

$\therefore$ $\{1+i, 1-i\}$ is linearly independent in $\mathbb{C}(\mathbb{R})$.

when $\{1+i, 1-i\} \in \mathbb{C}(\mathbb{C})$,

It is not linearly independent, Because

We can choose $c_1 = 1$, $c_2 = -i$, Then $c_1, c_2 \in \mathbb{C}$ and $c_1, c_2 \neq 0$. Such that

$$c_1(1+i) + c_2(1-i) = 1(1+i) - i(1-i) = 0.$$

$\therefore$ $\{1+i, 1-i\}$ is linearly dependent in $\mathbb{C}(\mathbb{C})$.

(f) $S = \{p_0, p_1, \ldots, p_m\} \subset \mathbb{P}_m$ such that

$$p_j(2) = 0 \quad \text{for } j = 0, 1, \ldots, m.$$

To check the linear dependency of S let us consider the relation

$$c_0 p_0(x) + c_1 p_1(x) + \cdots + c_m p_m(x) = 0 \quad \text{—} (1)$$

Now S will be linearly independent if (1) implies $c_0 = c_1 = \cdots = c_m = 0 \quad \forall x \in \mathbb{R}$.

But for $x = 2$ we can choose

$$c_0 = c_1 = \cdots = c_m = k \ (\neq 0)$$

such that (1) holds. Since $p_j(2) = 0$, $j = 0, \ldots, m$.

Hence S is NOT linearly independent.