Chapter – 6

FIRST – ORDER CIRCUITS

6.1 : INTRODUCTION

In the previous chapters, we have considered three passive elements resistors, capacitors and inductors individually. In this chapter, we will examine two types of circuits: a circuit comprising a resistor and a capacitor and a circuit comprising a resistor and inductor and these are called RC and RL circuits, respectively.

We carry out the analysis of RC and RL circuits by using Kirchhoff's laws and produces differential equations, which are more difficult to solve than algebraic equations. The differential equations resulting from analyzing RC and RL circuits are of the first order. Hence, the circuits are known as first-order circuits. A first-order circuit is characterized by a first-order differential equation.

There are two ways two excite the circuits. The first way to excite the circuits is by initial conditions of the storage elements. These circuits are called source-free circuits and we assume that the energy is initially stored in the capacitive or inductive element. The stored energy causes the current to flow in the circuit and gradually dissipated in the resistors. Source-free circuits are free of independent sources but they may have dependent sources.

The second way to excite the first-order circuits is by independent sources. In this chapter, we will consider dc independent sources.

6.2: SOURCE-FREE RC CIRCUIT

When dc source of a RC circuit is suddenly disconnected, a source-free RC circuit occurs. The energy already stored in the capacitor is gradually dissipated in the resistors.

Fig. 6.1 shows a series combination of a resistor and an initially charged capacitor. Main objective is to determine the circuit response.

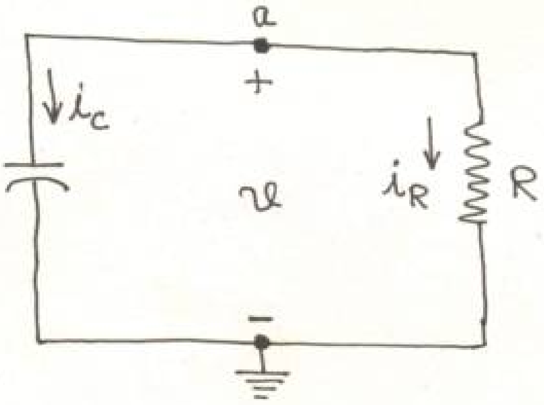


Fig. 6.1: A source-free circuit

Assume that the voltage across capacitor is $v(t)$. Since the capacitor is initially charged, we assume that, at time $t = 0$, the initial voltage is

$$v(0) = V_0 \quad \text{- - - -} \quad (6.1)$$

Stored energy in the capacitor is

$$w_c(0) = \frac{1}{2} C V_0^2 \quad \text{- - - -} \quad (6.2)$$

In Fig. 6.1, applying KCL at node a,

$$i_C + i_R = 0 \quad \text{- - - - - -} \quad (6.3)$$

By definition, we know $i_C = C \cdot \frac{dv}{dt}$ and $i_R = \frac{v}{R}$.

Thus,

$$\therefore \quad C \cdot \frac{dv}{dt} + \frac{v}{R} = 0 \quad \text{- - - -}$$

$$\therefore \quad \frac{dv}{dt} + \frac{v}{RC} = 0 \quad \text{- - - -} \quad (6.4)$$

This is a first-order differential equation, since only the first derivative of v is involved.

Eqn (6.4) can be written as,

$$\frac{dv}{v} = -\frac{1}{RC} \, dt \quad \text{- - - -} \quad (6.5)$$

Integrating both sides, we get,

$$\ln(v) = \frac{-t}{RC} + \ln(A) \quad \text{- - -} \quad (6.6)$$

where A is the integration constant. Thus,

$$\ln\left(\frac{v}{A}\right) = \frac{-t}{RC} \quad \text{- - - -} \quad (6.7)$$

Taking power of e produces

$$v(t) = A e^{-t/RC} \quad \text{- - - -} \quad (6.8)$$

But from the initial condition, when the time $t = 0$, $v(t) = v(0) = V_0$.

$$\therefore \quad v(0) = A$$

$$\therefore \quad A = V_0 \quad \text{- - - -} \quad (6.9)$$

Hence,

$$v(t) = V_0 e^{-t/RC} \quad \text{- - - - -} \quad (6.10)$$

Eqn (6.10) shows that that the voltage response of the RC circuit is an exponential decay of the initial voltage and it is called the <u>natural response</u> of the circuit because the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source.

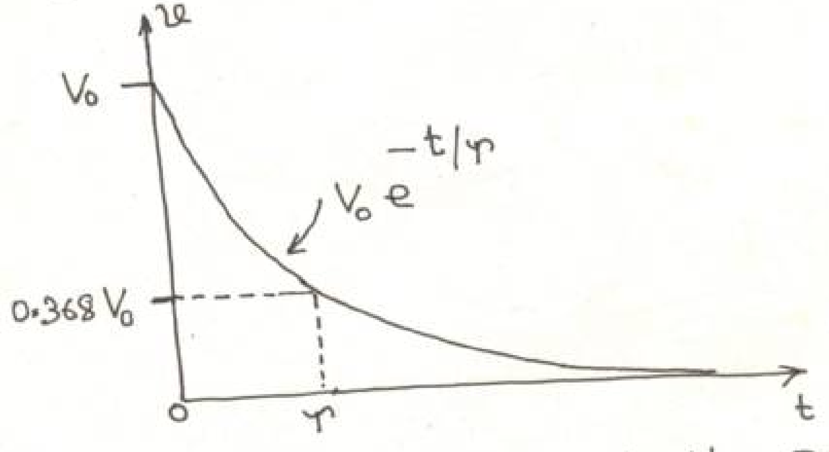


Fig. 6.2: Voltage response of the RC circuit.

Fig. 6.2 shows the natural response. At $t=0$, we have the correct initial condition $v(0) = V_0$ and as time t increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the time constant, denoted by τ and expressed as:

$$\tau = RC \quad --- \quad (6.11)$$

Therefore, Eqn. (6.10) can be expressed as:

$$v(t) = V_0 \, e^{-t/\tau} \quad ---- \quad (6.12)$$

At $t = \tau$,

$$v(\tau) = V_0 \, e^{-1} = 0.368 V_0 \quad --- \quad (6.13)$$

From eqn. (6.13), we can state that the time constant of a circuit is the time required for the response to decay 36.8 percent of its initial value.

Table-6.1 shows the value of $v(t)/V_0$. From Table-6.1, it is seen that the voltage $v(t)$ is less than 1 percent of v_0 after $t = 5\tau$. Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five time constants. In other words, it takes $t = 5\tau$ for the circuit to reach its steady state when no changes take place with time.

TABLE-6.1: Values of $v(t)/V_0 = e^{-t/\tau}$

t	$v(t)/V_0$
τ	0.3678
2τ	0.1353
3τ	0.0498
4τ	0.01832
5τ	0.0067

The time constant may be viewed from another perspective. Evaluating the derivative of $v(t)$ in Eqn.(6.12) at $t=0$, we get

$$\frac{d}{dt}\left(\frac{v}{V_0}\right)\Bigg|_{t=0} = -\frac{1}{\tau}e^{-t/\tau}\Bigg|_{t=0} = -\frac{1}{\tau} \quad \cdots (6.14)$$

From Eqn.(6.14), we can state that the time constant is the initial rate of decay or the time taken for v/V_0 to decay from unity to zero, assuming a constant rate of decay. This initial slope (at $t=0$) interpretation of the time constant is used in the laboratory to determine τ graphically from the response curve displayed on an oscilloscope. For determining τ from the response curve, draw the tangent to the curve at $t=0$ as shown in Fig. 6.3. The tangent meets the time axis at $t=\tau$.

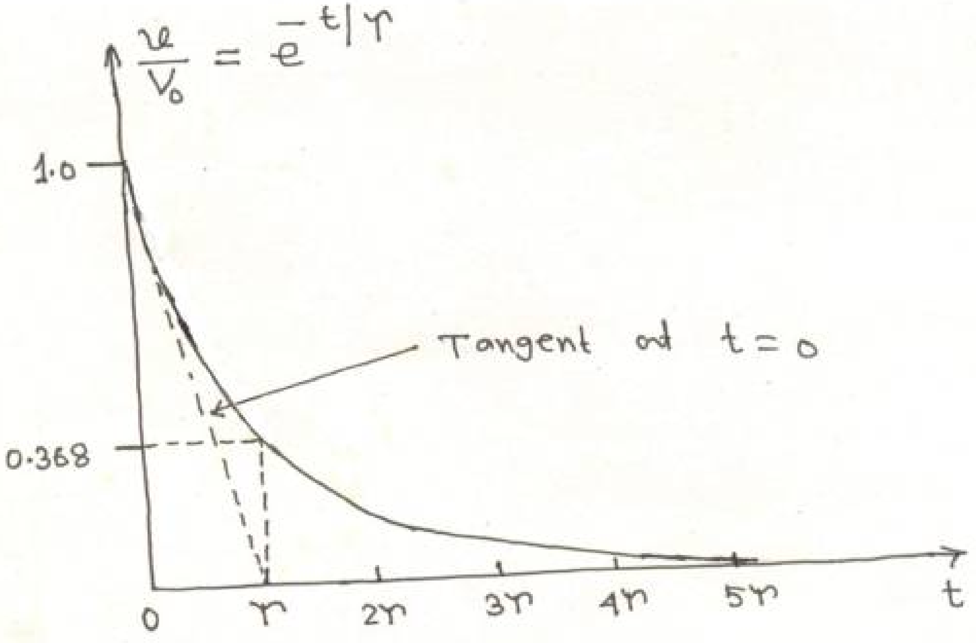


Fig. 6.3: Determination of time constant τ from the response curve.

It can be observed from eqn. (6.11) that the smaller the time constant, the faster the response, This is shown in Fig. 6.4.

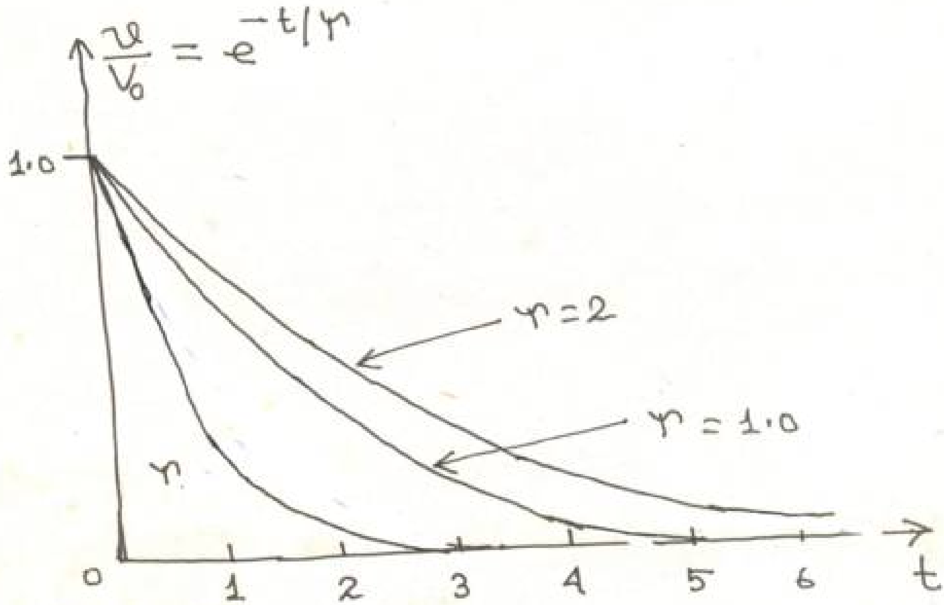


Fig. 6.4: Response curves for various values of the time constant τ.

At any rate, whether the time constant is small or large, the circuit reaches steady state at $t = 5\tau$.

Current $i_R(t)$ can be expressed as:

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau} \quad \text{- - - - (6.15)}$$

The power dissipated in the resistor is

$$p(t) = v(t)\, i_R(t) = \frac{V_0^2}{R} e^{-2t/\tau} \quad \text{- - - (6.16)}$$

The energy absorbed by the resistor up to time t, is,

$$\omega_R(t) = \int_0^t p(t)\, dt = \int_0^t \frac{V_0^2}{R} e^{-2t/\tau} dt$$

$$\therefore \omega_R(t) = \frac{1}{2} C V_0^2 \left(1 - e^{-2t/\tau}\right), \quad \tau = RC \quad \text{- - - (6.17)}$$

Note that as $t \to \infty$, $\omega_R(\infty) = \frac{1}{2} C V_0^2 = \omega_C(0)$.

Initial energy stored in the capacitor is eventually dissipated in the resistor.

Ex-6.1: In Fig. 6.5, $v_c(0) = 10$ Volt, determine v_c, v_x and i_x for $t > 0$.

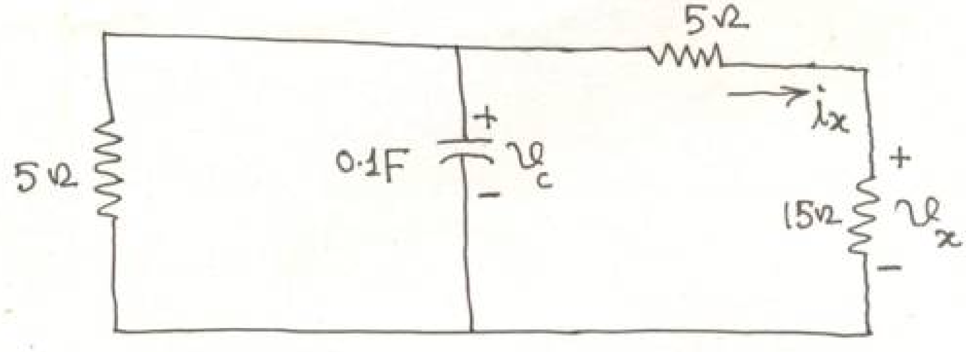


Fig. 6.5: Circuit for Ex-6.1.

Soln.

First we obtain the equivalent or Thevenin resistance across the capacitor terminals.

$$\therefore R_{eq} = R_{TH} = \frac{5 \times (5+15)}{5 + (5+15)} = 4\Omega$$

Equivalent circuit is shown in Fig. 6.6

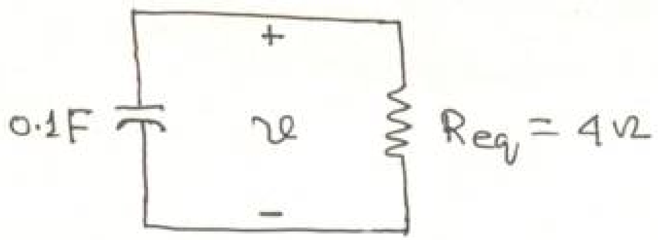


Fig. 6.6: Equivalent circuit for the circuit of Fig. 6.5.

The time constant is

$$\tau = R_{eq} C = 4 \times 0.1 = 0.4 \text{ sec.}$$

Thus

$$v = V_0 e^{-t/\tau}$$

$$V_0 = v(0) = v_c(0) = 10 \text{ Volt}, \quad \tau = 0.4 \text{ see}$$

$$\therefore v_c = v = 10 e^{-t/0.4} = 10 e^{-2.5t} \text{ Volt.}$$

From Fig. 6.5, we can use voltage division to get v_x

$$\therefore v_x = \frac{15}{(15+5)} v = 0.75 \left(10 e^{-2.5t}\right) = 7.5 e^{-2.5t} \text{ Volt}$$

and $$i_x = \frac{v_x}{15} = \frac{7.5 e^{-2.5t}}{15} = 0.5 e^{-2.5t} \text{ Amp.}$$

EX-6.2: The initial voltages on capacitors C_1 and C_2 in the circuit shown in Fig.6.7 have been established by sources not shown. The switch is closed at $t=0$.

(a) Determine $v_1(t)$, $v_2(t)$, $v(t)$ and $i(t)$ for $t \geq 0$

(b) Calculate the initial energy stored in the capacitors C_1 and C_2.

(c) Determine how much energy is stored in the capacitors as $t \to \infty$

(d) Show that the total energy delivered to the $250\,k\Omega$ resistor is the difference between the results obtained in (b) and (c).

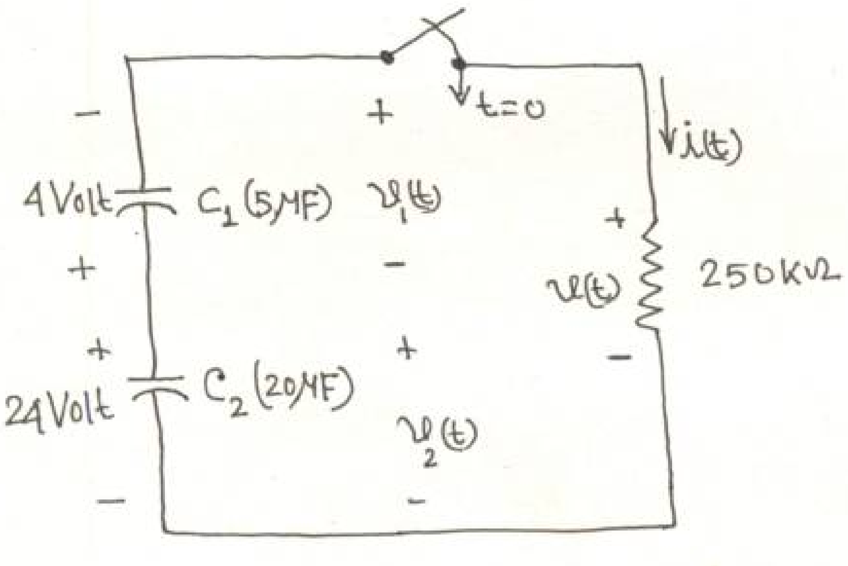


Fig.6.7: Circuit for EX-6.2.

Soln.

(a) From Fig. 6.7, C_1 and C_2 are in series, hence,

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = \left(\frac{1}{5} + \frac{1}{20}\right)^{-1} \mu F = 4\,\mu F$$

Given that,

$$v_{C1}(0) = 4 \text{ Volt} ; \quad v_{C2}(0) = 24 \text{ Volt},$$

$$\therefore v_{C_{eq}}(0) = v_{C2}(0) - v_{C1}(0) = 24 - 4 = 20 \text{ Volt}$$

$$\therefore v(0) = V_0 = v_{C_{eq}}(0) = 20 \text{ Volt}.$$

Equivalent circuit for the circuit of Fig. 6.7 is shown in Fig. 6.8.

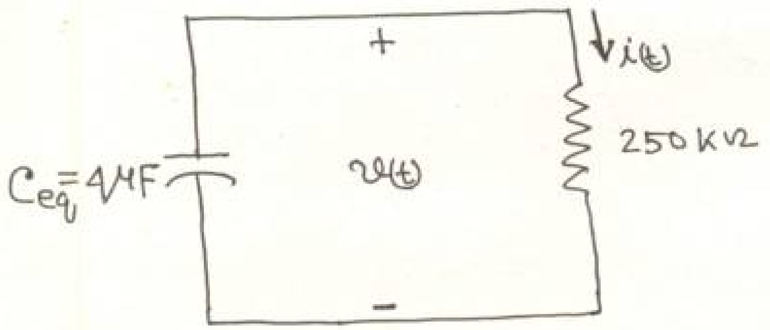


Fig. 6.8: Equivalent circuit for the circuit of Fig. 6.7.

The time constant is

$$\tau = RC_{eq} = 250 \times 10^{3} \times 4 \times 10^{-6} = 1 \text{ Sec}.$$

Thus the expression for $v(t)$ is

$$v(t) = V_0 e^{-t/\tau} = 20 e^{-t} \text{ Volt}.$$

and

$$i(t) = \frac{v(t)}{R} = \frac{20 e^{-t}}{250 \times 1000} = 80 e^{-t} \mu\text{Amp}.$$

By definition, we can calculate the expressions for $v_1(t)$ and $v_2(t)$: (12)

$$v_1(t) = -\frac{1}{C_1}\int_0^t 80\times10^{-6}e^{-t}\,dt - 4$$

$$\therefore v_1(t) = -\frac{10^6}{5}\int_0^t 80\times10^{-6}e^{-t}\,dt - 4$$

$$\therefore v_1(t) = (16e^{-t} - 20)\text{ Volt}$$

$$v_2(t) = \frac{-10^6}{20}\int_0^t 80\times10^{-6}e^{-t}\,dt \quad +24$$

$$\therefore v_2(t) = (4e^{-t} + 20)\text{ Volt}.$$

We also can obtain $v_2(t)$ using KVL,

$$v(t) = v_1(t) + v_2(t)$$

$$\therefore v_2(t) = v(t) - v_1(t)$$

$$\therefore v_2(t) = 20e^{-t} - 16e^{-t} + 20$$

$$\therefore v_2(t) = (4e^{-t} + 20)\text{ Volt}.$$

(b) The initial energy stored in C_1,

$$w_{C1}(0) = \frac{1}{2}C_1v_{C1}^2(0) = \frac{1}{2}\times5\times10^{-6}\times(4)^2 = 40\,\mu J$$

$$w_{C2}(0) = \frac{1}{2}C_2v_{C2}^2(0) = \frac{1}{2}\times20\times10^{-6}\times(24)^2 = 5760\,\mu J$$

The total energy stored in the two capacitors is

$$w_C(0) = 40 + 5760 = 5800\,\mu J$$

(c) As $t \to \infty$

$$v_1 = v_1(\infty) = -20 \text{ Volt}$$

$$v_2 = v_2(\infty) = 20 \text{ Volt}$$

Therefore the energy stored in the two capacitors is

$$\cancel{w_C(\infty) = \frac{1}{2}(5+20)\times 10^{-6} \times (400)}$$

$$w_c(\infty) = \frac{1}{2}(5+20)\times 10^{-6} \times (400) = 5000\,\mu J$$

(d) The total energy delivered to the 250 kΩ resistor is

$$w_R(\infty) = \int_0^\infty p(t)\,dt = \int_0^\infty 20e^{-t} \times 80\,e^{-t}\,dt \quad \mu J$$

$$\therefore w_R(\infty) = \int_0^\infty 1600\,e^{-2t}\,dt \;\mu J = 800\,\mu J$$

Comparing the results obtained in (b) and (c) shows that

$$800\,\mu J = (5800 - 5000)\,\mu J$$

The energy stored in the equivalent capacitor in Fig. 6.7 is $\frac{1}{2}(4\times 10^{-6})(20)^2$, or $800\,\mu J$. Because this capacitor predicts the terminal behaviour of the original series-connected capacitors, the energy stored in the equivalent capacitor is the energy delivered to the 250 kΩ resistor.

EX-6.3: The switch in the circuit in Fig. 6.9 has been closed for a long time and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Also calculate the initial energy stored in the capacitor.

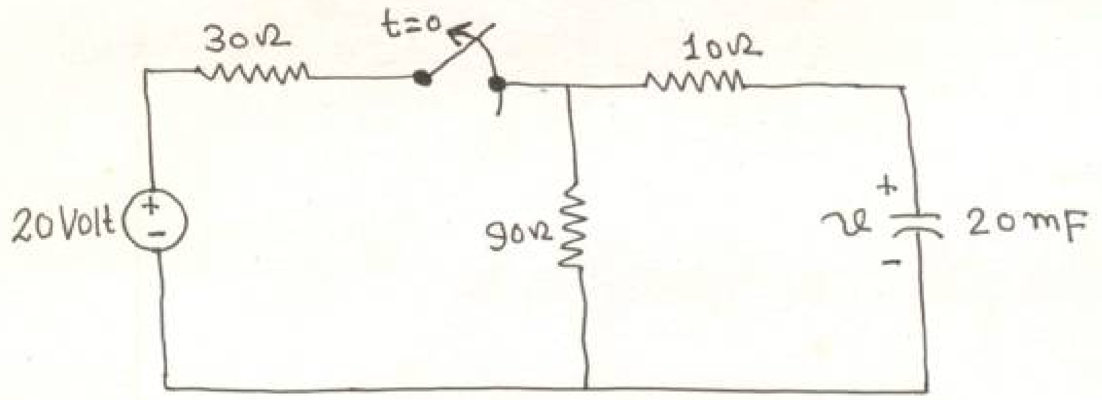


Fig. 6.9: Circuit for EX-6.3

Soln.

For $t < 0$, the switch is closed; the capacitor is an open circuit to dc as shown in Fig. 6.10.

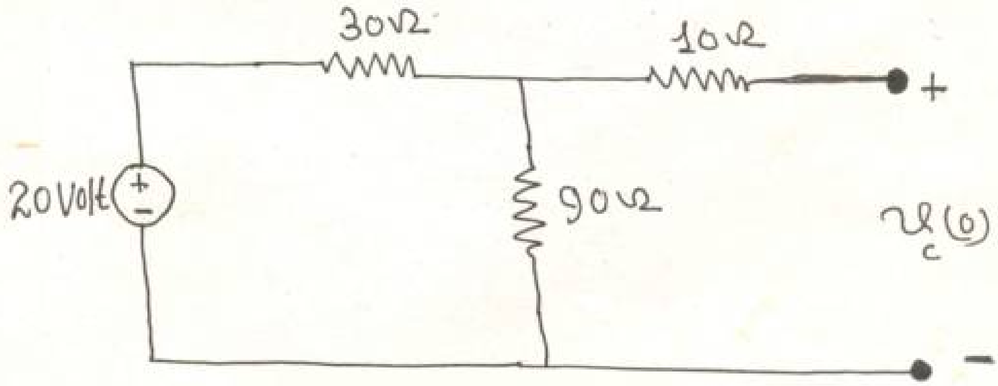


Fig. 6.10: Equivalent circuit of Fig. 6.9 for $t < 0$.

Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or

$$v_c(0) = V_0 = \frac{90}{(90+30)} \times 20 = 15 \text{ Volt.}$$

For $t > 0$, the switch is opened and the equivalent RC circuit is shown in Fig. 6.11. This is a source free circuit.

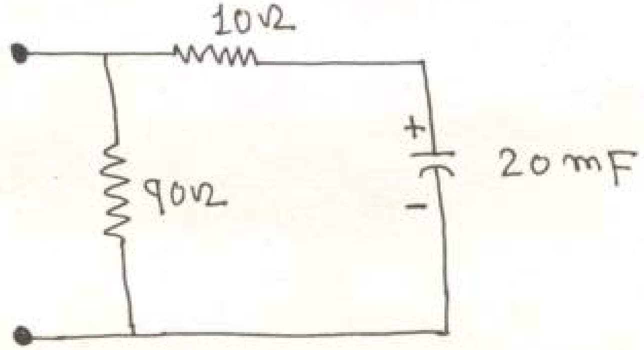


Fig. 6.11 : Equivalent circuit of Fig. 6.9 for $t > 0$.

$$R_{eq} = (10 + 90) = 100\,\Omega.$$

The time constant is

$$\tau = R_{eq} C = 100 \times 20 \times 10^{-3} \text{ sec} = 2 \text{ sec.}$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = V_0 e^{-t/\tau} = 15 e^{-0.5t}$$

and

$$w_c(0) = \frac{1}{2} C V_0^2 = \frac{1}{2} \times 20 \times 10^{-3} \times (15)^2 = 2.25 \text{ J}$$

6.3: SOURCE - FREE RL CIRCUIT

Fig. 6.12 shows the series connection of a resistor and an inductor. Basic objective is to obtain the circuit response, which we will assume to be the current $i(t)$ through the inductor.

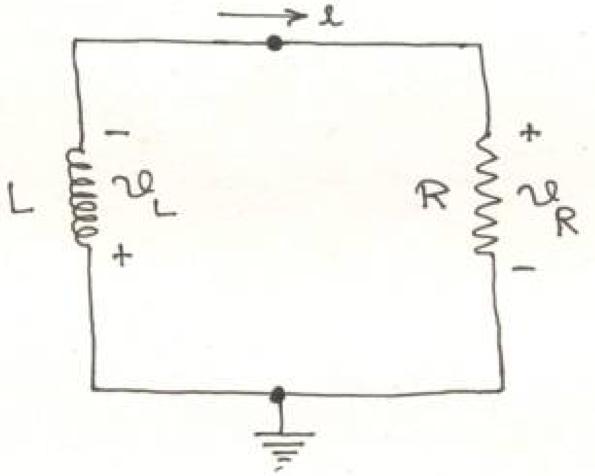


Fig. 6.12: Source - free RL circuit.

At time $t=0$, we assume that the inductor has an initial current I_0, or

$$i(0) = I_0 \quad --- \quad (6.18)$$

and the corresponding energy stored in the inductor is

$$w(0) = \frac{1}{2} L I_0^2 \quad ...... \quad (6.19)$$

In Fig. 6.12, Applying KVL around the loop, we get,

$$v_L + v_R = 0 \quad .... \quad (6.20)$$

But we know that $v_L = L\frac{di}{dt}$ and $v_R = iR$.

Thus,

$$L\frac{di}{dt} + iR = 0$$

$$\therefore \frac{di}{dt} + \frac{R}{L}i = 0 \quad \text{- - - -} \quad (6.21)$$

$$\therefore \frac{di}{i} = -\frac{R}{L}dt \quad \text{- - - -} \quad (6.22)$$

Integrating on both sides, we have,

$$\int_{I(0)}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}dt$$

$$\therefore \ln\left(\frac{i(t)}{I_0}\right) = -\frac{R}{L}t$$

$$\therefore i(t) = I_0 e^{-Rt/L} \quad \text{- - - -} \quad (6.23)$$

Eqn. (6.23) shows that the natural response of the RL circuit is an exponential decay of the current. The plot of $i(t)$ is shown in Fig. 6.13.

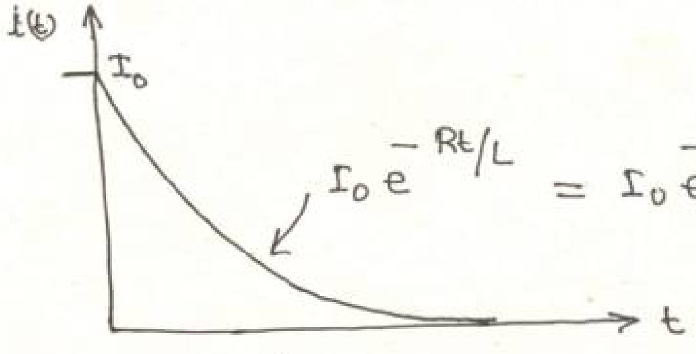


Fig. 6.13: Plot of $i(t)$ for RL circuit.

It is clear from eqn.(6.23) is that the time constant for the RL circuit is

$$\tau = \frac{L}{R} \quad \text{----} \quad (6.24)$$

Therefore, eqn.(6.23) can be written as:

$$i(t) = I_0 e^{-t/\tau} \quad \text{----} \quad (6.25)$$

Voltage across the resistor is

$$v_R(t) = iR = I_0 R e^{-t/\tau} \quad \text{---} (6.25)$$

Power dissipated in the resistor is

$$p(t) = v_R(t)\, i(t)$$

$$\therefore p(t) = \left(I_0 R e^{-t/\tau}\right)\left(I_0 e^{-t/\tau}\right)$$

$$\therefore p(t) = I_0^2 R e^{-2t/\tau} \quad \text{----} \quad (6.26)$$

The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p\, dt = \int_0^t I_0^2 R e^{-2t/\tau} dt$$

$$\therefore w_R(t) = \frac{1}{2} L I_0^2 \left(1 - e^{-2t/\tau}\right) \quad \text{-----} (6.27)$$

As $t \rightarrow \infty$, $w_R(\infty) \Rightarrow \frac{1}{2} L I_0^2 = w(0)$.

Again, the energy initially stored in the inductor is eventually dissipated in the resistor.

Ex- 6.4: In Fig. 6.14, determine $i(t)$ and $i_y(t)$.

Given that $i(0) = 1$ Amp.

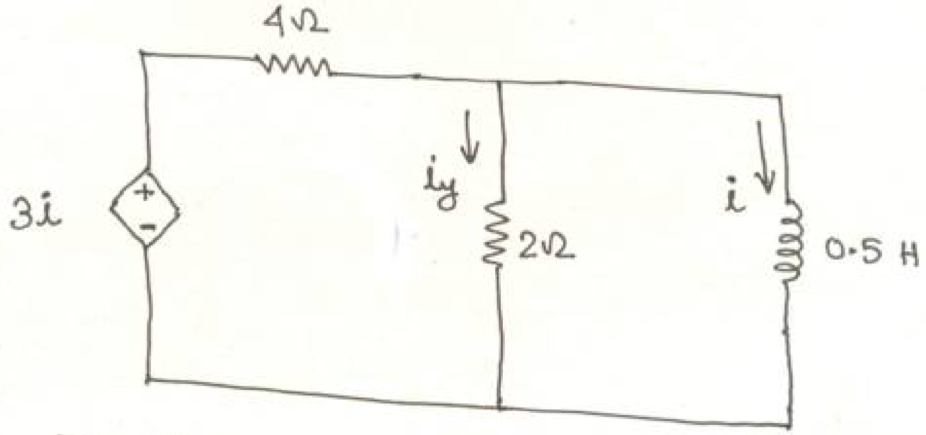


Fig. 6.14: circuit for Ex- 6.4.

Soln.

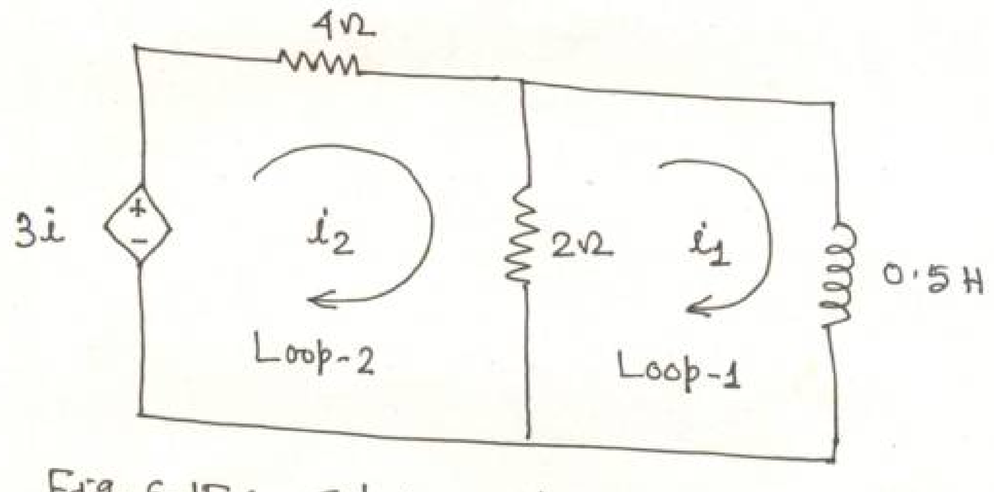


Fig. 6.15: Solving the circuit in Fig. 6.14.

$i_1 = i$ ---- (i)

$i_y = i_2 - i_1 = i_2 - i$ ---- (ii)

For loop-1,

$$0.5\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

$$\therefore \frac{di_1}{dt} + 4i_1 - 4i_2 = 0 \quad \text{---} \quad (iii)$$

For loop 2,

$$6i_2 - 2i_1 - 3i = 0 \quad \text{----} \quad (iv)$$

Since, $i = i_1$,

$$6i_2 - 2i_1 - 3i_1 = 0$$

$$\therefore i_2 = \frac{5}{6} i_1 \quad \text{-----} \quad (v)$$

From eqns. (iii) and (v), we get

$$\frac{di_1}{dt} + \frac{2}{3} i_1 = 0 \quad \text{---} \quad (vi)$$

But $i_1 = i$,

$$\therefore \frac{di}{dt} = -\frac{2}{3} i$$

$$\therefore \frac{di}{i} = -\frac{2}{3} dt \quad \text{---} \quad (vii)$$

Integrating on both sides, we get,

$$\int_{i(0)}^{i(t)} \frac{di}{i} = -\frac{2}{3}\int_0^t dt$$

$$\therefore i(t) = i(0)\, e^{-2t/3}, \quad t > 0 \quad \text{----} \quad (viii)$$

Given that $i(0) = 1$ Amp,

$$\therefore i(t) = e^{-2t/3} \quad \text{----} \quad (ix)$$

The voltage across the inductor is

$$v = L \cdot \frac{di}{dt} = 0.5\left(-\frac{2}{3}\right) e^{-2t/3}$$

$$\therefore v = -\frac{1}{3} e^{-2t/3} \text{ Volt.}$$

Since the inductor and the 2Ω resistor are in parallel,

$$i_y(t) = \frac{v}{2} = \frac{-1}{2\times 3} e^{-2t/3} \text{ Volt},$$

$$\therefore i_y(t) = -\frac{1}{6} e^{-2t/3} \text{ Volt.}$$

Note that we select the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously.

Ex-6.5: The switch in the circuit shown in Fig. 6.16, has been closed for a long time before it is opened at $t=0$. Determine

(a) $i_L(t)$ (b) $i(t)$ (c) $v(t)$

(d) the percentage of the total energy stored in the 0.2 H inductor that is dissipated in the 1Ω resistor.

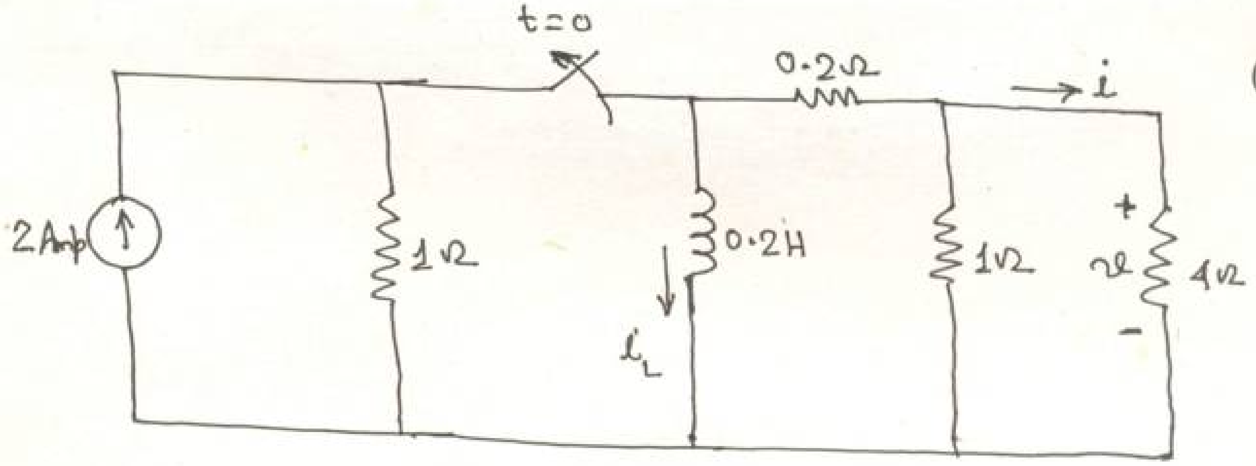


Fig. 6.16: Circuit for Ex-6.16

Soln.

(a) The switch has been closed for a long time prior to $t=0$, hence the voltage across the inductor must be zero at $t=0^-$. Therefore, initial current in the inductor is 2 Amp at $t=0^-$. Thus, $i_L(0) = 2$ Amp, because an instantaneous change in the current cannot occur in an inductor.

As the switch is open, we compute,

$$R_{eq} = 0.2 + \frac{1 \times 4}{1+4} = 1\,\Omega$$

$$\therefore\ \tau = L/R_{eq} = \frac{0.2}{1} = 0.2\ \text{sec.}$$

Therefore, expression for the inductor current is given as:

$$i_L(t) = i_L(0)\, e^{-t/\tau} = 2e^{-5t}\ \text{Amp},\quad t>0.$$

(b) Current i through 4Ω resistor can easily be obtained by current division, that is,

$$i(t) = -i_L(t) \cdot \frac{1}{1+4} = -\frac{1}{5} i_L(t)$$

$$\therefore\ i(t) = -\frac{2}{5} e^{-5t} = -0.4 e^{-5t} \text{ Amp}, \quad t > 0$$

(c) $$v = 4i = 4(-0.4) e^{-5t}.$$

$$\therefore\ v = -1.6 e^{-5t} \text{ Volt}, \quad t > 0$$

(d) The power dissipated in the 1 Ω resistor is

$$p(t) = \frac{v^2}{1} = \frac{2.56}{1} e^{-10t} = 2.56 e^{-10t} \text{ Watt}, \quad t > 0$$

Total energy dissipated in the 1Ω resistor is

$$w(t) = \int_0^\infty 2.56 e^{-10t}\, dt = 0.256 \text{ J}$$

The initial energy stored in the 0.5 H inductor is

$$w(0) = \frac{1}{2} L i_L^2(0) = \frac{1}{2} \times 0.2 \times (2)^2 = 0.4 \text{ J}$$

Therefore the percentage of energy dissipated in the 1Ω resistor is

$$\frac{0.256}{0.4} \times 100 = 64\,\%$$

<u>EX-6.6:</u> In the circuit shown in Fig. 6.17, the initial currents in inductors L_1 and L_2 have been established by sources not shown.

The switch is opened at $t>0$. ~~Determine i_{L1}, i_{L2} and i_3 for $t \geq 0$~~. Determine i_{L1}, i_{L2} and i_3 for $t>0$.

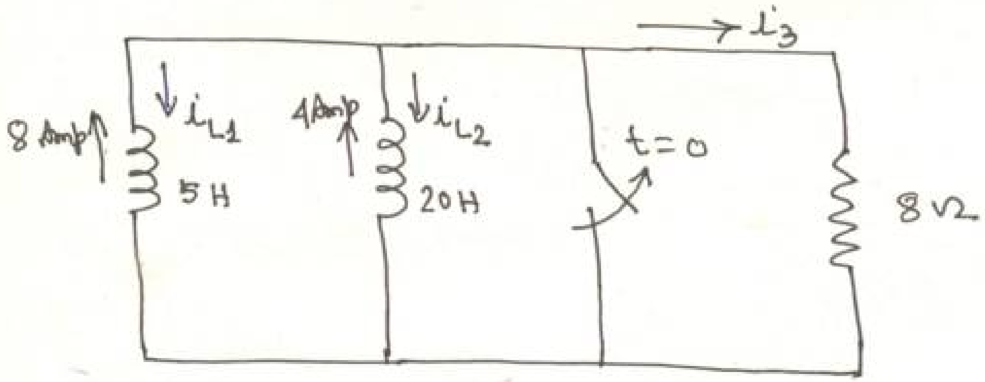


Fig. 6.17: Circuit for EX-6.6.

Soln.

5 H and 20 H inductors are in parallel. Hence,

$$L_{eq} = \frac{5 \times 20}{5+20} = 4\text{ H}.$$

$$i_{L_1}(0) = 8\text{ Amp}; \quad i_{L_2}(0) = 4\text{ Amp}.$$

$$\therefore \; i_{L_{eq}}(0) = i_{L_1}(0) + i_{L_2}(0) = 8+4 = 12\text{ Amp}.$$

Equivalent circuit is shown in Fig. 6.18.

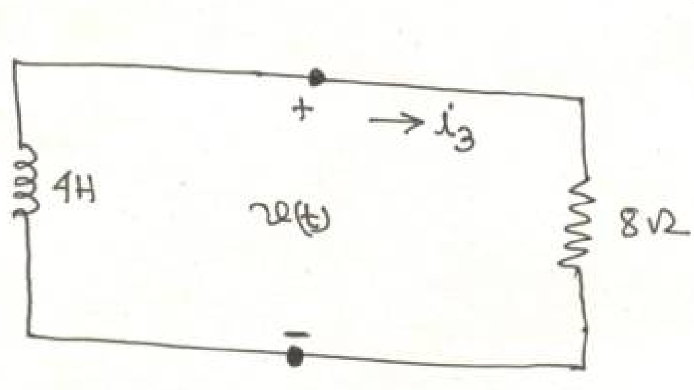


Fig. 6.18: Simplified circuit of Fig. 6.17.

$\tau = L_{eq}/R = 4/8 = 0.5$ sec.

Therefore,

$$i_3(t) = 12e^{-2t} \text{ Amp}, \quad t \geq 0^+$$

$$v(t) = 8i_3(t) = 96e^{-2t} \text{ Volt.}$$

at $t = 0$,

$$v(0) = 96 \text{ Volt.}$$

$$\therefore \; i_{L_1}(t) = \frac{1}{L_1}\int_0^t 96e^{-2t}\,dt - i_{L_1}(0)$$

$$\therefore \; i_{L_1}(t) = \frac{1}{5}\int_0^t 96e^{-2t}\,dt - 8$$

$$\therefore \; i_{L_1}(t) = (1.6 - 9.6e^{-2t}) \text{ Amp}, \quad t \geq 0$$

Similarly,

$$i_{L_2}(t) = \frac{1}{20}\int_0^t 96e^{-2t}\,dt - i_{L_2}(0)$$

$$\therefore \; i_{L_2}(t) = -(1.6 + 2.4e^{-2t}) \text{ Amp}, \quad t \geq 0.$$

<u>EX-6.7</u>: The switch in the circuit of Fig. 6.19 has been closed for a long time. At $t=0$, the switch is opened. Determine $i(t)$ for $t > 0$.

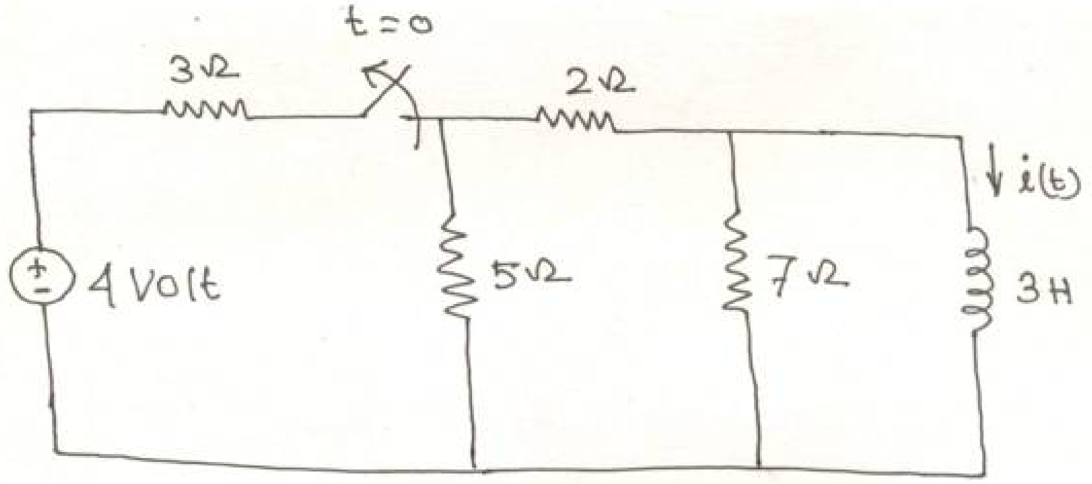


Fig. 6.19: Circuit for Ex-6.7.

Soln.

For $t < 0$, the switch was closed, and the the inductor acts as a short circuit to dc. The 7 Ω resistor is short circuited and the resulting circuit is shown in Fig. 6.20.

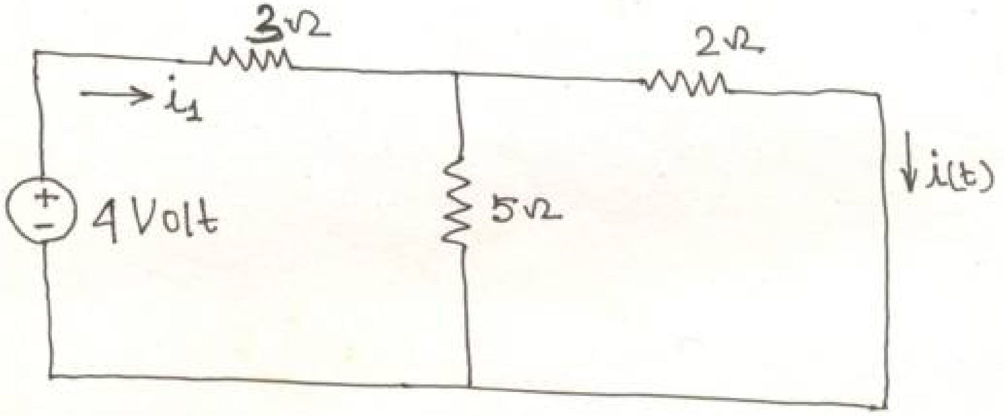


Fig. 6.20: Equivalent circuit of Fig. 6.19 for $t < 0$

For obtaining i_1 in Fig. 6.20, we combine the 2 Ω and 5 Ω resistors in parallel to get

$$\frac{2 \times 5}{2+5} = \frac{10}{7}\ \Omega$$

Hence, $i_1 = \dfrac{4}{\left(3 + \frac{10}{7}\right)} = \dfrac{28}{31}$ Amp

Thus $$i(t) = \frac{5}{(5+2)} i_1$$

$$\therefore \; i(t) = \frac{5}{7} \times \frac{28}{31} = \frac{20}{31} \text{ Amp}, \quad t < 0.$$

Since the current through an inductor cannot change instantaneously,

$$i(0) = i(0^-) = \frac{20}{31} \text{ Amp.}$$

For $t > 0$, the switch is open and the voltage source is disconnected and it is shown in Fig. 6.21. Fig. 6.21 is source free RL circuit.

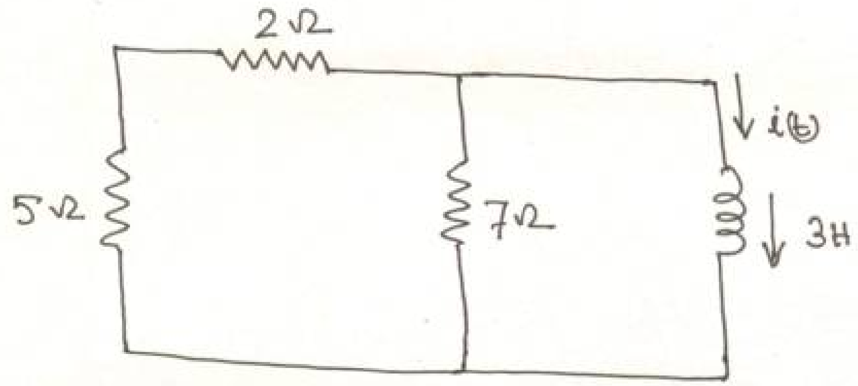


Fig. 6.21: Equivalent circuit of Fig. 6.19 for $t > 0$.

From Fig. 6.21;

$$R_{eq} = \frac{(5+2) \times 7}{(5+2) + 7} = \frac{49}{14} \, \Omega$$

The time-constant is

$$\tau = \frac{L}{R_{eq}} = \frac{3}{(49/14)} \text{ Sec} = \frac{6}{7} \text{ Sec.}$$

Thus,

$$i(t) = i(0)\, e^{-t/\tau} = \frac{20}{31}\, e^{-7t/6} \text{ Amp}$$

$$\therefore\ i(t) = 0.645\, e^{-7t/6} \text{ Amp}, \quad t > 0.$$

Ex-6.8: In the circuit shown in Fig. 6.22, determine i_x, v_x and i for all time. Assume that the switch was open for a long time.

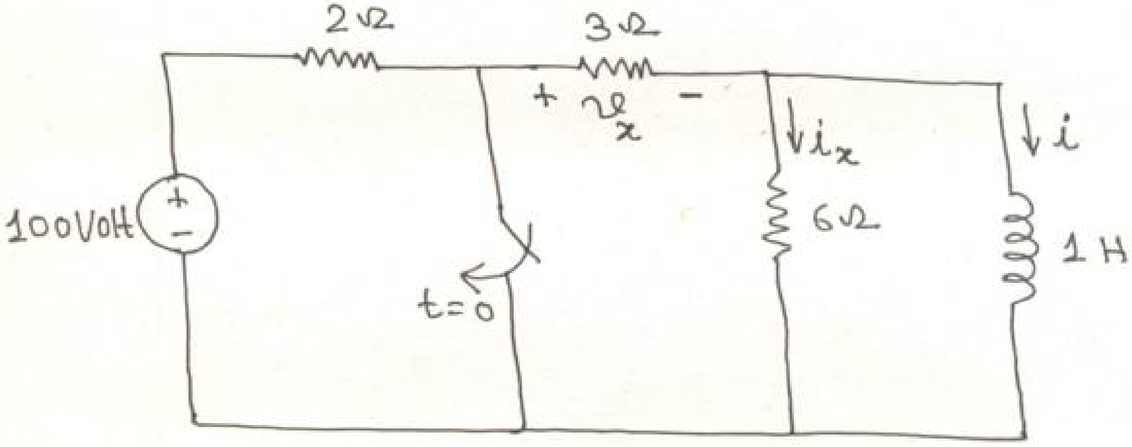


Fig. 6.22: Circuit for Ex-6.8

Solution

For $t < 0$, the switch was open. Since the inductor acts like a short circuit to dc, the 6 Ω resistor is short-circuited and the ~~equi~~ resulting circuit is shown in Fig. 6.23.

~~Hence~~,

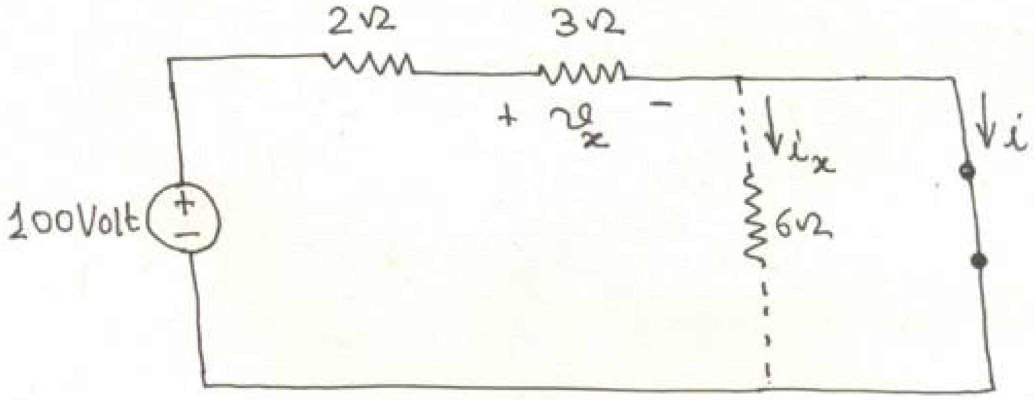


Fig.6.23: Resulting circuit of Fig.6.22 for $t<0$

From Fig. 6.23,

$$i_x = 0, \quad \text{and}$$

$$i(t) = \frac{100}{(2+3)} = 20 \text{ Amp}, \quad t<0$$

$$v_x(t) = 3\,i(t) = 3\times 20 = 60 \text{ Volt}, \quad t<0$$

Thus, $i(0) = 20$ Amp.

For $t>0$, the switch is closed, so that the Voltage source is short circuited. The Equivalent circuit is shown in Fig. 6.24, which is a source free RL circuit.

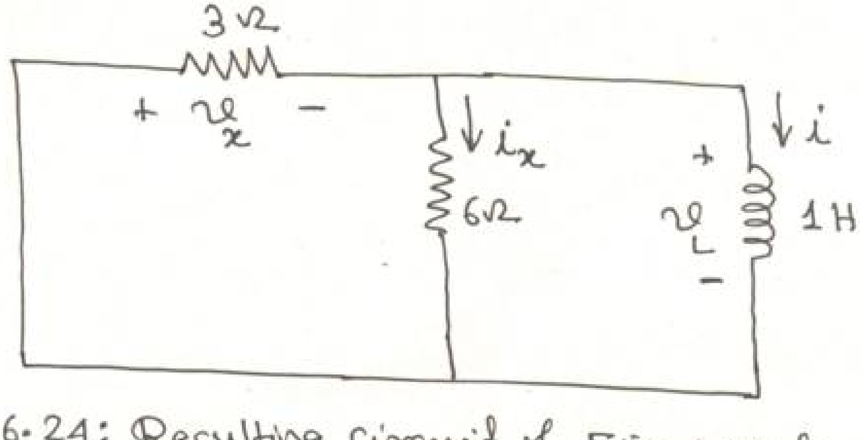


Fig. 6.24: Resulting circuit of Fig. 6.22 for $t>0$

At the inductor terminals,

$$R_{TH} = \frac{3 \times 6}{3+6} = 2\,\Omega$$

Thus,

$$\tau = \frac{L}{R_{TH}} = \frac{1}{2}\text{ sec.}$$

Hence,

$$i(t) = i(0)\,e^{-t/\tau} = 20\,e^{-2t} \text{ Amp}, \quad t > 0.$$

Applying KVL, we have,

$$v_x(t) + v_L(t) = 0$$

$$\therefore v_x(t) = -v_L(t) = -L\cdot\frac{di}{dt} = -(1)(20)(-2)\,e^{-2t}$$

$$\therefore v_x(t) = 40\,e^{-2t} \text{ Volt}, \quad t > 0$$

and

$$i_x(t) = \frac{v_L(t)}{6} = -\frac{40\,e^{-2t}}{6}$$

$$\therefore i_x(t) = -\frac{20}{3}\,e^{-2t}, \text{ Amp}, \quad t > 0$$

Thus for all time,

$$i_x(t) = \begin{cases} 0 \text{ Amp}, & t < 0 \\ -\frac{20}{3}\,e^{-2t} \text{ Amp}, & t > 0 \end{cases}$$

$$v_x(t) = \begin{cases} 60 \text{ Volt}, & t < 0 \\ 40\, e^{-2t} \text{ Volt}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 20 \text{ Amp}, & t < 0 \\ 20\, e^{-2t} \text{ Amp}, & t > 0 \end{cases}$$

6-4: SINGULARITY FUNCTIONS

Singularity functions are functions that either are discontinuous or have discontinuous derivatives. A basic understanding of singularity functions will help us to make sense of the response, especially the step response of RC or RL circuits. Singularity functions are also called switching functions and very useful in circuit analysis.

In circuit analysis, the three most widely used singularity functions are the unit step, the unit impulse and the unit ramp functions. Basic understanding of these three functions help to make sense of the first-order circuits following a

sudden application of an independent dc voltage
or current source.

6-4.1: UNIT STEP FUNCTION

The unit step function $u(t)$ is 0 for $t<0$ and 1 for $t>0$.

In mathematical forms,

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases} \quad \cdots \quad (6.28)$$

The unit step function is undefined at $t=0$, where it changes suddenly from 0 to 1. It is dimensionless quantity. Fig. 6.25 shows the unit step function.

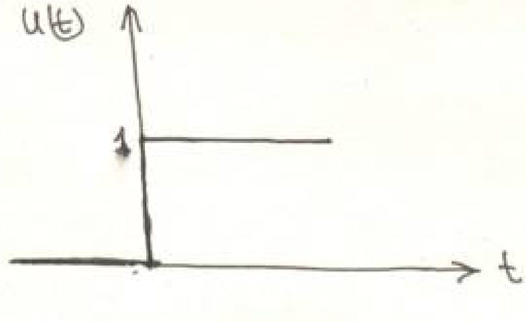


Fig. 6.25: The unit step function.

Instead of $t=0$, if the sudden change occurs at $t=t_0$ $(t_0>0)$, the unit step function can be expressed as

$$u(t-t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases} \quad \cdots (6.29)$$

Eqn.(6.29) indicates that $u(t)$ is delayed by t_0 seconds and is shown in Fig. 6.26

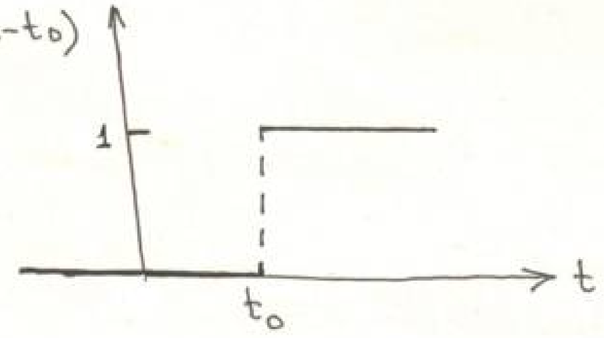


Fig. 6.26: The unit step function delayed by t_0

If the sudden change is at $t = -t_0$, the unit step function becomes

$$u(t+t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases} \quad \cdots (6.30)$$

Eqn.(6.30) indicates that $u(t)$ is advanced by t_0 seconds and is shown in Fig. 6.27.

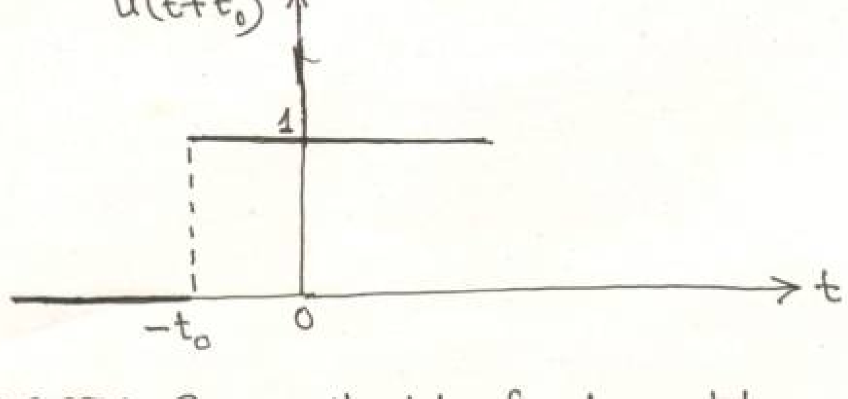


Fig. 6.27: The unit step function ~~delay~~ advanced by t_0.

step function can be used to represent an abrupt change in current or voltage, and this kind of abrupt changes occur in the circuits of digital computers and control systems.

For example, let us consider the voltage,

$$v(t) = \begin{cases} 0, & t < t_0 \\ v_0, & t > t_0 \end{cases} \quad \text{- - - (6.31)}$$

can be expressed in terms of the unit step function as:

$$v(t) = v_0\, u(t - t_0) \quad \text{- - - - (6.32)}$$

At $t_0 = 0$,

$$v(t) = v_0\, u(t) \quad \text{- - - - - (6.33)}$$

Fig. 6.28(a) shows a voltage source $v_0 u(t)$ and its equivalent circuit is shown in Fig. 6.28(b)

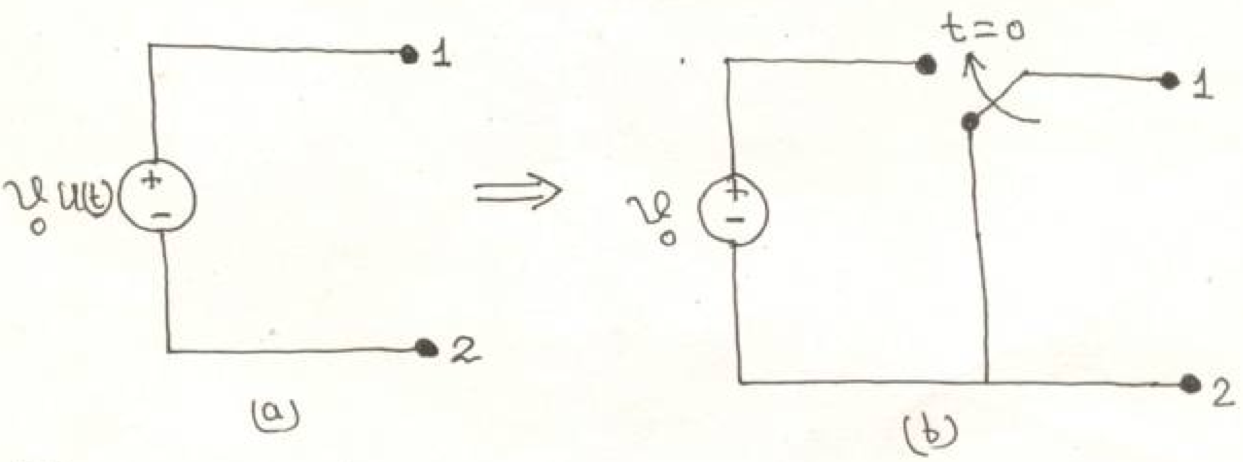


Fig. 6.28: (a) Voltage source of $v_0 u(t)$
(b) Equivalent circuit.

From Fig. 6.28(b), it is clear that the terminals 1–2 are short circuited, i.e. $v(t) = 0$ for $t < 0$ and for $t > 0$, $v(t) = v_0$ appears at the terminals 1–2, i.e., $v_{12} = v_0$.

Similarly, Fig. 6.29(a) shows a current source of $i_0 u(t)$ and its equivalent is shown in Fig. 6.29(b).

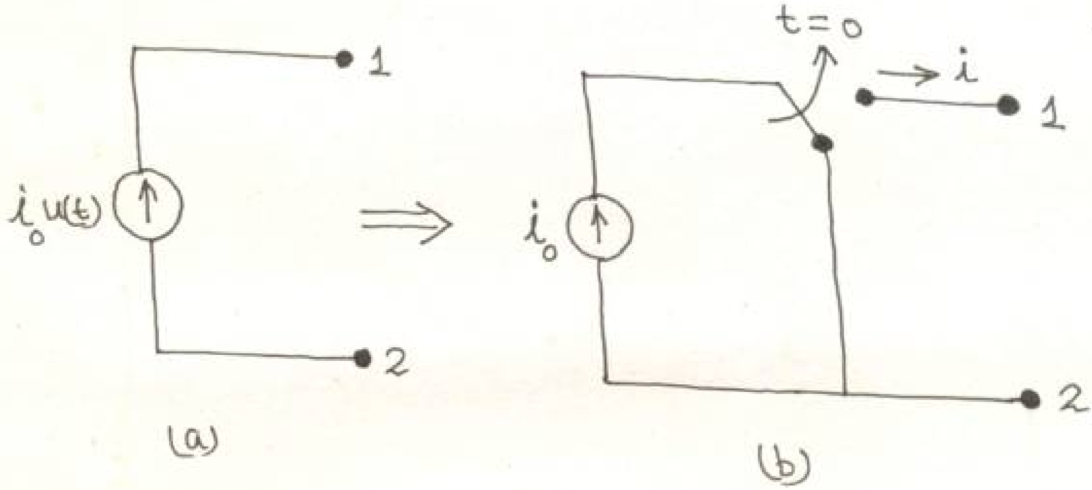


Fig. 6.29: (a) Current source of $i_0 u(t)$
(b) Equivalent circuit.

In Fig. 6.29(b), for $t < 0$, $i(t) = 0$ because it is an open circuit and for $t > 0$, $i(t) = i_0$.

6.4.2: UNIT IMPULSE FUNCTION

The derivative of the unit step function $u(t)$ is the unit impulse function.

Mathematically, it can be expressed as

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases} \quad \cdots (6.34)$$

The <u>unit impulse function</u> is also known as the <u>delta function</u> and is shown in Fig. 6.30.

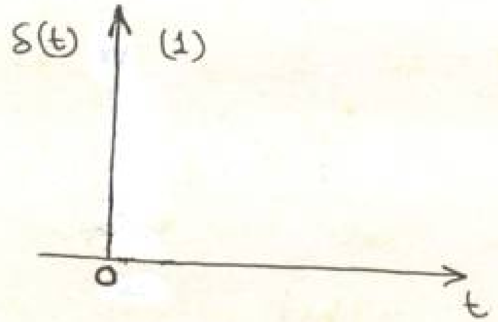


Fig. 6.30: Unit impulse function.

Impulsive voltage and currents occur in electric circuits as a result of switching operations or impulsive sources. Like ideal sources, ideal resistors, etc., the unit impulse function is not physically realizable but it is a very useful mathematical tool.

The unit impulse may be regarded as an applied or resulting shock and visualized as a very short duration pulse of unit area. Mathematically, it can be expressed as

$$\int_{0^-}^{0^+} \delta(t)\, dt = 1 \qquad \cdots (6.35)$$

where

$t = 0^- =$ time just before $t = 0$

$t = 0^+ =$ time just after $t = 0$

Due to this reason, it is customary to write 1 (denoting unit area) as in Fig. 6-30. The unit area is known as the <u>strength</u> of the impulse function and when an impulse function has a strength other than unity, the area of the impulse is equal to its strength. For example, an impulse function $8\delta(t)$ has an area of 8. Fig. 6.31 shows the impulse functions $4\delta(t+2)$, $8\delta(t)$, and $-4\delta(t-2)$

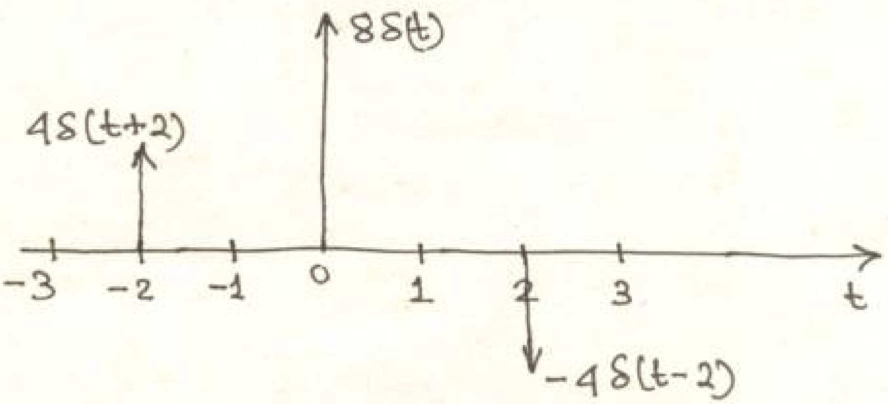


Fig. 6.31: Three impulse functions.

Impulse functions affects other functions and to illustrate this, let us evaluate the integral

$$\int_{\alpha}^{\beta} f(t)\,\delta(t-t_0)\,dt$$

where $\alpha < t_0 < \beta$. since $\delta(t-t_0) = 0$, except at $t = t_0$, the integrand is zero except at t_0.

Thus,

$$\int_{\alpha}^{\beta} f(t)\,\delta(t-t_0)\,dt = \int_{\alpha}^{\beta} f(t_0)\,\delta(t-t_0)\,dt$$

$$= f(t_0)\int_{\alpha}^{\beta} \delta(t-t_0)\,dt = f(t_0)$$

$$\therefore \int_{\alpha}^{\beta} f(t)\,\delta(t-t_0)\,dt = f(t_0) \quad \cdots\cdots \quad (6.36)$$

Eqn.(6.36) clearly shows that when a function is integrated with the impulse function, we get the value of the function at the point where the impulse occurs. This property of the impulse function is very useful and known as the <u>sampling</u> or <u>sifting</u> property. consider a special case for $t_0 = 0$. Then Eqn.(6.36) becomes

$$\int_{0^-}^{0^+} f(t)\,\delta(t)\,dt = f(0) \quad \cdots\cdots \quad (6.37)$$

6.4.3: UNIT RAMP FUNCTION

Unit ramp function $r(t)$ can be obtained by integrating the unit step function $u(t)$. We write

$$r(t) = \int_{-\infty}^{t} u(t)\,dt = t\,u(t) \quad \text{- - - - -} \quad (6.38)$$

or

$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases} \quad \text{- - - - -} \quad (6.39)$$

A ramp is a function that changes at a constant rate. The unit ramp function is zero for negative values of t and has a unit slope for positive values of t. Fig. 6.32 shows the unit ramp function.

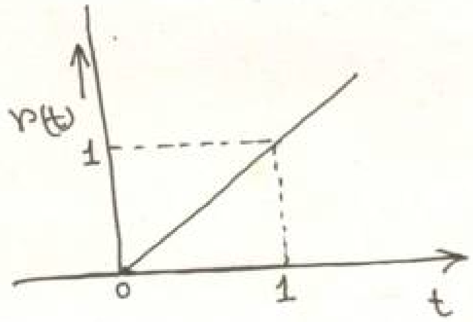


Fig. 6.32: Unit ramp function.

The unit ramp function may be advanced or delayed. For the advanced unit ramp function,

$$r(t+t_0) = \begin{cases} 0, & t \le -t_0 \\ t+t_0, & t \geq -t_0 \end{cases} \quad \cdots (6.40)$$

Fig. 6.33 shows the advanced unit ramp function.

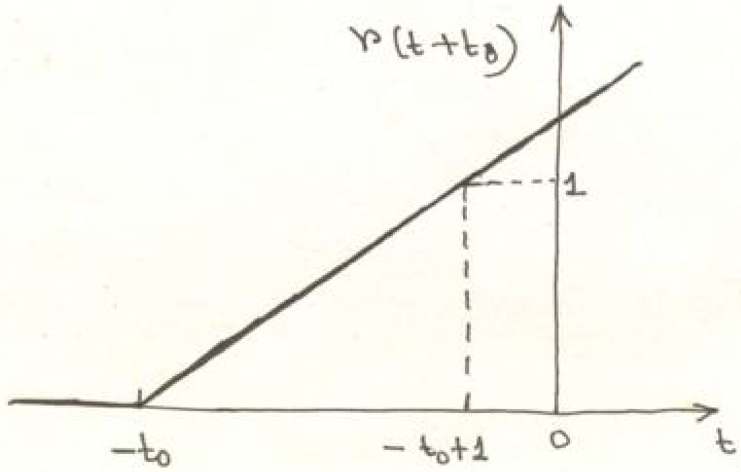


Fig. 6.33: Unit ramp function advanced by t_0.

For the delayed unit ramp function

$$r(t-t_0) = \begin{cases} 0, & t \le t_0 \\ t-t_0, & t \geq t_0 \end{cases} \quad \cdots (6.41)$$

Fig. 6.34 shows the delayed unit ramp function

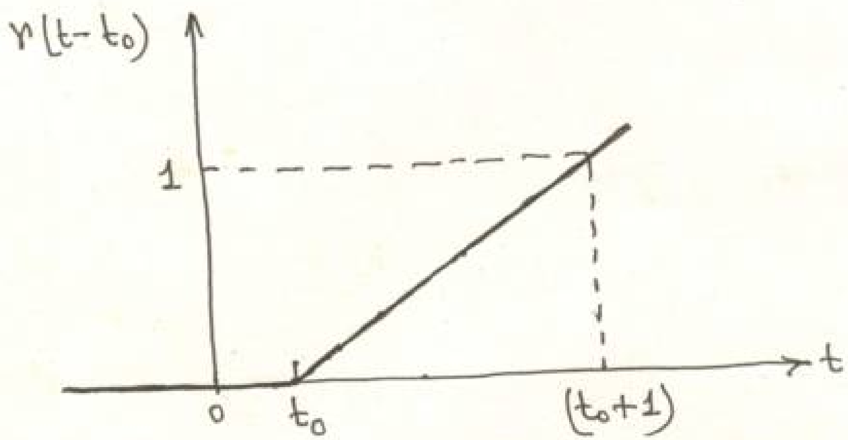


Fig. 6.34: Unit ramp function delayed by t_0.

Although there are many more singularity functions, (41) at this point, we are only interested in impulse function, unit step function and the ramp function. Note that the three singularity functions, impulse, step and ramp are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \quad u(t) = \frac{dr(t)}{dt} \quad \cdots\cdots (6.42)$$

or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(t)\,dt, \quad r(t) = \int_{-\infty}^{t} u(t)\,dt \quad \cdots (6.43)$$

EX-6.9 : Express the voltage pulse in Fig. 6.35 in terms of the unit step. Determine its derivative and sketch it.

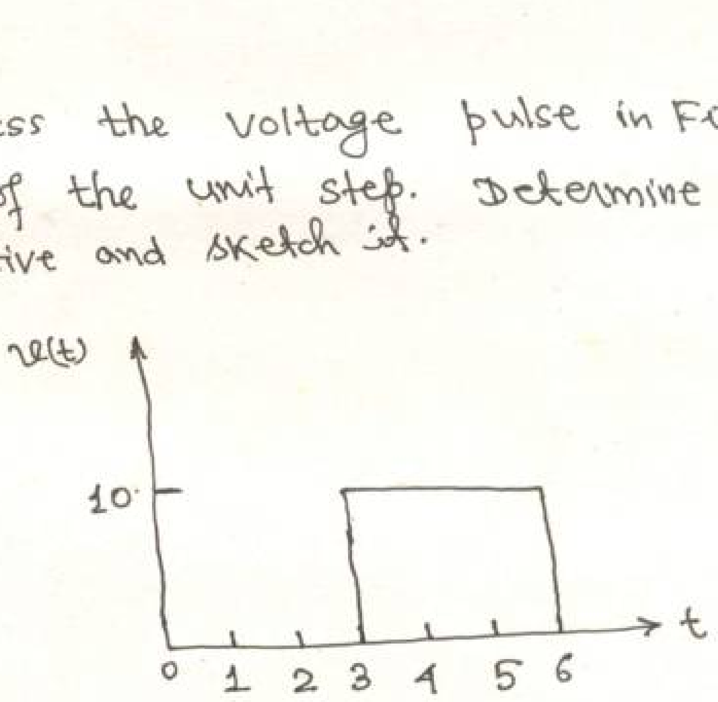


Fig. 6.35: Voltage pulse for Ex-6.9

Soln.

This type of pulse in Fig. 6.35 is called the gate function. It can be regarded as a step function that switches on at one value of t and switches off at another value of t. The gate function shown in Fig. 6.35, switches on at $t = 3$ sec and switches off at $t = 6$ sec. It consists the sum of two unit step functions as shown in Fig. 6.36.

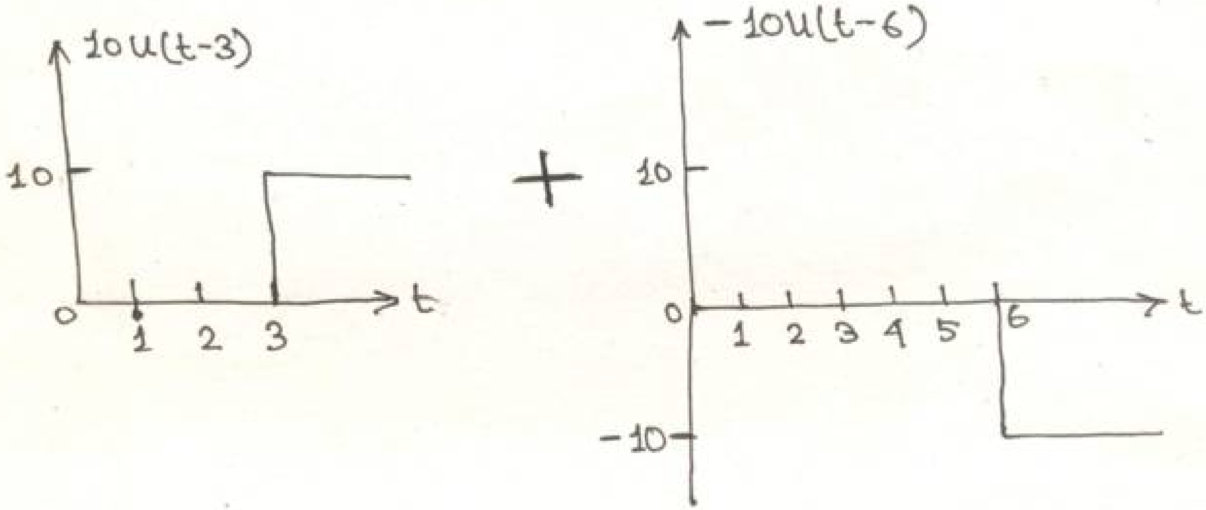


Fig. 6.36: Decomposition of the gate function shown in Fig. 6.35.

From Fig. 6.36, it is evident that

$$v(t) = 10u(t-3) - 10u(t-6) = 10\left[u(t-3) - u(t-6)\right]$$

Taking the derivative of this gives

$$\frac{dv}{dt} = 10\left[\delta(t-3) - \delta(t-6)\right]$$

Sketch of $\frac{dv}{dt}$ is shown in Fig. 6.37.

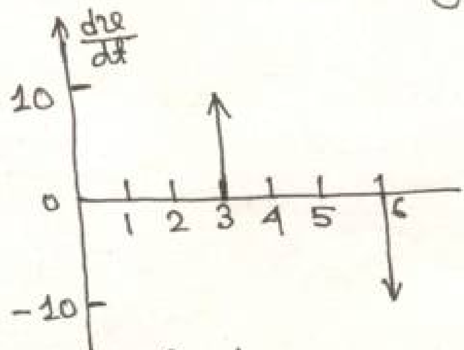


Fig. 6.37: Sketch of $\frac{dv}{dt}$.

EX-6.10: Fig. 6.38 shows a sawtooth function. Express this function in terms of singularity function.

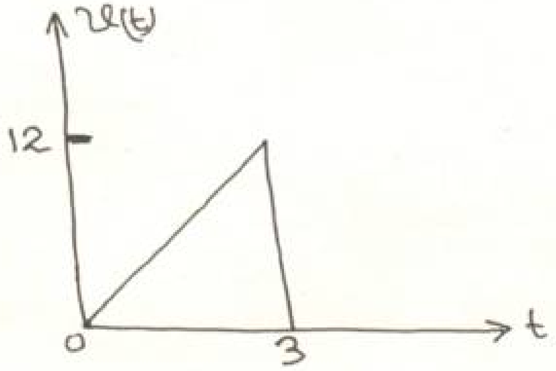


Fig. 6.38: Sawtooth function for Ex-6.10.

Soln.

A close observation of Fig. 6.38 reveals that $v(t)$ is multiplication of a ramp function and a gate function. Slope of the ramp function is 4. Thus

$$v(t) = 4t\left[u(t) - u(t-3)\right]$$

$$= 4t\,u(t) - 4t\,u(t-3)$$

$$= 4r(t) - 4(t-3+3)\,u(t-3)$$

$$= 4r(t) - 4(t-3)\,u(t-3) - 12u(t-3)$$

$$= 4r(t) - 4r(t-3) - 12u(t-3).$$

6.5: STEP RESPONSE OF AN RC CIRCUIT

The step response is the response of the circuit due to a sudden application of a dc current or voltage source.

Fig. 6.39 shows an RC circuit, where V_s is a constant dc voltage source. 44

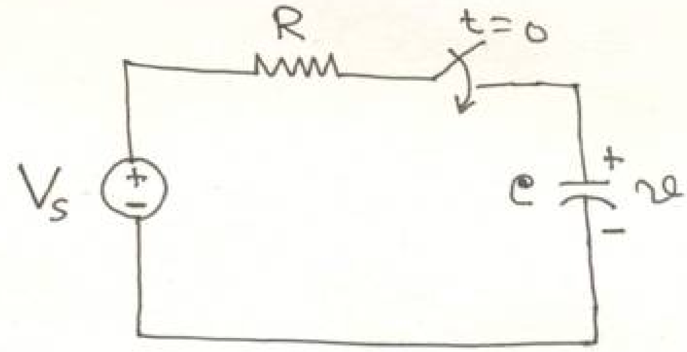


Fig. 6.39: An RC circuit.

Fig. 6.40 shows the circuit of Fig. 6.39 with Voltage step input ($V_s u(t)$).

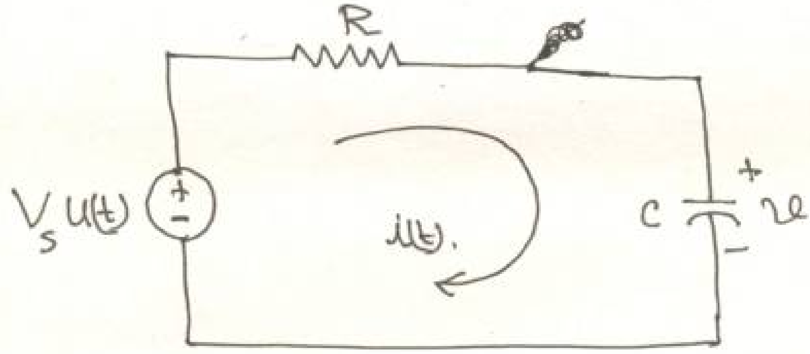


Fig. 6.40: An RC circuit with voltage step input.

Let us assume an initial voltage V_o on the capacitor, but this is not necessary for the step response. Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_o \quad \text{- - - -} \quad (6.44)$$

where

$v(0^-)$ = Voltage across capacitor just before switching

$v(0^+)$ = Voltage across capacitor just after switching.

By applying KCL in Fig. 6.40, we have

$$\frac{v - V_s u(t)}{R} + C.\frac{dv}{dt} = 0$$

$$\therefore \quad \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} u(t) \quad \cdots (6.45)$$

For $t > 0$, Eqn. (6.45) becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \quad \cdots$$

$$\therefore \quad \frac{dv}{dt} = -\frac{(v - V_s)}{RC}$$

$$\therefore \quad \frac{dv}{v - V_s} = -\frac{dt}{RC} \quad \cdots (6.46)$$

Integrating both sides

$$\int_{V_0}^{v(t)} \frac{dv}{(v - V_s)} = -\frac{1}{RC}\int_0^t dt$$

$$\therefore \quad \ln(v - V_s)\Big|_{V_0}^{v(t)} = \frac{-t}{RC}\Big|_0^t$$

$$\therefore \quad \ln\left(\frac{v - V_s}{V_0 - V_s}\right) = \frac{-t}{RC}$$

$$\therefore \quad \frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC \qquad \text{-(6.47)}$$

$$\therefore \quad v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0 \qquad \text{-(6.48)}$$

Thus

$$\boxed{v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}} \qquad \text{--(6.49)}$$

Eqn.(6.49) gives the total response of the RC circuit to a sudden application of dc voltage source, assuming the capacitor is initially charged.

Let $V_0 = 0$ (assuming that the capacitor is initially uncharged), then Eqn(6.49) reduces to

$$\boxed{v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}} \qquad \text{--(6.50)}$$

Alternatively, Eqn(6.50) can be written as:

$$v(t) = V_s\left(1 - e^{-t/\tau}\right)u(t) \quad \text{- - -} \quad (6.51)$$

Eqn.(6.51) gives the complete step response of the RC circuit when the capacitor is initially uncharged.

Current through the capacitor is obtained from Eqn.(6.50), i.e.,

$$i(t) = C.\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}, \qquad t > 0$$

$$\therefore i(t) = \frac{V_s}{R}e^{-t/\tau}\,u(t) \quad \text{- - - - -} \quad (6.52)$$

Now Eqn.(6.48) is rewritten as

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}$$

$$\therefore v(t) = v_{ss} + v_t \quad \text{- - - -} \quad (6.53)$$

Therefore,

~~Complete Response = Transient Response + Steady-state Response~~

~~where~~

Complete Response = Steady-state Response + Transient Response

$$= v_{ss} + v_t \quad \text{- - - -} \quad (6.54)$$

Where

$$v_{ss} = V_s \quad \text{- - -} \quad (6.55)$$

$$v_t = (V_0 - V_s)e^{-t/\tau} \quad \text{- - -} \quad (6.56)$$

The transient response is the temporary and it is the portion of the complete response that decays to zero as $t \to \infty$.

The steady-state response is the portion of the complete response that remains after the transient response has died out. Thus the steady-state response is the behaviour of the circuit a long time after an excitation is applied.

The complete response of Eqn.(6.48) may be written as

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \quad \ldots (6.57)$$

where

$v(0) =$ initial voltage at $t = 0^+$

$v(\infty) =$ final or steady-state value.

Note that if the switch changes position at time $t = t_0$ instead of $t = 0$, there is a time delay in the response so that Eqn(6.57) becomes

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau} \quad \ldots (6.58)$$

where

$v(t_0) =$ initial voltage at $t = t_0^+$

Ex-6.11: In Fig. 6.41, the switch has been in position A for a long time. At $t=0$, the switch changes its position to B. Obtain $v(t)$ for $t>0$ and obtain its value at $t = 0.5$ sec and $t = 3$ sec.

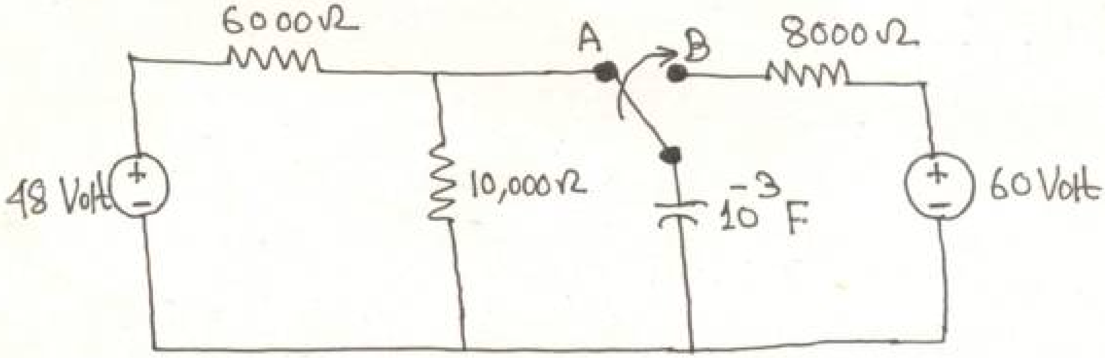


Fig. 6.41: Circuit for Ex-6.11.

Soln.

For $t<0$, the switch was at position A. The capacitor acts like an open circuit to dc. Hence, v is the same as the voltage across the 10000 Ω resistor. Hence, the voltage across the capacitor just before $t=0$ is obtained as

$$v(0^-) = \frac{10000}{(10000+6000)} \times 48 = 30 \text{ Volt}$$

Using the fact that the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = v(0^+) = 30 \text{ Volt}$$

For $t>0$, the switch is in position B. Hence,

$\tau = (8000 \times 10^{-3}) = 8$ sec.

Since the capacitor acts like an open circuit 50
to dc at steady state, $v(\infty) = 60$ Volt. Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$\therefore v(t) = 60 + [30 - 60]e^{-t/8} = (60 - 30e^{-t/8}) \text{ Volt.}$$

At $t = 0.5$ sec

$$v(t=0.5) = 60 - 30e^{-0.5/8} = 31.817 \text{ Volt}$$

At $t = 3$ sec,

$$v(t=3) = 60 - 30e^{-3/8} = 39.38 \text{ Volt.}$$

EX-6.12: In Fig. 6.42 switch S_1 has been closed for long time. At $t=0$, the switch S_2 is closed. Determine $v_c(0^+)$, $i_c(0^+)$, $v_c(\infty)$ and $i_c(\infty)$. Also derive an expression for $v_c(t)$.

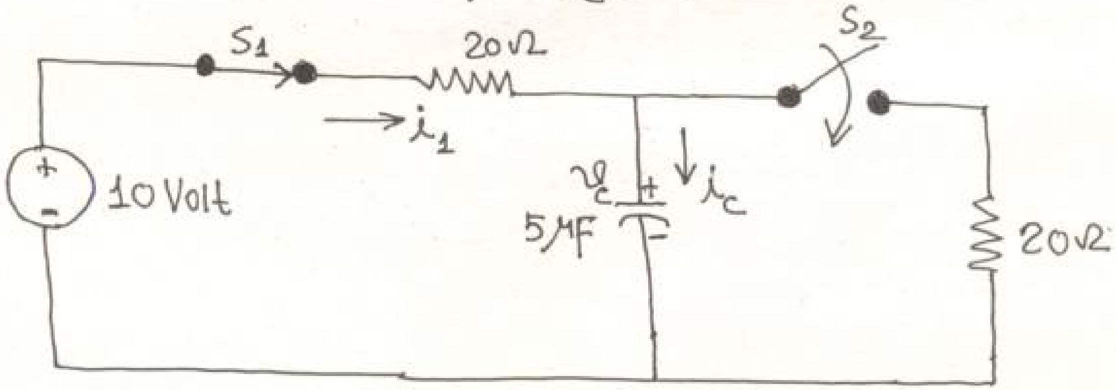


Fig. 6.42: Circuit for EX-6.12

Soln.

Switch S_1 was closed for long time. Hence, before S_2 is closed, the capacitor is fully charged.

Thus $v_c(0^+) = v_c(0^-) = 10$ Volt.

Fig. 6-43 shows the circuit when switch S_2 is closed.

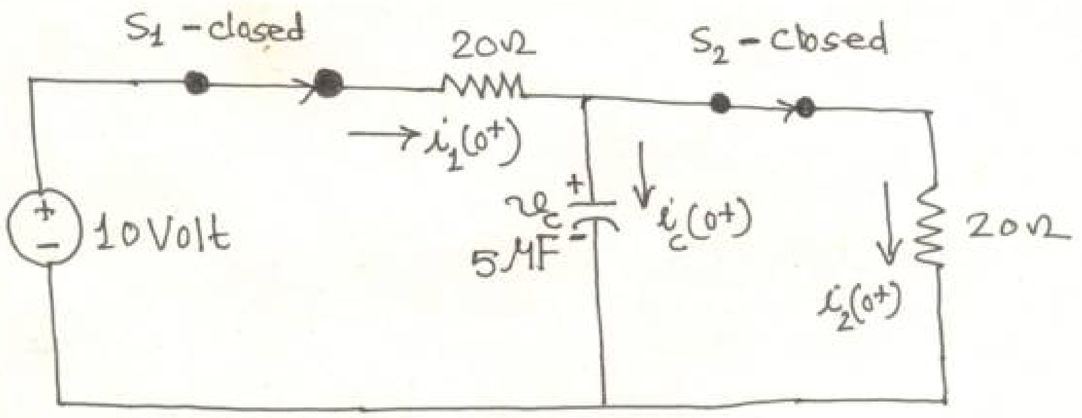


Fig. 6·43: Circuit for EX-6·12, when switch S_2 is closed

At $t = 0^+$, Applying KVL and KCL,

$$20\, i_1(0^+) + v_c(0^+) = 10$$

$$\therefore \quad 20\, i_1(0^+) + 10 = 10$$

$$\therefore \quad i_1(0^+) = 0 \text{ Amp}$$

$$i_2(0^+) = \frac{v_c(0^+)}{20} = \frac{10}{20} = 0.5 \text{ Amp}$$

Also,

$$i_1(0^+) = i_c(0^+) + i_2(0^+)$$

$$\therefore \quad i_c(0^+) + 0.5 = 0$$

$$\therefore \quad i_c(0^+) = -0.5 \text{ Amp.}$$

$$\tau = CR_{TH} = 5 \times 10^{-6} \times \frac{20 \times 20}{(20+20)} = 5 \times 10^{-5} \text{ Sec.}$$

For determining τ, voltage source was short circuited and Thevenin resistance seen by capacitor was obtained

At $t = \infty$, capacitor acts as open circuit, thus,

$$i_c(\infty) = 0$$

$$v_c(\infty) = 10 \times \frac{20}{40} = 5 \text{ Volt.}$$

We know

$$v_c(t) = v_c(\infty) + \left[v_c(0) - v_c(\infty)\right] e^{-t/\tau}$$

$$\therefore v_c(t) = 5 + (10-5)\, e^{-t/5\times10^{-5}}$$

$$\therefore v_c(t) = \left(5 + 5\, e^{-2\times10^{4} t}\right) \text{ Volt, } t>0$$

EX-6.13: In Fig. 6.44, determine $v_c(t)$ after the switch is closed. Given that $v_c(0^+) = 3$ Volt.

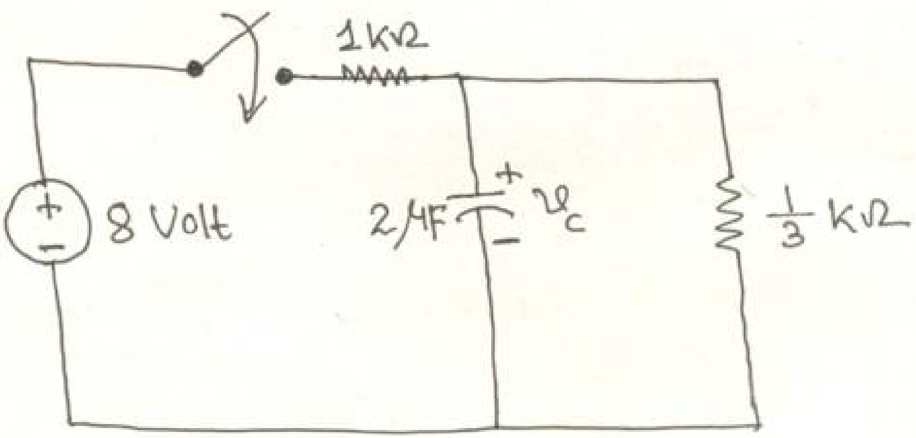


Fig. 6.44: Circuit for EX-6.13

Soln.

For determining the time constant close the switch and short circuit the voltage source, as shown in Fig. 6.45.

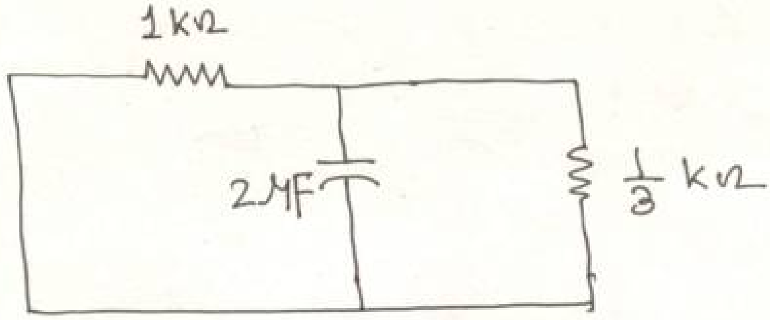


Fig. 6.45: Circuit for determing R_{TH}.

The Thevenin resistance seen by the capacitor is

$$R_{TH} = \frac{(1\,k\Omega) \times (\frac{1}{3}\,k\Omega)}{(1 + \frac{1}{3})\,k\Omega} = \frac{1}{4}\,k\Omega$$

$$\tau = CR_{TH} = 2 \times 10^{-6} \times \frac{1}{4} \times 10^{3} = \frac{1}{2} \times 10^{-3} \text{ Sec.}$$

Now

$$V_c(\infty) = \frac{8}{(1 + \frac{1}{3})} \times \frac{1}{3} = 8 \times \frac{3}{4} \times \frac{1}{3} = 2 \text{ Volt}$$

We know

$$v_c(t) = v_c(\infty) + [v_c(0) - v_c(\infty)] e^{-t/\tau}$$

$$\therefore v_c(t) = 2 + [3 - 2] e^{-2000t}$$

$$\therefore v_c(t) = (2 + e^{-2000t}) \text{ Volt.}, \quad t > 0$$

<u>Ex-6.14</u>: The switch in the circuit shown in Fig. 6.46 has been in position A for a long time. At t=0, the switch moves instantaneously to position B. Determine (a) $v_c(t)$, $v_0(t)$, $i_0(t)$ (b) the total energy dissipated in the 60kΩ resistor.

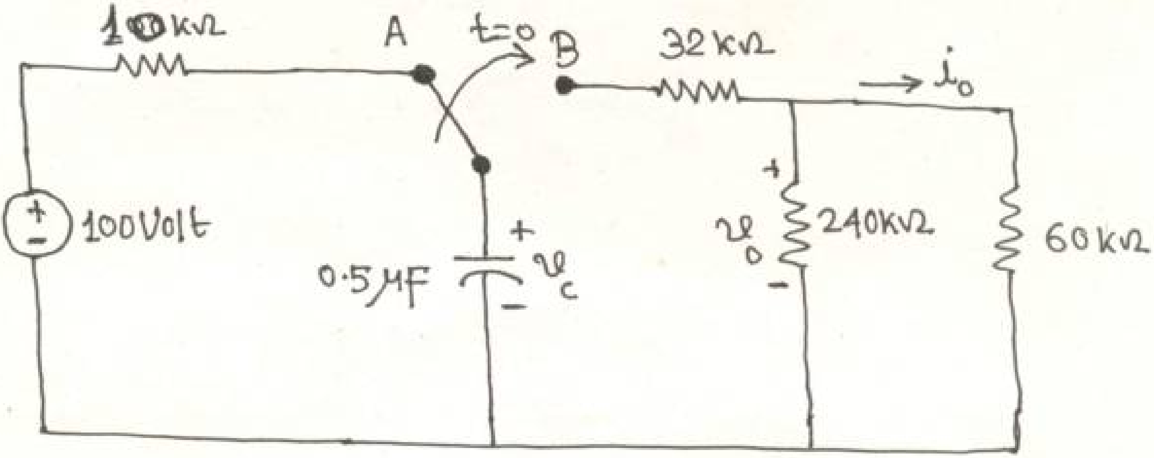


Fig. 6.46: Circuit for Ex- 6.14

Soln.

(a) Switch has been in position A for a long time. Thus, capacitor was opencircuited to dc. Hence $v_c(0) = 100$ Volt.

At $t = 0^+$, the switch moves from position A to B and the equivalent circuit is shown in Figs. 6.47(a) and 6.47(b).

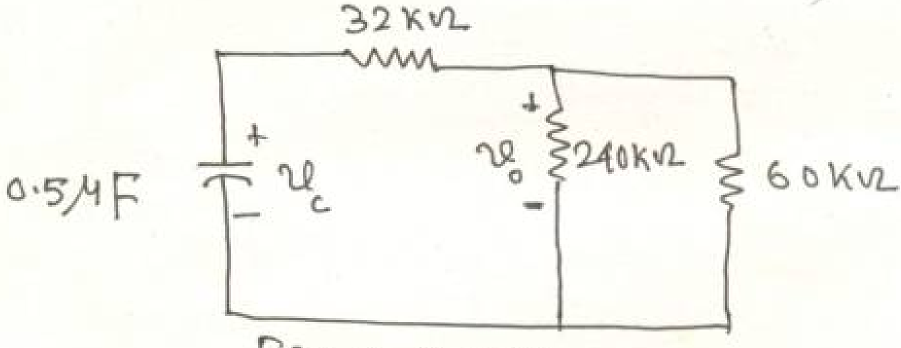
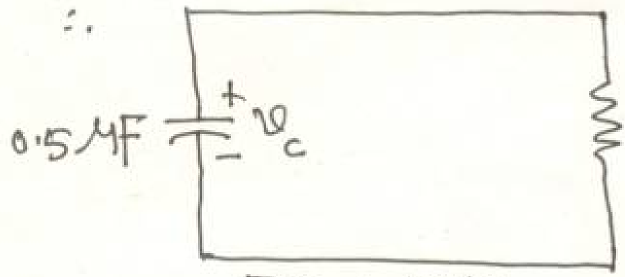


Fig. 6.47(a)

$\therefore$



$$R_{TH} = 32 + \frac{240 \times 60}{300}$$
$$= 80\ k\Omega.$$

Fig. 6.47(b)

$\therefore \tau = CR_{TH} = 0.5 \times 10^{-6} \times 80 \times 10^{3}$

$\therefore \tau = 0.04$ sec

Note that in Fig. 6.47(a), no dc voltage source ~~in the~~ is present. Hence $v_c(\infty) = 0$ Volt because energy stored in the capacitor will be dissipated in the resistors. Thus, we know

$$v_c(t) = v_c(\infty) + \left[v_c(0) - v_c(\infty)\right] e^{-t/\tau}$$

$$\therefore v_c(t) = 100\, e^{-25t} \text{ Volt}, \quad t > 0.$$

From Fig. 6.47(a),

$$v_o(t) = \frac{v_c(t)}{80} \times 48 = 60\, e^{-25t} \text{ Volt}, \ t > 0$$

From Fig. 6.46,

$$i_o(t) = \frac{v_o(t)}{60 \times 10^3} = \frac{60\, e^{-25t}}{60 \times 10^3} = e^{-25t} \text{ mA}, \ t > 0$$

(b) Total power dissipated in the 60 kΩ resistor is

$$P_{60} = i_o^2(t) \times 60 \times 10^3 = 60\, e^{-50t} \text{ mW}, \quad t > 0$$

$\therefore$ Energy dissipated in the 60 kΩ resistor is

$$W_{60} = \int_0^\infty i_o^2(t)\,(60 \times 10^3)\, dt = 1.2 \text{ mJ} \quad \text{Ans.}$$

EX-6.15: In Fig. 6.48, the switch is closed at $t=0$. Determine i for $t>0$. Given that $v_c(0) = 0.0$.

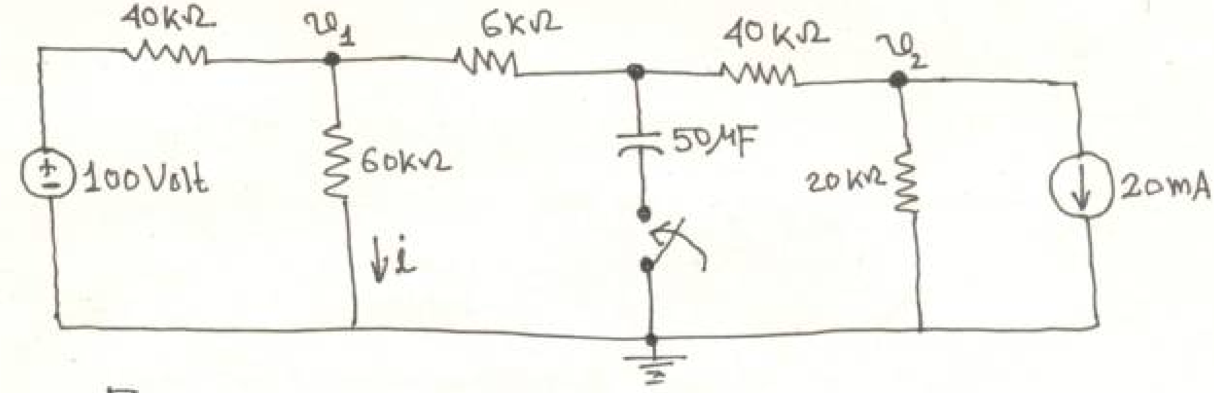


Fig. 6.48: Circuit for Ex-6.15

Soln.

At $t=0^+$, the short-circuiting action of the capacitor prevents the 20 mA current source from affecting $i(0^+)$. Also it places the 6 kΩ resistor in parallel with 60 kΩ resistor. Hence,

$$i(0^+) = \frac{100}{\left(40 + \frac{6 \times 60}{6+60}\right)} \times \frac{6}{(6+60)} = 0.2 \text{ mA}$$

As the switch remains close for long time, capacitor can be considered to be an open circuit. By nodal analysis

$$\left(\frac{1}{40} + \frac{1}{60} + \frac{1}{46}\right) v_1(\infty) - \frac{1}{46} v_2(\infty) = \frac{100}{40} \quad \text{---(i)}$$

$$-\frac{1}{46} v_1(\infty) + \left(\frac{1}{46} + \frac{1}{20}\right) v_2(\infty) = -20 \quad \text{---(ii)}$$

Solving eqns. (i) and (ii), we get,

$v_1(\infty) = -62.67$ Volt

Therefore,

$$i(\infty) = \frac{v_1(\infty)}{60\times10^3} = \frac{-62.67}{60\times10^3} = -1.04\,\text{mA}.$$

The Thevenin resistance at the capacitor terminals is

$$R_{TH} = (6+40)\|60)\|(40+20) = 20\,\text{K}\Omega$$

$$\therefore\ \tau = CR_{TH} = 50\times10^{-6}\times 20\times10^{3} = 1\ \text{sec}.$$

Using the relationship [Reader is asked to derive this]

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$$\therefore\ i(t) = -1.04 + (0.2+1.04)\,e^{-t/1.0}$$

$$\therefore\ i(t) = (-1.04 + 1.24e^{-t})\ \text{mA}. \qquad \text{Ans.}$$

EX-6.16: In Fig.6.49, the switch closes at $t=0$ sec. Determine $v_c(t)$ and $i(t)$ for $t>0$, Given that $v_c(0) = 100$ Volt.

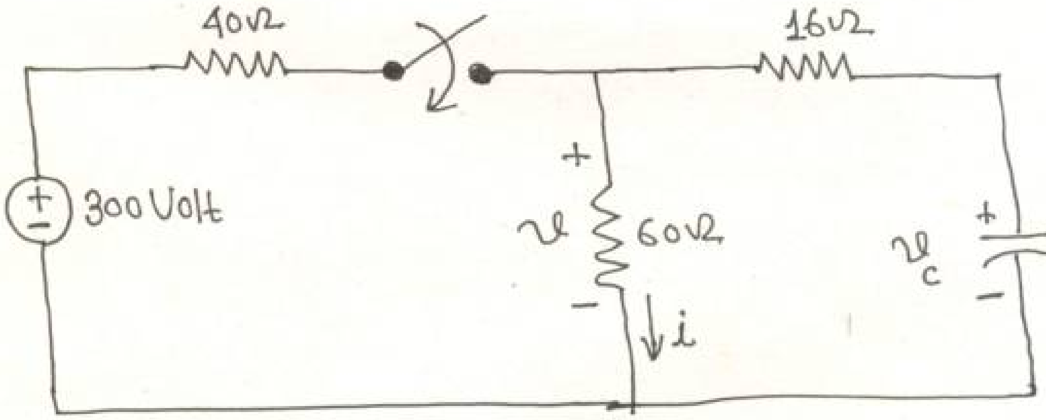


Fig.6.49: Circuit for Ex-6.16

Soln.

$$v_c(0) = 100 \text{ Volt.} = v_c(0^+)$$

The switch was closed for long time. Hence, capacitor acts an open circuit. Therefore,

$$v_c(\infty) = 300 \times \frac{60}{60+40} = 180 \text{ Volt.}$$

Also

$$i(\infty) = \frac{v_c(\infty)}{60} = \frac{180}{60} = 3 \text{ Amp.}$$

$v(0^+)$ can be easily obtained using nodal analysis, i.e.

$$\frac{v(0^+) - 300}{40} + \frac{v(0^+)}{60} + \frac{v(0^+) - 100}{16} = 0$$

$$\therefore\ v(0^+) = 132 \text{ Volt.}$$

$$\therefore\ i(0^+) = \frac{v(0^+)}{60} = \frac{132}{60} = 2.2 \text{ Amp.}$$

Thevenin resistance at the capacitor terminals

$$R_{TH} = 16 + \frac{60 \times 40}{(60+40)} = 40\Omega.$$

$$\therefore\ \tau = CR_{TH} = 2.5 \times 10^{-3} \times 40 \text{ sec} = 0.1 \text{ sec.}$$

We know

$$v_c(t) = v_c(\infty) + [v_c(0) - v_c(\infty)] e^{-t/\tau}$$

$$\therefore\ v_c(t) = 180 + (100 - 180) e^{-10t}$$

$$\therefore\ v_c(t) = (180 - 80 e^{-10t}) \text{ Volt.}, \quad t \geq 0$$

Similarly

$$i(t) = i(\infty) + \left[i(0) - i(\infty)\right] e^{-t/\tau}$$

$$\therefore\ i(t) = 3 + (2.2 - 3)\, e^{-10t}$$

$$\therefore\ i(t) = \left(3 - 0.8 e^{-10t}\right) \text{Amp.}, \quad t \geq 0$$

Ex-6.17: In Fig. 6.50, determine the indicated voltages and currents at $t = 0^+$ immediately after the switch closes. ~~Given that $v_c(0) = 0.0$~~. Also find these voltages and currents "a long time" after the switch closes. Given that capacitors are initially uncharged.

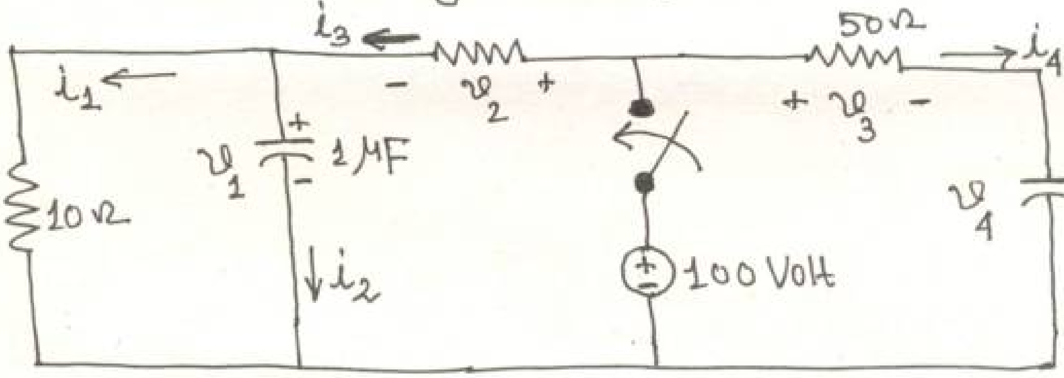


Fig. 6.50: Circuit for Ex-6.17.

Soln.

$$v_1(0) = v_4(0) = 0.0$$

With 0 Volt across the capacitors, they act like short circuits at $t = 0^+$. Hence,

$$v_2(0^+) = v_3(0^+) = 100 \text{ Volt.}$$

Therefore,

$$i_1(0^+) = \frac{0}{10} = 0 \text{ Amp},$$

$$i_3(0^+) = \frac{100}{25} = 4 \text{ Amp}.$$

$$i_4(0^+) = \frac{100}{50} = 2 \text{ Amp}.$$

$$i_2(0^+) = i_3(0^+) - i_1(0^+) = 4 - 0 = 4 \text{ Amp}.$$

A "long time" after the switch closes means the capacitors act like open circuits. Thus

$$i_2(\infty) = i_4(\infty) = 0 \text{ Amp}.$$

Also

$$i_1(\infty) = i_3(\infty) = \frac{100}{(10+25)} = 2.86 \text{ Amp}.$$

$$v_1(\infty) = 10 \times i_1(\infty) = 10 \times 2.86 = 28.6 \text{ Volt}.$$

$$v_2(\infty) = 25 \times 2.86 = 71.4 \text{ Volt}.$$

$$v_3(\infty) = 0 \times 50 = 0 \text{ Volt}.$$

Applying KVL to the right-hand mesh

$$v_4(\infty) = 100 - v_3(\infty) = 100 - 0 = 100 \text{ Volts}.$$

<u>EX-6.18</u>: In Fig. 6.51, the switch has been closed for a long time and is opened at $t=0$. Determine i and v for all time.

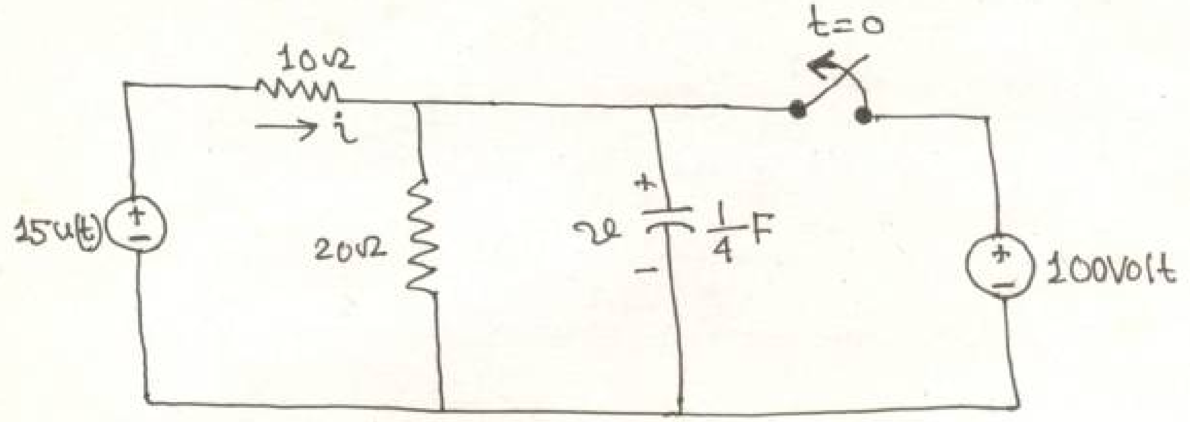


Fig.6-51: Circuit for EX- 6.18

Soln.

At $t=0$, current i through 10Ω resistor can be discontinuous while the capacitor voltage v cannot. Therefore it is better to find v and then obtain i from v.

By definition of the step function,

$$15\,u(t) = \begin{cases} 0, & t<0 \\ 15 & t>0 \end{cases}$$

For $t<0$, the switch is closed and $15u(t)=0$, so that the $15\,u(t)$ voltage source is replaced by a short circuit and contributing nothing to v. Since the switch has been closed for a long time, the capacitor voltage has reached steady state and capacitor acts like an open circuit. Hence for $t<0$, circuit becomes as shown in Fig.6.52.

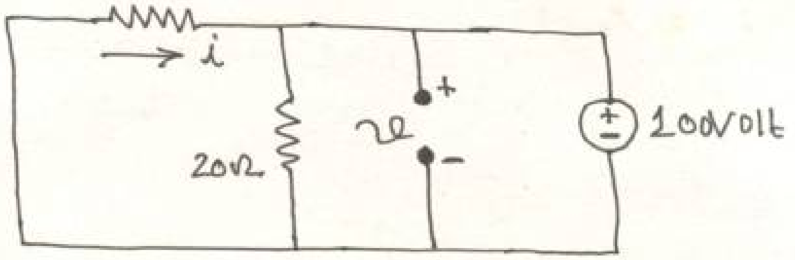


Fig. 6.52: Equivalent circuit for $t < 0$

From Fig. 6.52,

$$v = 100\text{Volt}, \quad i = -\frac{v}{10} = -10 \text{ Amp.}$$

Since the capacitor voltage cannot change instantaneously, $v(0) = v(0^-) = 100\text{Volt}$

For $t > 0$, the switch is opened and the 10 Volt Voltage source is disconnected from the circuit. The $15\,u(t)$ Voltage source is now operative and the circuit becomes as shown in ~~Fig. 6.~~ Fig. 6.53.

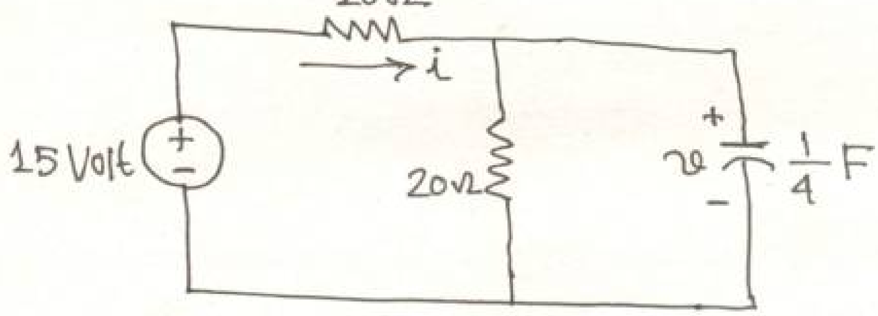


Fig. 6.53: Equivalent circuit for $t > 0$

After a long time, the circuit reaches steady-state and the capacitor acts like an open circuit again. We can easily obtain $v(\infty)$ by using voltage division.

Thus,

$$v(\infty) = \left(\frac{20}{20+10}\right) \times 15 = 10 \text{ Volt.}$$

Thevenins resistance at the capacitor terminals is

$$R_{TH} = \frac{10 \times 20}{(10+20)} = \frac{20}{3}\,\Omega$$

$$\therefore \tau = CR_{TH} = \frac{1}{4} \times \frac{20}{3} = \frac{5}{3} \text{ sec.}$$

We know

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$\therefore v(t) = 10 + (100 - 10)e^{-3t/5}$$

$$\therefore v(t) = \left(10 + 90e^{-3t/5}\right) \text{ volt.}$$

From Fig. 6.53,

$$i = \frac{v}{20} + C\frac{dv}{dt}$$

$$\therefore i = \frac{\left(10 + 90e^{-3t/5}\right)}{20} + \frac{1}{4}\left(-\frac{3}{5} \times 90e^{-3t/5}\right)$$

$$\therefore i = \frac{1}{2} + \frac{9}{2}e^{-3t/5} - \frac{27}{2}e^{-3t/5}$$

$$\therefore i = \left(\frac{1}{2} - 9e^{-3t/5}\right) \text{ Amp.}$$

~~Note that~~

Note that $v + 10i = 15$ is satisfied. Hence

$$v = \begin{cases} 100 \text{ Volt} & , t \le 0 \\ (10 + 90 e^{-3t/5}) \text{ Volt}, & t \ge 0 \end{cases}$$

$$i = \begin{cases} -10 \text{ Amp} & , t < 0 \\ (\frac{1}{2} - 9 e^{-3t/5}) \text{Amp}, & t > 0 \end{cases}$$

~~6.6: STEP RESPONSE OF AN RL CIRCUIT~~

Another way of looking at the complete response is to break into two-components - one natural ~~and other forced~~ response and other forced response. We can also write the total or complete response is

Complete Response = Natural response (stored energy) + Forced Response (independent source)

or, $v = v_n + v_f$ - - - - (6.59)

Now eqn(6.48) can be rewritten as

$$v(t) = V_0 e^{-t/\tau} + V_s(1 - e^{-t/\tau}) \quad \text{- - - (6.60)}$$

where

$$v_n = V_0 e^{-t/\tau} = \text{Natural Response} \quad \text{- (6.61)}$$

and

$$v_f = V_s(1 - e^{-t/\tau}) = \text{Forced Response.} \quad (6.62)$$

Natural response v_n is already expressed in Section-6.2. v_f is known as the forced response because it is produced by the circuit when an external "force" (in this case, a voltage source) is applied.

6.6: CURRENT RESPONSE OF PARALLEL RC CIRCUIT

Consider the circuit of Fig. 6.54.

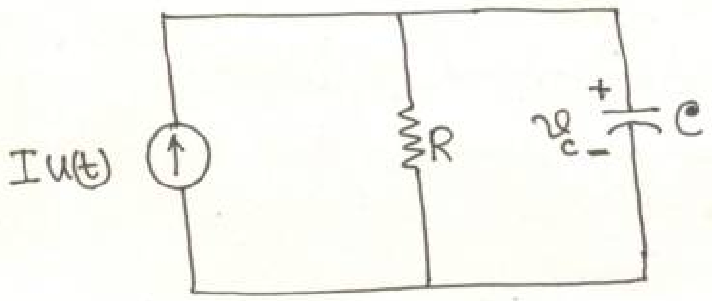


Fig. 6.54: Parallel RC circuit

Time constant $\tau = CR$

Let $v_c(0) = V_0$.

At steady state capacitor acts as open circuit. Therefore, $v_c(\infty) = IR$

By using eqn. (6.48), we obtain

$$v_c(t) = v_c(\infty) + [v_c(0) - v_c(\infty)]e^{-t/\tau}$$

$$\therefore v_c(t) = IR + (V_0 - IR)e^{-t/\tau}$$

$$\therefore v_c(t) = V_0 e^{-t/\tau} + IR(1 - e^{-t/\tau}) \quad \text{--- (6.63)}$$

where v_n = natural response = $V_0 e^{-t/\tau}$ --- (6.64)

v_f = forced response = $IR(1 - e^{-t/\tau})$ --- (6.65)

6.7: STEP RESPONSE OF AN RL CIRCUIT

Fig. 6.55 shows the RL circuit and may be replaced by the circuit shown in Fig. 6.56

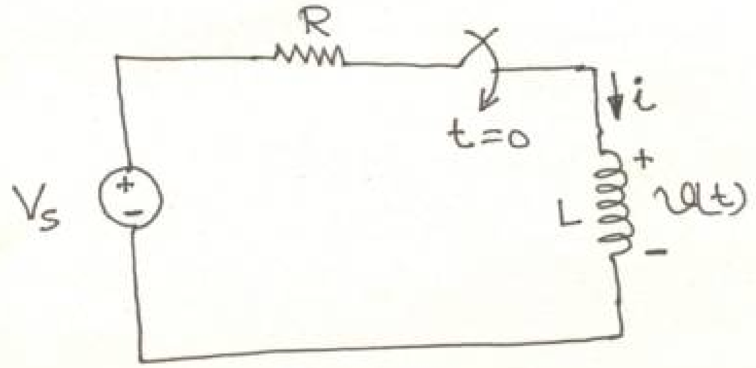


Fig. 6.55: An RL Circuit

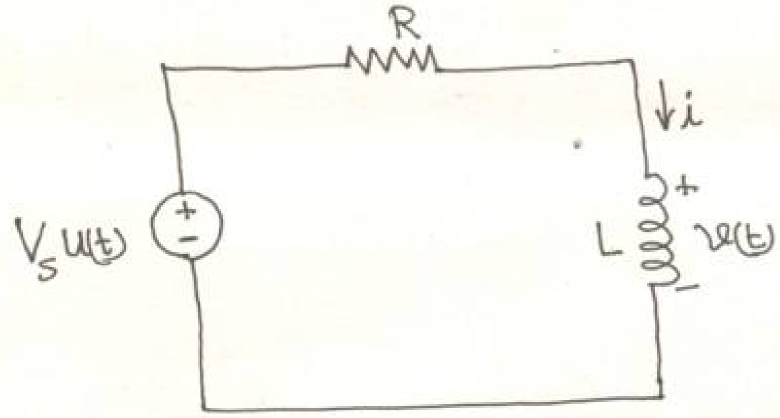


Fig. 6.56: An RL circuit with a step input Voltage.

our objective is to find the inductor current i as the circuit response. We will use the technique in eqns-(6.59) through (6.62). Let the response be the sum of the natural current and the forced current, i.e.,

$$i = i_n + i_f \quad \text{-----} \quad (6.66)$$

Natural response is always a decaying exponential, that is,

$$i_n = A e^{-t/\tau}, \quad \tau = \frac{L}{R} \quad ----(6.67)$$

where A is a constant to be determined.

The forced response is the value of the current a long time after the switch in Fig. 6.55 is closed. We also know that after five time constants (5τ), the natural response essentially dies out. At that time ($t \geq 5\tau$), the inductor becomes a short circuit and the voltage across it is zero. Under this condition, the entire source voltage V_s appears across the resistor R. Hence, the forced response is,

$$i_f = \frac{V_s}{R} \quad ----(6.68)$$

Substituting eqns. (6.67) and (6.68) into eqn. (6.66) gives,

$$i = A e^{-t/\tau} + \frac{V_s}{R} \quad ---(6.69)$$

Let I_0 be the initial current through the inductor, which may come from a source other than V_s. We know that current through inductor cannot change instantaneously, i.e.,

$$i(0^+) = i(0^-) = I_0 \quad ----(6.70)$$

Thus at $t=0$, $i = I_0$, Eqn. (6.69) becomes

$$I_0 = A + \frac{V_s}{R} \quad \cdots$$

$$\therefore \quad A = I_0 - \frac{V_s}{R} \quad \cdots (6.71)$$

Substituting for A in eqn. (6.69) we get

$$i(t) = \left(I_0 - \frac{V_s}{R}\right) e^{-t/\tau} + \frac{V_s}{R} \quad \cdots (6.72)$$

$$\therefore \quad i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right) e^{-t/\tau} \quad \cdots (6.73)$$

The response in eqn. (6.73) may be written as:

$$i(t) = i(\infty) + \left[i(0) - i(\infty)\right] e^{-t/\tau}, \quad (6.74)$$

where

$$i(0) = I_0 \quad \text{and} \quad i(\infty) = \frac{V_s}{R}.$$

Note that, if the switching takes place at time $t = t_0$ instead of $t = 0$, eqn. (6.71) becomes

$$i(t) = i(\infty) + \left[i(t_0) - i(\infty)\right] e^{-(t-t_0)/\tau} \quad \cdots (6.75)$$

If initial current, $I_0 = 0$, then,

$$i(t) = \begin{cases} 0, & t < 0 \\ \frac{V_s}{R}\left(1 - e^{-t/\tau}\right), & t > 0 \end{cases} \quad \cdots (6.76)$$

or

$$i(t) = \frac{V_s}{R}\left(1 - e^{-t/\tau}\right) u(t) \quad \cdots (6.77)$$

Eqn.(6.77) gives the step response of the RL circuit with $I_0 = 0$. The voltage across the inductor can be obtained from eqn.(6.77),

$$\therefore \; v(t) = L\frac{di}{dt} = \frac{V_s}{R}\cdot\frac{L}{\tau}\, e^{-t/\tau}$$

$$\therefore \; v(t) = \left(V_s\, e^{-t/\tau}\right)u(t) \quad \cdots\cdots (6.78)$$

Fig.6.57 shows the step response $i(t)$ and $v(t)$.

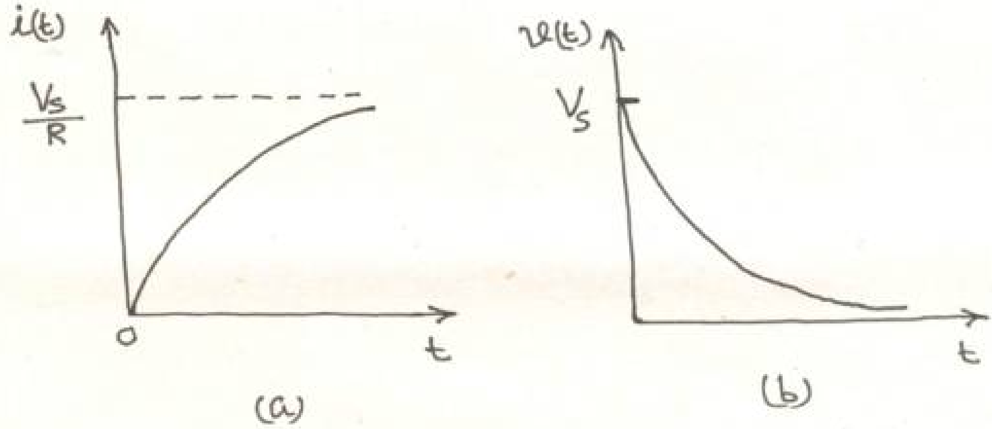


Fig.6.57: Step response of an RL circuit with $I_0 = 0$ (No initial inductor current): (a) Current Response (b) Voltage Response.

<u>EX-6.19:</u> In Fig.6.58, determine $i(t)$ for $t > 0$. Assume that switch has been closed for a long time.

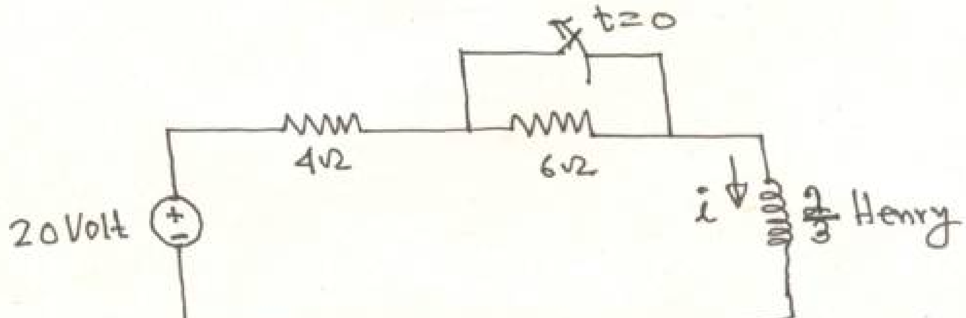


Fig.6.58: Circuit for EX-6.19.

Soln.

For $t<0$, the 6Ω resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t=0^-$ (i.e., just before $t=0$) is,

$$i(0^-) = \frac{20}{4} = 5 \text{ Amp}$$

We know that inductor current cannot change instantaneously, therefore,

$$i(0) = i(0^+) = i(0^-) = 5 \text{ Amp.}$$

For $t>0$, the switch is open. Therefore 4Ω and 6Ω resistors are in series, so that

$$i(\infty) = \frac{20}{4+6} = 2 \text{ Amp.}$$

The Thevenin resistance across the inductor terminals is

$$R_{TH} = (4+6) = 10\Omega$$

$$\tau = \text{time constant} = \frac{L}{R_{TH}} = \frac{2}{3\times 10} = \frac{1}{15} \text{ sec.}$$

We know

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$$\therefore i(t) = 2 + (5-2)e^{-15t}$$

$$\therefore i(t) = (2 + 3e^{-15t}) \text{ Amp.}$$

Ex-6.20: Determine i(t) for all the values of time in the circuit shown in Fig.6.59.

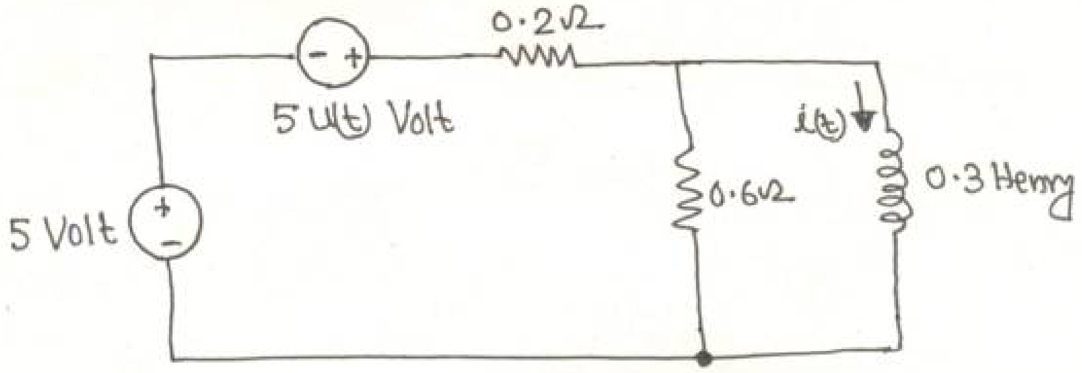


Fig.6.59: Circuit for Ex-6.20

Solns.

In Fig.6.59, circuit contains a dc voltage source as well as a step-voltage source.

The Thevenin resistance across the inductor terminals is

$$R_{TH} = \frac{0.2 \times 0.6}{0.2+0.6} = 0.15 \Omega$$

$$\therefore \quad \tau = \frac{L}{R_{TH}} = \frac{0.3}{0.15} = 2 \text{ sec.}$$

By definition of the step function

$$5\,u(t) = \begin{cases} 0 \,, & t < 0 \\ 5 \,, & t > 0 \end{cases}$$

Therefore, for $t < 0$,

$$i(0^-) = \frac{5}{0.2} = 25 \text{ Amp.}$$

$$\therefore \quad i(0) = i(0^+) = i(0^-) = 25 \text{ Amp.}$$

Again for $t > 0$,

$$i(\infty) = \frac{(5+5)}{0.2} = 50 \text{ Amp.}$$

We know for $t > 0$ [Eqn. 6.74]

$$i(t) = i(\infty) + [i(0) - i(\infty)] e^{-t/\tau}$$

$$\therefore \quad i(t) = 50 + (25 - 50) e^{-t/2}$$

$$\therefore \quad i(t) = (50 - 25 e^{-0.5t}) \text{ Amp}, \quad t > 0$$

Therefore

$$i(t) = \begin{cases} 25 \text{ Amp}, & t < 0 \\ (50 - 25 e^{-0.5t}) \text{ Amp} & t > 0 \end{cases}$$

or simply we can write

$$i(t) = 25 + (25 - 25 e^{-0.5t}) u(t)$$

$$\therefore \quad i(t) = 25 + 25 (1 - e^{-0.5t}) u(t) \text{ Amp.}$$

Note that for $t < 0$, $u(t) = 0$ and for $t > 0$, $u(t) = 1$.

EX-6.21: In Fig. 6.60, the switch has been in position 'a' for a long time. At $t = 0$, it is thrown to position "b". Determine (a) $i(t)$ for $t > 0$ (b) $v(0^-)$, $v(0^+)$ and $\frac{di}{dt}(0^+)$.

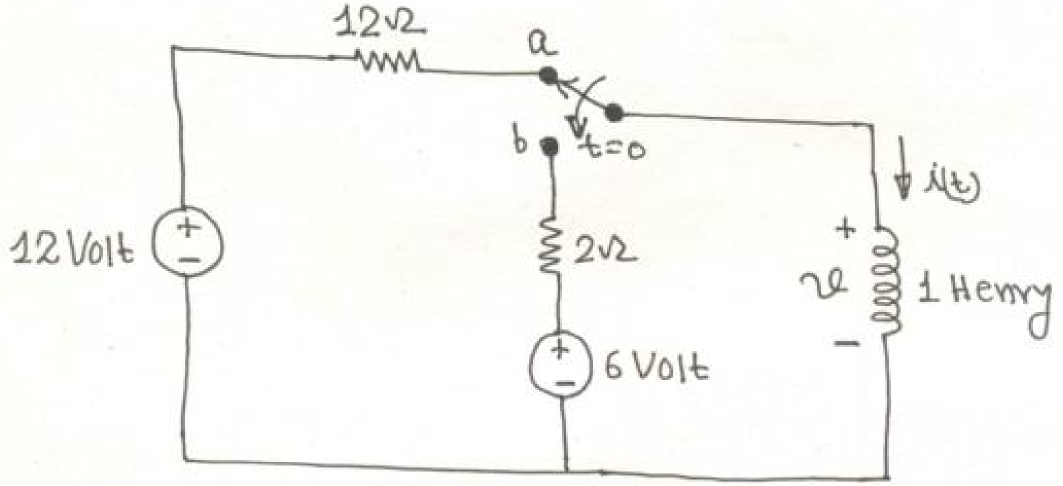


Fig.6.60: Circuit for EX-6.21

Soln.

(a) The switch was in position "a" for long time. Hence inductor acts as a short circuit. Therefore,

$$i(0^-) = \frac{12}{12} = 1 \text{ Amp.}$$

Since the inductor current cannot change instantaneously, we can write,

$$i(0) = i(0^-) = i(0^+) = 1 \text{ Amp.}$$

At $t=0$, switch is in position "b". The Thevenin resistance across inductor terminals $R_{TH} = 2\Omega$.

$$\tau = \frac{L}{R_{TH}} = \frac{1}{2} = 0.5 \text{ Sec}$$

Under this condition,

$i(\infty) = \frac{6}{2} = 3$ Amp. Because switch is in position 'b' and the circuit reached the steady state condition and the inductor acts as short circuit. For $t>0$, we know

$$i(t) = i(\infty) + \left[i(0) - i(\infty)\right] e^{-t/\tau}$$

$$\therefore\ i(t) = 3 + (1-3)e^{-2t}$$

$$\therefore\ i(t) = (3 - 2e^{-2t})\ \text{Amp.}, \quad t>0$$

~~(b)~~ $\cancel{\frac{v(0^-)}{L} =}$

(b) When switch was in position 'a', under steady state condition, inductor acts as a short circuit. Applying KVL, we get,

$$12 \times i(0^-) + v(0^-) - 12 = 0$$

$$\therefore\ v(0^-) = 12 - 12\times 1 = 0\ \text{Volt.}$$

~~When switch was in position "b", under steady state condition, inductor acts as a short-circuit. Applying KVL, we get,~~

At $t=0$, when switch was placed in position "b", we apply KVL,

$$2 \times i(0^+) + v(0^+) - 6 = 0$$

$$\therefore\ v(0^+) = 6 - 2\times 1 = 4\ \text{Volt.}$$

Also,

$$L\frac{di}{dt}(0^+) = v(0^+)$$

$$\therefore\ \frac{di}{dt}(0^+) = \frac{v(0^+)}{L} = \frac{4}{1} = 4\ \text{Amp/sec.}$$

Ex-6.22: In Fig.6.61, the switch has been open for a long time. If the switch is closed at $t=0$, determine $i(t)$.

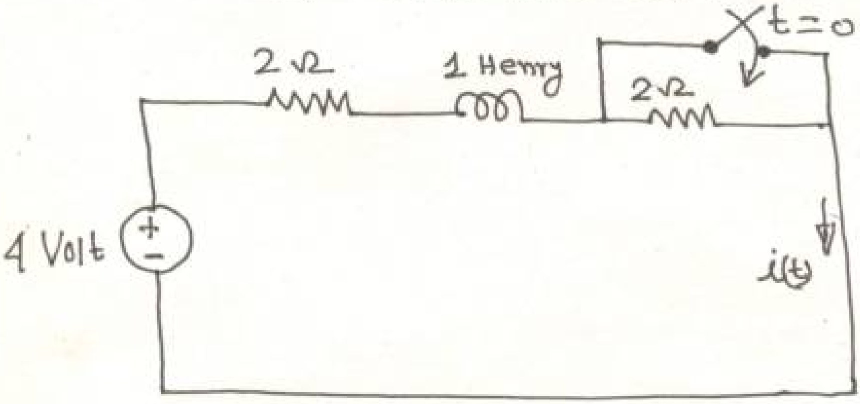


Fig.6.61: Circuit for Ex-6.22.

Soln.

The switch has been open for long time. Therefore,

$$i(0^-) = i(0) = i(0^+) = \frac{4}{4}\text{ Amp} = 1\text{ Amp}.$$

At $t=0$, switch is closed,

$$R_{TH} = 2\Omega, \qquad \tau = \frac{L}{R_{TH}} = \frac{1}{2}\text{ Sec.}$$

when $t \to \infty$ (the circuit reached the steady-state)

$$i(\infty) = \frac{4}{2} = 2\text{ Amp}.$$

We know for $t > 0$,

$$i(t) = i(\infty) + [i(0) - i(\infty)]\, e^{-t/\tau}$$

$$\therefore\ i(t) = 2 + (1-2)\, e^{-2t}$$

$$\therefore\ i(t) = (2 - e^{-2t})\text{ Amp}.$$

Now if we want $i(t)$ for all t, then

$$i(t) = \begin{cases} 1 \text{ Amp}, & t < 0 \\ (2 - e^{-2t}) \text{ Amp}, & t > 0 \end{cases}$$

or we can write

$$i(t) = 2 - e^{-2t} = 1 + (1 - e^{-2t})\,u(t).$$

Note that for $t < 0$, $u(t) = 0$ and for $t > 0$, $u(t) = 1$.

EX-6.23: In Fig. 6.62, at $t = 0$ switch S_1 is closed and 4 sec later, switch S_2 is closed. Determine $i(t)$. Calculate i for $t = 2$ sec and $t = 5$ sec.

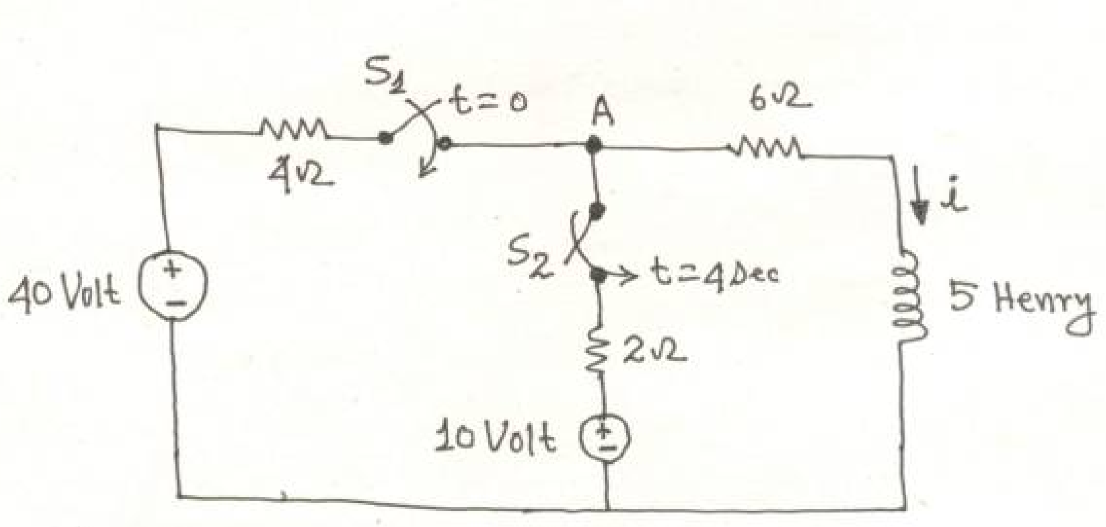


Fig. 6.62: Circuit for EX-6.23

Soln.

For $t<0$, S_1 and S_2 are open, so that $i=0$. Since the inductor current cannot change instantaneously,

$$i(0^-) = i(0) = i(0^+) = 0.$$

For $0 \le t \le 4$, S_1 is closed and S_2 is open. Hence, assuming for now that S_1 is closed forever,

$$i(\infty) = \frac{40}{(6+4)} = 4 \text{ Amp}$$

and $R_{TH} = (6+4) = 10\,\Omega.$

$$\therefore\ \tau = \frac{L}{R_{TH}} = \frac{5}{10} = \frac{1}{2} \text{ sec.}$$

We know

$$\cancel{i(\infty)}\ i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$$\therefore\ i(t) = 4 + (0-4)e^{-2t} = 4(1-e^{-2t}) \text{ Amp}$$

$$\therefore\ i(t) = 4(1-e^{-2t}) \text{ Amp}, \quad 0 \le t \le 4$$

For $t \geq 4$ sec, S_2 is closed. This means S_1 and S_2 are closed forever. Sudden closing of S_2 does not affect the inductor current because the current cannot change abruptly. Thus the initial current is

$$i(t=4) = i(4) = i(4^-) = 4(1-e^{-8}) \simeq 4 \text{ Amp}$$

Let v be the voltage at node A. Using KCL,

$$\frac{40-v}{4} + \frac{10-v}{2} = \frac{v}{6}$$

$$\therefore \quad v = \frac{180}{11} \text{ Volt.}$$

$$\therefore \quad i(\infty) = \frac{v}{6} = \frac{180}{11 \times 6} = 2.727 \text{ Amp.}$$

The Thevenin resistance at the inductor terminals is

$$R_{TH} = 6 + \frac{4 \times 2}{(4+2)} = \frac{22}{3}\,\Omega$$

$$\tau = \frac{L}{R_{TH}} = \frac{5}{\left(\frac{22}{3}\right)} = \frac{15}{22} \text{ Sec.}$$

We know [Eqn.(6.75)]

$$\cancel{i(t) = i(\infty) + [i(4) - i(\infty)]\, e}$$

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]\, e^{-(t-t_0)/\tau}$$

$$\therefore \quad i(t) = i(\infty) + [i(4) - i(\infty)]\, e^{-(t-4)/\tau}$$

$$\therefore \quad i(t) = 2.727 + (4 - 2.727)\, e^{-22(t-4)/15}$$

$$\therefore \quad i(t) = \left(2.727 + 1.273\, e^{-1.467(t-4)}\right) \text{Amp}, \quad t \geq 4$$

Putting all these together, we have

$$i(t) = \begin{cases} 0, & t \le 0 \\ 4(1-e^{-2t}), & 0 \le t \le 4 \\ 2.727 + 1.273\,e^{-1.467(t-4)}, & t \geq 4 \end{cases}$$

At $t = 2$ sec,

$$i(2) = 4(1 - e^{-4}) = 3.93 \text{ Amp.}$$

At $t = 5$ sec,

$$i(5) = \left(2.727 + 1.273\,e^{-1.467}\right) = 3.02 \text{ Amp.}$$

EX-6.24: Consider the circuit shown in Fig. 6.63, in which the switch opens at $t=0$. Determine expressions for $v(t)$, $i_R(t)$ and $i_L(t)$ for $t>0$. Assume that $i_L(t)$ is zero before the switch opens.

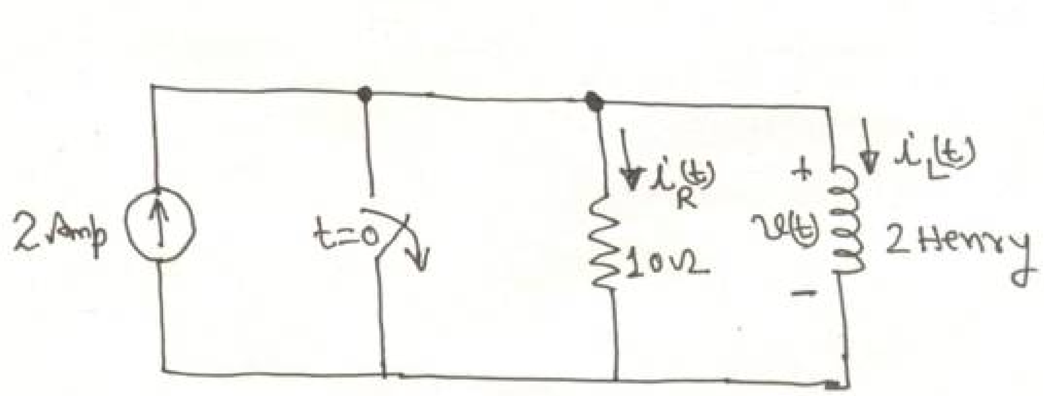


Fig. 6.63: Circuit for Ex-6.24.

Soln.

Given that $i_L(0^-) = i_L(0) = i_L(0^+) = 0$

At $t=0$, switch is opened.

~~Switch has been closed for long time~~ Hence,

$$v(0^-) = v(0) = v(0^+) = 2 \times 10 = 20 \text{ Volt.}$$

~~At $t=0$, switch is opened~~,

For $t \to \infty$ (circuit reached to the state)

$$i_L(\infty) = 2 \text{ Amp} \quad \text{and} \quad v(\infty) = 0$$

We know

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$\tau = \frac{L}{R_{TH}} = \frac{2}{10} = \frac{1}{5} \text{ sec.}$$

$$\therefore v(t) = 20e^{-5t} \text{ Volts.}$$

Also

$$i_L(t) = i_L(\infty) + [i_L(0) - i_L(\infty)]e^{-t/\tau}$$

$$\therefore i_L(t) = (2 - 2e^{-5t}) \text{ Amp.}$$

$$i_R(t) = \frac{v(t)}{R} = 2e^{-5t} \text{ Amp.}$$

EX- 6.25: In Fig. 6.64, The switch has been in position for a long time. At $t=0$, the switch moves from position '1' to position '2'. The switch is a make-before-break type; that is, the connection at position '2' is established before the connection at position 1 is broken, so there is no interruption of current through inductor. Determine

(a) the expression for $i(t)$ for $t \geq 0$

(b) the initial voltage across the inductor just after the switch has been moved to position '2'.

(c) Does this initial voltage make sense in terms of circuit behaviour

(d) How many milliseconds after the switch has been moved does the inductor voltage equal 30 volt.

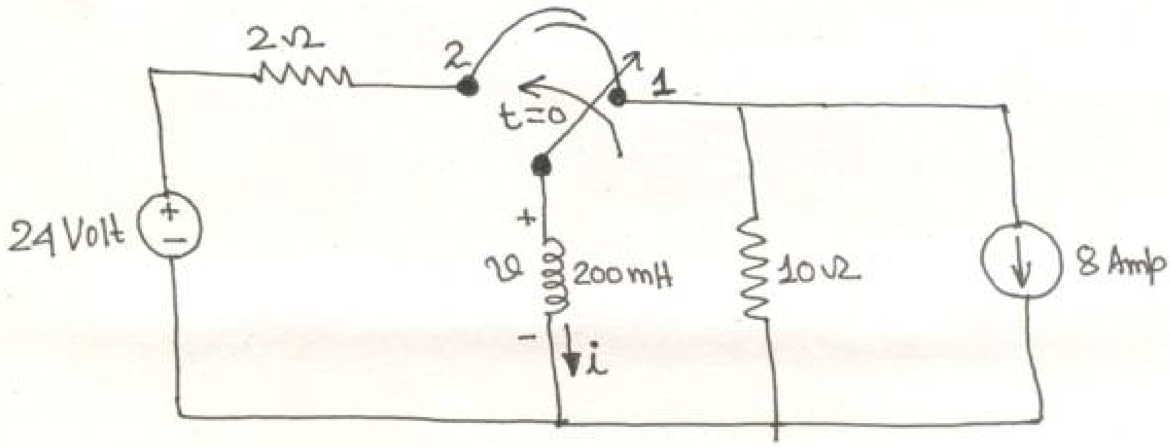


Fig.6.64: Circuit for Ex-6.25

Soln.

(a) The switch has been in position '1' for long time, so the 200 mH inductor is a short circuit across the 8 Amp current source. This current is oriented opposite to the reference direction for i;

Thus, $i(0^-) = i(0) = i(0^+) = 8$ Amp

When the switch is in position '2', the final value of i will be,

$$i(\infty) = \frac{24}{2} = 12 \text{ Amp.}$$

The time constant of the circuit is

$$\tau = \frac{L}{R_{TH}} = \frac{200}{2} = 100\text{ ms} = 0.1\text{ sec.}$$

We know,

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$$\therefore i(t) = 12 + (-8 - 12)e^{-10t}$$

$$\therefore i(t) = (12 - 20e^{-10t})\text{ Amp}; \quad t \geq 0$$

(b) The voltage across the inductor is

$$v = L\frac{di}{dt} = 200 \times 10^{-3}(200e^{-10t})$$

$$\therefore v = 40e^{-10t}\text{ Volt}; \quad t \geq 0^{+}$$

The initial inductor voltage is

$$v(0^{+}) = 40\text{ Volt.}$$

(c) Yes, in the instant after the switch has been moved to position '2', the inductor sustains a current of 8 Amp. counterclockwise around the newly formed closed path. This current causes a 16 Volt drop across the 2 Ω resistor. This voltage drop adds to the drop across the source, producing a 40 Volt drop across the inductor.

(d) We find the time at which the inductor voltage equals 30 Volt by solving the expression

$$30 = 40e^{-10t}$$

$$\therefore e^{10t} = \frac{4}{3}$$

$$\therefore t = \frac{1}{10}\ln\left(\frac{4}{3}\right) = 28.76 \text{ ms.}$$

EXERCISE-6

6.1: Assume that the switch in the circuit shown in Fig.6.64 [Ex-6.25] has been in position '2' for a long time and at $t=0$, it moves to position '1'. Determine (a) $i(0^+)$ (b) $v(0^+)$ (c) τ, $t>0$ (d) $i(t)$, $t \geq 0$ (e) $v(t)$, $t \geq 0$

Ans: (a) 12 Amp

(b) −200 Volt

(c) 20 ms

(d) $(-8 + 20e^{-50t})$ Amp

(e) $-200e^{-50t}$ Volt.