LTI Systems: Eigen functions and Frequency Response

Introduction: While studying SYSTEM SCIENCE one of our main objectives is to be able to predict the output of a system given any arbitrary input. We have seen previously that if we know the response (output) of an LTI (Linear Time-Invariant) system to the unit impulse ($\delta(t)$) excitation (input) then it is possible to compute the response (output) for any arbitrary input using the following formula (convolution)

$$y(t) = x(t) * h(t)$$

where $h(t)$ = unit impulse response.
$x(t)$ = any arbitrary input
$y(t)$ = the response to input $x(t)$

and $*$ denotes convolution.

So knowing $h(t)$ is equivalent to know all about an LTI system.

This above approach of analysing an LTI system is sometimes referred to as the time domain analysis [possibly because $x(t)$, $y(t)$ and $h(t)$ all are represented as functions of time in the equation $y(t)=x(t)*h(t)$]

In this note we shall discover a new approach of looking at the LTI systems (or analysing the LTI systems) which is called <u>frequency domain analysis</u>, the reason of such a name will be ~~cleared~~ clear soon.

Before that we will divert a bit and recall eigen values and eigen vectors of a matrix

EIGEN VECTORS & EIGEN VALUES OF A MATRIX

Suppose A is a 3×3 matrix.
If we multiply a 3×1 vector V with A, we get another 3×1 Vector as

$$\underset{(3\times 1)}{U} = \underset{(3\times 3)}{A}\,\underset{(3\times 1)}{V}$$

We can think as if A is transforming a physical vector to give another physical vector

For example consider

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

If we take any vector V in X-Y plane

and multiply with A then the result will be rotated by $90°$. Example: If $V = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ then $AV = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. So the vector V which was along X-axis produces a vector along Y-axis.

However if we take vector along Z axis as $V = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ then $AV = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix}$ which is also along the Z axis.

So we observe some vectors when multiplied by A gets their direction changed, but some vector keeps their direction unchanged after being multiplied by A (the magnitude or length may or may not change — we will not ~~consid.~~ bother about the magnitude in this context. The direction of a vector is important (for whatever reason) to us)

Vectors that keep their direction unchanged when multiplied by A are called the Eigen Vectors of A. [According to Google 'Eigen' is a Dutch word which means 'own'. Here Eigen vectors are somewhat special to the matrix A, since A changes the direction of all other vectors but allows its 'own' eigen vectors to retain their 'own' direction]

If we think V as the <u>input</u> to the matrix multiplication and $U=AV$ as the <u>output</u> of the multiplication, then eigen vectors are those <u>inputs</u> which retain their 'own' directions unchanged after the multiplication/ transformation. The magnitude / length of the output may get magnified / amplified though. And this factor of amplification is called the eigen value corresponding to that eigen vector.

<u>Example</u> for $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 10 \end{bmatrix}$ any vector along the Z-direction is an eigen vector and the corresponding eigen value is 10.

EIGEN FUNCTIONS OF A SYSTEM

With analogy to the Eigen Vectors of a matrix, we can think of Eigen functions of a system.

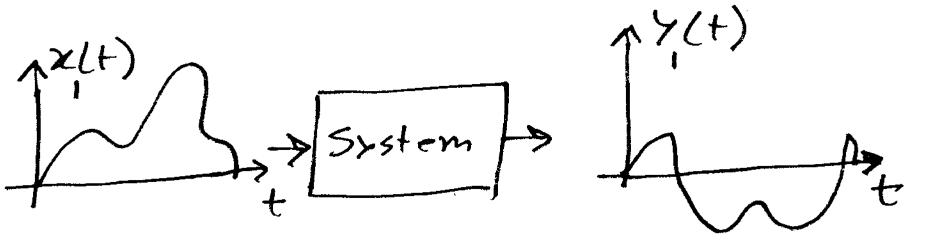


Figure 1

In figure 1, we see that the shape of the output function is not similar to the shape of the input function.

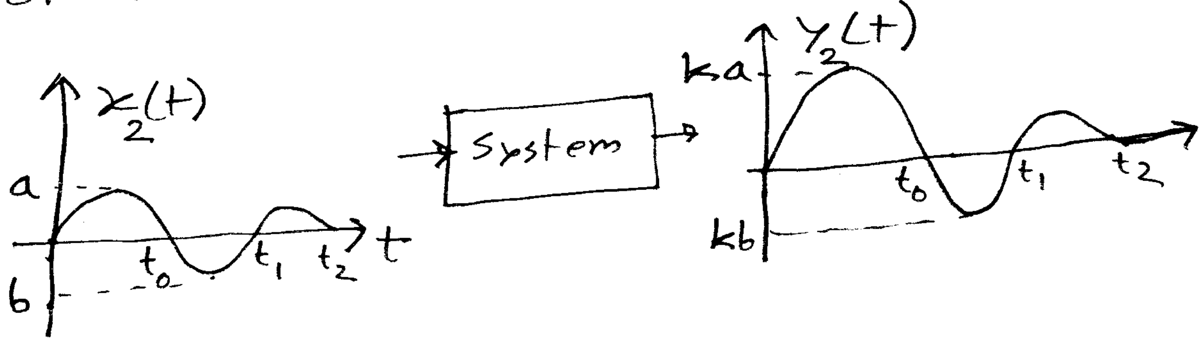


Figure 2

In figure 2, we see that the shape of the output is identical to the shape of the input. Except that the amplitude is increased / magnified by a factor of k at the output.

We will call $x_2(t)$ as an eigen function of this system. and K is the corresponding eigen value.

Examples

Consider a system which is described with a linear differential equation as

$$3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y(t) = x(t)$$

The system is linear and time invariant (please check yourselves)

Which of the following inputs eigen functions for this system.

(i) $x(t) = \delta(t)$

(ii) $x(t) = u(t)$

(iii) $x(t) = e^{\sigma t}$

(iv) $x(t) = e^{-\sigma t}$

(V) $x(t) = e^{Jwt}$

(vi) $x(t) = e^{-Jwt}$

(VII) $x(t) = \sin(wt)$

(viii) $x(t) = \cos(wt)$

Solution

(i) $x(t) = \delta(t)$

Assume that $\delta(t)$ is an eigen function. then for input $x(t) = \delta(t)$, output must be of the form $y(t) = k\,\delta(t)$ for some value of k. Now let us put $x(t) = \delta(t)$ and $y(t) = k\,\delta(t)$ in the system equation.

$$3k\frac{d^2}{dt^2}\delta(t) + 2k\frac{d}{dt}\delta(t) + k\,\delta(t) = \delta(t)$$

This equation can never be satisfied for any value of k. Since the LHS contains $\dot{\delta}(t)$, $\ddot{\delta}(t)$ which are not present on the right side.

Therefore, $\delta(t)$ is NOT an Eigen function

(ii) Similarly we can show that $x(t) = u(t)$ is also not an eigen function of the given system.

(iii) $x(t) = e^{\sigma t}$

Assume $x(t) = e^{\sigma t}$ is an eigen function.

Therefore when input $= x(t) = e^{\sigma t}$

output should be of the form $y(t) = k e^{\sigma t}$

Let's put these values in the system equation. We will get

$$3k \frac{d^2}{dt^2} e^{\sigma t} + 2k \frac{d}{dt} e^{\sigma t} + k e^{\sigma t} = e^{\sigma t}$$

$$\Rightarrow (3k\sigma^2 + 2k\sigma + k) e^{\sigma t} = e^{\sigma t}$$

$$\Rightarrow \cancel{3}k(3\sigma^2 + 2\sigma + 1) = 1$$

$$\Rightarrow k = \frac{1}{3\sigma^2 + 2\sigma + 1}$$

This means if we take $k = \dfrac{1}{3\sigma^2+2\sigma+1}$, then $y(t) = k e^{\sigma t}$ is actually the solution to the given differential equation. In other words for input $= x(t) = e^{\sigma t}$ the ouput

is $y(t) = \frac{1}{3\sigma^2+2\sigma+1} e^{\sigma t}$

$\therefore$ $x(t) = e^{\sigma t}$ is an eigen function of the given system.

Note that the eigen value or the magnification k depends on the value of σ. But our analysis is true for any value of σ: positive, negative, real, imaginary, complex anything.

Since we have not assumed anything about the value of σ in our analysis.

(iv), (v), (vi) It is left as an assignment for the students to show that the inputs in (iv), (v) and (vi) are all eigen functions. You may adopt the method used in part (iii).

After having an idea about the eigen functions of a system, we are now ready to state ~~the~~ one of the most important ~~the~~ properties about the eigen veetors of LTI systems.

Any and All exponenttal functions are eigen functions of all LTI (Linear and Time invariant) systems

$\boxed{\text{Proof}}$ Consider a LTI system whose unit-impulse response function is given as $h(t)$.

If we apply any input $x(t)$ to the system then the output $y(t)$ is given by

$$y(t) = x(t) * h(t)$$

Assume the input $x(t)$ is an exponential function $x(t) = e^{st}$

where s can be anything : real, imaginary, complex, positive, negative, zero.

then
$$y(t) = e^{st} * h(t)$$
$$= \int_{\tau=-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$
$$= e^{st} \int_{\tau=-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$= e^{st} \mathcal{L}\{h(t)\}$

$= e^{st} H(s)$

where $H(s) = \mathcal{L}\{h(t)\}$ = Laplace transform of $h(t)$.

So, the output $y(t) = e^{st} H(s)$ is also of the the same (exponential) shape as the input. It is only magnified by a factor $H(s)$

$\therefore$ The exponential function e^{st} is an eigen function for the LTI system for any value of s, and the corresponding ~~value~~ eigen value is given by the value of the Laplace transform of $h(t)$

<u>To Reemphasize this property</u>

If $h(t)$ is the impulse response of an LTI system.

$H(s) = \mathcal{L}\{h(t)\}$

Then for any input $x(t) = e^{at}$, where 'a' is an arbitrary real/complex number, the output will be

$$y(t) = e^{at} \times H(s)\big|_{s=a}$$

COMPLEX FREQUENCY

Consider the function $x(t) = e^{st}$

Here t has the dimension of time (or second). (st) should be dimensionless [Because $e^{5\,gram}$ or $e^{3\,meter}$ or $e^{2.2\,second}$ etc makes no sense]

Therefore (s) should have the dimension of $(time)^{-1}$ (or 1/second or Hertz)

Thus, the unit of s is same as the unit of frequency. Therefore, in $x(t) = e^{st}$, s is called a frequency.

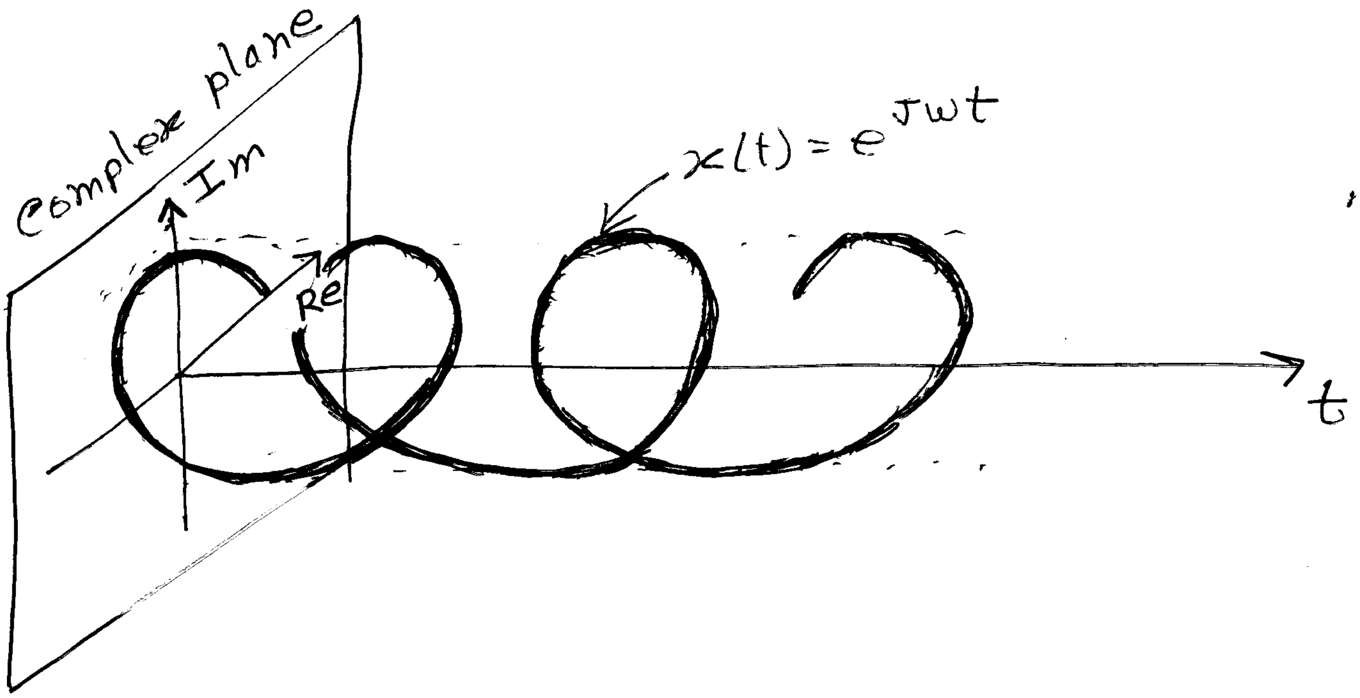
Here s can be real, imaginary or complex

Therefore we will call s as a complex frequency in general.

$\boxed{\text{Side note 1}}$: We are used to think frequency as a measure of how many times a function repeats a pattern per unit time. Complex frequency cannot be interpreted like that in general / always.

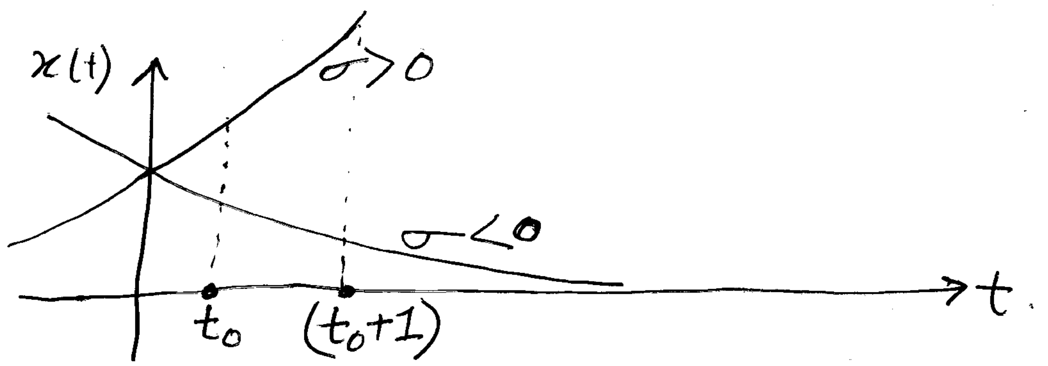
$\boxed{\text{Side note 2}}$: If in $x(t) = e^{st}$ s is purely imaginary i.e. $x(t) = e^{j\omega t}$

Then the function can be visualized as below



Here ω in $e^{j\omega t}$ is a measure of how many times the function rotates / revolves in the complex plane per unit time. ($f = \omega/2\pi$)

$\boxed{\text{Side note 3}}$ If in $x(t) = e^{st}$ s is real, i.e. $x(t) = e^{\sigma t}$ then the function can be visualized as below



If the value of the function at $t = t_o$ is $e^{\sigma t_o}$, then after 1 second the value will be

$$x(t_o+1) = e^{\sigma(t_o+1)} = e^{\sigma} x(t_o)$$

So no matter whatever is the value of ~~to~~ after 1 second the function is magnified by the factor of e^{σ}.

Also the property of the exponential function is that in a fixed interval of time, it is always ~~am~~ magnified by a fixed factor, like e^{σ} times in one second. So 'σ' here is a measure of how frequently the function is getting magnified.

WHAT IS THE USE OF ALL THESE

- As we have mentioned before, one of our main objectives is to predict the response of a system to any arbitrary input/excitation.
- The theory of eigen functions of an LTI system says that if the excitation is exponential then the response is also exponential with a (complex) magnitude gain.

• Also recall from our knowlege about Fourier / Laplace transform that ~~[illegible]~~ signals can be broken/decomposed into a set of exponential functions.

• The above two facts together provide us with a handy tool to compute the response of an LTI system to any (to be more precise, most of the practical) ~~signals~~ arbitrary excitation. For this we have to do the following —

(i) Get the Laplace transform of the impulse response $h(t)$ of the system. Call it $H(s)$. We will see that often we can find $H(s)$ of a system more easily than finding $h(t)$

(ii) $H(s)\big|_{s=J\omega} = H(J\omega)$ is the complex gain of the system while the input is of frequence $(J\omega)$ or $x(t) = e^{J\omega t}$.

~~(iii) Give~~ ~~Also note if the input~~

(iii) Given an arbitrary input $x(t)$, break it into exponential signals unsing Fourier transform.

(iv) For each component in the Fourier transform the gain can be found from $H(j\omega)$. Add (sum) the response to each component to get the total response.

Example:

RESPONSE OF AN LTI SYSTEM TO A SINUSOIDAL INPUT

Let the impulse response of the system be $h(t)$

Let the excitation be $x(t) = \cos(\omega t)$

$\cos(\omega t)$ can be broken into two exponentials as

$$x(t) = \cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

The response to $\frac{e^{j\omega t}}{2}$ will be

$$y_1(t) = \frac{e^{j\omega t}}{2} \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

similarly, response to $\frac{e^{-j\omega t}}{2}$ will be

$$y_2(t) = \frac{e^{-j\omega t}}{2} \int_{-\infty}^{\infty} h(t) e^{j\omega t} dt$$

Now since $h(t)$ must be real valued for all real system.

$$\int_{-\infty}^{\infty} h(t) e^{j\omega t} dt = \left[\int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \right]^*$$

where * denotes complex conjugate

So if $\int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt = M\angle\theta = Me^{j\theta}$

then $\int_{-\infty}^{\infty} h(t)\, e^{j\omega t}\, dt = M\angle{-\theta} = Me^{-j\theta}.$

$\therefore\ y_1 = \frac{e^{j\omega t}}{2} Me^{j\theta} = \frac{M}{2} e^{j(\omega t+\theta)}$

$y_2 = \frac{e^{-j\omega t}}{2} Me^{-j\theta}. = \frac{M}{2} e^{-j(\omega t+\theta)}$

$\therefore$ Total response $y = y_1 + y_2$

$= \frac{M}{2}\left(e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right)$

$= M \cos(\omega t+\theta)$

So if the input $x(t) = \cos(\omega t)$ then the o/p of an LTI system will be of the form $M\cos(\omega t+\theta)$. ~~where~~

So the input and output both will be sinusoidal (But it is not an eigen function since the output is not a constant time the input, but there is also a phase shift)

M is the amplification / gain / magnification. and θ is the phase shift (advancement)

Now we have already seen that

$$M\angle\theta = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt = H(s)\big|_{s=j\omega}$$

$$\therefore M = \left\| H(s)\big|_{s=j\omega} \right\|$$

$$\text{and } \theta = \angle H(s)\big|_{s=j\omega}$$

So in this example we see that the response ~~to sinusoid~~ of an LTI system to sinusoidal inputs is also sinusoidal with a magnification and phase shift.

TRANSFER FUNCTION

The Laplace transform of the impulse response $h(t)$ of an LTI system i.e. $H(s)$ is called / named as the transfer function ~~of~~ / system function of the system.

$H(s)$ plays a central role in the study of LTI systems. As we have seen that using $H(s)$ we can directly predict the output of the system to any exponential input, i.e. the eigen functions

MAGNITUDE AND PHASE RESPONSE PLOTS FROM POLE ZERO DIAGRAM

Finally, in this note, we shall learn how to find the magnitude gain and phase shift of cetrain class of LTI systems and how to represent that in the form of graphs.

We will consider those LTI system which can be described using a Linear differential equation as

$$a_n \frac{d^n}{dt^n} y + - a_2 \frac{d^2}{dt^2} y + a_1 \frac{dy}{dt} + a_0 y(t)$$
$$= b_0 x(t) + b_1 \frac{d}{dt} x + \cdots b_m \frac{d^m}{dt^m} x$$

[Note: • During the initial lectures by TKB sir, we have learned to prove this type of systems as Linear and time invariant. However, so far we have seen only $x(t)$ on the RHS. But now we are generalizing the system to a wider class where the RHS can contain derivatives of $x(t)$ as well]

First of all we need to know the Laplace transform of the impulse response of the system. One approach to do that is to put $x(t) = \delta(t)$ with the system being initially relaxed or $y(0^-) = \dot{y}(0^-) = \ddot{y}(0^-) = \cdots = \frac{d^{n-1}}{dt^{n-1}} y(0^-) = 0$. Then solve the differential equation to find $y(t)$. By definition this $y(t)$ will be same as $h(t)$. Then take the Laplace transform of $h(t)$.

However, there is a short cut to get $H(s)$ without explicitely computing $h(t)$. [we have mentioned earlier that often it is easier to find $H(s)$ than $h(t)$]. Here is the trick:

We know when $x(t) = \delta(t)$ and the system is initially relaxed, then $y(t) = h(t)$ (by definition)

So $\left(i/p = \delta(t),\ o/p = h(t)\right)$ pair must satisfy the system equation.

$$a_n \frac{d^n}{dt^n} h(t) + \cdots a_1 \frac{d}{dt} h(t) + a_0 h(t) = b_0 \delta(t) + b_1 \dot{\delta}(t) + \cdots + b_m \frac{d^m}{dt^m} \delta(t)$$

Now taking Laplace transform on both the sides (with initial conditions being zero) we have

$$a_n s^n H(s) + \cdots a_1 s H(s) + a_0 H(s)$$
$$= b_0 + b_1 s + \cdots b_m s^m$$

$$\Rightarrow (a_n s^n + \cdots a_1 s + a_0) H(s) = (b_m s^m + \cdots b_1 s + b_0)$$

$$\therefore H(s) = \frac{b_m s^m + \cdots \cdots + b_1 s + b_0}{a_n s^n + \cdots\cdots + a_1 s + a_0}$$

So in this case, when the system can be described with a Linear differential equation, the transfer function $H(s)$ can be written as the ratio of two polynomials of s.

For Your Information: The degree of the denominator polynomial is called the order of a system.

So we see that it is possible to find $H(s)$ directly even without finding $h(t)$

Next we will see how to get magnitude and phase response at different frequencies from $H(s)$

$$H(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

Using the fundamental theorem of algebra we can now factor the numerator & the denominator polynomials as

$$H(s) = k\,\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

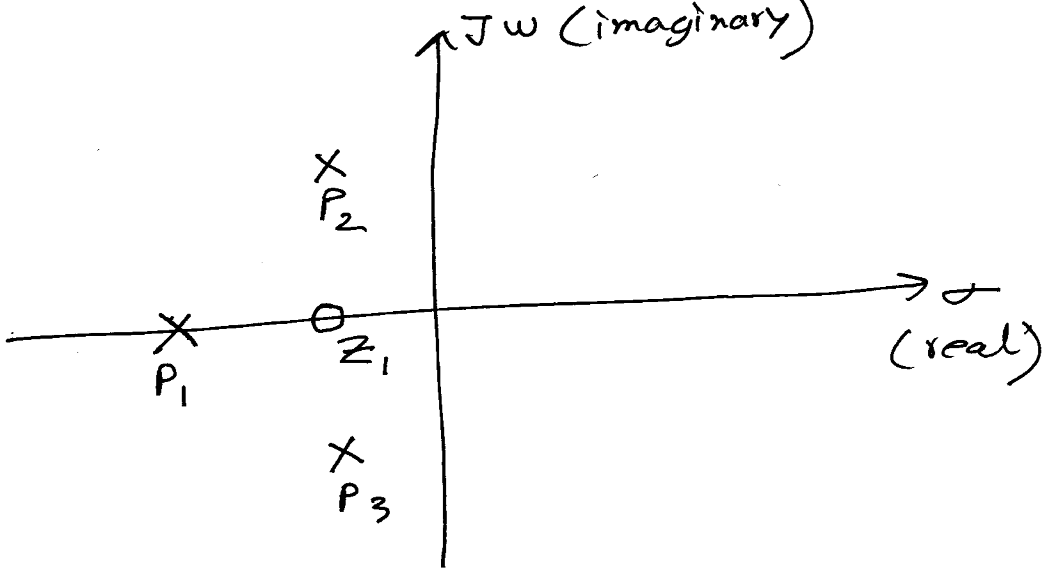
where $z_1, z_2, \cdots z_m$ are the roots of the numerator polynomials and people call them as 'zeros'. $p_1, p_2 \cdots p_n$ are the roots of the denominator polynomial & are known as poles.

Note that $z_1, z_2 \cdots z_m$ can be real, or complex. But complex zeros must come with its conjugate pair. That means if $z_i = a + jb$ for some i, then there must be $z_k = a - jb$ for some k.

This is because, the product $(s-z_1)(s-z_2)\cdots(s-z_m)$ must have only real coefficients, because $b_0, b_1, \cdots b_m$ are all real for a real system.

Similarly the poles can be real or complex, but complex poles must come with its conjugate pair.

Now we will plot the location of the zeros (z_1 --- z_m) & poles (P_1 --- P_n) in the complex plane as below.



where poles are denoted as X and zeros as O. Our final task is to learn how to get magnitude and phase response from this pole zero diagram.

We have discussed this in the class. If you have missed the class watch the lecture by Prof. Freeman starting from 40:42 min till 50:42 min. It is difficult to put down. ☺