R.K + BA

1 (a)

(1) $f(x,y) = \begin{cases} x^3 \sin\frac{1}{x^2} + y^3 \sin\frac{1}{y^2}, & \text{if } xy \neq 0 \\ 0 & \text{otherwise} \end{cases}$

Check the continuity & diff. of $f(x,y)$ at $(0,0)$.

Sol:-

$$\left| x^3 \sin\frac{1}{x^2} + y^3 \sin\frac{1}{y^2} \right| \leq |x^3| + |y^3|$$

$$\leq \left(\sqrt{(x^2+y^2)}\right)^3 + \left(\sqrt{x^2+y^2}\right)^3$$

$$= 2 \cdot \left(\sqrt{x^2+y^2}\right)^3$$

No marks for without justification of continuity

choose $\delta = \left(\frac{\epsilon}{2}\right)^{\frac{1}{3}}$

$$< 2(\delta^3)$$

$$= 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

$\therefore$ f is continuous at $(0,0)$

2

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} h^3 \sin\frac{1}{h^2}.$$

1

$$= 0$$

& $f_y(0,0) = 0$.

$$\underline{\Delta z = \epsilon_1 \Delta x + \epsilon_2 \Delta y} = \Delta x^3 \sin\frac{1}{\Delta x} + \Delta y^3 \sin\frac{1}{\Delta y}$$

$(f(x+\Delta x, y+\Delta y) - f(x,y))$

$$= \Delta x \left(\Delta x^2 \sin\frac{1}{\Delta x}\right) + \Delta y \left(\Delta y^2 \sin\frac{1}{\Delta y}\right)$$

$\epsilon_1 = \Delta x^2 \sin\frac{1}{\Delta x}$ $\quad \epsilon_2 = \Delta y^2 \sin\frac{1}{\Delta y}$

1

$, \epsilon_1 \to 0$ as $\Delta x, \Delta y \to 0$.

$\therefore f$ is differentiable at $(0,0)$.

1.(b) Find the distance from the point $(0,0,0)$ to the curve $z^2 = x^2+y^2$ & $x-2z=3$.

Sol:

$$\min \quad x^2+y^2+z^2$$
$$\text{s.t.} \quad x^2+y^2-z^2=0$$
$$x-2z-3=0$$

(1)

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

ie) $2x = 2x\lambda + \mu$

$2y = \lambda 2y$

$2z = -\lambda 2z - 2$

$\Rightarrow \lambda = 1$ (or) $y=0$

(1)

Case (1) $\lambda = 1$, $2x = 2x + \mu \Rightarrow \mu = 0$.

$2z = -2z - 2$

$\Rightarrow z = -\frac{1}{2}$

$x+1-3=0$

$\Rightarrow x = 2$

$4 + y^2 - \frac{1}{4} = 0$

$\Rightarrow y = \pm i \frac{\sqrt{15}}{2}$.

not possible

(1)

Case (2)

$y=0 \Rightarrow x = \pm z$.

(i) $x = z \Rightarrow x - 2z = 3$

$\Rightarrow x=-3, y=0, z=-3$ $(-3,0,-3)$

$\Rightarrow$ distance, $3\sqrt{2}$

(ii) $x=-z \Rightarrow 3x = 3 \Rightarrow x = 1, z=-1, y=0$

distance $= \sqrt{2}$. $(1,0,-1)$

minimum!

(1)

Prob: Evaluate $\lim_{x \to \infty} \left(\frac{1}{x} \frac{a^x - 1}{a-1}\right)^{1/x}$ where $a > 1$.

(c)

Soln: $\lim_{x \to \infty} \left(\frac{1}{x} \frac{a^x - 1}{a-1}\right)^{1/x}$

$= \lim_{x \to \infty} \left(\frac{1}{x} a^x \frac{1 - a^{-x}}{a-1}\right)^{1/x}$

$= \lim_{x \to \infty} \left(\frac{1}{x}\right)^{1/x} \cdot a \cdot \lim_{x \to \infty} \left(\frac{1 - a^{-x}}{a-1}\right)^{1/x}$

$= a \cdot \lim_{x \to \infty} \exp\left(\frac{1}{x} \ln \frac{1}{x}\right) \cdot \lim_{x \to \infty} \left(\frac{1}{a-1}\right)^{1/x}$

$\cdot \lim_{x \to \infty} \left(1 - \frac{1}{a^x}\right)^{1/x}$

$= a \cdot \left(\frac{1}{a-1}\right)^0 \cdot \lim_{x \to \infty} \exp\left(\frac{1}{x} \ln \frac{1}{x}\right)$

$= a \exp\left(\lim_{x \to \infty} -\frac{\ln x}{x}\right)$

$= a \exp(0)$ 2

$= a.$

No part marking.

Q2 (a) $M = x^3 + xy^2$ $\qquad$ $N = \alpha x^2 y + \beta x y^2$

$$\frac{\partial M}{\partial y} = 2xy \qquad \frac{\partial N}{\partial x} = 2\alpha xy + \beta y^2$$

Now $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow 2xy = 2\alpha xy + \beta y^2$

$\Rightarrow \alpha = 1$ and $\beta = 0$ [1]

$\therefore$ The eqn becomes $(x^3 + xy^2)dx + x^2 y\, dy = 0$

$$\Rightarrow \frac{x^4}{4} + \frac{x^2y^2}{2} = c \quad [1]$$

(b) $\frac{dy}{dx} - xy = x^3y^3$ $\qquad$ Put $v = y^{-2}$

$$\frac{dv}{dx} = -2y^{-3}\frac{dv}{dx}$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} - \frac{x}{y^2} = x^3$$

$$\Rightarrow \frac{dv}{dx} + 2xv = -2x^3 \quad [1]$$

$$\text{I.F} = e^{\int 2x\,dx} = e^{x^2}$$

$$\therefore v e^{x^2} = -\int e^{x^2} 2x^3\, dx \quad [1]$$

$$v e^z = -\int z e^z\, dz \qquad \text{Put } z = x^2$$

$$\Rightarrow v e^z = -z e^z + e^z + c$$

$$\Rightarrow \frac{1}{y^2} e^{x^2} = -x^2 e^{x^2} + e^{x^2} + c \quad [1]$$

$y(0) = \frac{1}{2} \Rightarrow c = 3 \quad \therefore \frac{1}{y^2} e^{x^2} = -x^2 e^{x^2} + e^{x^2} + 3$ [1]

Q2 (c): (4 M)

The char. eq. is $\lambda^3 - 2\lambda^2 - 7\lambda - 4 = 0$.

$\Rightarrow \quad \lambda = -1, -1, 4.$

$\therefore \quad y_c = (c_1 + c_2 x)e^{-x} + c_3 e^{4x}$. —— (1 M)

$y_p = ((D-4)(D+1)^2)^{-1} e^{-x} \sin 3x$

$= e^{-x} (D-5)^{-1} D^{-2} \sin 3x.$

$= e^{-x} (D-5)^{-1} \left(\frac{-1}{9}\right) \sin 3x.$ —— (1 M)

$= -\frac{e^{-x}}{9} (D+5) \left(\frac{-1}{34}\right) \sin 3x.$

$= \frac{e^{-x}}{306} (3\cos 3x + 5 \sin 3x)$ —— (1 M)

$\therefore$ The general solution is

$$y = (c_1 + c_2 x)e^{-x} + c_3 e^{4x} + \frac{e^{-x}}{306}(3\cos 3x + 5 \sin 3x).$$

—— (1 M)

OR: (Particular integral)

Suppose $y_p = e^{-x}(A \cos 3x + B \sin 3x)$.

we have $y_p''' - 2y_p'' - 7y_p' - 4y_p = e^{-x} \sin 3x.$

Then $A = \frac{3}{306}$ —— (1 M)

$B = \frac{5}{306}$ —— (1 M)

3a

$$\boxed{\begin{array}{l|l} (D+4)x + y = e^{3t} & x(0) = 2/15 \\ -2x + (D+1)y = e^{-4t} & y(0) = 1/15 \end{array}}$$

eliminating : x | y

$\boxed{(D^2+5D+6)y = 2e^{3t}}$ 1M $\boxed{(D^2+5D+6)x = 4e^{3t} - e^{-4t}}$

$\Rightarrow \boxed{y = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{15} e^{3t}}$ 1M $\boxed{x = c_1 e^{-2t} + c_2 e^{-3t} + \frac{2}{15} e^{3t} - \frac{1}{2} e^{-4t}}$

$y(0) = \frac{1}{15} = c_1 + c_2 + \frac{1}{15}$ | $x(0) = \frac{2}{15} = c_1 + c_2 + \frac{2}{15} - \frac{1}{2}$

$\Rightarrow \boxed{c_1 + c_2 = 0}$ | $\Rightarrow \boxed{c_1 + c_2 = 1/2}$

$x = \frac{1}{2}\left[(D+1)y - e^{-4t}\right]$ 1M $y = e^{3t} - (D+4)x$

$\boxed{x = \frac{1}{2}\left[-e^{-4t} - c_1 e^{-2t} - 2c_2 e^{-3t} + \frac{4}{15} e^{3t}\right]}$ | $\boxed{y = e^{3t} - 2c_1 e^{-2t} - c_2 e^{-3t} - \frac{14}{15} e^{3t}}$

$x(0) = \frac{2}{15} = \frac{1}{2}\left[-1 - c_1 - 2c_2 + \frac{4}{15}\right]$ | $y(0) = \frac{1}{15} = 1 - 2c_1 - c_2 - \frac{14}{15}$

$\Rightarrow \boxed{c_1 + 2c_2 = 1}$ | $\Rightarrow \boxed{2c_1 + c_2 = 0}$

$\Rightarrow c_1 = 1, c_2 = -1$ | $\Rightarrow c_1 = -1/2, c_2 = 1$

$$\left.\begin{array}{l} x(t) = -\frac{1}{2} e^{-2t} + e^{-3t} + \frac{2}{15} e^{3t} - \frac{1}{2} e^{-4t} \\ y(t) = \frac{1}{15} e^{3t} + e^{-2t} - e^{-3t} \end{array}\right\} \text{— 2M}$$

3b

$$x^2y'' - 2xy' + 2y = x^3 \sin x$$

$z = \ln x$ in the homogeneous eqn.

$$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = 0 \Rightarrow m = 1, 2$$

$$\boxed{y_h = c_1 x + c_2 x^2} \text{ — 2M}$$

$$\boxed{W(y_1, y_2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2} \text{ — 1M}$$

$$y_p = u(x) y_1 + v(x) y_2$$

$$u(x) = -\int \frac{y_2 \cdot r(x)}{W}$$

$$= -\int \frac{x^2 \; x \sin x}{x^2} \, dx = x\cos x - \sin x$$

$$v(x) = \int \frac{y_1 \, r(x)}{W} dx = \int \frac{x \; x \sin x}{x^2} dx = -\cos x$$

$$\boxed{y_p = x^2 \cos x - x \sin x - x^2 \cos x = -x \sin x} \text{ — 2M}$$

$$\boxed{y = c_1 x + c_2 x^2 - x \sin x} \text{ general soln.}$$

Note: If variation of parameters is not used after y_h, only 2 marks will be awarded.

4.9: $u(x,y) = e^{\alpha x}\cos y$

$u_x = \alpha e^{\alpha x}\cos y$, $u_y = -e^{\alpha x}\sin y$

$u_{xx} = \alpha^2 e^{\alpha x}\cos y$, $u_{yy} = -e^{\alpha x}\cos y$.

For $u(x,y)$ to be harmonic, $u_{xx}+u_{yy}=0$

$\Rightarrow (\alpha^2-1)\, e^{\alpha x}\cos y = 0, \ \forall x,y$

$\Rightarrow \alpha = \pm 1$.

The value of $\alpha > 0$ for which u is harmonic is that $\alpha = 1$. —— [1 mark]

Harmonic conjugate $v(x,y)$: —

C-R equations: $u_x = v_y$ & $u_y = -v_x$.

$v_x = -u_y = e^x \sin y$

$\Rightarrow v(x,y) = e^x \sin y + h(y)$

$\Rightarrow v_y = e^x \cos y + h'(y)$

$\because v_y = u_x \Rightarrow e^x\cos y + h'(y) = e^x \cos y$

$\Rightarrow h'(y) = 0$

$\Rightarrow h(y) = c$.

$\therefore v(x,y) = e^x \sin y + c$ —— [2 marks]

Now $f(z) = u + iv = \cancel{e^x\cos y + i(e^x\sin y + c)}$

$= e^x\cos y + i(e^x \sin y + c)$

$= e^x(\cos y + i\sin y) + ic$

$= e^z + ic$ —— [1 mark]

H.b $f(z) = 2x^2+y+i(y^2-x)$.

C-R equations: $u_x = v_y$ & $u_y = -v_x$.

$u_x = 4x$; $v_x = -1$,

$u_y = 1$ $v_y = 2y$.

Clearly $u_y = -v_x$

But $u_x = v_y$

$\Leftrightarrow 4x = 2y \Leftrightarrow y = 2x$

— [1 mark]

Hence $u_x = v_y$ is satisfied only on the line $y = 2x$. However for any z on this line $y = 2x$, there is no open disk about z in which C-R equations are satisfied.

$\therefore f$ is no where analytic.

— [2 marks]

* If some one writes that "Since Cauchy-Reimann equations are not satisfied, hence function is not analytic" No marks will be awarded.

4.c $\int_C (x^2+iy^2)\,dz$, $\quad C: z(t) = 3t + it^2, \; -1 \le t \le$

Sol $\quad x(t) = 3t, \quad y(t) = t^2, \quad -1 \le t \le 1.$

$dz(t) = (3 + i2t)\,dt.$

$$\therefore \int_C (x^2+iy^2)\,dz = \int_{-1}^{+1} \left[(3t)^2 + i(t^2)^2\right](3+i2t)\,dt$$

— [1 mark]

$$= \int_{-1}^{+1} \left[9t^2 + it^4\right](3+i2t)\,dt$$

$$= \int_{-1}^{+1} \left[(27t^2 - 2t^5) + i(18t^3 + 3t^4)\right]dt$$

$$= 27\cdot\left[\frac{t^3}{3}\right]_{-1}^{+1} - 0 + i\left[0 + 3\cdot\frac{t^5}{5}\right]_{-1}^{+1}$$

$$= \frac{27\times 2}{3} + i\,\frac{3}{5}\times 2$$

$$= 18 + i\,\frac{6}{5}$$

— [2 marks]

* If the final answer is not correct, two marks are deducted.

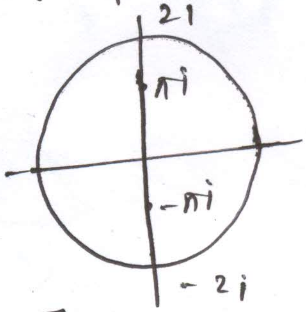
Dutta 25.10.19

5) (5a) a) Given $c: |z| = 4$, find the value of the integral

$$\oint_C \frac{e^z}{(z^2+\pi^2)^2}\, dz$$

using Cauchy's integral formula. (4M)

Sol^n: $z = \pm \pi i$ are inside $c: |z| = 4$



Now $\dfrac{1}{z^2+\pi^2} = \dfrac{1}{(z+i\pi)(z-i\pi)}$

$$= \frac{1}{2\pi i}\left[\frac{1}{z-i\pi} - \frac{1}{z+i\pi}\right]$$

$$\therefore \frac{1}{(z^2+\pi^2)^2} = -\frac{1}{4\pi^2}\left[\frac{1}{(z-i\pi)^2} + \frac{1}{(z+\pi i)^2} - \frac{2}{(z-\pi i)(z+\pi i)}\right]$$

$$= -\frac{1}{4\pi^2}\left[\frac{1}{(z-i\pi)^2} + \frac{1}{(z+\pi i)^2} - \frac{2}{2\pi i}\left\{\frac{1}{z-\pi i} - \frac{1}{z+\pi i}\right\}\right]$$

$$\therefore \oint_C \frac{e^z}{(z^2+\pi^2)^2}\, dz = -\frac{1}{4\pi^2}\left[\oint_C \frac{e^z}{(z-i\pi)^2}\, dz + \oint_C \frac{e^z}{(z+\pi i)^2}\, dz - \frac{1}{\pi i}\left\{\oint_C \frac{e^z}{z-\pi i}\, dz - \oint_C \frac{e^z}{z+\pi i}\, dz\right\}\right]$$

2 Marks

NOTE: Full marks has been awarded to those who used Cauchy residue thm to evaluate this integral

$$= -\frac{1}{4\pi^2}\left[2\pi i\left(\frac{d}{dz}e^z\right)_{z=\pi i} + 2\pi i\left(\frac{d}{dz}e^z\right)_{z=-\pi i} - \frac{1}{\pi i}2\pi i\,(e^z)_{z=\pi i} + \frac{1}{\pi i}2\pi i\,(e^z)_{z=-\pi i}\right]$$

by Cauchy's Integral formula (1 Mark)

$$= -\frac{1}{4\pi^2}\left[-2\pi i - 2\pi i + \cancel{2} - \cancel{2}\right]$$

(1 Mark)

$= + i/\pi$.

(5b)

Find Laurent's series expansion of

$$f(z) = \frac{z+5}{z^2-2z-3}$$

about $z=0$ in each of the following regions:

a) $|z|<1$, b) $1<|z|<3$, c) $|z|>3$ (3M)

Sol^n.

$$f(z) = \frac{z+5}{z^2-2z-3} = \frac{2}{z-3} - \frac{1}{z+1}$$

a) For $|z|<1$, we have

$$f(z) = -\frac{2}{3\left(1-\frac{z}{3}\right)} - \frac{1}{1-(-z)}$$

$$= -\frac{2}{3}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k - \sum_{k=0}^{\infty}(-z)^k$$

$$= -\sum_{k=0}^{\infty}\left(\frac{2}{3^{k+1}} + (-1)^k\right)z^k$$ (1 Mark)

b) For $1<|z|<3$, we have

$$f(z) = -\frac{2}{3}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k - \frac{1}{z\left(1+\frac{1}{z}\right)}$$

$$= -\sum_{k=0}^{\infty}\left(\frac{2}{3^{k+1}}\right)z^k - \frac{1}{z}\sum_{k=0}^{\infty}\frac{(-1)^k}{z^k}$$

$$= -\sum_{k=0}^{\infty}\left(\frac{2}{3^{k+1}}\right)z^k + \sum_{k=1}^{\infty}\frac{(-1)^k}{z^k}$$ (1 Mark)

c) For $|z|>3$, we have

$$f(z) = \frac{2}{z\left(1-\frac{3}{z}\right)} + \sum_{k=1}^{\infty}\frac{(-1)^k}{z^k}$$ (1 mark)

$$= \frac{2}{z}\sum_{k=0}^{\infty}\left(\frac{3}{z}\right)^k + \sum_{k=1}^{\infty}\frac{(-1)^k}{z^k} = \sum_{k=1}^{\infty}\left\{2\cdot 3^{k-1} + (-1)^k\right\}\frac{1}{z^k}$$

5C

(c) Classify the singularities of the following function in the finite complex plane:

$$f(z) = \frac{\pi \cot(\pi z)}{z^2}$$

In case the singularities are poles, then specify their order.

(3 Marks)

Sol^n:

Since $z^2 \sin(\pi z)$ has a zero at $z=0$ of order three, therefore $f(z)$ has poles at $z=0$ of order 3.

($1\frac{1}{2}$ Marks)

(for order of the pole $\frac{1}{2}$ mark)

Since $\sin(\pi z)$ has zeros at $z=\pm 1, \pm 2, \pm 3, \dots$ of order one, therefore, $f(z)$ has poles of order one at the pts $z = \pm 1, \pm 2, \dots$

($1\frac{1}{2}$ Marks)

(Note that $\lim_{z \to 0, \pm 1, \pm 2, \dots} f(z) = \infty$.)

for order of the pole $\frac{1}{2}$