# End Semester Examination

*IIT Kharagpur, CSE Dept., Autumn'15*

(CS41001) Theory of Computation (Full marks = 100)

Answer exactly 5 questions. In case of reasonable doubt, state your assumptions.

1. (a) Consider the function $f : \{a, b\}^* \rightarrow \{a, b\}^*$ such that
   $f(u) = a$, if $u$ contains even number of $a$'s and odd number of $b$'s;
   $= b$, otherwise.
   Show that $f$ is grammatically computable. Give a brief explanation of the computational steps. Indicate the fixed parenthetic strings $x, y, x', y'$; explain clearly the purpose of other nonterminal symbols used in the computation of $f$ by the grammar.

   (b) Show that the following functions are primitive recursive:
   (i) $rem(n, m)$ = the integer remainder when $n$ is divided by $m$; zero when $m$ is zero; use only the initial functions for this.
   (ii) $prime(n) = 0$ if $n$ is prime
   $= 1$ if $n$ is not a prime.

   (c) Prove that every grammatically computable function over strings is a $\mu$-recursive function. You may assume that the following predicates or functions, as the case may be, are given to be primitive recursive:
   (i) $\Sigma_0^* n$ ($\Sigma_1^* n$): $n$ is the Godel number of a string in the domain $\Sigma_0^*$ (codomain $\Sigma_1^*$) of the grammatically computable function,
   (ii) $Bpq$: ("the derivation $p$ begins with $q$") — $p$ is the Godel number of the derivation of the grammar (for computing the function) and $q$ is the Godel number of the first string of terminals and nonterminals in $p$,
   (iii) $Epq$: ("the derivation $p$ ends with $q$") — $p$ is as given in (ii) above and $q$ is the Godel number of the last string of terminals and nonterminals in $p$,
   (iv) $extract(p)$: extracts the Godel number of the string corresponding to the value $f(n)$ of the grammatically computable function from $q$ where $Epq$ holds.

   **[6 + (4 + 4) + 6 = 20]**

2. (a) Let $EQ_{REX}$ = { (R, S) | R and S are equivalent regular expressions}. Show that $EQ_{REX}$ is in PSPACE.

   (b) Let $BIPARTITE = \{G \mid G \text{ is a bipartite graph}\}$ . Show that, $BIPARTITE \in$ co-NL.

   (c) Let $GG = \{\langle G, b\rangle \mid$ Player 1 has winning strategy for the generalized geography game played on graph $G$ starting at node $b\}$. Prove that $GG$ is PSPACE-complete.

   (d) Show that, if every NP-hard language is also PSPACE-hard, then PSPACE = NP.

   **[5+5+7+3=20]**

3. (a) Formally define the complexity classes $\Sigma_2^p$ and $\Pi_2^p$. Prove that If $P =$ NP then PH $= P$.

   (b) Prove that $\Sigma_2^p \subseteq \text{NP}^{SAT}$. Prove that the emptyness problem of Buchi Automata can be checked in polynomial time.

   **[(4+6)+(5+5)=20]**

4. (a) Prove the equivalence of the following LTL formulae.

   i. $\neg\Diamond\phi \equiv \Box\neg\phi$

   ii. $\phi U \psi \equiv \psi \vee (\phi \wedge \bigcirc(\phi U \psi))$

   (b) Provide a formal definition of the *Buchi Acceptance Condition*. Construct a Buchi Automaton for the $\omega$-regular language $(ab)^* a (ba)^\omega$.

3S+ → SAT

$x \in \Sigma_i^P$ .

$\exists u_1 \forall$

$x, u_1 \in \Pi_{i-1}^P \rightarrow P$

$\forall u_1 \exists u_2$

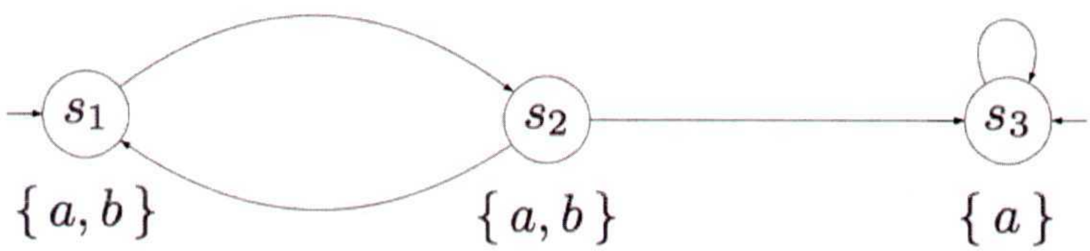
(c) Construct a Timed Automaton which captures the formal specification of a bus controller given as follows.
*"Consider a system with two processors $P_1$, $P_2$ connected by a bus. The bus controller grants exclusive access to the bus for $P_1, P_2$ using events $l_1, l_2$ respectively. The bus controller can request $P_1, P_2$ to release the bus using events $r_1, r_2$ respectively. Once granted access, a processor cannot entertain a release request within the next 2 seconds. If the release request comes within 2 seconds, the system cannot execute any further. If the release request comes after 2 seconds, then the processor releases the bus instantaneously. Once granted access, a processor can occupy the bus exclusively for not more than 5 seconds. If no release request comes within 5 seconds, then also system cannot execute any further. Two consecutive bus accesses by $P_1$ have to be separated by at least 10 seconds. Two consecutive bus accesses by $P_2$ have to be separated by at least 20 seconds."* - Note that $\Sigma = \{l_1, l_2, r_1, r_2\}$.

**[(3+3)+(4+4)+6=20]**

5. (a) Let MODEXP = $\{\langle$a,b,c,p$\rangle|$ a,b,c, and p are binary integers such that $a^b \equiv c(mod\ p)\}$. Show that MODEXP is in P.

   (b) For a graph $G = (V, E)$, a set of nodes $S \subseteq V$ is called independent if no two nodes in $S$ are connected by an edge $e \in E$. Let IND-SET be the language : IND-SET = $\{\langle G, k\rangle|$ Graph $G$ has an independent set of size $k\}$. Prove that IND-SET is NP-Complete.

   (c) Which of the following LTL formulea are true on all paths of the labeled transition system given




   i. $\Box a$
   ii. $\Box(\neg b \implies \Box(a \wedge \neg b))$
   iii. $\bigcirc(a \wedge b)$
   iv. $b\mathrm{U}(a \wedge \neg b)$

**[5+5+(2.5 X 4)=20]**

6. (a) Let $L_\infty = \{< M >|\ |L(M)| = \infty\}$. Prove that $L_\infty$ is not Turing-recognizable.

   (b) Let $TQBF = \{\langle\phi\rangle \mid \phi$ is a true fully quantified boolean formula$\}$. Prove that $TQBF$ is PSPACE-complete.

   (c) Let us define the class of problems **DP** as follows. A language $L$ is in the class **DP** if and only if there are two languages $L_1 \in$ **NP** and $L_2 \in co$**NP** such that $L = L_1 \cap L_2$. Now let $SAT - UNSAT = \{< \phi, \phi' >: \phi$ and $\phi'$ are boolean formulae in 3CNF and $\phi$ is satisfiable and $\phi'$ is not satisfiable$\}$. Prove that $SAT - UNSAT$ is **DP**-complete.

**[5+7+(3+5)= 20]**

7. (a) Let $L_{fin} = \{< M >|\ L(M)$ is finite$\}$. Prove that $L_{fin}$ is undecidable.

   (b) Construct a reduction from $Th(\mathbb{N}, <)$ to $Th(\mathbb{N}, +)$.

   (c) Prove that for $f(n) \geq \log n$, $ASPACE(f(n)) = TIME(2^{O(f(n))})$.

**[6+6+8=20]**