Problem Set - 10

Autumn 2019

MATHEMATICS-I (MA10001)

1. Find the following limits (if exists).

   (a) $\lim\limits_{z \to -i} \dfrac{iz^3 + 1}{z^2 + 1}$

   (b) $\lim\limits_{z \to 0} \dfrac{z}{\overline{z}}$

   (c) $\lim\limits_{z \to \infty} \dfrac{(az+b)^3}{(cz+d)^3},\ if\ c \neq 0$

   (d) $\lim\limits_{z \to 2i} \dfrac{\overline{z} + z^2}{1 - \overline{z}}$

2. Test the continuity of the following functions at $z = 0$.

   (a) $f(z) = \begin{cases} \dfrac{\text{Re }(z^3)}{|z|^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

   (b) $f(z) = \begin{cases} \dfrac{\overline{z}^3}{z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

   (c) $f(z) = \begin{cases} \dfrac{\text{Re}(z) - \text{Im(z)}}{|z|^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

3. Test the differentiability of the following functions at $z = 0$.

   (a) $f(z) = \overline{z}$

   (b) $f(z) = \text{Im(z)}$

   (c) $f(z) = |z|^2$

4. Let $f(z) = \begin{cases} \dfrac{z\ \text{Re}(z)}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$,

   Show that

   (a) $f(z)$ is continuous at $z = 0$.

   (b) The complex derivative $f'(0)$ does not exists.

5. Show that the function $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ satisfies C-R equations at the origin, but $f'(0)$ does not exist.

6. Let $f(z) = \begin{cases} \dfrac{\overline{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$. Show that

   (a) $f(z)$ is continuous everywhere on $\mathbb{C}$.

   (b) The complex derivative $f'(0)$ does not exists.

7. Show that the following functions are harmonic and find its harmonic conjugate.

   (a) $u(x,y) = 2x - x^3 + 3xy^2$

   (b) $u(x,y) = \log\sqrt{x^2+y^2}$

   (c) $u(x,y) = \dfrac{y}{x^2+y^2}$

   (d) $u(x,y) = \sinh x \sin y$

   (e) $u(x,y) = e^{-x}(x \sin y - y \cos y)$

8. Using Cauchy Riemann-equations, show that the following functions are nowhere analytic.

   (a) $f(z) = (\overline{z}+1)^3 - 3\overline{z}$

   (b) $f(z) = e^{\overline{z}}$

9. (a) If $f(z)$ is analytic at z_0. Prove that it must be continuous at z_0.

   (b) Give an example to show that the converse of (a) is not necessarily true.

10. If $f = u + iv$ is analytic in a region D and $v = u^2$ in D, then prove that f must be a constant in D.

11. If $u(x,y)$ is a harmonic function in a region D and $g(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$. Show that $g(z)$ is analytic in D.

12. For any complex function $f(z)$. If $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$, then prove that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\frac{\partial^2 f}{\partial z \partial \overline{z}}.$$

13. Given $v(x,y) = x^4 - 6x^2y^2 + y^4$. Find $f(z)$ in terms of z such that $f(z)$ is analytic.

14. Find the analytic function $f(z) = u + iv$ given that $u - v = e^x(\cos y - \sin y)$.

15. Prove the following statements:

    (a) Let f be an analytic function in a domain D. If $|f(z)| = K$, where K is a constant, then f is constant in D.

    (b) If $f(z)$ is a differentiable function, the C-R equations can be put in the form $\dfrac{\partial f}{\partial \overline{z}} = 0$.

    (c) If $f(z)$ and $\overline{f(z)}$ are analytic in a region D, show that $f(z)$ is constant in that region.

    (d) The functions $f(z)$ and $\overline{f(\overline{z})}$ are simultaneously analytic.

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