INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR

Date .....FN / AN *Time:* 3 Hrs. *Maximum Marks* 100 *No. of Students:* 05
End-Spring Semester:, 2011-12 Department: Computer Science and Engineering
Sub. No: CS31004
B. Tech.(Hons.), Dual Deg. Sub. Name: Theory of Computation

Instructions : Answer **ANY FIVE** questions

1. (a) Show that the universality problem
$UP = \{\langle M\rangle | M \text{ is a TM and } L(M) = \Sigma^*\}$
is not even partially decidable. You may assume that the uniform halting problem
$UHP = \{\langle M\rangle | M \text{ is a TM and } M \text{ halts on all } w \in \Sigma^*\}$
is not even partially decidable.

   (b) Show that the complement class of UP is not even partially decidable. Assume that
$\overline{A_{TM}} = \{\langle M, w\rangle | M \text{ is a TM and } M \text{ does not accept } w\}$
is not even partially decidable.

[8 + 12 = 20]

2. (a) Show that the acceptance problem of linear bounded automata (LBA)
$A_{LBA} = \{\langle M, w\rangle | M \text{ is an LBA and } M \text{ accepts } w\}$
is decidable.

   (b) Show that the emptiness problem of LBA
$E_{LBA} = \{\langle M\rangle | M \text{ is an LBA and } L(m) = \emptyset\}$
is undecidable. You may assume that $\overline{A_{TM}}$ is undecidable.

[10 + 10 = 20]

3. (a) Define the NP-complete class of problems.

   (b) How will you show that a new problem, Q, is NP-complete?

   (c) Which of the following problems are in NP? Give a brief justification.

      i. Deciding whether an array of integers is sorted

      ii. Deciding whether two Boolean formulas $f(x_1, \ldots, x_k)$ and $g(x_1, \ldots, x_k)$ are equivalent, that is, they have the same truth for all valuations of $x_1, \ldots, x_k$.

      iii. Deciding whether a graph has a path of length greater than k.

   (d) A *Hamiltonian path* in a graph G is a path containing each vertex of the graph exactly once. The task of deciding whether a given graph has a Hamiltonian path is known to be NP-complete. Use this information to show that $LPATH$ is NP-complete, where $LPATH = \{\langle G, a, b, k\rangle |\ G \text{ contains a path of length at least } k \text{ from vertex } a \text{ to vertex } b\}$. In other words, $LPATH(\langle G, a, b, k\rangle)$ is true if and only if the graph G contains a path of length k or more from vertex a to vertex b. Note that vertices cannot be repeated in a path (by definition) and each edge increases the path length by one.

[4 + 4 + 5 + 7 = 20]

4. (a) Suppose an algorithm A has time complexity $T(n)$ for inputs of size n. Under what conditions can we say that algorithm A has polynomial time complexity?

   (b) Answer the following questions with justification:

      i. Suppose $T(n) = 2T(n1) + 5$ and $T(1) = 1$. Is it correct to say that $T(n) = O(2^n)$?

      ii. Suppose $T(n) = 5T(n1) + 2$ and $T(1) = 1$. Is it correct to say that $T(n) = O(2^n)$?

iii. Suppose $T(n) = T(n1)/2 + 5$ and $T(1) = 1$. Is it correct to say that $T(n) = O(2^n)$?

iv. Suppose $T(n) = T(n/2) + 5$ and $T(1) = 1$. Give an asymptotic upper bound on $T(n)$

(c) In order to determine whether a given number N is prime, we check whether it is divisible by each number less than $N/2$. What is the time complexity of this algorithm? Does it have polynomial time complexity?

(d) What is space complexity? Define the PSPACE complexity class.

(e) Among the complexity classes P, NP and PSPACE, which is contained in which?

[4 + 6 + 4 + 3 + 3 = 20]

5. (a) Express the following sentences as propositional logic formula:

i. He must study hard, otherwise he will fail.

ii. He will go home only if it rains.

iii. If it rains, he will be at home; otherwise he will go to the market or school.

iv. If y is an integer then z is not real, provided that x is rational.

(b) Check if the following is a tautology or contradiction or neither:

$$(p \rightarrow r) \vee (q \rightarrow r) \leftrightarrow (p \wedge q \rightarrow r)$$

(c) Write down the rules of inference *modus ponens*, *unit resolution*, and *resolution*. Given (i) *if the program's input is an integer then the program outputs a boolean*, (ii) *the program's input is either an integer or a float*, and (iii) *the program's input is not a float*, show if you can apply one or more of the three rules to infer anything about the output of the program

[8 + 6 + 6 = 20]

6. (a) List the various primitives of First Order Logic, with one example of each.

(b) Express the following English sentences in First Order Logic:

i. Not all real numbers are rational numbers.

ii. There is no number such that no number is less than it.

iii. Any prime number with the property that all smaller numbers are prime is prime.

iv. There is a barber in town who shaves all men in town who do not shave themselves.

(c) Express in English what the following formula specifies:

$$\exists x \exists y(\neg(x = y) \wedge (\forall z((z = x) \vee (z = y))))$$

where $a = b$ corresponds to the predicate $equal(a, b)$ which is true if and only if $a = b$.

(d) Show (with an example) that the order of existential and universal quantifiers cannot be interchanged arbitrarily in a First Order Logic sentence.

[6 + 8 + 3 + 3 = 20]