System Properties

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1 Introduction

For a general system as shown in figure 1, the output $y[n]$ of a system for a given arbitrary input sequence $x[n]$ can be written as:

$$y[n] = \mathbf{T}\{x[n]\}$$

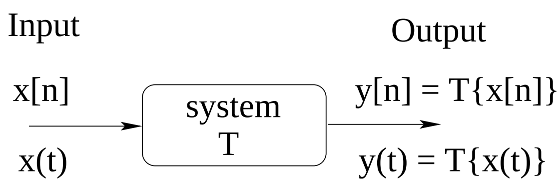

Figure 1: General System

A system when excited by an input signal $x(t)$ or $x[n]$, transforms (or maps) it to an output signal $y(t)$ or $y[n]$ as depicted in figure 1 . A system may be categorized as (i) a *memory* or *memory less* system, (ii) a *causal* or *non-causal* or *anti causal* system, (iii) a *stable* or *unstable* system system, (iv) a *linear* or *nonlinear* system.

1.1 System with memory or memory less

A system is said to be memory less if for every t_o and for every $x(t)$, output at $t = t_o$ i.e, $y(t_o)$ is decided only by the input at $t = t_o$ i.e, $x(t_o)$. However, if the output depends on past (or future!) values the system is said to have memory. A resistor is a memory less system while an inductor and capacitor belongs to system having memory. Presence of an energy storing element in a system make it a system with memory.

Examples of Memory less system

$$\begin{aligned} \text{Ex.1 } y(t) &= x^3(t) \\ \text{Ex.2 } y(t) &= \sin[x(t)] \\ \text{Ex.3 } y(t) &= x(t) + 3^{x(t)} \end{aligned}$$

Examples of systems with memory

$$\begin{aligned} \text{Ex.1 } y(t) &= x(t+3) \\ \text{Ex.2 } y(t) &= x(t-2) + x(t+2) \\ \text{Ex.3 } y(t) &= x^2(t+1) \end{aligned}$$

1.2 Causal system

In general , when the system output depends on past or present or both past & present values of input, the system is said to causal - other wise it is a non-causal system. In other words system can not anticipate what the future input will be. In real life we can realize or design only causal systems. Always remember we live in a *causal world*. Sometimes it is straight forward to declare whether the given system is causal or not by simply looking at mathematical relation between the output and the input. For example, if the output of a system is $y(t) = x(t + 1)$ when the input is $x(t)$. We can easily see that to get the current value of the output we require the future input value.

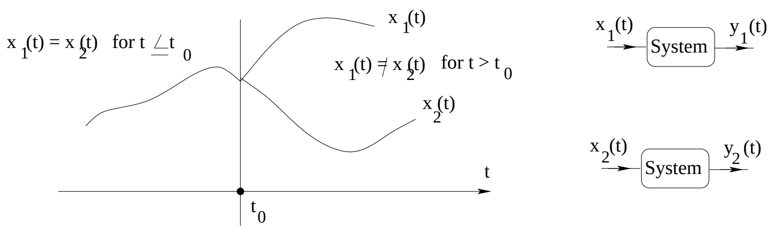

Figure 2: Testing causality

To check formally for causality, excite the given system separately by two signals $x_1(t)$ and $x_2(t)$ as depicted in figure 2. These input signals have the following description.

$$\begin{aligned} x_1(t) &= x_2(t) \text{ for } t \leq t_0 \\ \text{and } x_1(t) &\neq x_2(t) \text{ for } t > t_0 \end{aligned}$$

Suppose we are examining the output of the two cases: one with input $x_1(t)$ and then with input $x_2(t)$. If the output does not depend on future inputs we must expect $y_1(t) = y_2(t)$ for $t \leq t_0$. Of course $y_1(t)$ and $y_2(t)$ may differ for $t > t_0$. So causality demands that $y_1(t) = y_2(t)$ for $t \leq t_0$ and this must be true for all arbitrary chosen value t_0. If we find $y_1(t) \neq y_2(t)$ for $t \leq t_0$, conclusion will be the system is non-causal.

1.3 Linear Systems

To check whether a system is linear or not, follow the following steps.

1. If zero input i.e., $x(t) = 0$ produces non-zero output, the system is not linear. However if zero input produces zero output, the system may be linear. This is called homogeneity condition.

2. For a linear system if input is scaled by some factor α, output too will be scaled up by same factor i.e., $x(t) \rightarrow y(t)$ then $\alpha x(t) \rightarrow \alpha y(t)$.

3. A linear system must satisfy superposition theorem i.e., if $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$ then $x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$.

1.4 Time Invariant System

For a system to be time invariant if $x(t)$ produces $y(t)$ then if input is delayed by a, output should will be delayed also by same amount a; ie., $x(t-a)$ will produce $y(t-a)$.

To formally check for time invariance, refer to figure 3. First apply $x(t)$ to the given system and get $y(t)$ and then delay this output by an amount a to get $y(t-a)$ as shown. Now to the same system apply a delayed version of $x(t)$ i.e, $x(t-a)$ with output $y_1(t)$. Finally compare $y_1(t)$ with $y(t-a)$. If these two are same the system is time invariant.

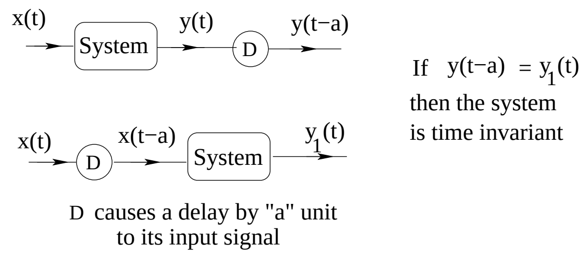

Figure 3: Testing Time invariance

1.5 Invertible System

Suppose a system produces $y(t)$ when the input is $x(t)$ i.e., $x(t) \rightarrow y(t)$. Suppose we now ask ourselves: can we find out another system whose input will be the output $y(t)$ of the first system and its output will be $x(t)$ i.e., $y(t) \rightarrow x(t)$? If second such system exists we say the system is invertible otherwise non-invertible.

1.6 Stable & unstable Systems

If the output $y(t)$ of a system remains bounded for a bounded input signal $x(t)$, the system is said to be stable. In short this is known as BIBO (Bounded Input Bounded Output) criteria. Mathematically Bounded Signal means:

1. A bounded input signal must satisfy : $|x(t)|$ for all t, is finite or less than ∞.
2. A bounded output signal must satisfy : $|y(t)|$ for all t, is finite or less than ∞.

2 Impulse response & its importance

If a system is excited by an impulse $\delta(t)$, the output (response) is called impulse response of the system and denoted by $h(t)$. We have seen that for a **LTI system** if $h(t)$ is known, then the output $y(t)$ of a system can be found out for any arbitrary input signal $x(t)$ as shown in the following equation.

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

The above integration operation involving $x(t)$ and $h(t)$ is more popularly known as convoluting the two signals or convolution between $x(t)$ and $h(t)$.

Apart from getting output response of a system for any input signal $x(t)$, from the nature impulse response $h(t)$, it is possible to conclude whether the system is (i) memory less, (ii) time invariant, (iii) invertible and (iv)stable.

2.1 $h(t)$ and memory less system

We know for a memory less system, present output will depend only on the value of present input i.e., $y(t) = kx(t)$ where k is a constant. Let $h(t)$ be the impulse response of the system which is memory less.

$$\begin{aligned}
\text{Now, output } y(t) &= kx(t) \\
\text{Also in terms of } h(t),\ y(t) &= \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
\therefore \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau &= kx(t) \\
\text{But we also know, } \int_{\tau=-\infty}^{\infty} \delta(\tau)x(t-\tau)d\tau &= x(t)
\end{aligned}$$

Comparing the last two equation above, we conclude that $h(t) = k\delta(t)$. Thus If the impulse response $h(t)$ of a system is an impulse, the system must be memory less.

2.2 $h(t)$ and Causal system

Recall that for a causal system, output $y(t)$ should depend on past and present value of $x(t)$. In other words, $y(t)$ should not depend future values of input i.e., $x(t+1)$, $x(t+2)$ etc. Let $h(t)$ be the impulse response of the system which is causal. Now output$y(t)$, for any input signal $x(t)$ is given by:

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Note in the above convolution integral, τ is the variable and t can be treated as constant. For example if you want to know $y(5)$, then,

$$y(5) = \int_{\tau=-\infty}^{\infty} h(\tau)x(5-\tau)d\tau$$

In the above general expression for $y(t)$, $x(t)$ require all the past, present and future values of $x(t)$ for computing the RHS integral. In the range $-\infty < \tau < 0$, all future values of input, $x(t+1)$, $x(t+2)$ etc will be required. Since future values of input should not decide the output of a **causal system**, we will demand the system to have an impulse response $h(t) = 0$ for $t < 0$. Also note present and past values of input will be required to evaluate the integral form $t = 0$ to $t = \infty$. So the conclusion is that if impulse response $h(t) = 0$ for $t < 0$, the system is causal and if $h(t) \neq 0$ for $t < 0$, the system must be causal.

2.3 $h(t)$ and Stable System

Recall that:
If the output $y(t)$ of a system remains bounded for a bounded input signal $x(t)$, the system is said to be stable. In short this is known as BIBO (Bounded Input Bounded Output) criteria. Mathematically Bounded Signal means:

1. A bounded input signal must satisfy : $|x(t)|$ for all t, is finite or less than ∞.
2. A bounded output signal must satisfy : $|y(t)|$ for all t, is finite or less than ∞.

Assume $x(t)$ to be bounded i.e., $|x(t)| =$ finite $< \infty$. In the light of impulse response we know:

$$\begin{aligned}
y(t) &= \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
\text{To check for bounded output} &\quad \text{take modulus of both sides} \\
|y(t)| &\leq \int_{\tau=-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau \\
\text{or, } |y(t)| &\leq \int_{\tau=-\infty}^{\infty} |h(\tau)|d\tau \\
\text{Since } x(t) \text{ is bounded} &\quad x(t-\tau) \text{ too will be bounded} \\
\therefore \int_{\tau=-\infty}^{\infty} |h(\tau)|d\tau &= \text{must be finite or } < \infty
\end{aligned}$$

2.4 $h(t)$ and Invertible System

Suppose a system produces $y(t)$ when the input is $x(t)$ i.e., $x(t) \rightarrow y(t)$. Suppose we now ask ourselves: can we find out another system whose input will be the output $y(t)$ of the first system and its output will be $x(t)$ i.e., $y(t) \rightarrow x(t)$? If second such system exists we say the system is invertible otherwise non-invertible. This is pictorially demonstrated in figure 4(a) Let us assume

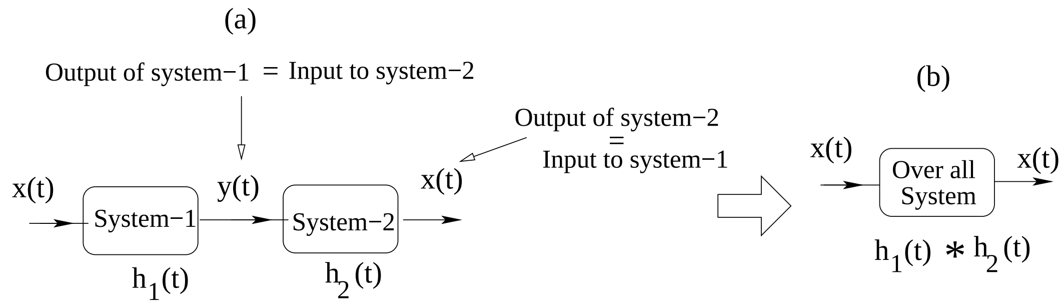

Figure 4: Testing Invertibility

that the impulse response of the first and second systems be respectively $h_1(t)$ and $h_2(t)$ Let us try to understand what should be the nature of $h_1(t)$ and $h_2(t)$, so that the system will be invertible. Referring to figure 4(a), we can write the following:

$$\begin{aligned} y(t) &= x(t) * h_1(t) \text{ for the system-1} \\ \text{and } x(t) &= y(t) * h_1(t) \text{ for the system-2} \\ \text{or } x(t) &= [x(t) * h_1(t)] * h_1(t) \\ \text{or } x(t) &= x(t) * [h_1(t) * h_1(t)] \end{aligned}$$

Which suggest that the impulse response $h(t)$, of the over all system is $h_1(t) * h_1(t)$. Now,

$$\begin{aligned} x(t) &= x(t) * [h_1(t) * h_2(t)] \\ \text{we also know } x(t) &= x(t) * \delta(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \\ \text{Comparing the above two equation, we conclude } h_1(t) * h_1(t) &= \delta(t) \end{aligned}$$

Finally we now know if the two systems (with impulse responses $h_1(t)$ and $h_2(t)$) are to be invertible then $h_1(t) * h_1(t) = \delta(t)$