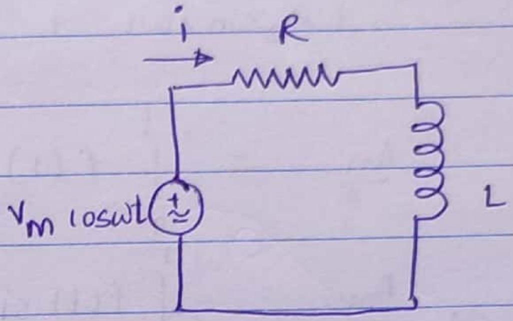
SINUSOIDAL STEADY-STATE ANALYSIS

① Forced response to sinusoidal functions

$$L\frac{di}{dt} + Ri = V_m \cos \omega t \quad \text{—— ①}$$



$i(t) = C.F + P.I$

Complimentary function (transient) $= Ae^{-\frac{Rt}{L}}$

Particular Integral (steady state part).

To find P.I

(Assume linear function)

$$i(t) = I_1 \cos \omega t + I_2 \sin \omega t \quad \text{—— ②}$$

Putting in equation ① and solving $I_1, I_2 \rightarrow$

$$I_1 = R\frac{V_m}{R^2+\omega^2L^2} \qquad I_2 = \omega L\frac{V_m}{R^2+\omega^2L^2}$$

let $i(t) = A\cos(\omega t - \theta)$

$$= A\cos\theta\cos\omega t + A\sin\theta\sin\omega t$$

$$\Rightarrow \theta = \tan^{-1}\frac{\omega L}{R} \; ; \; A = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}$$

$$\Rightarrow i(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cdot \cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

SIBA SMARAK NOTES

* Fourier Series

$$f(t) = A_0 + A_1\cos\omega t + A_2\cos 2\omega t + \cdots + B_1\sin\omega t + B_2\sin 2\omega t + \cdots$$

$$\frac{A_n}{2} = \frac{1}{T}\int_0^T f(t)\cos n\omega t\,dt$$

$$\frac{B_m}{2} = \frac{1}{T}\int_0^T f(t)\sin m\omega t\,dt$$

Orthogonality

(I) $\int_0^T \cos n\omega t\,\sin m\omega t\,dt = 0$

(II) $\int_0^T \cos n\omega t\,\cos m\omega t\,dt = 0$

(III) $\int_0^T \sin n\omega t\,\sin m\omega t\,dt$

$$f(t) = A_0 + \frac{A_1}{2}\left[e^{i\omega t} + e^{-i\omega t}\right] + \frac{A_2}{2}\left[e^{i2\omega t} + e^{-i2\omega t}\right] + \cdots + \frac{B_1}{2i}\left[e^{i\omega t} - e^{-i\omega t}\right] + \frac{B_2}{2i}\left[e^{2i\omega t} - e^{-2i\omega t}\right] + \cdots$$

$$= \sum_{n=0}^{\infty} C_n e^{in\omega t} + C_n^* e^{-in\omega t}$$

$$C_n = (A_n - iB_n)/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

half-period

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

full period.

Siba Smarak Notes

OSCILLATIONS

① SHM (Free, undamped)

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \quad , \quad \omega_0^2 = \frac{k}{m}$$

a) General solution $\rightarrow$

$$\boxed{x = c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}}$$

$$\Rightarrow \boxed{x = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t}$$

Case 1 : $x = a_0$, $\frac{dx}{dt} = 0$ at $t = 0$

$$\boxed{x = a_0 \cos\omega_0 t}$$

Case 2 : $x = 0$, $\frac{dx}{dt} = V_0$ at $t = 0$

$$\boxed{x = \frac{V_0}{\omega_0} \sin(\omega_0 t)}$$

Case 3 : $x = a$, $\frac{dx}{dt} = V$ at $t = 0$

$$\boxed{x = a_0 \cos(\omega_0 t + \alpha)} \quad ; \quad a_0 = \sqrt{a^2 + \left(\frac{V}{\omega_0}\right)^2}$$

$$\alpha = \tan^{-1}\left(-\frac{V_0}{\omega_0 a}\right)$$

SIBA SMARAK NOTES

(2) Damped Harmonic Motion

Eqn: $\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = 0$; $2\beta = \frac{r}{m}$ and

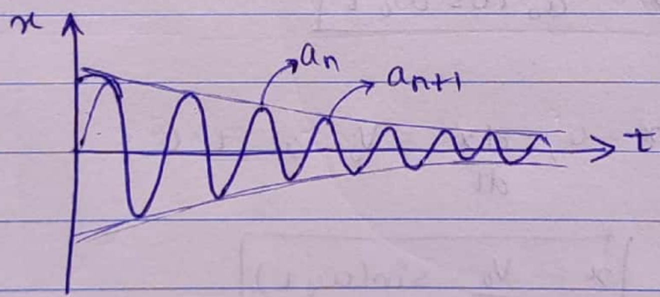
$\omega_0^2 = \frac{k}{m}$.

$(x = e^{pt})$

$p_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2} = -\beta \pm i\omega$ $\boxed{\omega = \sqrt{\omega_0^2 - \beta^2}}$

(a) Underdamped motion $(\beta^2 - \omega_0^2 < 0)$

$x = e^{-\beta t}(c_1 e^{i\omega t} + c_2 e^{-i\omega t})$



Case 1: $x = a$, $\frac{dx}{dt} = 0$ at $t = 0$

$\Rightarrow \boxed{x = a_0 e^{-\beta t}\cos(\omega t + \alpha)}$

$\rightarrow a_0 = a\sqrt{1 + \frac{\beta^2}{\omega^2}}$; $\alpha = \tan^{-1}\left(-\frac{\beta}{\omega}\right)$

$\rightarrow \lambda = \ln\frac{a_n}{a_{n+1}} = \beta T$ = logarithmic decrement.

$\rightarrow$ Quality factor $= Q = \frac{\pi}{\lambda} = \left(\frac{\omega}{2\beta}\right)$ $(\omega = \sqrt{\omega_0^2 - \beta^2})$

Case 2 : $x=0,\ \frac{dx}{dt}=V_0\ ,\ t=0$

$$\boxed{x=\frac{V_0}{\omega}\, e^{-\beta t}\sin\omega t}$$

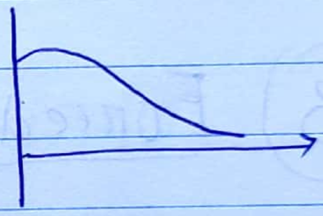
→ maxm. displacement of the mass occurs at time : $\boxed{t=\frac{1}{\omega}\tan^{-1}\left(\frac{\omega}{\beta}\right)}$

(b) Overdamped motion $(\beta^2-\omega_0^2>0)$

$P_{1,2}=-\beta\pm\gamma$ ..... $\gamma=\sqrt{\beta^2-\omega_0^2}$

Case 1 : $x=a,\ \frac{dx}{dt}=0$, at $t=0$

$$\boxed{x=a_0\, e^{-\beta t}\cosh\gamma t}$$



Case 2 : $x=0\ ,\ \frac{dx}{dt}=V_0$ at $t=0$

$$\boxed{x=\frac{V_0}{\gamma}\, e^{-\beta t}\sinh\gamma t}$$

Case 3 : Heavily overdamped $(\beta\gg\omega_0)$

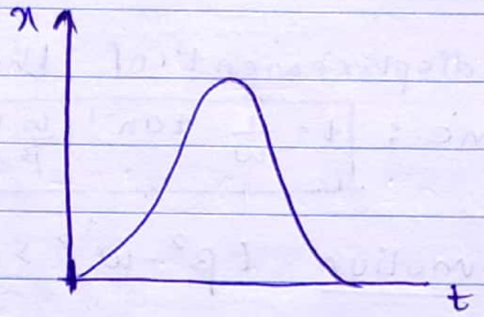
$$\boxed{x=c_1\, e^{-\left(\frac{\omega_0^2}{2\beta}\right)t}+c_2\, e^{-2\beta t}}$$

(c) Critically Damped Motion $(\beta^2-\omega_0^2=0)$

$$\boxed{x=e^{-\beta t}(c_1+c_2 t)}$$

Case 1 : $x = 0, \frac{dx}{dt} = V_0$ at $t = 0$.

$$\boxed{x = V_0 t e^{-\beta t}}$$



→ Fastest possible return to equilibrium position

Maximum displacement $x = \frac{V_0}{\beta e}$ at $t = \frac{1}{\beta}$

3 Forced Oscillations and Resonance

$$\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_0^2 x = f_0 \cos \omega t$$

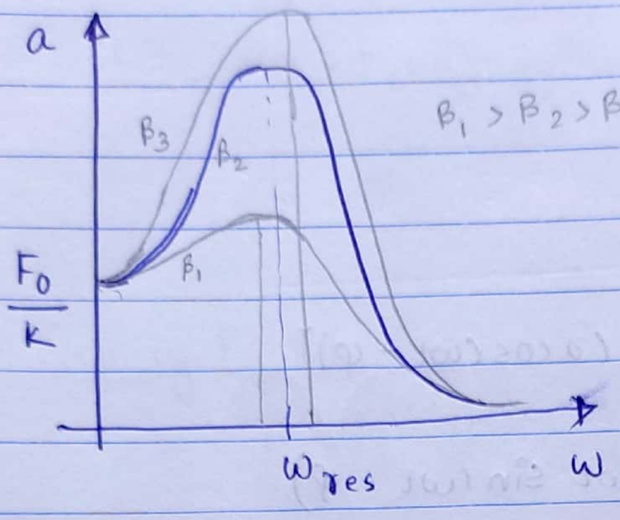
$$\left(f_0 = \frac{F_0}{m} \right)$$

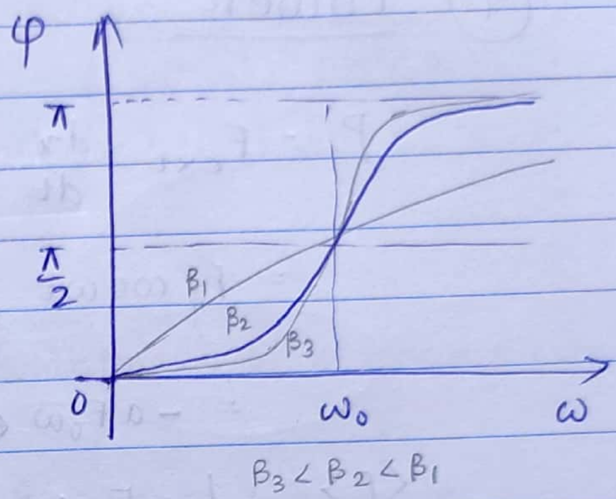
→ Steady-state : particular integral

$$\boxed{x = a \cos(\omega t - \varphi)}$$

$$a = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\varphi = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2} \right)$$





Low frequencies (stiffness -controlled)

$$a \longrightarrow \frac{F_0}{m\omega_0^2} \qquad \varphi \longrightarrow 0$$

High frequencies (mass - controlled)

$$a \longrightarrow \frac{F_0}{m\omega^2} \qquad \varphi \rightarrow \pi$$

Mid - frequencies (damping - controlled).

$$\omega_{res} = \sqrt{\omega_0^2 - 2\beta^2}$$

$$a_{max} = \frac{f_0}{2\beta\omega} = \frac{F_0}{2\gamma\omega} \qquad \phi \rightarrow \frac{\pi}{2}$$

$$\frac{f_0}{2\beta\sqrt{\omega_0^2 - \beta^2}}$$

(4) Power

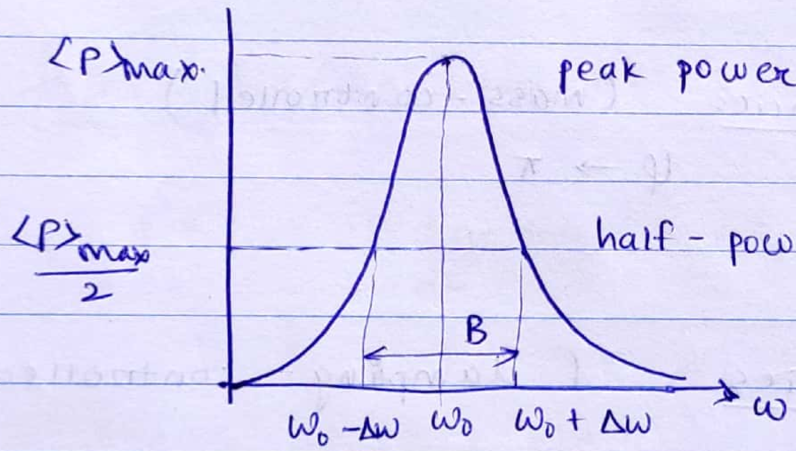
$$P = F_{ext} \cdot \frac{dx}{dt}$$

$$= F_0 \cos \omega t \; \frac{d}{dt}\left(a \cos(\omega t - \varphi)\right)$$

$$= -a F_0 \omega \cos \omega t \, \sin(\omega t - \varphi)$$

$$\langle P \rangle = \frac{1}{2} a F_0 \,\omega \sin\varphi$$ (to prove this use $\sin(\omega t + \phi)$)

$$= \frac{F_0^2 \gamma}{2m^2} \cdot \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$



(Lorentzian)

$$\boxed{\langle P \rangle_{max} = \frac{F_0^2}{2\gamma}} \qquad \boxed{B = 2\beta = \frac{\gamma}{m}}$$

$\rightarrow \langle E \rangle =$ stored energy $= \dfrac{m f_0^2}{4} \dfrac{(\omega_0^2 + \omega^2)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$

$= \dfrac{m}{4} (\omega_0^2 + \omega^2)\, a^2$

Quality Factor

$\rightarrow$ quantifies the sharpness of resonance curve.

$Q_0 = \dfrac{\omega_0}{2\beta} = \dfrac{\pi}{\lambda} \approx \dfrac{\text{maxm. resonance amplitude}}{\text{amplitude at low frequencies}}$

$= 2\pi \left(\dfrac{\text{energy stored as pe + ke}}{\text{energy dissipated against friction}} \right)$

* Superposition of two SHM (in perpendicular dirns.)

$x(t) = a\cos(\omega_1 t + \phi_1)$

$y(t) = b\cos(\omega_2 t + \phi_2)$

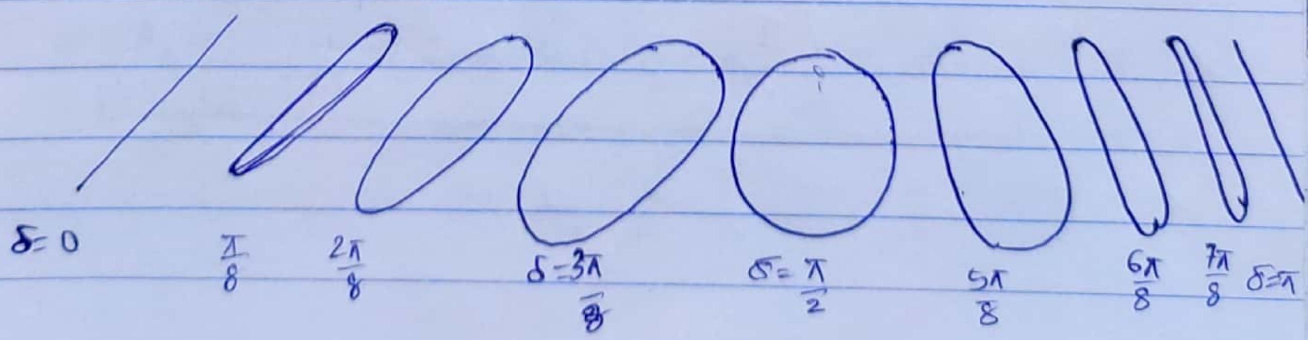
Case I) $\omega_1 = \omega_2 = \omega$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos(\phi_2 - \phi_1) = \sin^2(\phi_2 - \phi_1)}$$

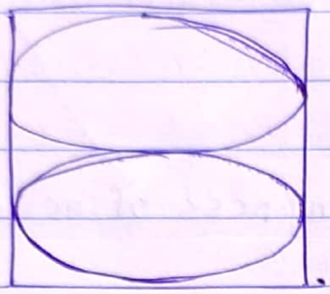
$\delta = \phi_2 - \phi_1 = \pm n\pi$ — straight line

$\delta = \pi/2$ and $a = b$ — circle

ellipse otherwise.

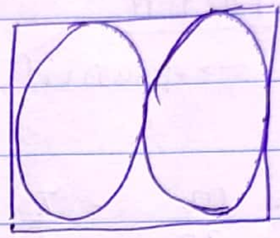


Case II): $\omega_1 \neq \omega_2$: (Lissajous figures)



$\frac{\omega_x}{\omega_y} = 2$

$\frac{\omega_y}{\omega_x} = 2$



- A phase space is a space in which all possible states of a system are represented.

For mechanical systems, the phase space usually consists of all possible values of position and momentum variables.

$$E = \frac{1}{2}KA^2 = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

or $\boxed{1 = \frac{p^2}{m\omega_0^2 A^2} + \frac{x^2}{A^2}}$ as E increases, A also increases.

WAVES

1 Coupled Oscillations

→ Eigenstates (stationary states / Normal modes) are defined as those states of the oscillator in which the entire system oscillates with a single frequency.

Eigenfreq. — Eigenstate

$q_2 = 0$; ① $\omega_1 = \sqrt{\frac{k}{m}}$ (slow mode) ⇒ $(A/B) = 1$ (synchronously)
↳ $\xi_1(t) = \xi_2(t)$

$q_1 = 0$; COM at rest ② $\omega_2 = \sqrt{\frac{k+2k_0}{m}}$ ⇒ $(A/B) = -1$ (Antichronously)
↳ fast mode (breathing mode) ↳ $\xi_1(t) = -\xi_2(t)$.

General solution

From above eigenstates –

Assume a linear combination $\xi_1 = \frac{1}{\sqrt{2}}(q_1 - q_2)$ and $\xi_2 = \frac{1}{\sqrt{2}}(q_1 + q_2)$

↓

$q_1 = a_1 \cos(\omega_1 t + \alpha_1)$ $q_2 = a_2 \cos(\omega_2 t + \alpha_2)$

$\left(\omega_1 = -\omega_2 = \sqrt{\frac{\sqrt{2}k}{m}}\right)$

↓

$$\boxed{\begin{aligned} \xi_1(t) &= \frac{1}{\sqrt{2}}\left[a_1 \cos(\omega_1 t + \alpha_1) - a_2 \cos(\omega_2 t + \alpha_2)\right] \\ \xi_2(t) &= \frac{1}{\sqrt{2}}\left[a_1 \cos(\omega_1 t + \alpha_1) + a_2 \cos(\omega_2 t + \alpha_2)\right] \end{aligned}}$$

Case 1): (Normal mode 1)

Pulling each of two masses to right by same amount A and leaving them from rest →

$\xi_1(0) = \xi_2(0) = A$ & $\dot{\xi}_1(0) = \dot{\xi}_2(0) = 0$

$$\boxed{\xi_1(t) = \xi_2(t) = A\cos(\omega_1 t)}$$

Case 2): (N M 2)

$\xi_1(0) = -\xi_2(0) = A$ & $\dot{\xi}_1(0) = \dot{\xi}_2(0) = 0$

$$\boxed{\xi_1(t) = -\xi_2(t) = A\cos(\omega_2 t)}$$

Case 3): Resonance (One mass pulled right by A other rest).

$\xi_1(0) = A$

$\xi_2(0) = \dot{\xi}_1(0) = \dot{\xi}_2(0) = 0$

✓ $\xi_1(t) = \frac{A}{2}(\cos\omega_1 t + \cos\omega_2 t)$

✓ $\xi_2(t) = \frac{A}{2}(\cos\omega_1 t - \cos\omega_2 t)$.

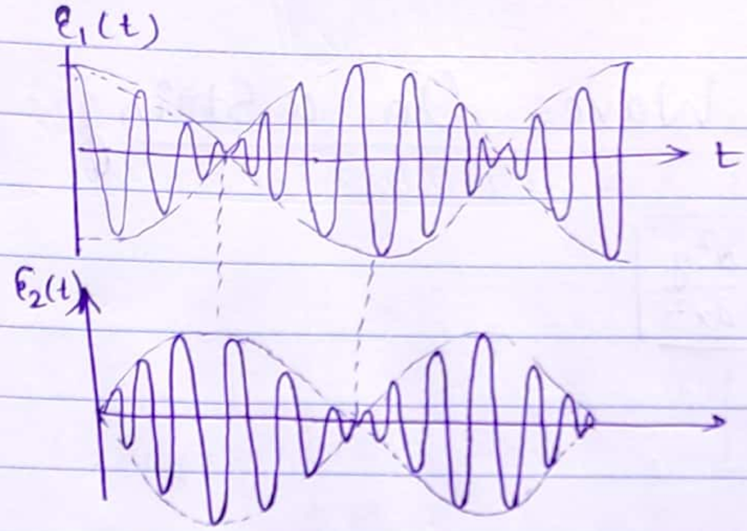
If coupling of middle spring is weak;

$k_0 = \eta k$; $\eta \ll 1$

$\omega_2 \approx \omega_1 = \omega$

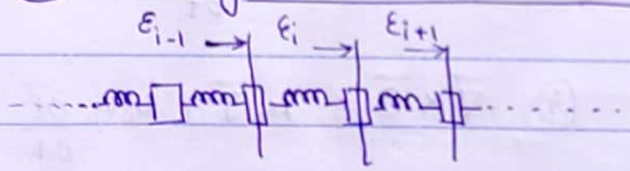
✓ $\xi_1(t) = A\cos\left(\frac{\eta}{2}\omega t\right)\cos\omega t$

✓ $\xi_2(t) = A\sin\left(\frac{\eta}{2}\omega t\right)\sin\omega t$



→ The periodic shuttling of energy of oscillations between the two masses is referred to as <u>resonance</u>.

② Longitudinal Waves in a Rod



$$m\frac{d^2\xi_i}{dt^2} = k(\xi_{i+1} - \xi_i) - k(\xi_i - \xi_{i-1})$$

Take $m \rightarrow \rho A(\Delta x)$ $\quad\quad$ $\xi_{i-1}(t) = \xi(x - \Delta x, t)$

$\xi_i(t) \rightarrow \xi(x,t)$. $\quad\quad$ $k \rightarrow \left(\frac{YA}{\Delta x}\right)$

$\xi_{i+1}(t) \rightarrow \xi(x + \Delta x, t)$

$$\boxed{\frac{\partial^2\xi}{\partial t^2} = \frac{Y}{\rho}\frac{\partial^2\xi}{\partial x^2}} \Rightarrow c^2 = \frac{Y}{\rho} \rightarrow \text{Young's Modulus}$$

$|||^{ly}$ for fluid flowing through pipe →

$$\boxed{\frac{\partial^2\xi}{\partial t^2} = \frac{B_a}{\rho}\frac{\partial^2\xi}{\partial t^2}} \Rightarrow \boxed{c^2 = \frac{B_a}{\rho}} \rightarrow \text{adiabatic Bulk modulus.}$$

(3) Transverse Waves On a String

$$\boxed{\frac{\partial^2 y}{\partial t^2} = \left(\frac{T}{\mu}\right)\frac{\partial^2 y}{\partial x^2}}$$

$$\Rightarrow \boxed{c^2 = \frac{T}{\mu}}$$

(4) Electromagnetic Waves

let $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = (E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k})\, e^{i(k_x x + k_y y + k_z z - \omega t)}$

Now from Maxwell's eqns –

(1) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ (3) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

(2) $\vec{\nabla}\cdot\vec{B} = 0$ (4) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

In free space : $\vec{\nabla}\cdot\vec{B} = 0$ & $\vec{\nabla}\cdot\vec{E} = 0$

$$\vec{\nabla}\cdot\vec{E} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\vec{E}$$

$$= \frac{\partial}{\partial x}E_{0x}e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \frac{\partial}{\partial y}E_{0y}e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \frac{\partial}{\partial z}E_{0z}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= i\big(\underbrace{k_x E_{0x} + k_y E_{0y} + k_z E_{0z}}_{\vec{k}\cdot\vec{E}}\big)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= 0 \qquad |\vec{k}| = \frac{2\pi}{\lambda}$$

$\Rightarrow i\,\vec{k}\cdot\vec{E} = 0 \Rightarrow \vec{E}\perp\vec{k}$ (i.e. transverse wave)

$\vec{k}$ — propagation vector or wave

Siba Smarak Notes

(A) Faraday's Law -

$$\oint_c \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint_s \vec{B}\cdot d\vec{s}$$

$$\Rightarrow \boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}} \Rightarrow \frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial^2 B_z}{\partial x\,\partial t}$$

(B) Ampere's Law -

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\iint_s \vec{E}\cdot d\vec{s}$$

$$\Rightarrow \boxed{-\frac{\partial B_z}{\partial x} = \mu_0\varepsilon_0 \frac{\partial E_y}{\partial t}} \Rightarrow -\frac{\partial^2 B_z}{\partial t\,\partial x} = \mu_0\varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

$$\Rightarrow \boxed{\left(\frac{\partial^2 E_y}{\partial t^2}\right)\mu_0\varepsilon_0 = \left(\frac{\partial^2 E_y}{\partial x^2}\right)} \Rightarrow \boxed{c^2 = \frac{1}{\mu_0\varepsilon_0}}$$

$$\boxed{\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}}$$

(5) Maxwell's Eqn.

$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ (divergence of E)

$\nabla\cdot\vec{B} = 0$ (divergence of B)

$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ (curl of E)

$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (curl of B).

$$\left(\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).$$

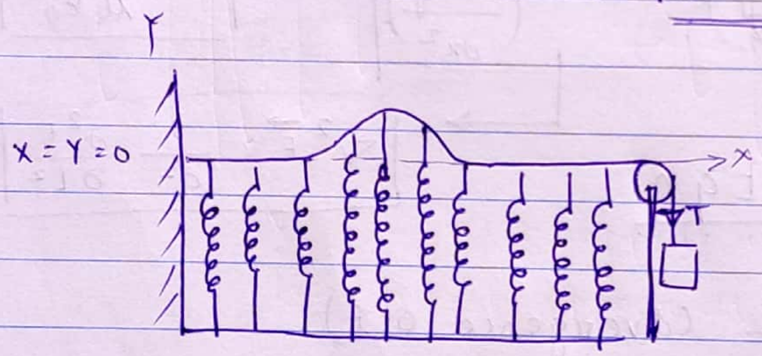
6 Dispersive Propagation And Dispersion Relation

→ The relation between the frequency ω and the wave number k is shown as the dispersion relation

$(\omega = ck) \longrightarrow$ for non-dispersive propagn.

$c = \frac{\omega}{k}$ is independent of frequency or the wavelength of the wave

→ this is characteristic feature of non-dispersive propagn.



(Dispersive)

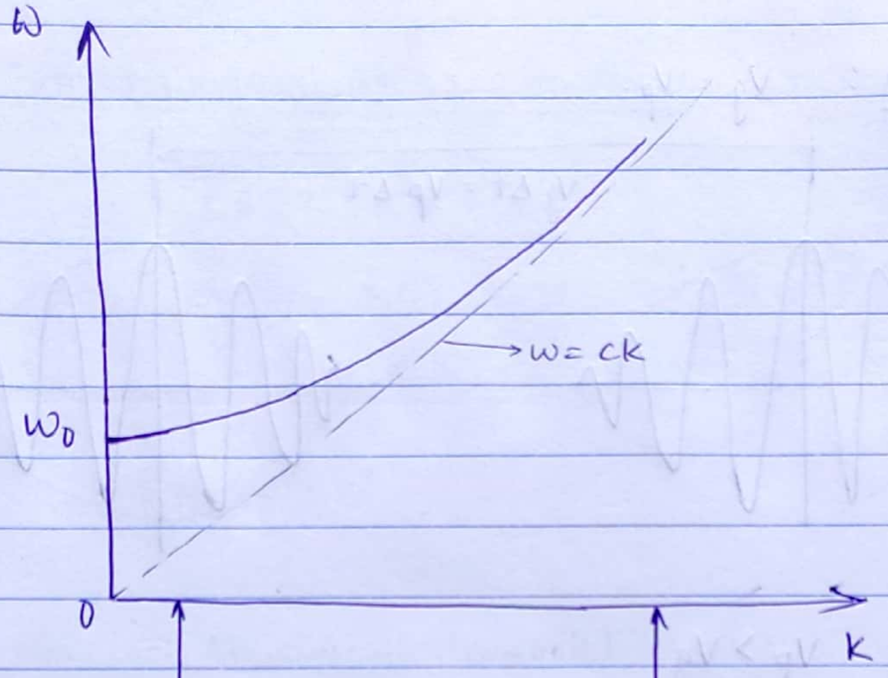
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \omega_0^2 y \ ; \ c^2 = \frac{T}{\mu} \ ; \ \omega_0^2 = \frac{K}{\mu}$$

Put $y = Ae^{i(kx - \omega t)}$ (monochromatic soln) to find dispersion relation

$$\Downarrow$$

$$\boxed{\omega = \sqrt{c^2 k^2 + \omega_0^2}}$$

(dispersive dispersion relation)



|low k| (high λ)

$\omega = \omega_0$

(cutoff frequency)

below which no waves can propagate

|high k| (low λ)

$\omega = ck$

7 Phase velocity and Group velocity

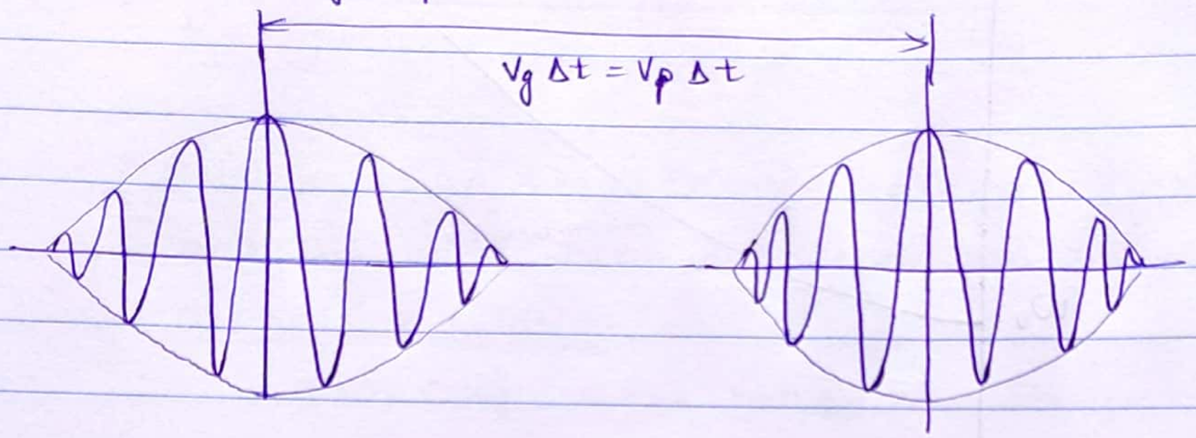
- Phase velocity $= V_p = \dfrac{\omega}{k}$

  ↳ each particle constitutes a phase

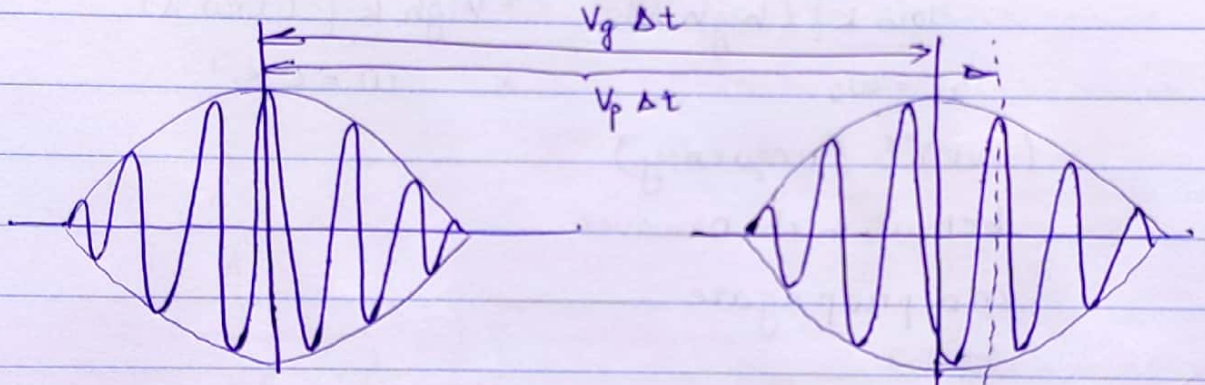
- Group velocity $= V_g = \dfrac{\partial \omega}{\partial k}$

  ↳ the entire bulk

Case A : $V_g = V_p$.



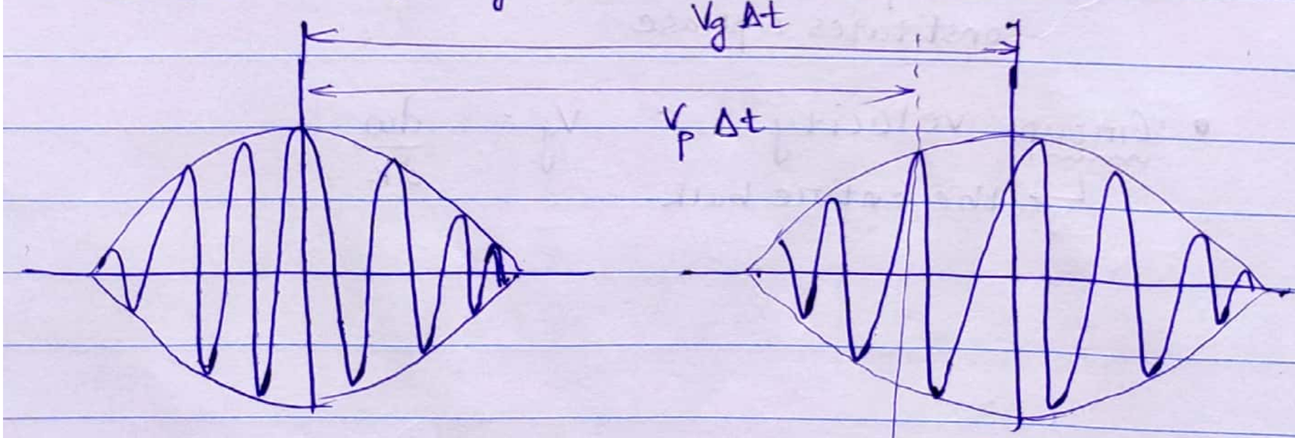
Case B : $V_p > V_g$ (Normal dispersion)



→ New wavelets appear at back

→ Wavelets disappear at front end.

Case C : $V_p < V_g$ (Anomalous dispersion)



→ Wavelets appear at front end.

→ wavelets disappear at back.

1. Wave on strings ; de Broglie waves

$$\omega = \sqrt{c^2 k^2 + \omega_0^2}$$

$$\rightarrow v_p = \frac{\omega}{k} > c$$

$$v_g = \frac{d\omega}{dk} < c$$

$$v_p \cdot v_g = c^2$$

2. Ripples on liquid surfaces

$$v_p = \sqrt{\frac{g}{k} + \frac{2\pi S}{\lambda \rho}} = \sqrt{\frac{g}{k} + \frac{kS}{\rho}}$$

$v_g = \frac{v_p}{2}$ (long waves in deep water)

$$v_p = \sqrt{\frac{Sk}{\rho}}$$ (short waves under surface tension S)

$v_g = \frac{3v_p}{2}$

Long waves in shallow (Tsunami)

$$v_p^2 = \left(\frac{g}{k} + \frac{kS}{\rho}\right) \tan h(kh)$$

$v_g \approx v_p$

$$\omega^2 = \left(gk + \frac{Tk^3}{\rho}\right) \underbrace{\tanh(kh)}_{\approx kh}$$ if $(kh \ll 1)$

$K' = \frac{1}{2} m v^2$

- KE density on standing waves $= \frac{1}{2} \rho \left(\frac{\partial \xi}{\partial t}\right)^2$
- PE density on standing waves $= \frac{1}{2} T \left(\frac{\partial \xi}{\partial x}\right)^2$

$$F = -\frac{\partial U}{\partial r}$$

$$= \frac{1}{2} \rho c^2 \left(\frac{\partial \xi}{\partial x}\right)^2$$

8 Reflection And Transmission

$$\boxed{Z = \frac{T}{c} = \rho c}$$ $(T = \rho c^2)$

(Characteristic Impedance)

$$y_i = A_1 e^{i(\omega t - k_1 x)}$$
$$y_r = B_1 e^{i(\omega t + k_1 x)}$$
$$y_t = A_2 e^{i(\omega t - k_2 x)}$$

Boundary conditions –

a) displacement immediately to the left $(y_i + y_r)$
= displacement immediately to right (y_t)

b) $T\left(\frac{\partial y}{\partial x}\right)$ must be continuous.

$$y_i + y_r = y_t$$
$$\Rightarrow \boxed{A_1 + B_1 = A_2} \quad (\because x = 0)$$

$$T \frac{\partial}{\partial x}(y_i + y_r) = T \frac{\partial}{\partial x}(y_t)$$

at $x = 0$

$$-k_1 T A_1 + k_1 T B_1 = -k_2 T A_2$$
$$-\omega \frac{T}{c_1} A_1 + \omega \frac{T}{c_1} B_1 = -\omega \frac{T}{c_2} A_2$$
$$\Rightarrow \boxed{Z_1 (A_1 - B_1) = Z_2 A_2}$$

$$\longrightarrow r_{12} = \frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

$\longrightarrow t_{12} = \frac{A_2}{A_1} = \frac{2Z_1}{Z_1+Z_2}$

ⓐ if $Z_2 = \infty$; fixed point (end)

$\frac{B_1}{A_1} = -1$; reflection with phase change π.

ⓑ if $Z_2 = 0$; open end / free end

$\frac{B_1}{A_1} = 1$; $\frac{A_2}{A_1} = 2$ (flick at end of whip from free end).

ⓒ $\frac{\text{Reflected Energy}}{\text{Incident energy}} = \frac{Z_1 B_1^2}{Z_1 A_1^2} = \left(\frac{B_1}{A_1}\right)^2 = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$

ⓓ $\frac{\text{Transmitted Energy}}{\text{Incident energy}} = \frac{Z_2 A_2^2}{Z_1 A_1^2} = \frac{4Z_1Z_2}{(Z_1+Z_2)^2}$

(*) Standing waves from H.J. Pain (Nicely explained)

$\longrightarrow$ Standing wave Ratio (SWR) $= \frac{1+r}{1-r} = \frac{A_1+B_1}{A_1-B_1}$

$\longrightarrow$ standing waves : $y = a e^{i(\omega t - kx)} + b e^{i(\omega t + kx)}$

$x = 0$ and $x = l \Rightarrow y = 0$

$\boxed{y = (-2i)\, a\, e^{i\omega t} \sin kx}$ $\left(\because k = \frac{\omega_n}{c} = \frac{n\pi}{l}\right)$

$E_n \text{ (kinetic)} = \frac{1}{2}\int_0^l \rho\, \dot{y}_n^2\, dx$

$E_n \text{ (potential)} = \frac{1}{2} T \int_0^l \left(\frac{\partial y_n}{\partial x}\right)^2 dx$

9 The Uncertainty Product

$$y(t) = \int \cos \omega t \; dA(\omega)$$

$$f(\omega) = \frac{dA(\omega)}{d\omega} = \text{amplitude spectral density}$$

$$= \int \cos \omega t \; f(\omega)\, d\omega$$

$$= \int_{\omega - \frac{\Delta\omega}{2}}^{\omega_0 + \frac{\Delta\omega}{2}} f_0 \cos \omega t \, d\omega$$

$$= \frac{f_0}{t} \left| \sin \omega t \right|_{\omega_0 - \frac{\Delta\omega}{2}}^{\omega_0 + \frac{\Delta\omega}{2}}$$

$$= \frac{f_0}{t} \left[\sin\left(\omega_0 - \frac{\Delta\omega}{2}\right) t + \sin\left(\omega_0 + \frac{\Delta\omega}{2}\right) t \right]$$

$$= \frac{f_0 \, 2 \cos \omega_0 t \, \sin\left(\frac{\Delta\omega t}{2}\right)}{t \, \Delta\omega} \, \Delta\omega$$

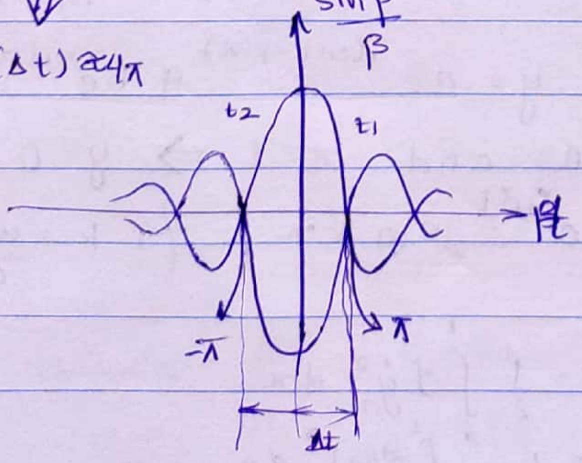
$$= (f_0 \, \Delta\omega) \, \frac{\sin\beta}{\beta} \, (\cos \omega_0 t)$$

Uncertainty product : $\boxed{\underbrace{(\Delta\omega)}_{\text{spread in frequency.}} \underbrace{(\Delta t)}_{\text{spread in time}} \approx 4\pi}$

$\frac{\Delta\omega t_1}{2} = \pi \;/ -\pi$

$\Downarrow$

$(\Delta\omega)(\Delta t) \approx 4\pi$



$$\Delta\beta = 2\pi = \frac{\Delta\omega \, \Delta t}{2}$$

$$\boxed{\Delta\omega \Delta t \approx 4\pi}$$

Siba Smarak Notes

(10) Stoke's relations

$\longrightarrow r_{12} = -r_{21}$ and $1 - r_{12}^2 = t_{12} \cdot t_{21}$

Note: Reflactance of intensity $R_{12} = r_{12}^2$

Transmittance of intensity $T_{12} = 1 - R_{12}$

EM WAVES

① $U_{em} = \frac{1}{2}\int_V \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau$; $u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$

② $\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \vec{S}\cdot d\vec{a}$

$$= -\frac{d}{dt}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau - \frac{1}{\mu_0}\oint_S (\vec{E}\times\vec{B})\cdot d\vec{a}$$

③ $\vec{S}$ = poynting vector

$$\boxed{\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})} \qquad \boxed{|\vec{S}| = uc}$$

$|\vec{S}|_{avg}$ = intensity $= \frac{1}{2} c^2 \epsilon_0 E_0^2$

④ Poynting Theorem:

→ The first integral on the right is the total energy stored in the fields, U_{em}.

→ The second term evidently represents the rate at which energy is carried out of volume V, across its boundary surface, by EM fields.

⑤ Radiation Pressure : P = Energy density of the wave

$$P = u_E + u_B$$

$$\rightarrow P = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

INTERFERENCE

(1) Addition Law of Phasors

$e_n(x,t) = E_n \cos(\omega t - kx + \alpha_n)$.

Let superposition at $x = x_0$, $t = t_0$ –

$e(x_0, t_0) = \sum e_n(x_0, t_0) = \sum E_n \cos(\omega t_0 - kx_0 + \alpha_n)$

$= \sum \mathrm{Re}\, E_n e^{i(\omega t_0 - kx_0 + \alpha_n)}$

$= \mathrm{Re}\left[e^{i(\omega t_0 - kx_0)} \underbrace{\sum E_n e^{i\alpha_n}}_{\text{phasors}} \right]$

$$\boxed{P = \sum E_n e^{i\alpha_n} = E e^{i\phi}}$$

$\Rightarrow e(x_0, t_0) = E(\cos(\omega t_0 - kx_0 + \phi))$

→ "Add the individual phasors by the polygon law to get the resultant phasor."

* Intensity rule

$$\boxed{I = E^2 = PP^*}$$

(a) Two beam interference:

$P = E + Ee^{i\phi} = E(1 + e^{i\phi})$

$I = PP^* = E^2(1 + e^{i\phi})(1 + e^{-i\phi})$

$= 4E^2 \cos^2\frac{\phi}{2} = I_M \cos^2\frac{\phi}{2}$

$$\boxed{I = I_M \cos^2\frac{\phi}{2}}$$

→ YDSE

→ Michelson - interferometer

Siba Smarak Notes

(b) Multiple beam interference (identical amplitude, regular phase increments)

$$P = E + Ee^{i\phi} + Fe^{i(2\phi)} + \cdots + fe^{i(n-1)\phi}$$

$$= \frac{E(1-e^{in\phi})}{(1-e^{i\phi})}$$

$$\boxed{I = PP^* = I_0 \frac{\sin^2 \frac{n\phi}{2}}{\sin^2 \frac{\phi}{2}}} \quad (I_0 = E^2)$$

→ Diffraction grating, Radio interferometry

(c) Multiple BI (continuous infinity of beams)

$n \to \infty$, $\phi \to 0$ but $n\phi \to 2\beta$

$E \to 0$ but $nE \to E_0 = \sqrt{I_m}$

$$\boxed{I = I_m \frac{\sin^2 \beta}{\beta^2}}$$

→ Fraunhofer diffraction.

(d) Multiple BI (regular amplitude decrements, phase increments)

$$P = E(1 + \rho e^{i\phi} + \rho^2 e^{i2\phi} + \cdots)$$

$$= E\left(\frac{1}{1-\rho e^{i\phi}}\right)$$

$$\boxed{I = PP^* = \frac{I_0}{1+\rho^2 - 2\rho\cos\phi}}$$

→ Thin films

→ Fabry - Perot Interferometer

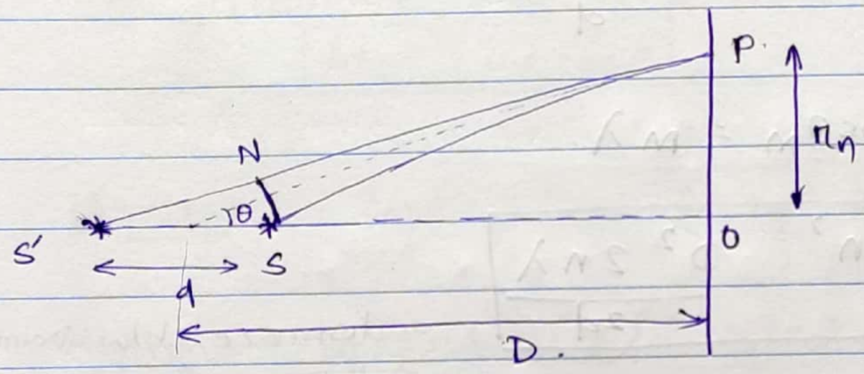
(e) Coherent superposition (Laser)

$$I = PP^* = (E + E + \cdots)^2 = n^2 I_0$$

(f) Incoherent superposition (Bulb)

$$I = PP^*$$
$$= E^2(e^{i\phi_1} + e^{i\phi_2} + \cdots)(e^{-i\phi_1} + e^{-i\phi_2} + \cdots)$$
$$= n I_0 .$$

2 Circular fringes and Michelson Interferometer



$$\boxed{d = m_0 \lambda}$$

$$\Delta x = S'N = \underline{d\cos\theta_m = m\lambda}$$

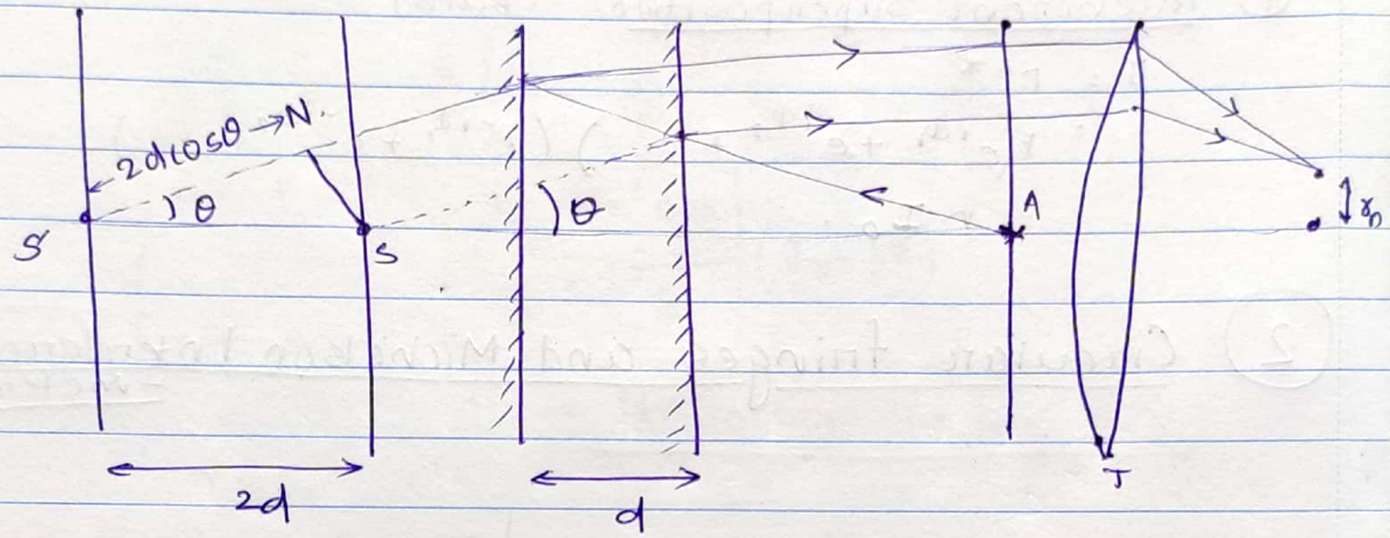
$$d\cos\theta_m = d\left(1 - \frac{\theta_m^2}{2}\right) = m\lambda$$

$$\theta_m^2 = \frac{2(m_0 - m)\lambda}{d} = \frac{2n\lambda}{d}$$

$$\boxed{r_n^2 = D^2\theta_m^2 = D^2 \frac{2n\lambda}{d}}$$

SIBA SMARAK NOTES

* Michelson Interferometer



$$(2d)\cos\theta_m = m\lambda.$$

$$\Rightarrow \boxed{r_n^2 = \frac{D^2\, 2n\lambda}{2d}}$$ → distance b/w mirror is d
⇒ distance b/w sources is '2d'.

→ Measurement of wavelength –

$2d = m_0\lambda.$

* for each shift of $\left(\frac{\lambda}{2}\right)$, one more bright fringe appears.

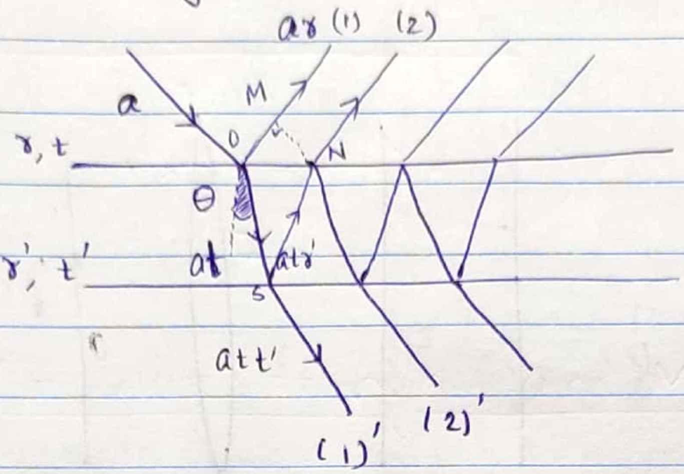
→ Wavelength separation of doublet –

$$\frac{2d}{\lambda_1} - \frac{2d}{\lambda_2} = \frac{1}{2} \text{ or } \frac{3}{2}\ldots$$ for consecutive disappearance of interference pattern

$$\frac{2d}{\lambda_1} - \frac{2d}{\lambda_2} = 1 \text{ or } 2\ldots$$ for consecutive ~~concordance~~ same interference pattern appearances

$$\boxed{\Delta\lambda \approx \frac{\lambda^2}{2(d_2 - d_1)}}$$

3 Multiple - Beam Interference (thin films) and Fabry - Perot Interferometer



Path difference b/w (1) and (2)

$$\boxed{\Delta x = 2\mu d \cos\theta}$$

Maxima : $2\mu d \cos\theta_m = (m + \frac{1}{2})\lambda$ } Reflection

Minima : $2\mu d \cos\theta_m = m\lambda$.

Maxima: $2\mu d \cos\theta_m = m\lambda$ } Transmission

Minima : $2\mu d \cos\theta_m = (m+\frac{1}{2})\lambda$

* Fabry - Perot

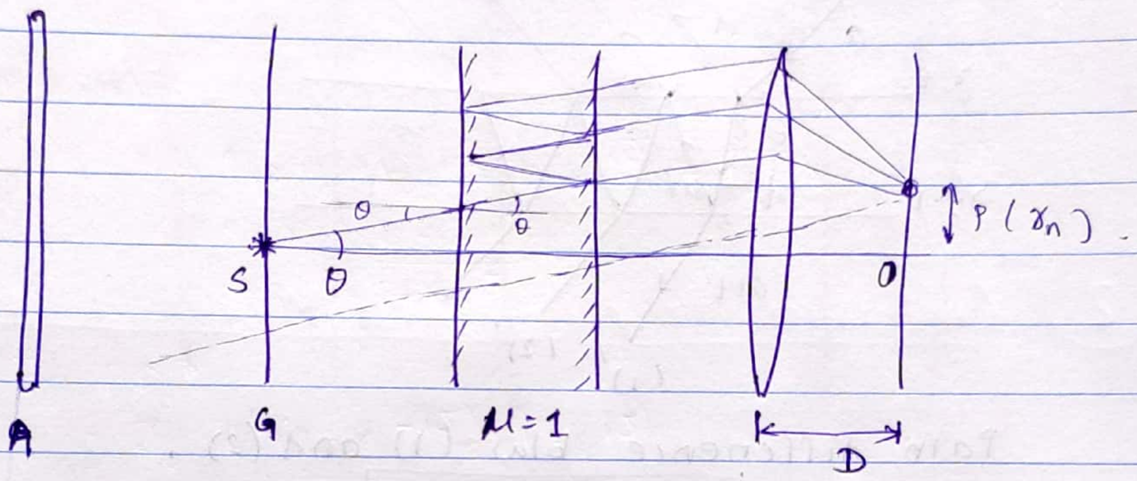
d = distance b/w M_1 and M_2

$\Rightarrow$ 2d = distance b/w sources.

$$\boxed{2d\cos\theta_m = m\lambda}$$ ($\because$ θ = angle of refraction from m.)

(bright fringes - transmission)

$$\boxed{r_n^2 = \frac{D^2 n\lambda}{d}}$$ D = focal length



- **Intensity of fringes**

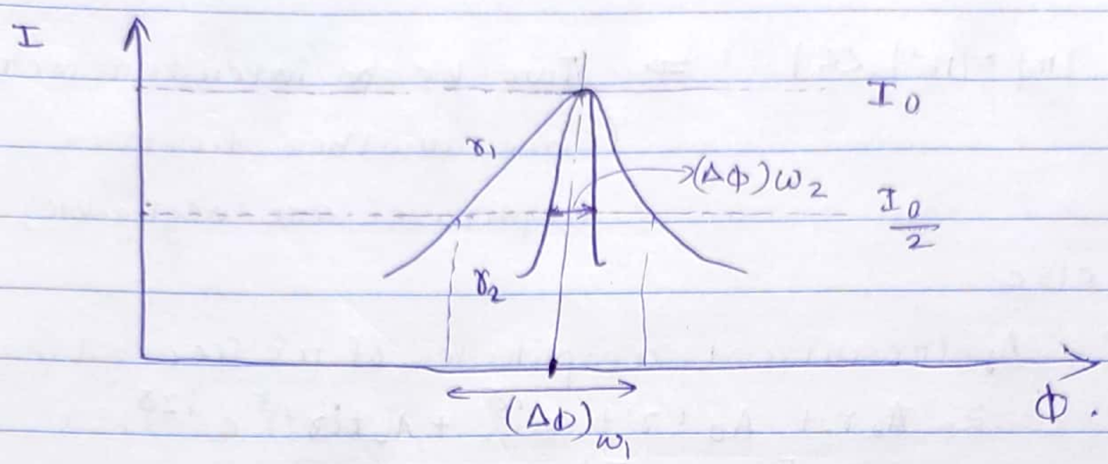
$$P = a t t' (1 + r^2 e^{i\phi} + r^4 e^{i2\phi} + \cdots)$$
$$= a' E (1 + f e^{i\phi} + f^2 e^{i2\phi} + \cdots)$$

$$I = PP^* = \frac{I_0}{1 + f^2 \sin^2\frac{\phi}{2}}$$

f = coefficient of finesse

$$\boxed{f = \frac{2r}{1-r^2}}$$ — rapidity of the fall in intensity is dictated by 'f'.

$r \to 1$; more polished mirror.
$\Rightarrow$ f increases. $\Rightarrow$ sharp rings.



④ Multiple Beam Interferometry

→ δ = Phase difference between successive waves emanating from the film $= \frac{2\pi}{\lambda}\Delta$

$$= \frac{2\pi}{\lambda}(2\mu t \cos\theta_r)$$

→ $r, t \Rightarrow$ amplitude reflection & transmission coefficient. (of wave travelling from surrounding medium into thin film).

$r', t' \Rightarrow$ corresponding quantities for wave from film to surrounding.

$$\left.\begin{aligned} r &= -r' \\ 1 - tt' &= r^2 = r'^2 \end{aligned}\right\}$$

$R = r^2$ = energy reflectivity | reflectivity of film surface.

$T = tt'$ = " transmittivity | " " " transmitivity

$$R + T = 1 \}$$

If $|r| \,\&\, |r'| << 1$ $\Rightarrow$ Two beam interferometry (Since all other further amplitudes are negligible)

~~wt~~. else

$P = A_r$ (resultant amplitude of reflected wave)

$$= A_0 r + A_0 t r' t' e^{i\delta} + A_0 tt'(r')^3 e^{i2\delta} + \ldots$$

$$= A_0 r \left[\frac{1 - e^{i\delta}}{1 - Re^{i\delta}}\right]$$

$$I_r = PP^*$$

$$= I_0 R \frac{(1 - e^{-i\delta})(1 - e^{i\delta})}{(1 - Re^{i\delta})(1 - Re^{-i\delta})}$$

$$= I_0 R \frac{(2 - 2\cos\delta)}{1 + R^2 - 2R\cos\delta}$$

$$\boxed{I_r = I_0 \frac{F \sin^2 \delta/2}{1 + F\sin^2\delta/2}}$$

$$\boxed{I_t = I_0 \frac{1}{1 + F\sin^2\delta/2}}$$

$$I_r + I_t = I_0 = |A_0|^2$$

F = coefficient of Finesse

$$= \left(\frac{2r}{1 - r^2}\right)^2 = \frac{4R}{(1-R)^2}$$

($\because$ If absorption considered $I_t = \left(1 - \frac{A}{1-R}\right)^2 \frac{I_0}{1 + F\sin^2\delta/2}$)

$R + T + A = 1$

→ Half - width $= \frac{4}{\sqrt{F}}$ (for T vs δ).

As F increases ; the fringes become more sharper.

● Fabry - Perot Interferometry → every formula same with MI.

→ The fringe (circular) pattern is sharper than Michelson - Interferometer.

The intensity is $I_t = \frac{I_0}{1 + F\sin^2 \delta/2}$

$$\delta = \frac{2\pi}{\lambda}(2\mu t \cos\theta_r).$$

$(\because \mu \approx 1$ (as air b/w two reflecting surface))

$$\Delta\delta = \frac{4\pi h \cos\theta_r \cdot \Delta\lambda}{\lambda^2}$$

$$I_{total} = \frac{I_0}{1 + F\sin^2\delta/2} + \frac{I_0}{1 + F\sin^2\left(\frac{\delta - \Delta\delta}{2}\right)}$$

. . . .

. . . .

$$\text{Resolving Power} = \frac{\lambda}{\Delta\lambda} = \frac{2\pi\sqrt{F}}{4.147} m$$

$$= 1.515\, m\sqrt{F}$$

$$= 1.515\sqrt{F}\left(\frac{2h\cos\theta_r}{\lambda}\right)$$

Better Use

$$CRP \equiv \left(\frac{\lambda}{\Delta\lambda}\right)_{bare} = \frac{\pi m r}{1 - r^2} = \frac{\pi m f}{2}$$

5 Coherence

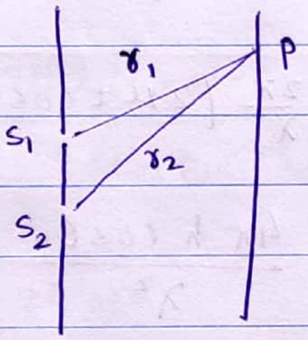
τ_c = coherence time (temporal coherence)

At a given point, the electric field at times t and $(t+\Delta t)$ will, have a definite phase relationship if $\boxed{\Delta t \ll \tau_c}$ and will never have any phase relationship if $\boxed{\Delta t \gg \tau_c}$

(a) YDSE

→ definite phase relationship of two waves at P if

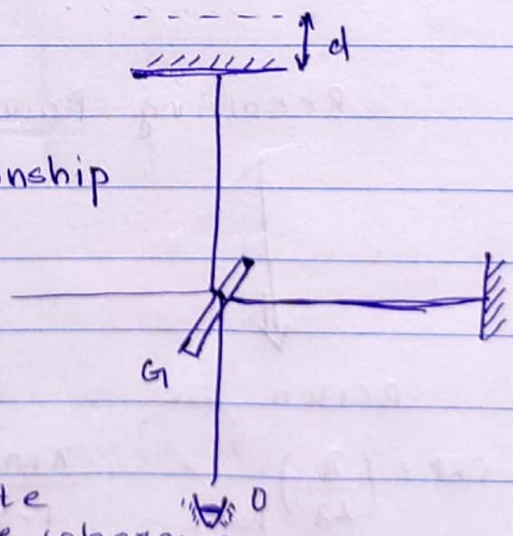
$$\left(t - \frac{r_1}{c}\right) - \left(t - \frac{r_2}{c}\right) = \frac{r_2 - r_1}{c} \ll \tau_c$$



(Pg-350)
if the path difference due to a slit $\frac{(\mu-1)t}{c}$ is $\ll \tau_c$ then the pattern remains the same nearly

(b) MICHELSON

→ definite phase relationship at O if $\frac{2d}{c} \ll \tau_c$.



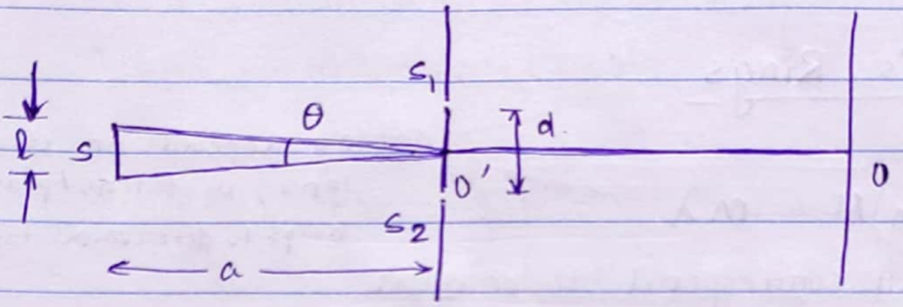
→ $\Delta\lambda \sim \frac{\lambda^2}{L} = \frac{\lambda^2}{c\tau_c}$ → finite time coherence

(spectral width)

→ $\Delta\nu \sim \frac{1}{\tau_c}$ → $\frac{\Delta\nu}{\nu}$ = monochromaticity

SIBA SMARAK NOTES

- <u>Spatial Coherence</u>



For fringes to disappear; $\boxed{l \simeq \frac{\lambda a}{2d}}$

Now; as separation between pinholes is increased from zero, the interference fringes disappear for $d = 1.22 \frac{\lambda}{\theta}$, as d is increased further the fringes reappear with poor contrast and then disappear again for $d = 2.25 \frac{\lambda}{\theta}$ ....

$\boxed{l_w = \frac{\lambda}{\theta}}$ (lateral coherence width)

gives the distance over which beam may be assumed to be spatially coherent and is referred to as <u>lateral coherence width</u>

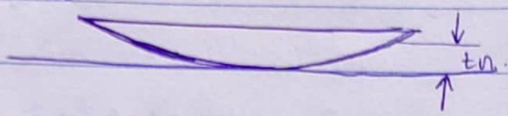
(6) Haidinger's Bands, Fizeau Fringes, Newton's Rings

(A) Newton's Rings

i) $(2t_n)\mu = m\lambda$ ~~for~~

doesn't depends on μ of lens; μ of lens/plate only helps to determine formula.

will correspond to minima



ii) $2\mu t_n = \left(m+\frac{1}{2}\right)\lambda$

will correspond to ~~minima~~ maxima.

$$\frac{r_n^2}{2R} \approx t_n \quad \text{(sagitta formula)}$$

$$r_n = \sqrt{2Rt_n}$$

$$\boxed{r_n = \sqrt{m\lambda R}} \quad \text{(for dark fringes)}$$

→ bright: $r_n = \sqrt{\left(m+\frac{1}{2}\right)\lambda R}$

Now $r^2_{m+p} - r^2_{m} = p\lambda R$

$$\Rightarrow \boxed{\lambda = \frac{D^2_{m+p} - D^2_m}{4pR}}$$

→ Like in sodium lamp; λ_1 and λ_2

for some value of t

$$2t = m\lambda_1 = \left(m+\frac{1}{2}\right)\lambda_2$$

$$\Rightarrow \boxed{\frac{2t}{\lambda_2} - \frac{2t}{\lambda_1} = \frac{1}{2}}$$

(around that point fringes will completely disappear)

$$\Rightarrow \boxed{2t = \frac{1}{2}\frac{\lambda_1\lambda_2}{\Delta\lambda}}$$

→ Further in the above case, instead the plano-convex lens in raised up to a value t_0, such that

$$2t_0 = m\lambda_1 = \left(m+\tfrac{1}{2}\right)\lambda_2$$

$$\Rightarrow \boxed{\frac{2t_0}{\lambda_2} - \frac{2t_0}{\lambda_1} = \frac{1}{2}}$$

(the fringes at that point will disappear)

(B) Haidinger's Bands

- **Rossi's dictum** : "Localized sources produce non-localized fringes; while non-localized sources produce localized fringes"

→ YDSE, Llyod's Mirror, biprism – localized sources (small) ; the resulting fringe systems are non-localized and can be observed at any distance.

→ Newton's Rings, MI, F-P – non-localized sources (big) ; (as such ground glass plate illuminated by Na-flame) produces localized circular fringes at infinity and straight fringes localized at site of the image mirrors.

circles in (Newton's Rings || straight line in wedge fringe)

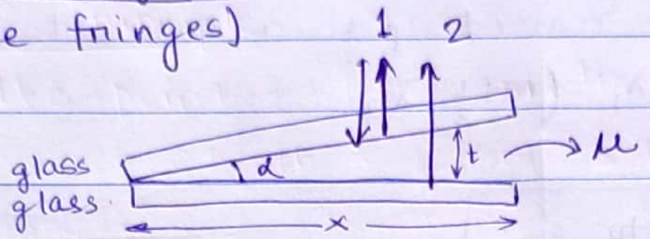
counterpart constant thickness (d = constant)

→ Fringes formed due to constant inclination is Haidinger's bands (MI, Fabry Perot) $(2d\cos\theta_m = m\lambda)$

non-constant (d) constant (m)

(c) Fizeau Fringes

(Wedge fringes)



Path difference = $2\mu t$

$2\mu t = (n+\frac{1}{2})\lambda$ (maxima)

$2\mu t = n\lambda$ (minima)

fringes of equal thickness. (straight line fringes)

$t = x\alpha$

$$x_m = \left(\frac{m+1/2}{2\alpha}\right)\lambda \rightarrow \text{fizeau fringes}$$

Δx = separation of consecutive fringes

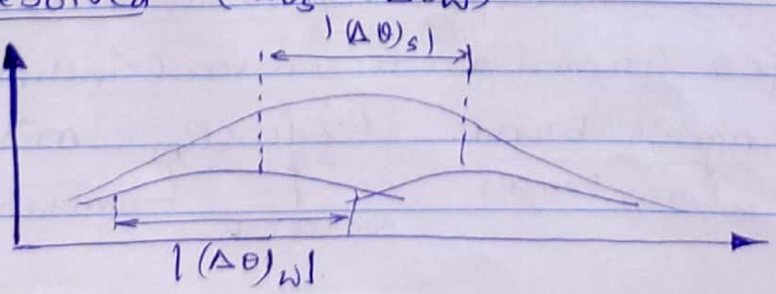
$$= \frac{\lambda}{2\alpha}.$$

(7) Chromatic Resolving Power (CRP)

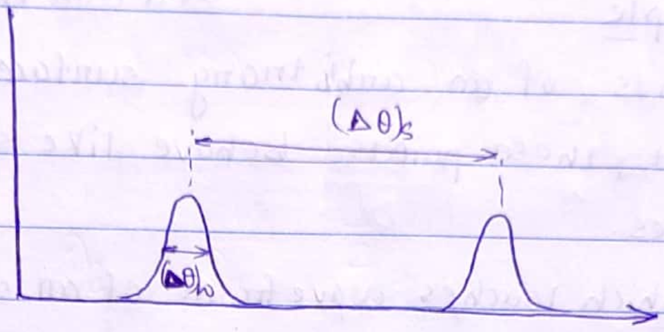
$(\Delta\theta)_s \rightarrow$ angular distance that separates two closely lying peaks.

$(\Delta\theta)_w \rightarrow$ width of each peak.

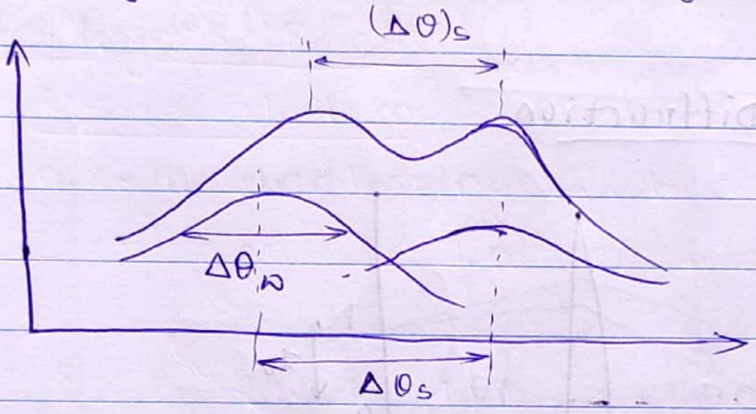
(i) Unresolved $(\Delta\theta_s < \Delta\theta_w)$



② Well resolved $(\Delta\theta)_s > (\Delta\theta)_w$



③ Barely resolved $(\Delta\theta)_s = (\Delta\theta)_w$



Instrument capable of barely resolving doublet interval $(\Delta\lambda)$ —

then $CRP \equiv \left(\frac{\lambda}{\Delta\lambda}\right)_{bare}$

$$2d\cos\theta_m = m\lambda$$

$$2d\sin\theta_m \, \Delta\theta_m \simeq m\,\Delta\lambda$$

$$|\Delta\theta_m| = \frac{m\,\Delta\lambda}{2d\sin\theta_m}$$

FRAUNHOFER DIFFRACTION

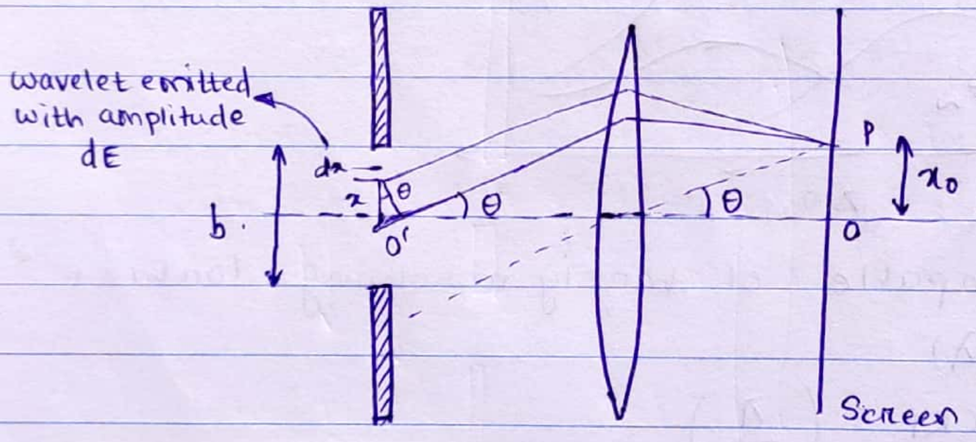
(to bend round obstacles and leak through apertures)

* Huygen's Principle

a) When the points of an arbitrary surface are reached by a wavefront, these points behave like sources of secondary waves.

b) The envelope (which touches wavefront of all secondary waves) of all such secondary waves at a later instant give the position of the wavefront at that instant.

① Single Slit Diffraction



- Intensity distribution formula

a) $dE = \sigma dx$

phase difference of wavelet emitted by element at x

$$\varphi = \frac{2\pi}{\lambda} \cdot \Omega = \frac{2\pi}{\lambda}(x\sin\theta)$$

$$dP = dE\, e^{i\varphi} = (\sigma dx)\, e^{i\frac{2\pi}{\lambda}(x\sin\theta)}$$

$$P(\theta) = \int_{-b/2}^{b/2} dP = \sigma \int_{-b/2}^{b/2} e^{i\frac{2\pi}{\lambda}(x\sin\theta)}\, dx$$

Siba Smarak Notes

$I(\theta)$ at point $P = P(\theta)\, P(\theta)^*$

$$\boxed{I(\theta) = I_0 \frac{\sin^2\beta}{\beta^2}}$$

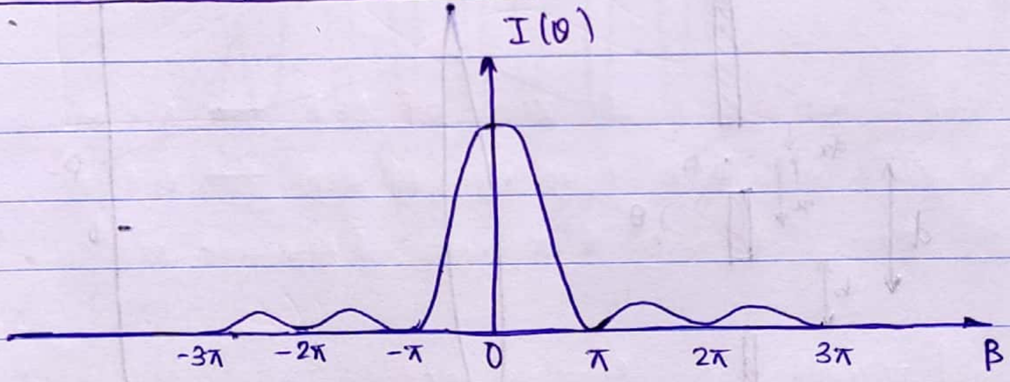
where $\beta = \frac{\pi b \sin\theta}{\lambda}$ $\qquad I_0 = (\sigma . b)^2$

b) $E = a[\cos\omega t + \cos(\omega t - \phi) + \cdots + \cos(\omega t - (n-1)\phi)]$

$(\phi = \varphi)$

$E = E_\theta \cos(\omega t - \frac{1}{2}(n-1)\phi)$

- The maxima and minima



central minima : $b \sin\theta_m = \pm m\lambda$

secondary maxima : at positions where $I(\theta)$ is max.

i.e. $\boxed{\tan\beta = \beta}$

→ The half-width of central smeared envelope = $\boxed{(\Delta\theta)_w = \frac{\lambda}{b}}$

→ Smearing effect of diffraction = forming a fringe due to slit, absence of slit would have formed a point.

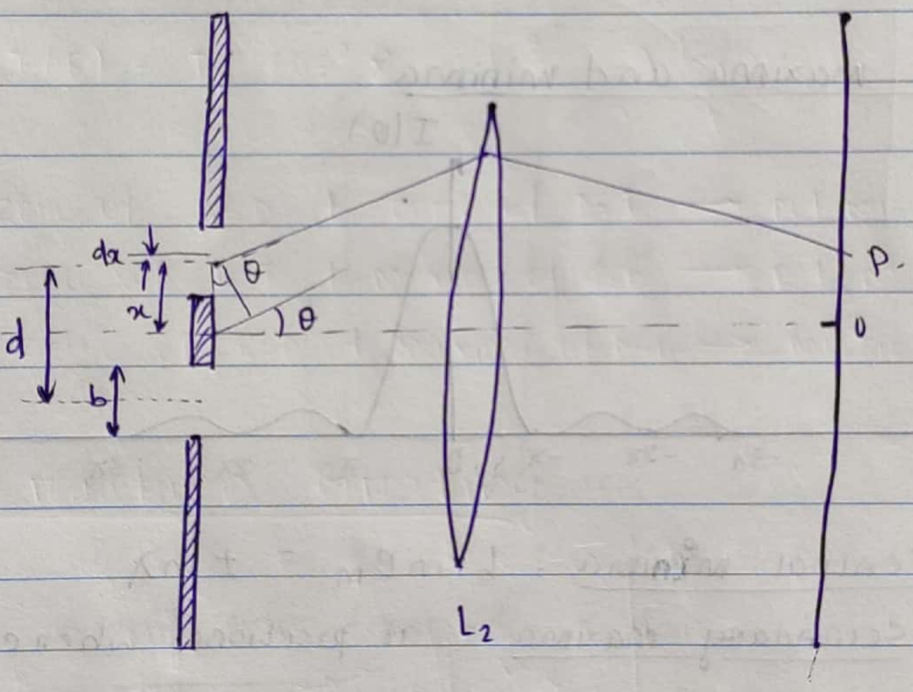
(envelope size) $\boxed{(\Delta x_0) \approx \frac{f\lambda}{b}}$

→ If we insert a circular hole (of diameter D) instead, the smeared image would be broadened into a circular patch (Airy's disk) of width.

$$(\Delta\theta)_w = \frac{1.22\lambda}{D}$$

$$(\Delta x_0) = \frac{1.22\lambda f}{D}$$

(2) Double Slit Diffraction



• Intensity distribution

$$P(\theta) = \sigma\left[\left(\int_{-\frac{d}{2}-\frac{b}{2}}^{-\frac{d}{2}+\frac{b}{2}} + \int_{\frac{d}{2}-\frac{b}{2}}^{\frac{d}{2}+\frac{b}{2}}\right) e^{i\frac{2\pi x\sin\theta}{\lambda}} dx\right]$$

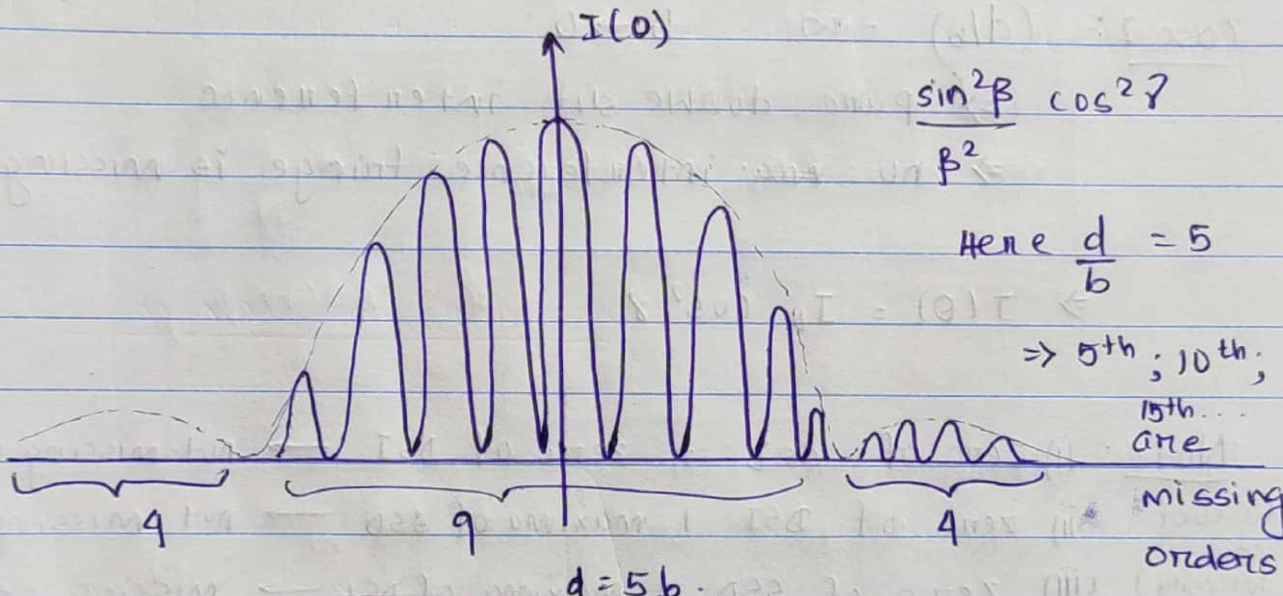
single slit diffr

$$I(\theta) = P(\theta)\,P(\theta)^* = I_0 \frac{\sin^2\beta}{\beta^2} \cos^2\gamma$$

→ double slit interference term

$$\beta = \frac{\pi b \sin\theta}{\lambda} \qquad \gamma = \frac{\pi d \sin\theta}{\lambda}$$

- Maxima and Minima



$\frac{\sin^2\beta}{\beta^2} \cos^2\gamma$

Here $\frac{d}{b} = 5$

$\Rightarrow$ 5th; 10th; 15th... are missing orders.

→ The missing orders

$b\sin\theta = m\lambda$ (for minima : singleslit diffraction)

$d\sin\theta' = n\lambda$ (for maxima : double slit interference)

when $\theta = \theta'$: $\frac{b}{d} = \frac{m}{n}$

$\Rightarrow$ Then the nth order interference fringe will have a zero intensity and will therefore be missing.

Case 1: $(d/b) = 1$ $\Rightarrow$ all interference maxima will be missing except the central one

$$\Rightarrow I(\theta) = I_0 \left(\frac{\sin(2\beta)}{(2\beta)}\right)^2$$ = single - slit diffraction with double width.

Case 2: $(d/b) = \infty$ $b \to 0$

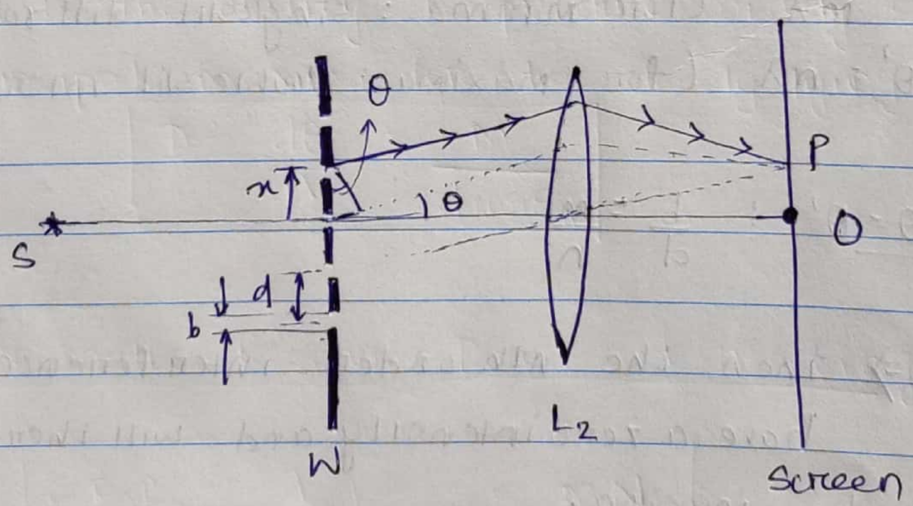
$\Rightarrow$ pure double slit interference

$\Rightarrow$ no ~~ma~~ interference fringe is missing

$$\Rightarrow I(\theta) = I_0 \cos^2 \gamma$$

Note: (cases of zero intensity)
(i) zero of SSD + zero of DSI $\longrightarrow$ not missing order
(ii) zero of DSI + maxima of SSD $\longrightarrow$ not missing order
(iii) zero of SSD + maxima of DSI $\longrightarrow$ missing order

3 Diffraction Grating



- The intensity distribution formula

→ Diffraction Pattern of a Grating = (single slit Diffraction Pattern) × (N - slit Interference Pattern)

$$I(\theta) = I_0 \frac{\sin^2\beta}{\beta^2} \cdot \frac{\sin^2 N\gamma}{\sin^2\gamma}$$

$$\beta = \frac{\pi b \sin\theta}{\lambda} \quad ; \quad \gamma = \frac{\pi d \sin\theta}{\lambda}$$

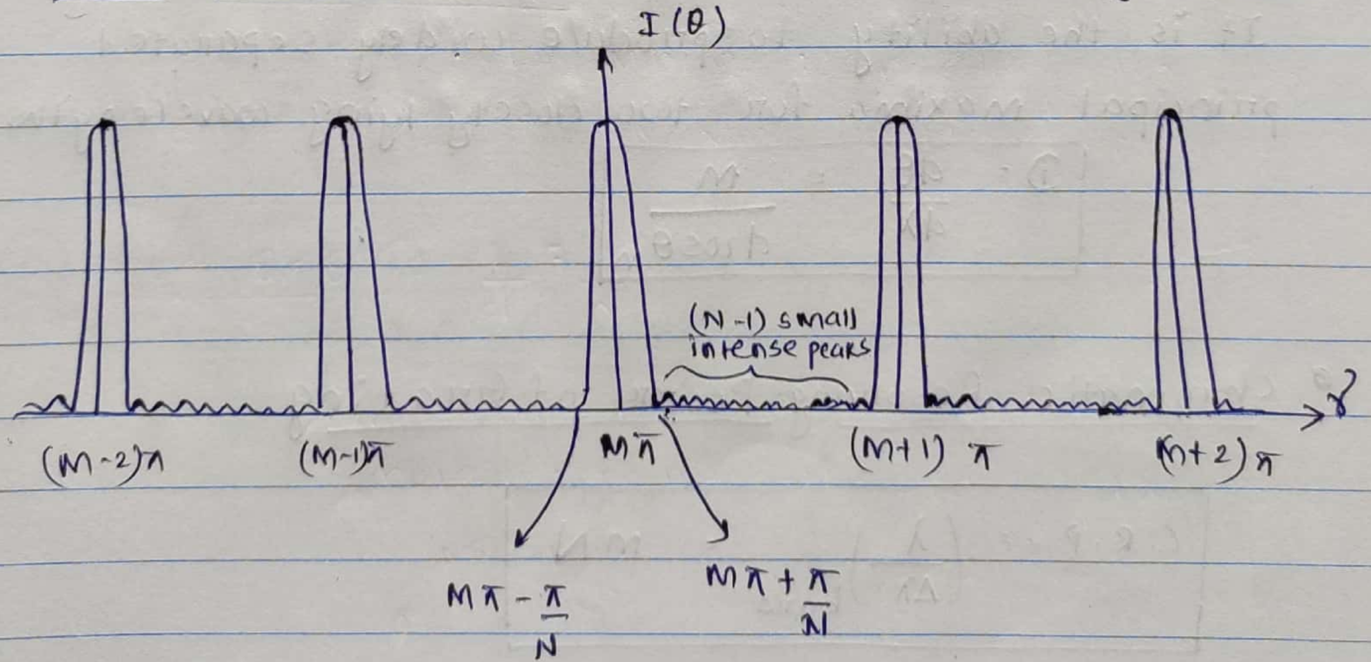
Now if width of each slit is very small $\Rightarrow \beta \approx 0$

$$\Rightarrow \boxed{I(\theta) = I_0 \frac{\sin^2 N\gamma}{\sin^2\gamma}}$$

goes to zero more often.

- Almost same overall intensity for different orders.
- No missing orders.

- Principal maxima, minima and secondary maxima



→ Principal maxima :

$$\boxed{N\gamma = mN\pi} \Leftrightarrow \boxed{d\sin\theta_m = m\lambda} \; ; \; m = 0, 1, 2 \ldots$$

(N disappears i.e., the maxima occurs exactly at the same angles of double slit).

The minimum that neighbours the mth order maximum satisfies -

$$\boxed{\begin{aligned} & N\gamma = (mN+1)\pi \\ \Leftrightarrow \; & d\sin(\theta_m + (\Delta\theta)_w) = m\lambda + \frac{\lambda}{N} \end{aligned}}$$

$$\left(\because \gamma = \frac{\pi d \sin\theta}{\lambda} \right)$$

From above equations;

half-width of mth order $= \boxed{(\Delta\theta)_w \approx \dfrac{\lambda}{N d \cos\theta_m}}$

- Dispersive Power

It is the ability to produce widely separated principal maxima for two closely lying wavelengths.

$$\boxed{D = \frac{d\theta}{d\lambda} = \frac{m}{d\cos\theta_m}}$$

- Chromatic Resolving Power of Grating

$$\boxed{C.R.P. = \left(\frac{\lambda}{\Delta\lambda}\right)_{bare} = mN}$$

→ Blazing : Modifying the profile of its rulings so that there is intensity redistribution

→ Normal spectrum : if $(\Delta\theta) \propto (\Delta\lambda)$

(4) Resolving Power (CRP) of prism

→ At limit of resolution ;

$$\boxed{(\Delta\theta)_s = (\Delta\theta)_w = \frac{\lambda}{b}}$$

→ $$\boxed{C.R.P. = \left(\frac{\lambda}{\Delta\lambda}\right)_{bare} = B \left(\frac{d\mu}{d\lambda}\right)}$$

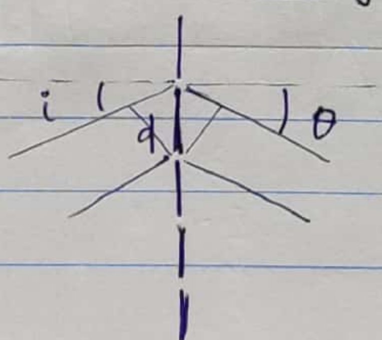
dispersive power of prism.

base of the prism

→ Also the formula for dispersive power is such that the increase in deviation angle is not proportional to $(\Delta\lambda)$. So, a prism doesn't produce a convenient 'normal' spectrum.

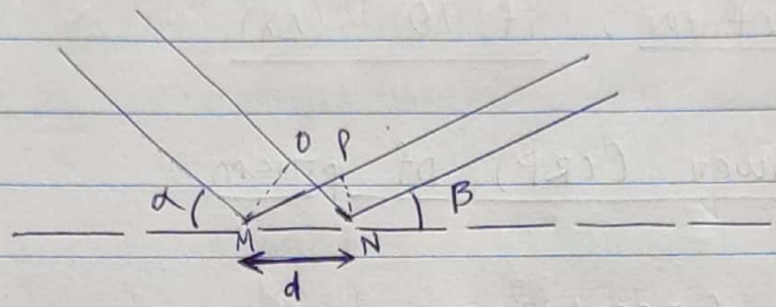
(5) Oblique Incidence

- Transmission Grating



$$\boxed{d(\sin i + \sin\theta_m) = m\lambda}$$

• Reflection grating –

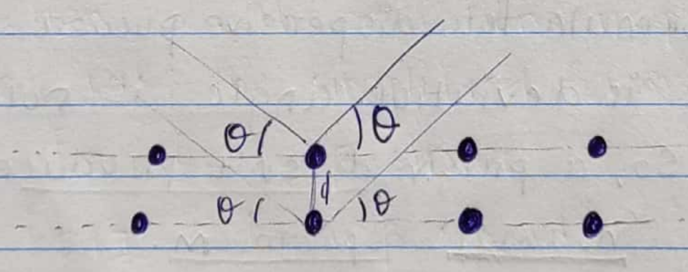


$$\Delta = ON - PM = d(\cos\alpha - \cos\beta)$$

→ Bragg's Law (x-Ray diffraction (crystals))

$$\boxed{2d\sin\theta = n\lambda}$$ ✓

d = inter-crystal layer distance.
θ = conjugate of incident angle



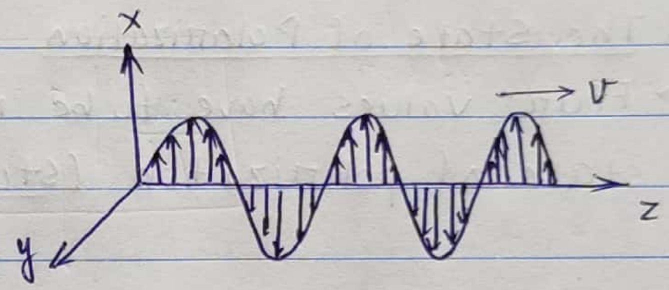
POLARIZATION (TBC)

→ When a transverse wave travels through a medium, the plane in which the displacement of the medium occurs is called the plane of polarisation of the wave.

Polarized light waves

→ Linearly (plane) polarized waves:

1) x - polarised
2) z - polarised

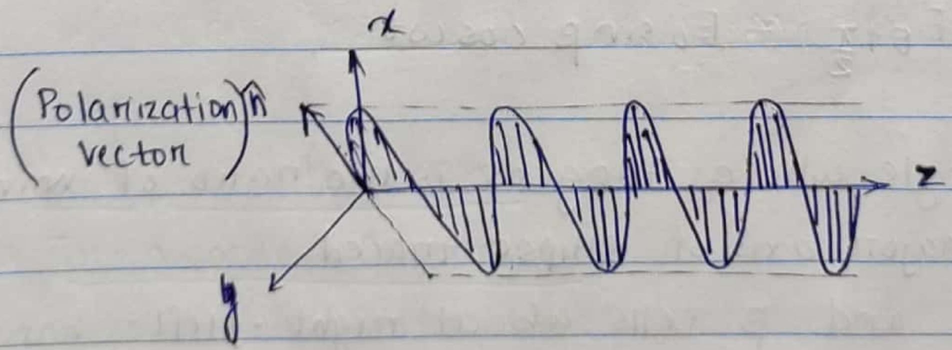


→ Circularly polarized waves

If string rotates on the circumference of the circle

$$x(z,t) = a\cos(kz - \omega t + \phi)$$

$$y(z,t) = a\cos(kz - \omega t + \phi)$$



* Elliptically Polarized Waves

Jones concept:

$$E_x = E_{0x} \cos \omega t \qquad E_y = E_{0y} \cos(\omega t + \varepsilon)$$

$$\Rightarrow \frac{E_x}{E_{0x}} = \cos \omega t \quad \text{and} \quad \sin \omega t = \sqrt{1 - \frac{E_x^2}{E_{0x}^2}}$$

$$\Rightarrow \left[\frac{E_x}{E_{0x}}\right]^2 + \left[\frac{E_y}{E_{0y}}\right]^2 - 2\left[\frac{E_x}{E_{0x}}\right]\left[\frac{E_y}{E_{0y}}\right] \cos \varepsilon = \sin^2 \varepsilon$$

(Lissajous figures)

* The State of Polarization

→ Four values have to be measured to identify the state of polarization (Stokes Formalism)

(when z is direction of propagn)

→ If the principal axes of ellipse described by (E_x, E_y) are in directions making an angle θ and $(\theta + \frac{\pi}{2})$ to the direction x, the equations representing the vibration take the simplified forms -

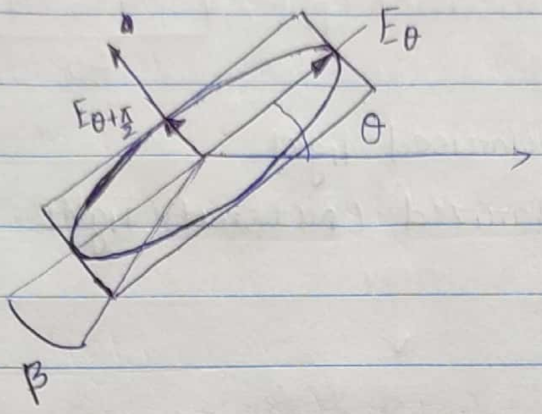
$$E_\theta = E_0 \cos\beta \sin\omega t$$

$$E_{\theta + \frac{\pi}{2}} = E_0 \sin\beta \cos\omega t$$

β = angle whose tangent is the ratio of minor to major axis of ellipse traced.

$0 < \beta < \frac{\pi}{2}$ and β tells about right-/left- handed polarization.

a) $I = E_0^2 = E_{0x}^2 + E_{0y}^2 = I_x + I_y$



(I is twice that measured by detector).

b) $S_0 = I = I_x + I_y$

c) $S_1 = I_x - I_y = I \cos 2\beta \cos 2\theta$

d) $S_2 = (I_x - I_y) \tan 2\theta = I \cos 2\beta \sin 2\theta$

e) $S_3 = (I_x - I_y) \dfrac{\tan 2\beta}{\cos 2\theta} = I \sin 2\beta$

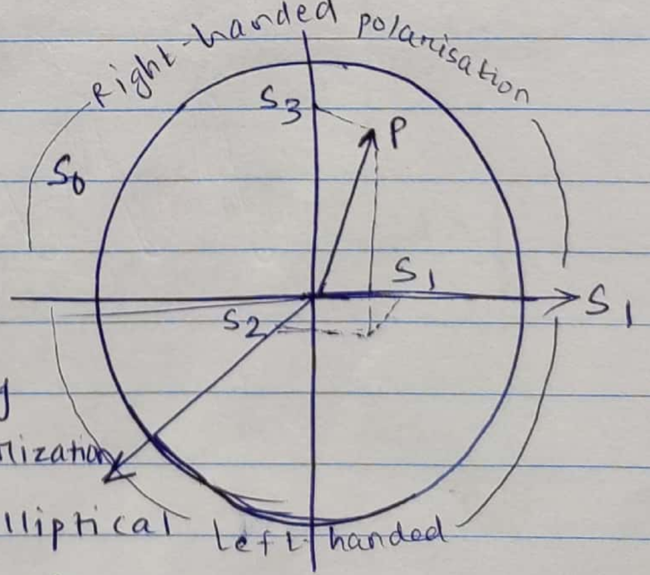
$\left.\begin{matrix} S_1 \\ S_2 \\ S_3 \end{matrix}\right\} \in (-S_0, S_0)$

f) $S_0^2 = S_1^2 + S_2^2 + S_3^2$ (valid for fully polarized light)

g) $\tan 2\theta = \dfrac{S_2}{S_1}$

h) $\sin 2\beta = \dfrac{S_3}{\sqrt{S_1^2 + S_2^2 + S_3^2}}$

i) Right-handed circular polarization is represented by <u>north-pole</u>; left by <u>south-pole</u>, linear polarization by equatorial plane, elliptical states by between the planes.



j) For partially polarized light:

$$\text{Degree of polarisation } (p) = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}$$

$p = 1 \Rightarrow$ fully polarised light

$p = 0 \Rightarrow$ non-polarized (natural) light

Quantum Mechanics

* Black Body Radiation

→ Black body is an empty cavity whose walls are maintained at a given temperature T sufficiently long for the radiation inside it to attain thermodynamic equilibrium.

Rayleigh - Jean's Law :

— There are infinite number of normal modes, hence at a given finite temperature; there can be infinite energy inside black body.

$$\boxed{f(\nu)\,d\nu = \frac{8\pi \nu^2 kT}{c^3}\,d\nu}$$

$$\Rightarrow \left(f(\nu) \propto \nu^2\right).$$

From Planck's Law -

$$\boxed{f(\nu)\,d\nu = \frac{8\pi h}{c^3} \cdot \frac{\nu^3}{e^{h\nu/kT} - 1}\,d\nu}$$

$$\Rightarrow \boxed{f(\nu)\,d\nu = \frac{8\pi h}{\lambda^5} \frac{c}{\left(e^{hc/\lambda kT} - 1\right)}\,d\lambda}$$

$$\because \boxed{E = nh\nu}$$

NOTE: Photon has two states of polarization

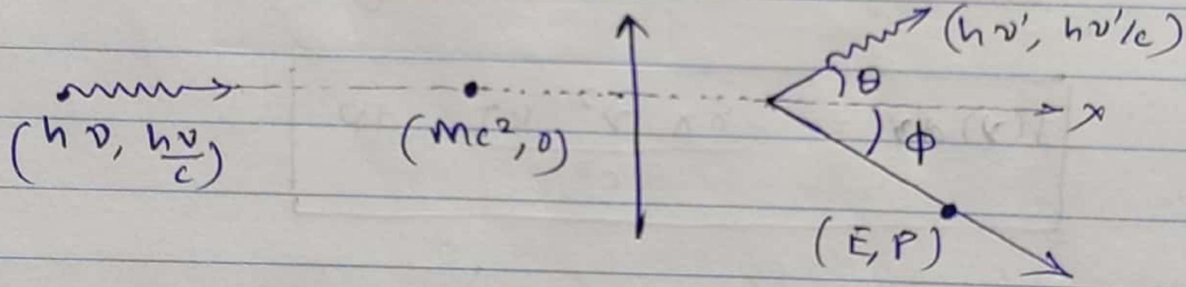
* Compton Scattering

Compton shift:

$$\boxed{\Delta\lambda = (\lambda' - \lambda) = \left(\frac{h}{mc}\right)(1 - \cos\theta)}$$

$\frac{h}{mc}$ ↓ Compton wavelength of a particle, having rest mass 'm'.

$= 2.4\ \text{pm}$ for $e^{\ominus}$



BEFORE   AFTER

* Stern - Gerlach Experiment

(Electron Spin - Check Youtube).

① $\vec{\mu} = \frac{e}{m_0}\vec{S}$
$(\vec{M})$

② $F_z = \frac{\partial}{\partial z}(\vec{\mu}\cdot\vec{B}) = M_z \frac{\partial B_z}{\partial z}$.

* De-Broglie Hypothesis

$$\boxed{\lambda = \frac{h}{p}}$$ p = momentum

* Heisenberg's Uncertainty Principle

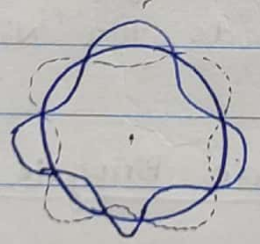
1) $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$

2) $\Delta E \cdot \Delta t \geq \hbar/2$

* Bohr's Quantisation Principle

$$\boxed{mvr = \frac{nh}{2\pi}} \iff \boxed{2\pi r = n\lambda}$$

(Formation of standing electron waves)

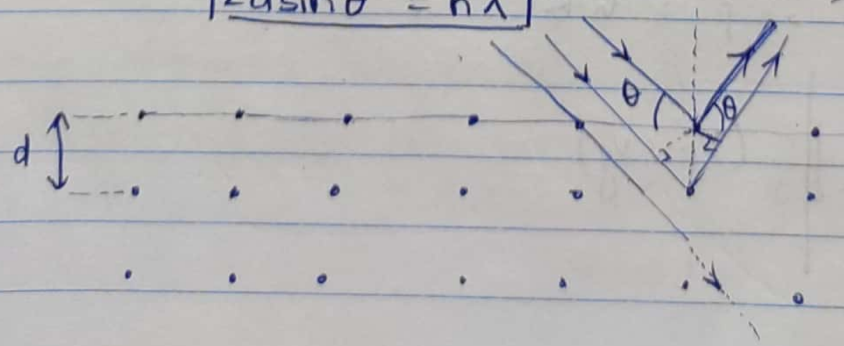


$n = 8$ (all have equal distance)

* Bragg's Law and x-ray diffraction

$$\boxed{2d\sin\theta = n\lambda}$$

θ = glancing angle



direction of incidence

SIBA SMARAK NOTES

* Davisson - Germer experiment and electron diffraction

→ The results showed a smooth and uneventful angular distribution. During the course of the experiment, however, air leaked into the evacuated set-up and oxidized the surface of the nickel block. (The heating converted polycrystalline nickel into a single crystal of nickel)

→ electron diffraction pattern, that varied with the accelerated voltage V -

$$\boxed{\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV}}}$$

→ was exactly agreeing with the 'λ' determined by Bragg's law. (means equating the 'd' found by both methods for 'Ni')

* Dispersion Relation of de Broglie waves

$E^2 = p^2c^2 + m_0^2c^4$ (Relativistic)

$E = \hbar\omega$; $p = \hbar K$

$v_g = \frac{d\omega}{dK}$ | $(v_p \neq v_g)$

$v_p = \frac{\omega}{K}$

SIBA SMARAK NOTES

<u>Non-relativistic</u>

$$E = \frac{p^2}{2m} = \hbar\omega$$

$$p = \hbar k$$

$$\omega = \frac{\hbar k^2}{2m} \rightarrow v_p = \omega/k = p/2m = \frac{v}{2}$$

$$\rightarrow v_g = \frac{d\omega}{dk} = \frac{p}{m} = v$$

* Ground States

(A) Harmonic Oscillator

$$E = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 = \frac{\hbar^2}{2m\,\Delta x} + \frac{1}{2}k(\Delta x)^2$$

$x \sim \Delta x$ $\qquad p_x \sim \Delta p_x \sim \frac{\hbar}{2\Delta x}$

$$\frac{\partial E}{\partial \Delta x} = 0 \Rightarrow \boxed{(\Delta x)^2 = \frac{\hbar}{2\sqrt{mk}}}$$

$$\Rightarrow E = \frac{\hbar}{2}\sqrt{\frac{k}{m}} = \frac{\hbar}{2}\omega = \frac{h\nu}{2}$$

(Zero-point energy $\neq 0$)

(B) Hydrogen atom

$$E = \frac{p^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r} = \frac{\hbar^2}{2m(\Delta r)^2} - \frac{e^2}{4\pi\epsilon_0 \Delta r}$$

$r \sim \Delta r$ $\qquad p \sim \Delta p \sim \frac{\hbar}{\Delta r}$

$$\frac{\partial E}{\partial \Delta r} = 0 \Rightarrow \boxed{\Delta r = \frac{4\pi\epsilon_0 \hbar^2}{me^2}} \sim r = 0.529\,Å$$

$$\Rightarrow E = -13.6\ eV$$

SIBA SMARAK NOTES

*Quantum Formalism

$\because \quad \omega = \frac{\hbar}{2m} k^2$

let $\psi(x,t) = Ce^{i(kx-\omega t)}$ → plane-wave

$$\frac{\partial \psi}{\partial t} = -i\omega\psi \qquad \frac{\partial^2 \psi}{\partial x^2} = -k^2\psi$$

$$\Rightarrow \frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2}$$

$$\Rightarrow \underbrace{\left(i\hbar\frac{\partial}{\partial t}\right)}_{\text{energy operator } (E_{op})}\psi = \frac{1}{2m}\underbrace{\left(-i\hbar\frac{\partial}{\partial x}\right)}_{\text{momentum operator } (p_{op})}^2\psi$$

→ Eigen-equation

$$\underbrace{A_{op}}_{\text{operator}}\ \underbrace{\psi}_{\text{eigenfunction}} = \underbrace{a}_{\text{eigen value}}\psi$$

→ Hamiltonian ($H_{classical}$)

$$= KE + PE = \frac{p^2}{2m} + V(x)$$

$$H_{op} = i\hbar\frac{\partial}{\partial t}$$

$$P_{op} = -i\hbar\frac{\partial}{\partial x}$$

$$x_{op} = x$$

SIBA SMARAK NOTES

- <u>Schrodinger equation</u>

$$\boxed{i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \cdot \frac{\partial^2 \psi}{\partial x^2} + V(x).\psi}$$

$\psi \rightarrow$ Schrodinger wave function ψ

→ The real quantity $\boxed{dP = \psi^* \psi \, dx}$ is the probability that the particle will be found between x and $x+dx$ when a measurement of its position is made.

Now $P = \int dP = \int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$

if the value instead of unity is M, then we divide the ψ by $\frac{1}{\sqrt{M}} = N$ i.e., N is the normalization constant.

→ Thus the physical significance of the normalized wave function ψ is that it is a probability amplitude.

* <u>Operator Algebra</u>

→ Operators act on wave functions (such as ψ) and convert them into new wave functions (such as η):

$$\boxed{A\psi = \eta}$$

Siba Smarak Notes

(a) $A + B = B + A$

(b) Commutator of two operators A & B:

$$\boxed{[A,B] = AB - BA}$$

→ If the commutator vanishes then operators are said to commute ⇒ $[A,B] = 0$

⇒ $AB = BA$

→ $[A,B] = -[B,A]$

→ $[AB, C] = A[B,C] + [A,C]B$ & $[A, BC] = B[A,C] + [A,B]C$

(c) Fundamental commutation relation -

$$[x, p]\psi = \left[x, -i\hbar\frac{\partial}{\partial x}\right]\psi$$

$$= i\hbar\psi$$

$$\Rightarrow \boxed{[x,p] = i\hbar}$$

(d) Scalar product of wave functions

→ (ψ_1, ψ_2) = scalar product of ψ_1 and ψ_2

$$= \int_{-\infty}^{\infty} \psi_1^*(x)\, \psi_2(x)\, dx$$

→ $(\psi_1, \psi_2)^* = (\psi_2, \psi_1)$

→ $(\psi_1, \psi_1) = (\psi_1, \psi_1)^* > 0$

→ The scalar product of two normalized wave functions $\psi_1(x)$ and $\psi_2(x)$ is called the overlap integral.

→ Overlap integral represents the extent to which the two functions plotted against x overlap each other.

(i) if $(\psi_1, \psi_2) = 0 \Rightarrow \psi_1$ and ψ_2 are orthogonal

→ A set of functions ψ_i ; each of which is normalized and each orthogonal to the other is said to form an orthonormal set / basis.

ⓔ Adjoint operator

The (Hermitian) adjoint A^+ of an operator A is defined as:

$$\boxed{(\varphi, A\psi) = (A^+\varphi, \psi)}$$

↓

scalar product

$$\left(\frac{d}{dx}\right)^+ = \left(-\frac{d}{dx}\right)$$

→ An operator A, whose adjoint is identical with itself ($A^+ = A$) is known as self-adjoint or Hermitian operator.

Siba Smarak Notes

* Stationary States

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = H_{op}\, \psi(x,t)$$

let $\psi(x,t) = u(x) \cdot T(t)$

$$\Rightarrow \boxed{i\hbar \underbrace{\frac{1}{T(t)} \frac{dT}{dt}}_{\text{time}} = \underbrace{\frac{1}{u(x)} H_{op} \cdot u(x)}_{\text{space}}}$$

$$\Rightarrow i\hbar \frac{1}{T(t)} \frac{dT}{dt} = E = \frac{1}{u(x)} H_{op}\, u(x) \qquad (E = \text{constant})$$

Solve to get solutions.

(a) Time independent SE

$$\frac{1}{u(x)} H_{op}\, u(x) = E$$

$$\Rightarrow \boxed{H_{op}\, u(x) = E \cdot u(x)}$$

→ second order differential equation

↓

Obtain $u_n(x)$ for different eigenvalues $E_n(x)$.

↓

$|u_n(x)|^2$ gives the pictorial representation of probability distribution or 'orbital'

(b) Time dependent SE

$$\frac{dT}{dt} = -\frac{i}{\hbar} E_n T(t)$$

$$\Rightarrow T(t) = c_n e^{-\frac{iE_n t}{\hbar}}$$

Thus

$$\psi_n(x,t) = u_n(x) \cdot T_n(t)$$

$$\Rightarrow \boxed{\psi_n(x,t) = c_n u_n(x) e^{-iE_nt/\hbar}} \quad \text{—— } (*)$$

(c) Stationary States

→ time t enters only as a phase factor. Overall phase factors do not effect the probability distribution or expectation values.

- $$(\psi_n, \psi_n) = \int_{-\infty}^{\infty} \psi_n^*(x,t) \cdot \psi_n(x,t)\, dx$$
$$= |c_n|^2 \int_{-\infty}^{\infty} u_n^*(x) \cdot u_n(x)\, dx \neq f(x).$$

- $$\langle A \rangle_{\psi_n} = (\psi_n(x,t), A\psi_n(x,t)) = (u_n(x), Au_n(x))$$

→ since time t disappears from measurable quantities like, probability densities and expectation values, the eigenstates of (*) of time-dependent Schrodinger equation (*) are called ~~eigen~~ stationary states.

$$\Rightarrow \psi(x,t) = \sum_n c_n \psi_n(x,t) = \sum_n c_n u_n(x) e^{-i\frac{E_nt}{\hbar}}$$

is a well-behaved system.

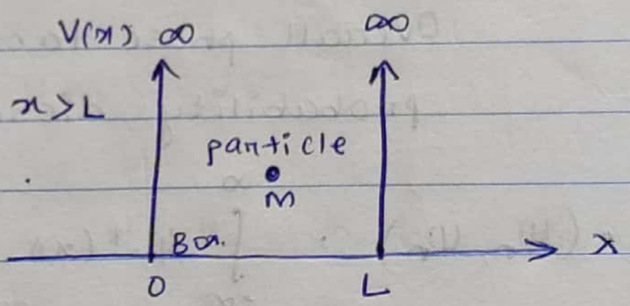
Siba Smarak Notes

(d) Degeneracy

→ for each E_n, if only one solution $u_n(x)$ then such eigenvalues are called non-degenerate.

→ If there are more than one solution to a given eigenvalue, then eigenvalue is degenerate. If number 'k' of linearly independent solutions of a degenerate Eigenval. then k is degree of degeneracy (k-fold degenerate).

* Particle In a box (one dimension)



$V(x) \to \infty$, for $x<0 \,||\, x>L$

$V(x) = 0$, for $0<x<L$.

$$H = \frac{p^2}{2m} + V(x) = \frac{p^2}{2m}$$

$$p \to p_{op} = -i\hbar \frac{\partial}{\partial x}$$

$$H_{op} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$\Rightarrow \quad H_{op}\, u(x) = E\, u(x)$ (Time independent SE)

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\frac{\partial^2 u(x)}{\partial x^2} = E u(x).$$

$$k^2 = \frac{2mE}{\hbar^2} \Rightarrow u(x) = A\cos kx + B\sin kx$$

$u(0) = 0$ & $u(L) = 0$

$$\Rightarrow \boxed{u(x) = B \sin\left(\frac{n\pi x}{L}\right)} \quad \left(\because k = \frac{n\pi}{L}\right)$$

Now to normalize -

$$\int_{-\infty}^{\infty} u^*(x) \cdot u(x)\, dx = 1$$

$$\Rightarrow B^2 \frac{L}{2} = 1 \Rightarrow \boxed{B = \sqrt{\frac{2}{L}}}$$

$$\boxed{u_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}}$$

- Energy levels

$$E_n = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

$$= \frac{n^2 \hbar^2 \pi^2}{2mL^2}$$

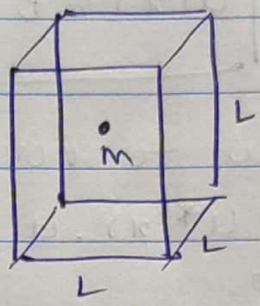
$$\boxed{E_n = \frac{n^2 h^2}{8mL^2}} \rightarrow \text{(stationary states)}$$

$$\boxed{\psi_n(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{iE_n t}{\hbar}}}$$

NOTE : $n \frac{\lambda_n}{2} = L$; $p_n = \frac{h}{\lambda_n} = \frac{nh}{2L}$; ground state energy corresponds to $n = 1$.

* Particle In a Box (Three - Dimensional)

$V(x, y, z) = 0$ inside box.



$$H_{op} = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

$\Rightarrow \quad H_{op}\, u(x,y,z) = E\,(u(x,y,z))$

$\Rightarrow \quad -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(x,y,z) = E \cdot u(x,y,z)$

$\Rightarrow$ Let $u(x,y,z) = \underbrace{X(x) \cdot Y(y) \cdot Z(z)}$

Eigenfunctions of separated independent equations.

$\Rightarrow \quad \frac{d^2 X(x)}{dx^2} + K_x^2 X(x) = 0$

$\frac{d^2 Y(y)}{dy^2} + K_y^2 Y(y) = 0$

$\frac{d^2 Z(z)}{dz^2} + K_z^2 Z(z) = 0$

$K_x = \frac{n_x \pi}{L}$

$K_y = \frac{n_y \pi}{L}$

$K_z = \frac{n_z \pi}{L}$

$\Rightarrow \quad K^2 = K_x^2 + K_y^2 + K_z^2$

$\Rightarrow \quad u_{n_x, n_y, n_z}(x, y, z) = B \sin\left(\frac{n_x \pi}{L} x\right) \sin\left(\frac{n_y \pi}{L} y\right) \sin\left(\frac{n_z \pi}{L} \cdot z\right)$

$\Rightarrow$ Energy levels

$$E_{n_x, n_y, n_z} = \frac{\hbar^2}{2m}\left(K_x^2 + K_y^2 + K_z^2\right) = \frac{\hbar^2 \pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

⇒ Ground state corresponds to $n_x = n_y = n_z = 1$

First excited state ,, to $n_x = n_y = 1$ & $n_z = 2$

OR $n_x = n_z = 1$ & $n_y = 2$

OR $n_y = n_z = 1$ & $n_x = 2$

All three corresponds to same energy (i.e. degenerate)

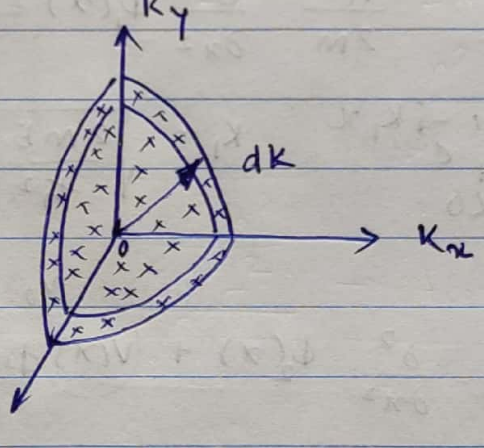
- **Density of states in momentum space**

→ A plot of k_x, k_y, k_z as the cartesian axes is called 'momentum space'.

$$k_x = n_x \frac{\pi}{L} \; ; \; k_y = n_y \left(\frac{\pi}{L}\right) \; ; \; k_z = n_z \frac{\pi}{L}$$

Lattice translational parameter $= \pi/L$

$n_x = 1, 2, 3 \ldots \quad n_y = 1, 2, 3 \ldots \quad n_z = 1, 2, 3 \ldots$



$\left(\vec{K} = \frac{\vec{P}}{\hbar}\right)$

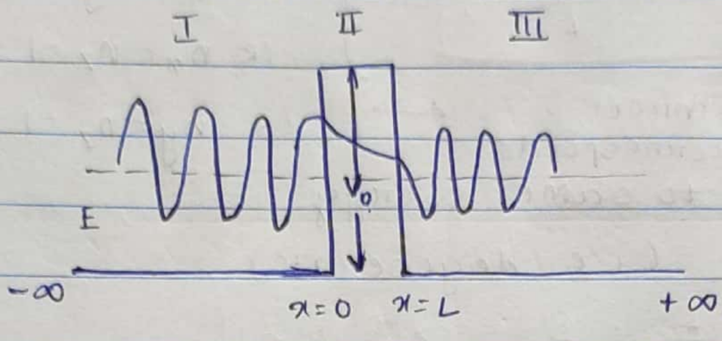
→ A cube of volume $\left(\frac{\pi}{L}\right)^3$ contains one lattice point.

→ Hence number of lattice points per unit volume $= \left(\frac{L}{\pi}\right)^3$ = density of states in the momentum space

→ number of states with their 'k' vectors lying between k and k+dk $= dn = \frac{(4\pi k^2)}{8} dk \cdot \left(\frac{L}{\pi}\right)^3$

$$\boxed{dn = \frac{L^3}{2\pi^2} k^2 dk}$$

*Quantum Mechanical Tunneling



$V(x)=0$ for $x<0$ & $x>L$

$V(x)=V_0$ for $0<x<L$

→ At boundary wavefunctions and corresponding derivatives must be continuous

mass = m ; energy = E

⇒ Time-independent SE :

$$-\frac{\hbar^2}{2m}\underbrace{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)}_{\times}\phi(x)+V(x)\,\phi(x)=E\,\phi(x)$$

For region I :

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_1(x)=E\,\phi_1(x)$$

$$\phi_1(x)=A_1e^{ik_1x}+A_1'e^{-ik_1x} \qquad k_1=\sqrt{\frac{2mE}{\hbar^2}}$$

$x<0$.

R-II :

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_2(x)+\overset{V_0}{V(x)}\phi(x)=E\,\phi_1(x)$$

$$\Rightarrow \phi_2(x)=A_2e^{ik_2x}+A_2'e^{-ik_2x}$$

$$k_2=\sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

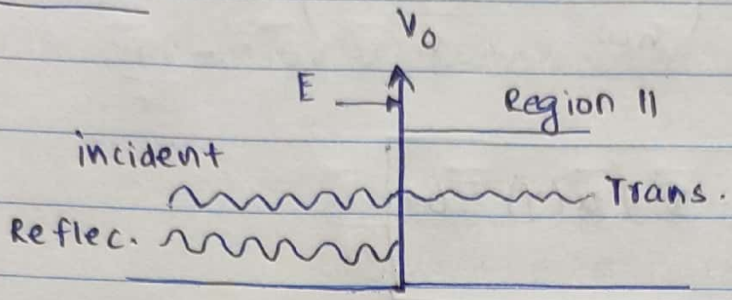
R-III :

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_3(x)+0=E\,\phi_3(x)$$

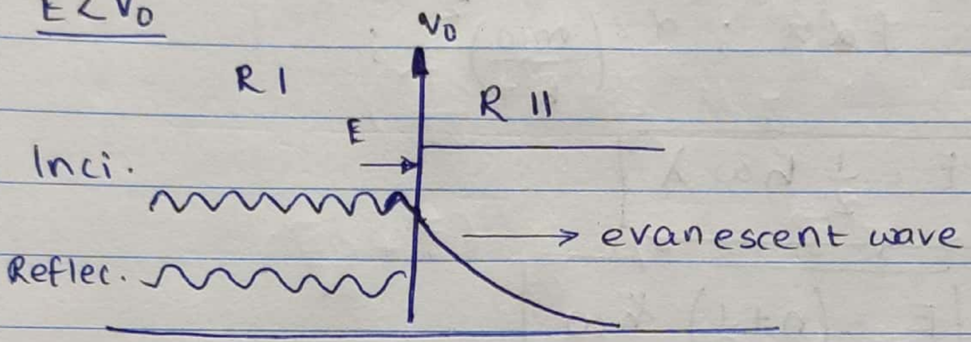
$$\Rightarrow \phi_3(x)=A_3e^{ik_3x}+A_3'e^{-ik_3x}$$

$$k_3=k_1=\sqrt{\frac{2mE}{\hbar^2}}$$

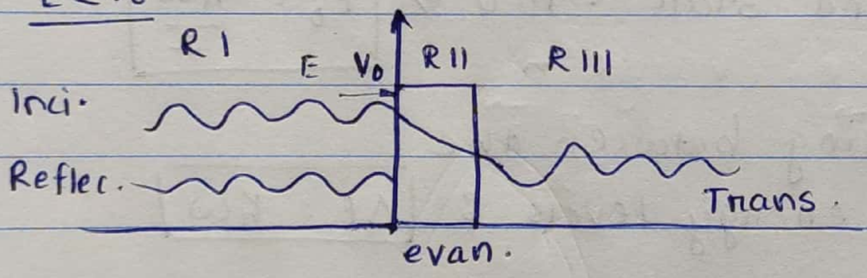
$E > V_0$



$E < V_0$



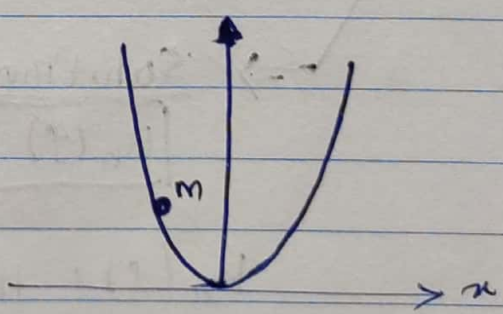
$E < V_0$



* Harmonic Oscillator

$V(x) = \frac{1}{2} kx^2$

KE $= \frac{p^2}{2m}$



Time independent SE:

$H_{op}\, u(x) = E\, u(x)$

$\Rightarrow -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} u(x) + \left(E - \frac{1}{2} m\omega^2 x^2\right) u(x) = 0$

$$\Rightarrow \frac{\hbar}{m\omega}\frac{d^2 u(x)}{dx^2} + \left(\frac{2E}{\hbar\omega} - \frac{m\omega}{\hbar}x^2\right)u(x) = 0$$

$$\Rightarrow \boxed{\frac{d^2 u(\rho)}{d\rho^2} + (\lambda - \rho^2)\,u(\rho) = 0}$$

where

$\rho = \alpha x$; $\alpha = \left(\frac{m\omega}{\hbar}\right)^{2}$;

$$\boxed{E = \frac{1}{2}\hbar\omega\lambda}$$

$$\Rightarrow \boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega}$$

Ground state : $n=0 \Rightarrow \boxed{E_0 = \frac{\hbar\omega}{2}}$

Spacing between the energy levels : $\boxed{\Delta E = \hbar\omega}$

$\rightarrow$ Solutions :

$$\boxed{u_n(\rho) \propto H_n(\rho)\, e^{-\rho^2/2}}$$

$H_n(\rho)$ = Hermite polynomials

$H_0(\rho) = 1$

$H_1(\rho) = 2\rho$

$H_2(\rho) = 4\rho^2 - 2$

$H_3(\rho) = 8\rho^3 - 12\rho = H_3(\rho)$

NOTE :

✓ Uncertainty in operator $\hat{A}$

$$= \Delta\hat{A} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$$

$$= \sqrt{\left(\hat{A} - \langle\hat{A}\rangle\right)^2}$$

= Standard deviation in A

$$\boxed{\Delta\hat{A} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}}$$

SIBA SMARAK NOTES

EM Waves

① Faraday's Laws of Induction : $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Ampere's Law : $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$= -\mu \frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

$$\vec{\nabla}\left(\frac{\rho}{\varepsilon_0}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right)$$

For no charge -

$$\boxed{\nabla^2 \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$$

(Doubt slide 4)

$$I \equiv \langle S \rangle = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2\mu_0 c} = \frac{c B_0^2}{2\mu_0}$$

→ The instantenous pressure that would be exerted on a perfectly absorbing surface by a normally incident beam $= \boxed{P = \frac{I}{c}} = \frac{\langle S \rangle}{c}$

Siba Smarak Notes

→ There is a pressure in the direction normal to the waves and numerically equal to energy in a unit volume.

$$P = \frac{S}{c} \Rightarrow \boxed{S = uc}$$

$\because P = u = u_E + U_B$

$= \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2$

Also $S = \frac{U}{A\Delta t} = \frac{u(c\Delta t)A}{A\Delta t} = uc$ ✓

$$I = \langle S \rangle_T = \frac{c^2 \epsilon_0}{2} |\vec{E_0} \times \vec{B_0}|$$

$$= \frac{1}{2}\epsilon_0 c E_0^2 \checkmark$$

→ The time rate of flow of radiant energy is the optical power of radiant flux (Watts)

→ Poynting vector (instantaneous) $= \vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = c^2\epsilon_0 (\vec{E} \times \vec{B})$

$$r_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$r_{\parallel} = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

$R = r^2 = R_{\parallel} = R_\perp$

$= \left(\frac{n_t - n_i}{n_t + n_i}\right)^2$

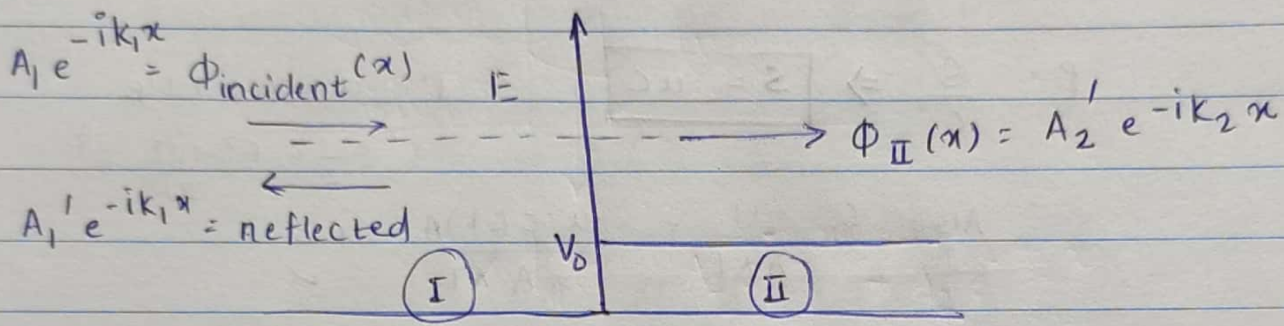
$$t_\perp = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$T = T_{\parallel} = T_\perp = \frac{4 n_t n_i}{(n_t + n_i)^2}$

$$t_{\parallel} = \frac{2n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i}$$

Tunneling

① $E > V_0$



$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_I(x) = E\phi_1(x)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\phi_1(x) + \frac{2mE}{\hbar^2}\phi_1(x) = 0$$

$$\Rightarrow \phi_1(x) = A_1 e^{-ik_1x} + A_1' e^{-ik_1x} \checkmark$$

Also

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi_{II}(x)}{\partial x^2} + V_0\phi_{II}(x) = E\,\phi_{II}(x)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\phi_{II}(x) + (E - V_0)\frac{2m}{\hbar^2}\phi_{II}(x) = 0$$

$$\Rightarrow \phi_{II}(x) = \cancel{A_2 e^{ik_2x}} + A_2' e^{-ik_2x}$$

$$= A_2' e^{-ik_2x} \checkmark$$

$\phi_1(0) = \phi_{II}(0) \Rightarrow A_1 + A_1' = A_2'$

$\phi_{II}'(0) = \phi_{II}'(1) \Rightarrow A_1 - A_1' = \frac{k_2}{k_1}A_2'$

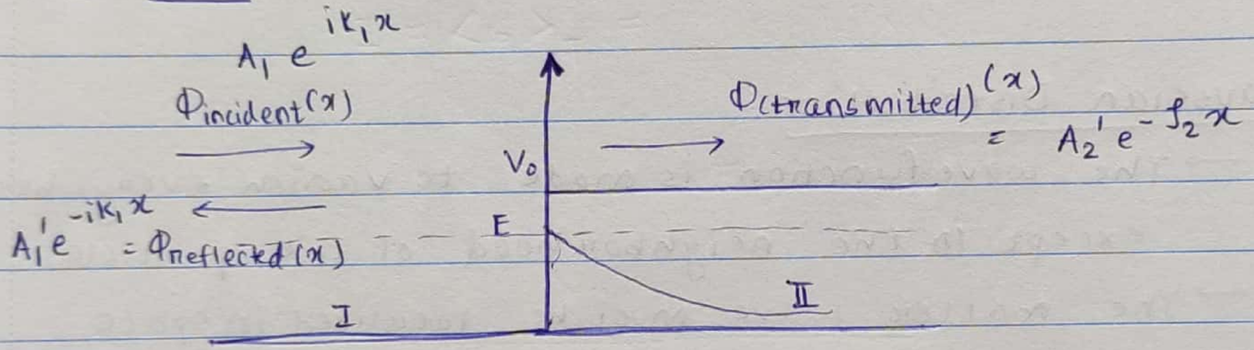
$$\Rightarrow \frac{A_1'}{A_1} = \frac{k_1 - k_2}{k_1 + k_2} \qquad \Rightarrow R = \left|\frac{A_1'}{A_1}\right|^2$$

$$R = \frac{|k_1 - k_2|^2}{|k_1 + k_2|^2}$$

and $\frac{A_2'}{A_1} = \frac{2k_1}{k_1 + k_2}$

$$T = 1 - R = \frac{k_2}{k_1}\left|\frac{2k_1}{k_1 + k_2}\right|^2$$

(2) $E < V_0$



$$-\frac{\hbar^2}{2m}\frac{\partial^2 \phi_1(x)}{\partial x^2} = E\phi_1(x)$$

$$\frac{\partial^2}{\partial x^2}\phi_1(x) + \overbrace{\frac{2mE}{\hbar^2}}^{k^2}\phi_1(x) = 0$$

$$\Rightarrow \phi_1(x) = A_1 e^{-ik_1x} + A_1' e^{-ik_1x}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_{II}(x) + V_0\phi_{II}(x) = E\phi_{II}(x).$$

$$\frac{\partial^2}{\partial x^2}\phi_{II}(x) + \frac{2m(E - V_0)}{\hbar^2}\phi_{II}(x) = 0.$$

$$\frac{\partial^2}{\partial x^2}\phi_{II}(x) - \rho_2^2\phi_{II}(x) = 0$$

$$\Rightarrow \phi_{II}(x) = A_2 \cancel{e^{\rho_2 x}} + A_2' e^{-\rho_2 x}$$

At $x = 0$ -

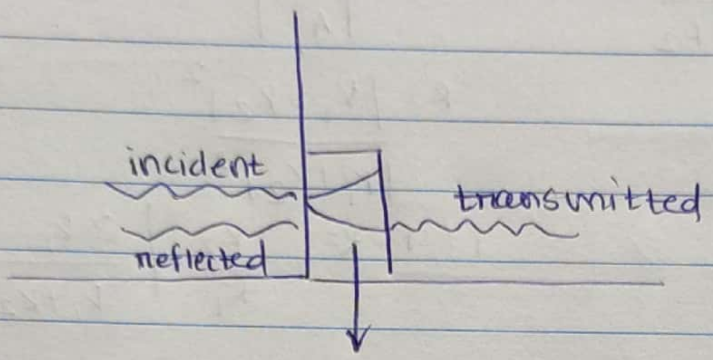
$\phi_1(0) = \phi_{II}(0) \Rightarrow A_1 + A_1' = A_2'$

$\phi_1'(0) = \phi_{II}'(0) \Rightarrow A_1 - A_1' = -i\frac{\rho_2}{k_1} A$

$$\frac{A_1'}{A} = \frac{1 + i\rho_2/k_1}{1 - i\rho_2/k_1} \quad ; \quad R = \left|\frac{A_1'}{A_1}\right|^2 = 1 \quad \Rightarrow T = 1 - R = 0.$$

$$A_2'/A_1 = 2/(1 - i\rho_2/k_1).$$

If barrier is thin -



$$\checkmark\ \phi_{II}(x) = A_2 e^{f_2 x} + A_2' e^{-f_2 x}$$

* Gaussian Distribution

→ The wave function is made to vanish everywhere except in the neighborhood of the particle.

→ The matter wave must be localized in space within which the particle is confined.

→ Localized wavefunction is called "wave-packet".

$$\psi(x,t) = \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx-\omega t)}\, dx \quad \times \frac{1}{\sqrt{2\pi}}$$

$$\int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dx$$

Amplitude of the wave packet (Normalize first). $\int_{-\infty}^{\infty} \phi^* \phi \, dk = 1$

$\psi_0(x) = \psi(x,t)$ at $t=0$

$\psi_0(x) =$ Wave function probability amplitude for finding the particle at position 'x'.

→ $|\psi_0(x)|^2$ → probability density

$P(x)\,dx = [\psi_0(x)]^2\,dx$ (Probability between x and $x+dx$).

$$\int_{-\infty}^{\infty} e^{-\frac{a^2}{2} z^2} dz = \frac{\sqrt{2\pi}}{a}$$