① (a) Using Rolle's theorem, Show that the function $f(x) = 3x^5 - 2x^3 + 12x - 8$ is an one-one function on $(-\infty, \infty)$.

Soln:

Let $x_1, x_2 \in (-\infty, \infty)$ be such that $(x_1 < x_2)$

$f(x_1) = f(x_2)$. Then, by Rolle's theorem, we have

$$f'(c) = 0 \quad \text{for some } c \in (x_1, x_2)$$

(1 mark)

Now,

$$f'(x) = 3 \cdot \cancel{5} \cdot x^4 - 2 \cdot 3x^2 + 12$$

$$= 15x^4 - 6x^2 + 12.$$

For $|x| < 1$, $15x^4 - 6x^2 + 12 > 0$

as $-6x^2 + 12 > 0$;

and, for $|x| \geq 1$, $15x^4 - 6x^2 + 12 > 0$

as $15x^4 - 6x^2 > 0$.

$\therefore f'(x) > 0 \quad \forall x \in (-\infty, \infty)$.

$\therefore f(x_1) = f(x_2)$ for $x_1 \neq x_2$ is not possible. Thus the function is one-one.

[2 marks]

Remarks:-

① If $f(\alpha) = f(\beta) = 0$ is assumed, then no marks shall be awarded.

As the function has only one real root, $f(\alpha) = f(\beta) = 0$ is not possible.

No marks for the following argument

② $\because f'(x) > 0$, so f is increasing and hence f is one-one.

③ If the Rolle's theorem is applied for this problem with $g(x_1) = f(x_1) - k$, $g(x_2) = f(x_2) - k$, then 3 marks awarded.

ie) You are allowed to use the following version of Rolle's theorem.

If f is differentiable on (a, b) & f is continuous on $[a, b]$, $f(a) = f(b) = 0$, then $f'(c) = 0$ for some $c \in (a, b)$.

R.N.

Q1 (b)

$$f(x) = \frac{1}{(1+x)^{\sqrt{3}}}$$

$$f^{(n)}(x) = \frac{(-1)^n \sqrt{3}(\sqrt{3}+1)\cdots(\sqrt{3}+n-1)}{(1+x)^{\sqrt{3}+n}} \quad \forall n \geq 1.$$

(1 M) — $$f^{(n)}(1) = \frac{(-1)^n \sqrt{3}(\sqrt{3}+1)\cdots(\sqrt{3}+n-1)}{2^{\sqrt{3}+n}} \quad \forall n \geq 1.$$

The Taylor polynomial of $f(x)$ about the point $x=1$ ~~of~~ degree n is

$$P_n(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{1!}(x-1)^2 + \cdots$$
$$\cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n.$$

& $$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1} \quad \text{for some } c,\ 1 < c < x.$$

(1) — $$P_n(x) = \frac{1}{2^{\sqrt{3}}} - \frac{\sqrt{3}}{2^{\sqrt{3}+1}}(x-1) + \cdots + \frac{(-1)^n\sqrt{3}(\sqrt{3}+1)\cdots(\sqrt{3}+n-1)}{n!\ 2^{\sqrt{3}+n}}(x-1)^n$$

(1) — & $$R_n(x) = \frac{(-1)^{n+1}\sqrt{3}(\sqrt{3}+1)\cdots(\sqrt{3}+n)}{(n+1)!\ (2+(x-1)\theta)^{\sqrt{3}+n+1}}(x-1)^{n+1}$$

Lagrange form. for some θ, $0 < \theta < 1$.

Remark! (1) If the remainder $R_n(x)$ is not in the Lagrange form, then NO marks awarded.

(2) NO marks if the n^{th} derivative $f^{(n)}$ is not computed.

Question 2(a) Consider

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{x^2+y^2-2x+1}$$

Substitute $x = 1 + r\cos\theta$
$y = r\sin\theta$

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{x^2+y^2-2x+1}$$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ any value}}} \frac{r\cos\theta \cdot r\sin\theta}{r^2(\cos^2\theta+\sin^2\theta)}$$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ any value}}} \cos\theta\,\sin\theta$$

$\Rightarrow$ Limit does NOT exist [1 mark]

$\Rightarrow$ Function is NOT continuous at $(1,0)$ and for no real number c, function can be made continuous at $(1,0)$. [1 mark]

NOTE: In order to show limit does NOT exist, path should be chosen so that the point $(1,0)$ can be approached. Thus for path $y = mx$ no marks will be awarded as one cannot reach point $(1,0)$ by letting $x \to 0$ or $y \to 0$.

Question 2(b)

Consider the function

$$f(x,y) = \begin{cases} x^3 \log(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

To check continuity of f_x and f_y at $(0,0)$.

<u>Step 1</u>: Evaluate f_x.

When $(x,y) \neq (0,0)$

$$f_x = \frac{\partial}{\partial x}\left(x^3 \log(x^2+y^2)\right)$$

$$f_x = 3x^2 \log(x^2+y^2) + \frac{2x^4}{x^2+y^2}$$ [½ mark]

At $(0,0)$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} h^2 \log h^2$$

$$= \lim_{h \to 0} \frac{1/h^2 \cdot 2h}{-2/h^3}$$ [L'Hôpital rule]

$$= \lim_{h \to 0} -h^2$$

$$= 0$$ [½ mark]

Thus

$$f_x(x,y) = \begin{cases} 3x^2 \log(x^2+y^2) + \dfrac{2x^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$ (3)

To check continuity of $f_x(x,y)$ at $(0,0)$

Polar-co.ordinate substitution:

$x = r\cos\theta \,;\; y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} f_x(x,y)$$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ any value}}} \left\{ 3r^2\cos^2\theta \log(r^2) + \frac{2r^4\cos^4\theta}{r^2} \right\}$$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ any value}}} (r^2 \log r^2)(3\cos^2\theta) + \lim_{\substack{r\to 0 \\ \theta \text{ any value}}} r^2(2\cos^4\theta)$$

Note that $(3\cos^2\theta)$ and $(2\cos^4\theta)$ are bounded functions of θ

& $\lim_{r\to 0} r^2 \log r^2 = 0$ (L'Höpital rule)

$\Rightarrow \lim_{(x,y)\to(0,0)} f_x(x,y) = 0 = f_x(0,0)$

[1 mark]

$\Rightarrow f_x$ is continuous at $(0,0)$

Step 2: Evaluate f_y

when $(x,y) \neq (0,0)$

$$f_y = \frac{2x^3 y}{x^2+y^2}$$ [½ mark]

when $(x,y) = (0,0)$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{0 - 0}{k}$$

$$= 0$$ [½ mark]

Thus,

$$f_y(x,y) = \begin{cases} \dfrac{2x^3 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

To check continuity of f_y at $(0,0)$

Polar co-ordinate substitution:

$x = r\cos\theta$; $y = r\sin\theta$

$$\lim_{(x,y) \to (0,0)} f_y(x,y) = \lim_{\substack{r \to 0 \\ \theta \text{ any} \\ \text{value}}} \frac{2r^3 \cos^3\theta \, r \sin\theta}{r^2}$$

$$= \lim_{\substack{\theta \text{ any} \\ \text{value} \\ r \to 0}} r^2 \left(2\cos^3\theta \sin\theta\right)$$

Note that $2\cos^3\theta \sin\theta$ is a bounded function of θ

$\therefore \lim_{(x,y)\to(0,0)} f_y(x,y) = 0 = f_y(0,0)$

$\therefore$ f_y is continuous at $(0,0)$

[1 mark]

NOTE:

1) Full marks given if continuity is proved using $\varepsilon-\delta$ method. However $f_x(0,0)$ and $f_y(0,0)$ need to be calculated. If not done, 1 mark is deducted.

2) Only calculation of $f_x(0,0)$, $f_y(0,0)$ does not prove continuity of f_x & f_y at $(0,0)$.

3 a). $f(x,y) = \begin{cases} \left(\dfrac{2x^2-3y^2}{x^2+y^2}\right)\sin xy, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y)-f(0,y)}{h}$$

$$= \lim_{h\to 0} \left(\frac{2h^2-3y^2}{h^2+y^2}\right)\frac{\sin hy}{h}\cdot\frac{y}{y} = -3y \quad \text{— 1M}$$

$$f_y(x,0) = \lim_{k\to 0} \frac{f(x,k)-f(x,0)}{k}$$

$$= \lim_{k\to 0} \left(\frac{2x^2-3k^2}{x^2+k^2}\right)\frac{\sin xk}{k}\cdot\frac{x}{x} = 2x \quad \text{— 1M}$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{-3k-0}{k} = -3 \quad \text{— 1M}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2h-0}{h} = 2 \quad \text{— 1M}$$

Remarks ①. If the calculation is shown without proper definition of derivatives and the rest is correct: 1 Mark deducted

②. If the entire procedure and the answer is correct but $f_x(0,0)=0$, $f_y(0,0)=0$ not shown: 1 Mark deducted

③. If f_{yx} and f_{xy} are computed at $(x,y)\neq(0,0)$ and arrived at $f_{yx}=-3$, $f_{xy}=2$: No marks awarded.

(b). $u = 5x - 2y$; $v = 4y - z$; $w = z + 2x$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$= 5f_u + 2f_w$$

Using similar definitions

$$\frac{\partial f}{\partial y} = -2f_u + 4f_v$$

$$\frac{\partial f}{\partial z} = -f_v + f_w$$

Correct chain rule definitions and coefficients - 1 Mark

Correct final answer : 1 Mark

$$\therefore \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 3(f_u + f_v + f_w) = 9$$

Remarks: (1) If improper chain rule is used and some how arrived at the correct answer: No marks awarded.

Q. 4(a)

Test differentiability of the following function at the origin

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

(3 Marks)

Soln :-

- Compute $f_x(0,0)$ & $f_y(0,0)$:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

(1 M)

and $$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

- Therefore $$\Delta f = f(0+h, 0+k) - f(0,0) = \frac{h^3 k}{h^4+k^2}$$

$$df = h f_x(0,0) + k f_y(0,0) = 0$$

Now, $$\lim_{\Delta\rho\to 0} \frac{\Delta f - df}{\Delta \rho}$$ (here $\Delta\rho = \sqrt{h^2+k^2}$)

1(M)

$$= \lim_{(h,k)\to(0,0)} \frac{h^3 k}{(h^4+k^2)\sqrt{h^2+k^2}}.$$

This limit must be zero if f is differentiable.

- Considering the path $k = mh^2$, we see that

$$\lim_{\Delta\rho\to 0} \frac{\Delta f - df}{\Delta\rho} = \lim_{h\to 0} \frac{h^3 \times mh^2}{(h^4+m^2h^4)\sqrt{h^2+m^2h^4}}$$

(1 M)

$$= \lim_{h\to 0} \frac{m}{(1+m^2)\sqrt{1+m^2h^2}}$$

$$= \frac{m}{1+m^2},$$ which is different for different values of m. Therefore, the given function is NOT differentiable at (0,0).

Q.4(a)

(Alternate solution)

- Compute $f_x(0,0)$ & $f_y(0,0)$ [1 M]
- f is differentiable at $(0,0)$ if and only if

$$\Delta f = f(h,k) - f(0,0) = h f_x(0,0) + k f_y(0,0) + \varepsilon_1 h + \varepsilon_2 k$$

where ε_1 & ε_2 are functions of (h,k) such that they tend to zero as $(h,k) \to (0,0)$.

Thus,
$$\frac{h^3 k}{h^4 + k^2} = \varepsilon_1 h + \varepsilon_2 k$$
where $\varepsilon_1, \varepsilon_2 \to 0$ as $(h,k) \to (0,0)$. 1M

- Taking $h^2 = k$ path, we see that

$$\frac{h^3 \cdot h^2}{h^4 + h^4} = \varepsilon_1 h + \varepsilon_2 h^2$$

$$\Rightarrow \frac{h}{2} = \varepsilon_1 h + \varepsilon_2 h^2$$

$$\Rightarrow \frac{1}{2} = \varepsilon_1 + \varepsilon_2 h$$

Now taking limit as $(h,k) \to (0,0)$ we see that

LHS $= \frac{1}{2} \neq$ RHS $= 0$ 1M

This shows that the given function is <u>not</u> differentiable at $(0,0)$.

Q 4(b)

$$f(x,y) = \sqrt{x^2+y^3} \quad , \quad (x,y) = (1,2)$$

$$f((1,2)) = 3$$

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^3}} \quad , \quad f_x((1,2)) = \frac{1}{3}$$

$$f_y(x,y) = \frac{3y^2}{2\sqrt{x^2+y^3}} \quad , \quad f_y((1,2)) = 2$$

$$f_{xx}(x,y) = \frac{y^3}{(x^2+y^3)^{3/2}} \quad , \quad f_{xx}((1,2)) = \frac{8}{27}$$

$$f_{yy}(x,y) = \frac{12x^2y+3y^4}{4(x^2+y^3)^{3/2}} \quad , \quad f_{yy}((1,2)) = \frac{2}{3}$$

$$f_{xy}(x,y) = \frac{-3xy^2}{2(x^2+y^3)^{3/2}} \quad , \quad f_{xy}((1,2)) = -\frac{2}{9}$$

1 mark

Then $f(1+h, 2+k) \cong 3 + \frac{1}{3}h + 2k + \frac{1}{2!}\left[\frac{8}{27}h^2 + 2\left(-\frac{2}{9}\right)hk + \frac{2}{3}k^2\right]$ 1 mark

Setting $x = 1+h$, $y = 2+k$

$$f(x,y) = 3 + \frac{1}{3}(x-1) + 2(y-2) + \frac{4}{27}(x-1)^2 - \frac{2}{9}(x-1)(y-2) + \frac{1}{3}(y-2)^2$$

Then $\sqrt{(1.02)^2 + (1.97)^3} = f(1+0.02,\ 2-0.03)$

$$\cong 3 + \frac{1}{3}(0.02) + 2(-0.03) + \frac{4}{7}(0.02)^2 - \frac{2}{9}(0.02)(-0.03) + \frac{1}{3}(-0.03)^2$$

1 mark

$$= 2.9476$$

5(a) Let $H(x,y)$ be a homogeneous fn. of x and y of degree n having continuous partial derivatives and $u(x,y) = (x^2+y^2)^{-n/2}$, $n>0$. Determine the constant α satisfying

$$\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right) = \alpha.$$

Dutta: 25.9.19

Soln.

$$xH_x + yH_y = nH \quad \text{———} \quad (1M)$$

$$u = (x^2+y^2)^{-n/2}$$

$$u_x = -nx(x^2+y^2)^{-(n+2)/2}, \quad u_y = -ny(x^2+y^2)^{-(n+2)/2}$$

$$u_{xx} = -n(x^2+y^2)^{-(n+2)/2} + n(n+2).x^2(x^2+y^2)^{-\frac{n}{2}-2}$$

$$u_{yy} = -n(x^2+y^2)^{-(n+2)/2} + n(n+2).y^2(x^2+y^2)^{-\frac{n}{2}-2}$$

$$\therefore \frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right)$$

[2 Marks]

$$= \frac{\partial H}{\partial x}\cdot\frac{\partial u}{\partial x} + H\frac{\partial^2 u}{\partial x^2} + \frac{\partial H}{\partial y}\cdot\frac{\partial u}{\partial y} + H\frac{\partial^2 u}{\partial y^2}$$

$$= -nx(x^2+y^2)^{-(n+2)/2}\frac{\partial H}{\partial x} - ny(x^2+y^2)^{-(n+2)/2}\frac{\partial H}{\partial y}$$

$$+ H\left[-2n\cancel{(x^2+y^2)}^{-(n+2)/2} + n(n+\cancel{2})(x^2+y^2)^{-(n+2)/2}\right]$$

$$= -n(x^2+y^2)^{-(n+2)/2}\left[x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y}\right] + n^2H(x^2+y^2)^{-(n+2)/2}$$

$$= -n\cancel{(x^2+y^2)}^{-(n+2)/2}.\,nH + \cancel{n^2H(x^2+y^2)}^{-(n+2)/2}$$

$$= 0 \quad \text{———}$$

5-b Find all Critical Points of the following function

$$f(x,y) = e^x (x^2-y^2), \quad \forall (x,y) \in R^2.$$

Also classify them for the local maximum, local minimum, and saddle points. [3 mark]

Sol: $f_x = e^x(x^2+2x-y^2)$

$f_y = -2y e^x$

For critical points: $f_x = 0$ & $f_y = 0$

implies $y = 0$ & $x = 0, -2$

$\therefore$ $(0,0)$ & $(-2,0)$ are critical points [1 mark]

Classification: —

$f_{xx} = e^x(x^2+4x-y^2+2)$, $f_{yy} = -2e^x$, $f_{xy} = -2ye^x$.

$f_{xx}(0,0) f_{yy}(0,0) - f_{xy}^2(0,0) = -4 < 0$

$\therefore$ $(0,0)$ is a saddle point. – [1 mark]

$f_{xx}(-2,0) f_{yy}(-2,0) - f_{xy}^2(-2,0) = \frac{4}{e^4} > 0$

and $f_{xx}(-2,0) = -2/e^2 < 0$

$\therefore$ $(-2,0)$ is a point of local maximum [1 mark]