Problem Set - 3

SPRING 2020

MATHEMATICS-II (MA1002)

1. (a) Prove that if $\lambda(\neq 0)$ be an eigenvalue of a non-singular matrix A, then $\dfrac{|A|}{\lambda}$ is an eigenvalue of $adj A$.

   (b) Prove that if A and B be two square invertible matrices, then AB and BA have same characteristic roots.

   (c) Prove that if λ be an eigenvalue of algebraic multiplicity r of A, then 0 is an eigenvalue of algebraic multiplicity r of the matrix $A - \lambda I_n$.

2. For each of the following matrices, find all the eigenvalues and the corresponding eigenvectors.

   (a) $\begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

3. $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. Use Cayley-Hamilton theorem to express $2A^5 - 3A^4 + A^2 - 5I$ as a linear polynomial in A.

4. Let, $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, show that for every integer $(n \geq 3)$ $A^n = A^{n-2} + A^2 - I$. Hence evaluate A^{50}.

5. Let, $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$. If $A = P^{-1}DP$ then find the diagonal matrix D.

6. The square matrix A is defined as $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$. Find a diagonal matrix D and an invertible matrix P such that $A = P^{-1}DP$.

7. Find two different 2×2 matrices A and B, such that both have same eigenvalues $\lambda_1 = \lambda_2 = 2$ and both have the same eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponding to 2.

8. (a) Show that $A = \begin{bmatrix} -i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$ is Skew-Hermitian.

   (b) Diagonalize $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$ and compute A^{2020}.

9. When $a + b = c + d$ show that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and find the eigenvalues.

10. (a) Show that if $0 < \theta < \pi$, then $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has no real eigenvalues.

    (b) Show that if λ is an eigenvalue of an orthogonal matrix, then $\dfrac{1}{\lambda}$ is also an eigenvalue of it.

11. Examine whether A is similar to B or not, where

(a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -1 \\ 4 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$.

12. If A and B are two unitary matrices, show that AB is a unitary matrix.

13. Express the matrix $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ as the sum of a Hermitian and a skew Hermitian matrix.

14. If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ is a matrix, then show that $(I-N)(I+N)^{-1}$ is a unitary matrix, where I is the identity matrix of order 2.

15. If $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{pmatrix}$ where $a = e^{\frac{2i\pi}{3}}$, then prove that $M^{-1} = \frac{1}{3}\bar{M}$.