Fourier Transform-I

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1 Fourier Series to Fourier Transform

So far we have seen that a periodic function $f(t)$ with finite period T can be represented with a sum of fundamental frequency $\omega = 2\pi/T$ component along with higher order harmonics ω, 2ω, 3ω etc. The amplitudes of the harmonics progressively decrease. In other words by carrying out Fourier transform of a periodic signal we can say about the predominant harmonic components present and predict about the output of a system when such a periodic signal happens to be the input. This opens up a new way of looking at a system called frequency response. The idea as such is quite straight forward. Now we ask ourselves, if the function (or signal) is not periodic in nature, is it possible to say about the different frequency components which characterize the signal?

1.1 Fourier analysis of periodic pulses

Consider a periodic pulse signal of amplitude A and time period T as shown in figure 1 Mathematical

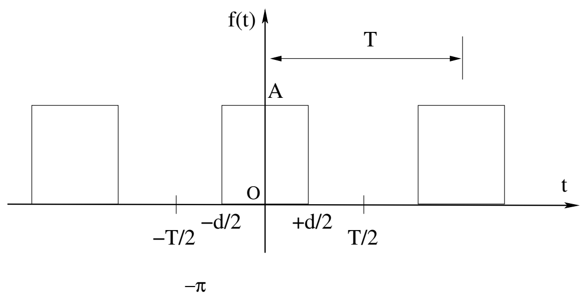


Figure 1: Train of square wave pulses

description of the signal is:

$$f(t) = \begin{cases} 0 & \text{for } -T/2 < t < -d/2 \\ A & \text{for } -d/2 < t < d/2 \\ 0 & \text{for } d/2 < t < T/2 \end{cases}$$

Now we know.

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega t} \\
\text{and } c_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega t}\, dt \\
&= \frac{1}{T}\int_{-d/2}^{d/2} A\, e^{-jn\omega t}\, dt \\
&= \frac{A}{T}\left(\frac{e^{-jn\omega t}}{-jn\omega}\right)\Bigg|_{-d/2}^{d/2} \\
&= \frac{jA}{n\omega T}\left(e^{-jn\omega t}\right)\Bigg|_{-d/2}^{d/2} \\
&= \frac{jA}{n\omega T}\left[e^{-jn\omega d/2} - e^{jn\omega d/2}\right] \\
&= \frac{jA}{n\omega T}\left[-2jn\omega d/2\right] \\
&= \frac{2A}{n\omega T}\sin\frac{n\omega d}{2} \\
&= \frac{2A}{n\omega T}\frac{n\omega d}{2}\frac{\sin\frac{n\omega d}{2}}{\frac{n\omega d}{2}} \\
\text{Finally, } c_n &= \frac{Ad}{T}\left(\frac{\sin\frac{n\omega d}{2}}{\frac{n\omega d}{2}}\right) \quad \text{Even function, imaginary part is zero} \\
\therefore\ \frac{a_n}{2} &= c_n = \frac{Ad}{T}\left(\frac{\sin\frac{n\omega d}{2}}{\frac{n\omega d}{2}}\right) \\
\text{or, } a_n &= \frac{2Ad}{T}\left(\frac{\sin\frac{n\omega d}{2}}{\frac{n\omega d}{2}}\right)
\end{aligned}
$$

The magnitudes of c_n (Fourier coefficients), as a function of $n\omega$ is shown in figure 2. It is noted that the envelop of the curve is a sinc function and the consecutive spectral lines are separated by ω where ω is the fundamental frequency ($\omega = 2\pi/T$).
If the time period T of the above periodic pulses is made higher then the separation between the spectral lines will reduce as $\omega = 2\pi/T$.

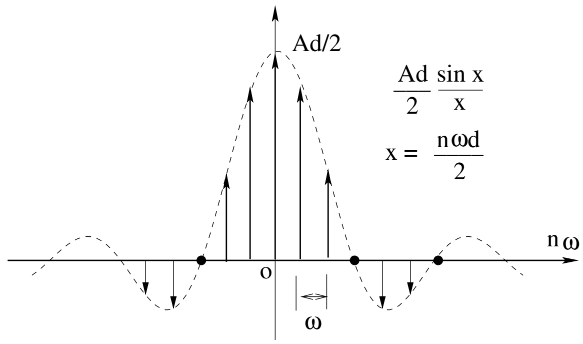


Figure 2: Train of square wave pulses

1.2 Fourier analysis of a single pulse

Shown in figure 3, a single pulse function $f(t)$ of width d. We ask ourselves whether such a function $f(t)$ which is aperiodic in nature can have a Fourier series expansion? Apparently the answer is no as there is no time period T for this single pulse and calculation of the Fourier coefficients can not be carried out.

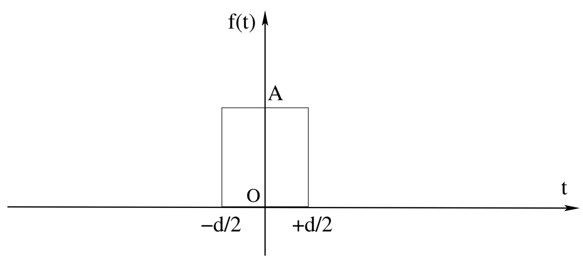


Figure 3: A single pulse

However, let us pretend that the single pulse $f(t)$ of figure 3 is a periodic wave with $T = \infty$. Then the fundamental frequency $\omega = 2\pi/T$ becomes zero and the Fourier coefficient c_n too will become zero. The end results of this exercise is not going to reveal any meaningful information of the pulse regarding the fundamental frequency & harmonics and their magnitudes as everything collapse to zero as $T = \infty$.

One can avoid such an unpleasant situation by avoiding, replacement of T by ∞ blindly. Instead what we can do is to allow time period to approach infinity i.e., $T \to \infty$. Under this condition, let us first concentrate on Fourier coefficients c_n We know,

$$
\begin{aligned}
c_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega t}\, dt \\
c_n &= \frac{1}{T}\int_{-\infty}^{\infty} f(t)\, e^{-jn\omega t}\, dt
\end{aligned}
$$

In the above, the limits of integration are replaced with ease, but what to do with $1/T$ and the fundamental frequency $\omega = 2\pi/T$. As $T \to \infty$,the fundamental frequency $\omega \to 0$. So fundamental frequency can be represented as $\Delta\omega \to 0$. Also we note that the separation between the consecutive spectral lines will tend to zero, meaning that the c_n values will exist for all frequencies. Thus c_n becomes a continuous function of ω. So far so good, however, we note that because of the presence of the term $1/T$ before the integral, c_n values will become zero. Now to avoid this, bring the $1/T = f$ term to the left so as to get c_n/f on the left hand side. The above equation can be written as:

$$
\begin{aligned}
c_n &= \frac{1}{T}\int_{-\infty}^{\infty} f(t)\, e^{-jn\,\Delta\omega\, t}\, dt \\
\frac{c_n}{f} &= \int_{-\infty}^{\infty} f(t)\, e^{-jn\,\Delta\omega\, t}\, dt
\end{aligned}
$$

n losses its importance and $n\Delta\omega$ can be replaced by ω. Finally we get,

$$
\frac{c}{f} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt = F(\omega)
$$

Thus for an aperiodic wave, no point in talking about a discrete value of Fourier coefficient c_n as all frequencies are involved. In language, the left hand side is to be read as *Fourier coefficient density* (per Hz). The integration of the right hand side will return a function of ω, shown as $F(\omega)$. $F(\omega)$ on the right hand side is called the **Fourier transform of the function** $f(t)$.

We now know that given a function $f(t)$, its Fourier transform $F(\omega)$ can be calculated by using the above equation. Naturally question comes, if $F(\omega)$ is known, can we find out the corresponding

$f(t)$. The answer to this is yes and we proceed as follows to get the result.

$$\begin{aligned}
\text{Now, } f(t) &= \sum_{n=-\infty}^{\infty} c_n \, e^{jn\Delta\omega t} \\
\text{we know, } F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt = c/f \\
\text{or, } c &= f F(\omega) = \frac{1}{T} F(\omega)
\end{aligned}$$

Put this c in the first equation and get,

$$\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T} F(\omega) \;\; e^{jn\Delta\omega t} \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{2\pi}{T} F(\omega) \;\; e^{jn\Delta\omega t} \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \Delta\omega \, F(\omega) \;\; e^{j\omega t} \\
\text{Finally, } f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega
\end{aligned}$$

We reproduce the following two final results which will often be used to find Fourier transform for a given function or to find out the function if Fourier transform is know.

$$\begin{aligned}
\text{Fourier transform of } f(t) : F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt \\
\text{Inverse Fourier transform of } F(\omega) : f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega
\end{aligned}$$

Note that $f(t)$ and $F(\omega)$ form a transform pair.

$$\mathcal{F}\{f(t)\} \leftrightarrow F(\omega) \;\text{ and conversely, }\; \mathcal{F}^{-1}\{F(\omega)\} \leftrightarrow f(t)$$

Loosely the transform pair is written as: $f(t) \leftrightarrow F(\omega)$

2 Properties of Fourier Transform

1. If Fourier transform of $f(t) \leftrightarrow F(\omega)$, then $f(at) \leftrightarrow \frac{1}{a}F(\frac{\omega}{a})$; $a > 0$
   Proof:

$$\begin{aligned}
\text{Now, } \mathcal{F}[f(at)] &= \int_{-\infty}^{\infty} f(at)\, e^{-j\omega\, t}\, dt \\
\text{Put} &\quad at = \tau \;\therefore\; a\, dt = d\tau \\
\therefore\; \mathcal{F}[f(at)] &= \frac{1}{a}\int_{-\infty}^{\infty} f(\tau)\, e^{-j\,\frac{\omega}{a}}\, d\tau \\
\therefore\; \mathcal{F}[f(at)] &= \frac{1}{a}F\left(\frac{\omega}{a}\right) \\
\text{Finally, } \mathcal{F}[f(at)] &\leftrightarrow \frac{1}{a}F\left(\frac{\omega}{a}\right)
\end{aligned}$$

2. If Fourier transform of $f(t) \leftrightarrow F(\omega)$, then $\frac{df}{dt} \leftrightarrow j\omega F(\omega)$;
   Proof:
   To prove this we take help of the inverse relation of Fourier transform.

$$\begin{aligned}
\text{Now } f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \\
\text{Differentiating both sides, } \frac{df(t)}{dt} &= \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \\
\text{or, } \frac{df(t)}{dt} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, \frac{de^{j\omega t}}{dt}\, d\omega \\
\text{or, } \frac{df(t)}{dt} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega F(\omega)]\, e^{j\omega t}\, d\omega \\
\therefore\; \frac{df}{dt} &\leftrightarrow j\omega F(\omega)
\end{aligned}$$

3. If Fourier transform of $f(t) \leftrightarrow F(\omega)$, then $f(t)e^{-at} \leftrightarrow F(j\omega + a)$; $a > 0$
   Proof:

$$\begin{aligned}
\text{Now, } \mathcal{F}[f(t)e^{-at}] &= \int_{-\infty}^{\infty} f(t)e^{-at}\, e^{-j\omega\, t}\, dt \\
\text{or, } \mathcal{F}[f(t)e^{-at}] &= \int_{-\infty}^{\infty} f(t)\, e^{-(j\omega+a)t}\, dt \\
\therefore\; \mathcal{F}[f(t)e^{-at}] &= F(j\omega + a);\;\; a > 0
\end{aligned}$$

4. What is $\mathcal{F}[e^{-at}u(t)]$?
   Answer:

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\omega\, t}\, dt \\
\text{or, } F(\omega) &= \int_{0}^{\infty} e^{-at}\, e^{-j\omega\, t}\, dt = \int_{0}^{\infty} e^{-(j\omega\, + a)t}\, dt \\
\text{or, } F(\omega) &= \frac{e^{-(j\omega\, + a)t}}{-(j\omega\ + a)}\Big|_0^\infty = \frac{1}{(j\omega\ + a)} \\
\therefore\ \mathcal{F}[e^{-at}u(t)] &= \frac{1}{(j\omega\ + a)}
\end{aligned}$$

5. **If $f(t)$ is even then $F(\omega)$ too will be even and real.**
   Proof:

$$\begin{aligned}
\text{Since } f(t) \text{ is even, } f(t) &= f(-t) \\
\text{Now, } F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt \\
\text{Put } t = -\tau \quad & \quad \text{so, } dt = -d\tau \\
\therefore\ F(\omega) &= \int_{\infty}^{-\infty} f(-\tau)\, e^{-j\omega\,(-\tau)}\, (-d\tau) \\
\text{or, } F(\omega) &= \int_{-\infty}^{\infty} f(-\tau)\, e^{-j(-\omega)\,\tau}\, (d\tau) \\
\text{or, } F(\omega) &= \int_{-\infty}^{\infty} f(\tau)\, e^{-j(-\omega)\,\tau}\, (d\tau)\ ;\ f(t) \text{ is even} \\
\therefore\ F(\omega) &= F(-\omega)\ ; \text{So } F(\omega) \text{ is even} \\
\text{Also, } F(\omega) &= F^{\star}(\omega)\ ; \text{Signifying } F(\omega) \text{ is real}
\end{aligned}$$

6. **If $f(t)$ is odd then $F(\omega)$ too will be odd and imaginary.**
   Proof:

$$
\begin{aligned}
\text{Since } f(t) \text{ is odd, } f(t) &= -f(-t) \\
\text{Now, } F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt \\
\text{Put } t = -\tau \quad & \quad \text{so, } dt = -d\tau \\
\therefore\ F(\omega) &= \int_{\infty}^{-\infty} f(-\tau)\, e^{-j\omega\,(-\tau)}\,(-d\tau) \\
\text{or, } F(\omega) &= \int_{-\infty}^{\infty} f(-\tau)\, e^{-j(-\omega)\,\tau}\,(d\tau) \\
\text{or, } F(\omega) &= \int_{-\infty}^{\infty} -f(\tau)\, e^{-j(-\omega)\,\tau}\,(d\tau)\ ;\ f(t) \text{ is odd} \\
\therefore\ F(\omega) &= -F(-\omega)\ ;\ \text{So } F(\omega) \text{ is odd} \\
\text{Also, } F(\omega) &= -F^{\star}(\omega)\ ;\ \text{Signifying } F(\omega) \text{ is imaginary}
\end{aligned}
$$

7. **If** $f(t) \leftrightarrow F(\omega)$ **then** $F(t) \leftrightarrow 2\pi f(-\omega)$
   Proof The proof is trivial and involves mere interesting substitution and change of variables name.

$$\begin{aligned}
\text{Now, } F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt \\
\text{or, } F(\omega) &= \int_{-\infty}^{\infty} f(z)\, e^{-j\omega\, z}\, dz \ \text{ ; replacing } t \text{ by } z \\
\text{or, } F(t) &= \int_{-\infty}^{\infty} f(z)\, e^{-jt\, z}\, dz \ \text{ ; replacing } \omega \text{ by } t \text{ on both sides} \\
\text{or, } F(t) &= \int_{\infty}^{-\infty} f(-\omega)\, e^{-jt\,(-\omega)}\, d(-\omega) \ \text{ ; replacing } z \text{ by } -\omega \\
\text{or, } F(t) &= \int_{-\infty}^{\infty} f(-\omega)\, e^{-jt\,(-\omega)}\, d(\omega) \\
\text{or, } F(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi f(-\omega)\, e^{j\,\omega t}\, d(\omega) \ \text{ ; multiply \& divide by } 2\pi
\end{aligned}$$

Therefore $F(t)$ and $2\pi f(-\omega)$ forms a Fourier transform pair i.e.,

$$F(t) \leftrightarrow 2\pi f(-\omega)$$

3 Some examples of Fourier Transform

3.1 Fourier transform of an Impulse $\delta(t)$

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt \\
\therefore\ \mathcal{F}[\delta(t)] &= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega\, t}\, dt \\
&= \int_{0^-}^{0^+} \delta(t)\, e^{-j\omega\, t}\, dt \\
&= \int_{0^-}^{0^+} \delta(t)\, dt \\
\text{Finally, } \mathcal{F}[\delta(t)] &= 1
\end{aligned}$$

3.2 Fourier transform of a rectangular pulse

Here we find out the Fourier transform of a single rectangular pulse shown in figure 3. The function can be mathematically described as:

$$f(t) = \begin{cases} A & \text{for } -d/2 < t < d/2 \\ 0 & \text{elsewhere} \end{cases}$$

Now,

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \\ &= \int_{-d/2}^{d/2} A\, e^{-j\omega t}\, dt \\ &= \frac{A}{-j\omega}\left(e^{-j\omega\, d/2} - e^{j\omega\, d/2}\right) \\ &= \frac{2A\, \sin\omega d/2}{\omega} \\ \therefore\; F(\omega) &= Ad\,\frac{\sin\omega d/2}{\omega d/2} \end{aligned}$$

The single rectangular pulse function and its Fourier transform are shown in figure 4 We note that

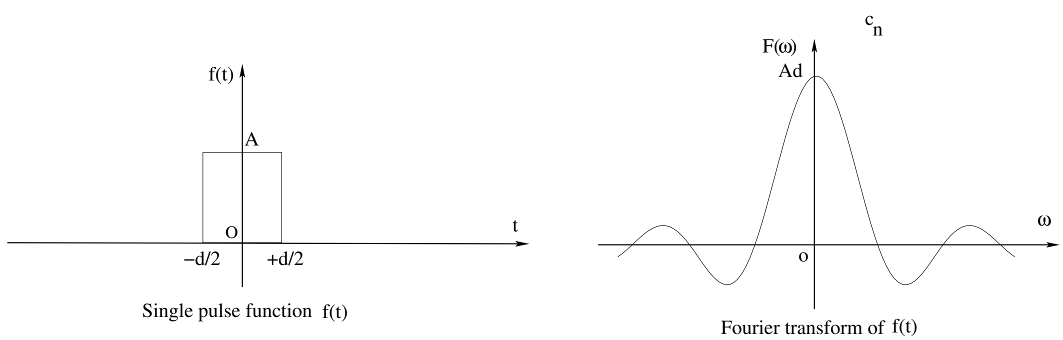


Figure 4: A single pulse & its Fourier transform

since $f(t)$ is even $F(\omega)$ is also even and has no imaginary part.

Interesting side result

Let us assume, for the pulse $A = 1$ and $d = 2$, then

$$\begin{aligned}
F(\omega) &= 2\,\frac{\sin\omega}{\omega} \\
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega \\
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin\omega}{\omega}\,e^{j\omega t}\,d\omega \\
f(t) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\,e^{j\omega t}\,d\omega \\
f(0) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\,d\omega \\
f(0) &= \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin\omega}{\omega}\,d\omega \\
1 &= \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin\omega}{\omega}\,d\omega \\
\int_{0}^{\infty} \frac{\sin\omega}{\omega}\,d\omega &= \frac{\pi}{2} \\
\text{Obviously,}\ \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\,d\omega &= \pi
\end{aligned}$$

Total area under the sinc function is π.

3.3 Fourier transform of a constant

Let us try to find out the Fourier transform of a constant number A. Here $f(t) = A$ for all time i.e., $-\infty < t < \infty$. Using the formula for $F(\omega)$ we get,

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)\,e^{-j\omega\,t}\,dt \\
&= \int_{-\infty}^{\infty} A\,e^{-j\omega\,t}\,dt \\
F(\omega) &= A\int_{-\infty}^{\infty} e^{-j\omega\,t}\,dt
\end{aligned}$$

Instead of trying to integrate RHS, let us use the inverse relation and guess the result.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \qquad (1)$$

$$A = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \qquad (2)$$

$$(3)$$

Guess of $F(\omega)$ is straight forward if we know the properties of impulse function. If $F(\omega) = 2\pi\, A\, \delta(\omega)$, then :

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, A\, \delta(\omega)\, e^{j\omega t}\, d\omega = A\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega = A$$

Thus we get,

$$A = \frac{1}{2\pi}\int_{-\infty}^{\infty} [2\pi\, A\, \delta(\omega)]\, e^{j\omega t}\, d\omega$$

So the term inside the square bracket must be $F(\omega)$. Hence,

$$\mathcal{F}\{A\} \leftrightarrow 2\pi\, A\, \delta(\omega)$$

From the physical consideration the result is fine because the function in time domain is constant throughout which means that only zero frequency term will be present and it should be very large (impulse) indicating all the energy is concentrated at $\omega = 0$.

3.4 Fourier transform of $u(t)$

Find the Fourier transform of the unit step function $u(t)$.
Solution

$$\mathcal{F}[u(t)] = \int_{-\infty}^{\infty} u(t)\, e^{-j\omega\, t}\, dt$$

$$\text{or, } \mathcal{F}[u(t)] = \int_{0}^{\infty} e^{-j\omega\, t}\, dt = \frac{1}{j\omega} \text{ ; But this can not be correct.}$$

Because the result $\frac{1}{j\omega}$, obtained is purely imaginary which is only possible if the original time function is odd. However $u(t)$ is neither odd nor even. We also know that a general function can be broken up into an even and odd parts. So, $u(t)$ too can be written as sum of even and odd parts. Thus $\mathcal{F}[u(t)]$ must be complex in nature having both real and imaginary parts.

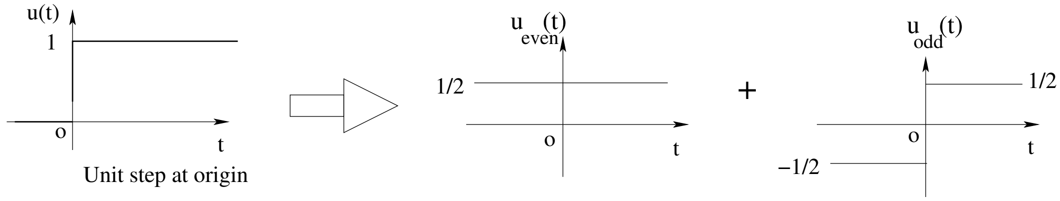


Figure 5: $u(t)$ as sum of even and odd parts

$$\therefore\ \mathcal{F}[u(t)] = \mathcal{F}[u_{\text{even}}(t)] + \mathcal{F}[u_{\text{odd}}(t)]$$

$$\text{Now, } \mathcal{F}[u_{\text{even}}(t)] = \mathcal{F}\left[\frac{1}{2}\right] = 2\pi\frac{1}{2}\,\delta(\omega) = \pi\,\delta(\omega)$$

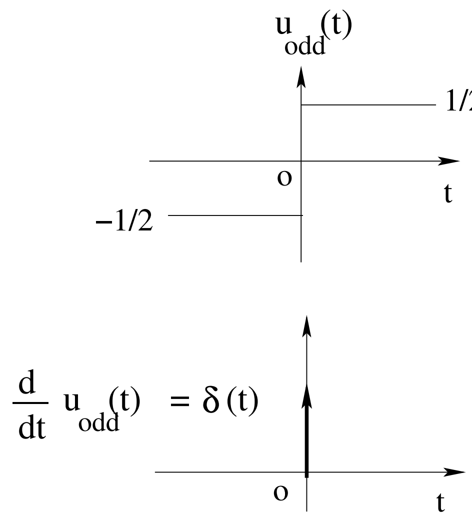


Figure 6: Differentiation of u_{odd}

Now we have to calculate $\mathcal{F}[u_{\text{odd}}(t)]$. From figure 6, we see that,

$$\begin{aligned}
\frac{d}{dt}(u_{\text{odd}}) &= \delta(t) \\
\therefore\ \mathcal{F}[\frac{d}{dt}(u_{\text{odd}})] &= \mathcal{F}[\delta(t)] = 1 \\
\text{or, } j\omega\mathcal{F}(u_{\text{odd}}) &= 1 \\
\therefore\ \mathcal{F}(u_{\text{odd}}) &= \frac{1}{j\omega}\ ;\text{ Result is consistent} \\
\text{Finally, } \mathcal{F}[u(t)] &= \mathcal{F}[u_{\text{even}}(t)] + \mathcal{F}[u_{\text{odd}}(t)] \\
\text{or, } \mathcal{F}[u(t)] &= \pi\,\delta(\omega) + \frac{1}{j\omega}
\end{aligned}$$

Fourier Series to Fourier Transform

- We have seen that for a <u>periodic</u> f^n. $x(t)$ satisfying Dirichlet condition :-

$$x(t) = \sum_{K=-\infty}^{\infty} c_K e^{jK\omega t}$$ Fourier series

where $c_K = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jK\omega t} dt$

T = time period (fundamental) and $\omega = \frac{2\pi}{T}$ = fundamental frequency.

- The question is can an aperiodic function too can be resolved into different frequency. The answer is yes!

An aperiodic signal $x(t)$ can be considered to be <u>a periodic signal</u> with time period $T \to \infty$ and fundamental frequency $\Delta\omega = \frac{2\pi}{T} \to 0$.

Example is a single pulse.

Under the condition $T \to \infty$ & $\Delta\omega \to 0$ let us examine the above formulas.

$$c_K = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jK\Delta\omega t} dt \quad \text{with } T \to \infty,\ \Delta\omega \to 0$$

$\therefore$ It looks like all $c_K \to 0$ & no useful information can be obtained.

However

$$\frac{c_K}{(1/T)} = \int_{-T/2}^{T/2} x(t) e^{-jK\Delta\omega t} dt$$ may converge to some value giving Fourier Co-eff density

Replace $K\Delta\omega$ by ω

$$\frac{C_K}{1/T} = \int_{-T/2}^{T/2} x(t)\, e^{-j\omega t}\, dt \xrightarrow{\lim_{T\to\infty}} X(\omega)$$

Fourier co-eff. density.

$$\boxed{X(\omega) = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

Now the other formula:-

$$x(t) = \sum_{K=-\infty}^{\infty} C_K\, e^{jK\Delta\omega t} \qquad T = \frac{2\pi}{\Delta\omega}$$

$$= \sum_{K=-\infty}^{\infty} \left(\frac{C_K}{1/T}\right) \frac{1}{T}\, e^{j\underline{K\Delta\omega} t}$$

$$= \int_{\omega=-\infty}^{\infty} F(\omega)\, \frac{\Delta\omega}{2\pi}\, e^{j\omega t}$$

$$\boxed{x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega}$$

Fourier Transform pair

$$x(t) \longleftrightarrow F(\omega)$$

Fourier Transform of some standard signals and properties :— $\quad f(t) \longleftrightarrow F(\omega)$

We know

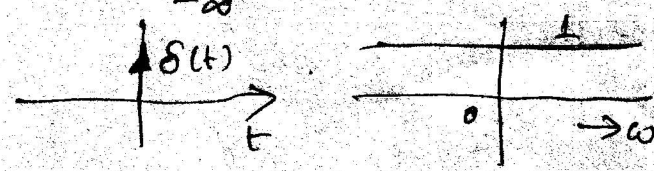
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \quad \& \quad F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

① Fourier transform of $\delta(t)$

$x(t) = \delta(t)$

$$\therefore \ \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\therefore \ \boxed{\mathcal{F}[\delta(t)] = 1}$$



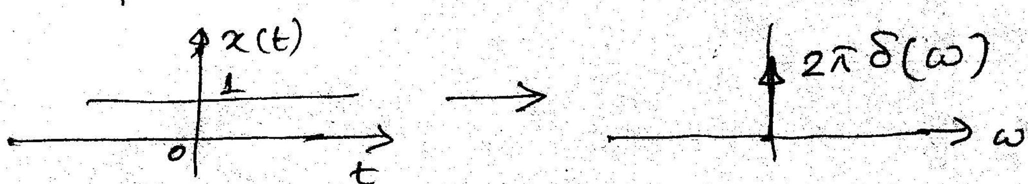
② If $x(t) = 1 \qquad X(\omega) = ?$

$$X(1) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} 1 e^{-j\omega t}\, dt$$

$$x(t) = 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega = 1$$

Guess $\cancel{X(\omega)}$ $= \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} 2\pi\,\delta(\omega)\, e^{-j\omega t}\, d\omega = 1$

$$\therefore \ \boxed{1 \longleftrightarrow 2\pi\,\delta(\omega)}$$



③ Time shifting property:-

If $f(t) \leftrightarrow F(\omega)$ then $f(t-t_0) \leftrightarrow e^{-j\omega t_0} F(\omega)$

Now $\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$ put $t - t_0 = \tau$, $dt = d\tau$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_0)} d\tau = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

$$= e^{-j\omega t_0} X(\omega)$$

$\therefore$ $\boxed{\begin{array}{l} x(t) \leftrightarrow X(\omega) \\ x(t-t_0) \leftrightarrow e^{-j\omega t_0} X(\omega) \end{array}}$

④ If $x(t) \leftrightarrow X(\omega)$ Then $x(t) e^{j\omega_c t} \leftrightarrow X(\omega - \omega_c)$

$$\mathcal{F}\{x(t) e^{j\omega_c t}\} = \int_{-\infty}^{\infty} x(t) e^{j\omega_c t} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_c)t} dt = X(\omega - \omega_c)$$

$\therefore$

⑤ $x(t) \leftrightarrow X(\omega)$ then $\frac{dx}{dt} \leftrightarrow j\omega X(\omega)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\therefore \frac{dx}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(\omega) e^{j\omega t} d\omega$$

$\therefore$ $\boxed{\frac{dx}{dt} \leftrightarrow j\omega X(\omega)}$

(6) $x(t) \longleftrightarrow X(\omega)$ Then $t f(t) \longleftrightarrow j \dfrac{dX}{d\omega}$.

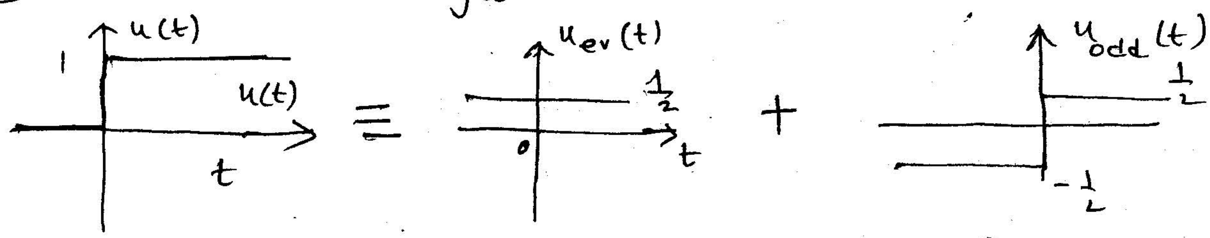
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\therefore \frac{dX}{d\omega} = \int_{-\infty}^{\infty} (-jt)\, x(t)\, e^{-j\omega t}\, dt$$

$$\therefore j\frac{dX}{d\omega} = \int_{-\infty}^{\infty} t\, x(t)\, e^{-j\omega t}\, dt$$

$$\therefore t\, x(t) \longleftrightarrow j\frac{dX}{d\omega}$$

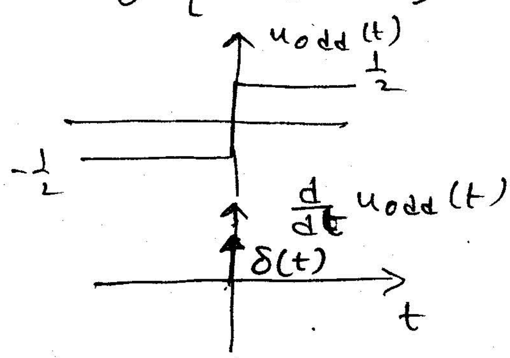
(7) $u(t) \longleftrightarrow \dfrac{1}{j\omega} + \pi\delta(\omega)$



$$\therefore \mathcal{F}\{u(t)\} = \mathcal{F}\{u_{ev}(t)\} + \mathcal{F}\{u_{odd}(t)\}.$$

Now $\mathcal{F}\{u_{ev}(t)\} = \mathcal{F}\left(\frac{1}{2}\right) = \frac{1}{2} \times 2\pi\delta(\omega) = \pi\delta(\omega)$

$\mathcal{F}\{u_{odd}(t)\} = ?$



$$\mathcal{F}\left\{\frac{d}{dt}u_{odd}(t)\right\} = \mathcal{F}\{\delta(t)\} = 1$$

$$\therefore j\omega\, \mathcal{F}\{u_{odd}(t)\} = 1$$

$$\therefore \mathcal{F}\{u_{odd}(t)\} = \frac{1}{j\omega}$$

$$\boxed{\therefore \mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega)}$$

(8) very important property

If $z(t) = x(t) * y(t)$ Then $Z(\omega) = X(\omega) \times Y(\omega)$

Now

$$\mathcal{F}\{z(t)\} = Z(\omega) = \int_{t=-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

Now $$z(t) = x(t) * y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$\therefore Z(\omega) = \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \; e^{-j\omega t}\, dt$$

Now changing the order of integration.

$$Z(\omega) = \int_{\tau=-\infty}^{\infty} x(\tau) \int_{t=-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt \; d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\; \mathcal{F}\{y(t-\tau)\}\, d\tau$$

but $y(t) \leftrightarrow Y(\omega)$

$y(t-\tau) \leftrightarrow Y(\omega)\, e^{-j\omega\tau}$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\; Y(\omega)\, e^{-j\omega\tau}\, d\tau = Y(\omega) \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

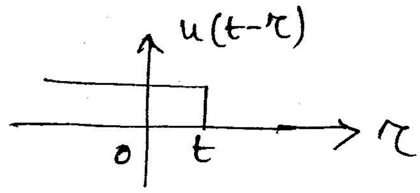
$$= Y(\omega)\, X(\omega) = X(\omega)\, Y(\omega)$$

(9) Integration property.

$$x(t) \longleftrightarrow X(\omega) \text{ Then } \int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(\omega)\delta(\omega)$$

Now

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{t} x(\tau)\,d\tau + \int_{t}^{\infty} 0\,d\tau$$



$$= \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau = x(t) * u(t)$$

$$\therefore \quad \mathcal{F}\left\{\int_{0}^{t} x(\tau)\,d\tau\right\} = \mathcal{F}\left\{x(t) * u(t)\right\}$$

$$= X(\omega)\,U(\omega)$$

$$= X(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= \frac{X(\omega)}{j\omega} + \pi X(\omega)\delta(\omega)$$

$$\boxed{\mathcal{F}\left\{\int_{0}^{t} x(\tau)\,d\tau\right\} = \frac{X(\omega)}{j\omega} + \pi X(0)\,\delta(\omega)}$$

~~Some~~ One important observations:-

⑨ If $x(t)$ is a real function.

and $x(t) \longleftrightarrow X(\omega)$

Then $X(-\omega) = X^*(\omega)$

Proof:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) \cos\omega t \, dt - j\int_{-\infty}^{\infty} x(t) \sin\omega t \, dt$$

$$\therefore X(\omega) = \int_{-\infty}^{\infty} \underbrace{x(t)}_{\text{Real}} \cos\omega t \, dt - j\int_{-\infty}^{\infty} x(t) \sin\omega t \, dt$$

$$\therefore X(-\omega) = \int_{-\infty}^{\infty} x(t) \cos\omega t \, dt + j\int_{-\infty}^{\infty} x(t) \sin\omega t \, dt.$$

$$\therefore \boxed{X(-\omega) = X^*(\omega)} \quad \text{for real } x(t)$$

⑩ If $x(t)$ is even & Real and $x(t) \longleftrightarrow X(\omega)$

Then $X(\omega)$ also will be real and even.

Proof :-

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

put $t = -\tau \qquad dt = -d\tau$

$$\therefore X(\omega) = \int_{+\infty}^{-\infty} x(\tau) e^{-j\omega(-\tau)} (-d\tau) = \int_{-\infty}^{\infty} x(\tau) e^{-j(-\omega)\tau} d\tau$$

$\because x(\tau) = x(-\tau)$ even.

$X(\omega) = X(-\omega) \qquad \therefore X(\omega)$ even.

but for real $x(t) \quad X(-\omega) = X^*(\omega)$

$\therefore X(\omega) = X(-\omega) = X^*(\omega)$

$\therefore X(\omega)$ must be real (no imaginary component).

(17) If $x(t)$ is odd and real

& $x(t) \longleftrightarrow X(\omega)$

Then $X(\omega)$ will be also odd and will be purely imaginary.

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

put $t = -\tau$

$$\therefore\ X(\omega) = \int_{+\infty}^{-\infty} x(-\tau)\, e^{-j\omega(-\tau)}\, (-d\tau)$$

$\because$ $x(t)$ is odd $\therefore$ $x(-\tau) = -x(\tau)$

$$X(\omega) = \int_{+\infty}^{-\infty} -x(\tau)\, e^{-j\omega(-\tau)}\, (-d\tau)$$

$$= -\int_{-\infty}^{\infty} x(\tau)\, e^{-j(-\omega)\tau}\, d\tau = -X(-\omega)$$

$\therefore$ $X(\omega) = -X(-\omega)$ $\therefore$ $\underline{X(\omega) \text{ is odd}}$

or $\boxed{X(\omega) = -X^*(\omega)}$ $\because$ $X(-\omega) = X^*(\omega)$ for real $x(t)$

This is possible if $X(\omega)$ is purely imaginary.

(12) $x(t) \longleftrightarrow X(\omega)$ Then $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

Case(i) $a>0$ i,e $a=+ve$

Now

$$\mathcal{F}\{x(at)\} = \int_{t=-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $\tau = at$ $\therefore d\tau = a\,dt$

$$\therefore \mathcal{F}\{x(at)\} = \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega \frac{\tau}{a}}\, \frac{d\tau}{a} = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau$$

$$\therefore \boxed{\mathcal{F}\{x(at)\} = \frac{1}{a} X\left(\frac{\omega}{a}\right)}$$

Case (ii) $a<0$ a $-ve$ let $b=-a$ $b>0$

$$\mathcal{F}\{x(at)\} = \int_{t=-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

$$= \int_{t=-\infty}^{\infty} x(-bt)\, e^{-j\omega t}\, dt$$

put $\tau = -bt$ $d\tau = -b\,dt$

$$\mathcal{F}\{x(at)\} = \int_{\tau=+\infty}^{-\infty} x(\tau)\, e^{-j\omega \frac{\tau}{(-b)}}\, \frac{d\tau}{(-b)}$$

$$= \frac{1}{b}\int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\frac{\omega}{a}\tau}\, d\tau$$

$$= \frac{1}{b} X\left(\frac{\omega}{a}\right)$$

$$\mathcal{F}\{at\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

(13) Principle of Duality:

If $x(t) \longleftrightarrow X(\omega)$

Then $X(t) \longleftrightarrow 2\pi\, x(-\omega)$.

Now
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Replacing t by y.

$$X(\omega) = \int_{-\infty}^{\infty} x(y)\, e^{-j\omega y}\, dy$$

* Now replace ω by t on both sides.

$$X(t) = \int_{-\infty}^{\infty} x(y)\, e^{-jty}\, dy$$

Now replace y by $-\omega$ $\quad y = -\omega$

$$X(t) = \int_{\infty}^{-\infty} x(-\omega)\, e^{-j(-\omega)t}\, (-d\omega) = \int_{-\infty}^{\infty} x(-\omega)\, e^{j\omega t}\, d\omega$$

or
$$X(t) = 2\pi \times \frac{1}{2\pi}\int_{-\infty}^{\infty} x(-\omega)\, e^{j\omega t}\, d\omega.$$

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, x(-\omega)\, e^{j\omega t}\, d\omega$$

$$\therefore \boxed{X(t) \longleftrightarrow 2\pi\, x(-\omega)}$$

Pictorially

time	frequency
$x(t) \longleftrightarrow$	$X(\omega)$
$X(t) \longleftrightarrow$	$2\pi x(-\omega)$

Alternative proof o Duality

$x(t) \longleftrightarrow X(\omega)$

then $X(t) \longleftrightarrow 2\pi\, x(-\omega)$

Proof:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

and

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Replace t by $-t$ in both the sides

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

$$\therefore\; 2\pi\, x(-t) = \cancel{\frac{1}{2\pi}}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Now replace t by ω
and ω by t

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt = \mathcal{F}\{X(t)\}.$$

$$\therefore\; \boxed{X(t) \longleftrightarrow 2\pi\, x(-\omega)}$$ proved.

Application

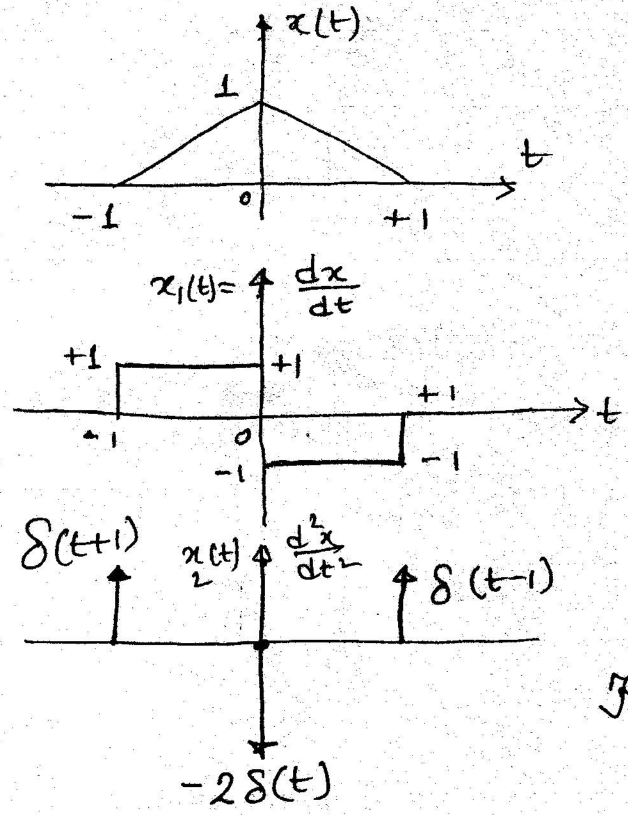
time $\quad$ frequency

$\delta(t) \longleftrightarrow 1$

$1 \longleftrightarrow 2\pi\,\delta(-\omega) = 2\pi\,\delta(\omega)$

$\because\; \delta(-\omega) = \delta(\omega)$.

Fourier Transform of a Δ-pulse.



$\mathcal{F}\{x(t)\} = ?$

To get this we first calculate

$$\mathcal{F}\{x_2(t)\} = \mathcal{F}\left\{\frac{d^2x}{dt^2}\right\}.$$

$$= \mathcal{F}\{\delta(t+1)\} - 2\mathcal{F}\{\delta(t)\} + \mathcal{F}\,\delta(t-1)$$

$$\therefore \mathcal{F}\left\{\frac{d^2x}{dt^2}\right\} = e^{j\omega} - 2 + e^{-j\omega}$$

$$\mathcal{F}\left\{\frac{d^2x}{dt^2}\right\} = \left(e^{j\omega} + e^{-j\omega}\right) - 2$$

$$= 2\cos\omega - 2$$

$$= 2\left(1 - 2\sin^2\frac{\omega}{2} - 1\right).$$

$$(j\omega)^2\,\mathcal{F}\{x(t)\} = -4\sin^2\frac{\omega}{2}$$

$$\therefore -\omega^2\,\mathcal{F}\{x(t)\} = -4\sin^2\frac{\omega}{2}$$

$$\therefore \mathcal{F}\{x(t)\} = \frac{4}{\omega^2}\sin^2\frac{\omega}{2} = \left(\frac{\sin\frac{\omega}{2}}{\omega/2}\right)^2$$

Also note

$$j\omega\,\mathcal{F}\{x_1(t)\} = j\omega\mathcal{F}\left(\frac{dx}{dt}\right) = \left(\frac{\sin\omega/2}{\omega/2}\right)^2$$

$$\therefore \mathcal{F}\left\{\frac{dx}{dt}\right\} = \frac{1}{j\omega}\left(\frac{\sin\omega/2}{\omega/2}\right)^2 = \mathcal{F}\{x_1(t)\}.$$

Area under $x(t)$

and Area under $X(\omega)$

Now $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

put $\omega = 0$ on both sides.

$$\therefore X(0) = \int_{-\infty}^{\infty} x(t)\,dt = \text{Area under } x(t).$$

Also $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

put $t = 0$ on both sides:-

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega$$

$$\therefore \int_{-\infty}^{\infty} X(\omega)\, d\omega = 2\pi\, x(0) = \text{area under the } f^n\ X(\omega).$$