1. Let A and B be uncountable sets with $A \subseteq B$. Prove or disprove: A and B are equinumerous.

$$B = 2^{\mathbb{R}} \leftarrow \text{larger than } \mathbb{R}$$

$$A = \{ \{x\} \mid x \in \mathbb{R} \} \leftarrow \text{equinumerous with } \mathbb{R}$$

2. Let A be an uncountable set and B a countably infinite subset of A. Prove/Disprove: A is equinumerous with A−B.

<u>True</u>

$|A-B| \leq |A|$ ✓

$|A| \overset{?}{\leq} |A-B|$

$A-B$ uncountable

$\Rightarrow A-B$ is infinite.

$f : A \to A-B$ injective

$C \subseteq A-B$

countable

$B = \{b_1, b_2, b_3, \ldots\}$

$C = \{c_1, c_2, c_3, \ldots\}$

$$f(a) = \begin{cases} c_{2n-1} & \text{if } a = c_n \\ c_{2n} & \text{if } a = b_n \\ a & \text{otherwise} \end{cases}$$

f is a bijection.

3. Prove that the real interval $[0,1)$ is equinumerous with the unit square $[0,1) \times [0,1)$.

$$\mathbb{F} = \mathbb{Q} \cap [0,1)$$

$$A = [0,1) - \mathbb{F}$$

$$B = [0,1) \times [0,1) - \mathbb{F}^2$$

$$f : B \to A$$

$$(0.a_1a_2a_3\cdots,\ 0.b_1b_2b_3\cdots)$$

$$\mapsto (0.a_1b_1a_2b_2a_3b_3\cdots)$$

4. Define a relation ~ on IR such that a ~ b if and only if a - b ∈ Q.

(a) Prove that ~ is an equivalence relation.

(b) Is the set IR/~ of all equivalence classes of ~ countable?

$$[x] = \{x + a \mid a \in \mathbb{Q}\}$$

equinumerous with $\mathbb{Q}$

countable

uncountable

5. Let Z[x] denote the set of all univariate polynomials with integer coefficients. Prove that Z[x] is countable.

$\deg(f) = d$

d constant

$a_0 + a_1 x + \cdots + a_d x^d$

countable

$$\mathbb{Z}[x] = \left(\bigcup_{d \geq 0} \{ f \mid \deg(f) = d \} \right) \cup \{0\}$$

6. A real or complex number a is called algebraic if f(a) = 0 for some non-zero f(x) ϵ Z[x]. Let A denote the set of all algebraic numbers. Prove that A is countable.

$\deg(f) = d \Rightarrow f$ can have at most d roots.

A = a countable union of finite sets

7. Let Z[x,y] be the set of all bivariate polynomials with integer coefficients.

(a) Prove that Z[x,y] is countable. similar to $\mathbb{Z}[x]$

(b) Let

V = {(a,b) ϵ C × C | f(a,b) = 0 for some nonzero f(x,y) ϵ Z[x,y]}

Is V countable?

$x - y$

V contains the uncountable subset

$\{(a,a) \mid a \in \mathbb{C}\}$

8. A set $S \subseteq \mathbb{R}$ is called bounded if S has both a lower bound and an upper bound. Countable/Uncountable?

(a) The set of all bounded subsets of Z.
(b) The set of all bounded subsets of Q.

Let A be a bounded set. Let l be a lower bound, and u an upper bound. Let $B = [l, u] \bigcap S$. A can be any subset of B. For $S = Z$, B is finite. For $S = Q$, B is countable (and infinite if $l < b$).