Prove that every unhappy number ends up in the loop containing 4.

By induction on $n \geqslant 1$, prove that either n reduces to 1 or n enters the loop involving 4.

$$\boxed{n \longrightarrow \text{sosd}(n)}$$

Show this for $1 \leq n \leq 99$.

For $n \geqslant 100$, $\text{sosd}(n) < n$.

$n = (abc)_{10}$

$$n - \text{sosd}(n)$$

$$= 100a + 10b + c - a^2 - b^2 - c^2$$

$$= (100 - a)a + (10 - b)b + (c - c^2)$$

$$\geqslant (100 - 1) \times 1 \qquad \geqslant 0 \qquad -72$$

$$= 99$$

Clarification about the iterated logarithm function

$$n \to \log n \to \log\log n \to \log\log\log n \to \dots \to \leq 1$$

$\log^* n \to$ # times log is taken

1 $\quad \log^* 1 = 0$

$2 \to 1$ $\quad \log^* 2 = 1$

$3 \to 1 \dots \to 0 \dots$ $\quad \log^* 3 = 2$

$4 \to 2 \to 1$ $\quad \log^* 4 = 2$

$5 - 15$ $\quad 3$

$16 \to 4 \to 2 \to 1$ $\quad \log^* 16 = 3$

$17 - (2^{16} - 1)$ $\quad 4$

$2^{16} \to 16 \to 4 \to 2 \to 1$ $\quad \log^* 2^{16} = 4$

$2^{16} + 1$ ---- 5

$2^{2^{16}}$ ---- 5

$2^{2^{16}} + 1$ ---- 6

1. Consider the language

L = {a^n b^n c^n | n ≥ 0}

over the alphabet {a,b,c}. Characterize all the strings in the complement ~L.

$$\{a^n b^n c^n \mid n \geq 0\} \subseteq \{a,b,c\}^*$$

$$\sim L = \overline{L} = \{a^i b^j c^k \mid i \neq j \text{ or } j \neq k,\ i,j,k \geq 0\}$$

$$\cup \{\text{strings containing } ba\}$$

$$\cup \{\text{strings containing } cb\}$$

$$\cup \{\text{strings containing } ca\}$$

2. Give an example of two infinite languages A and B over the alphabet {a,b} such that $A \cap B = \varnothing$ and $AB = BA$, or prove that no such languages exist.

$$A = \{\alpha \in \{a,b\}^* \mid |\alpha| \text{ is even}\}$$

$$B = \{\alpha \in \{a,b\}^* \mid |\alpha| \text{ is odd}\}$$

$$\boxed{AB = BA = B}$$

3. Give an example of a language L (over any alphabet of your choice) such that

$$\sim(L^*) = (\sim L)^*,$$

or prove that no such language can exist.

For any language A, we have $\varepsilon \in A^*$

(Even if $A = \emptyset$)

L^* contains ε $\Rightarrow$ $\sim(L^*)$ does not contain ε

$(\sim L)^*$ contain ε

Can never be equal

4. For two languages A and B over the same alphabet, define the language

$$A / B = \{x \mid xy \in A \text{ for some } y \in B\}.$$

Give examples of infinite languages A and B over the alphabet $\{a,b\}$ such that:

(a) $A / B = A$

$$\left.\begin{array}{l} A = B = \Sigma^* \\ A / B = \Sigma^* \end{array}\right\} \text{any } \Sigma$$

$A = \{ab, b\}$

$B = \{b\}$

$A / B = \{a, \varepsilon\}$

(b) $A / B \neq A$

$\Sigma = \{a, b\}$

$A = \{a^n \mid n \geq 0\}$, $B = \{b^n \mid n > 0\}$

$A / B = \emptyset \neq A$.

5. Let A be a language over the alphabet $\{a,b\}$. Define the language

$$B = \{xy \mid xay \in A\}.$$

If A is non-empty and not contained in $\{a\}^*$, prove that $B \neq A$.

X Not correct

Case 1 : $A \subseteq \{b\}^* \Rightarrow B = \emptyset \neq A$

Case 2 : A contains strings containing a.

Take a string $w \in A$ s.t. $\#_a(w)$ is +ve but as small as possible.

$A = \{ba^n \mid n \geq 0\}$ $\boxed{B = A}$

6. Let B be a language over some alphabet. We call B transitive if $BB \subseteq B$. We call B reflexive if $\varepsilon \in B$. Prove that for any language A, the smallest transitive and reflexive language containing A is A^*.

- A^* is reflexive, transitive, and contains A.

$$A^* A^* \overset{\subseteq}{=} A^*$$

- $A \subseteq B$ and B is reflexive and transitive, then $\boxed{A^* \subseteq B}$.

• $A \subseteq B \Rightarrow A^n \subseteq B^n$ for all $n \geq 0 \Rightarrow A^* \subseteq B^*$

• B is ref and tran $\Rightarrow B^* = B$.

$BB \subseteq B$, $BB \supseteq B$ $\Rightarrow B^2 = B$

Ind: $B^n = B \ \forall n \geq 1$

$\Rightarrow \boxed{A^* \subseteq B}$