Queries and Explanations

Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, x, y, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$ (where x, y are the 24th, 25th lowercase letters of the alphabet and do not represent anything else, such as 3, 5, or $\{1, 2\}$). Then $|\mathcal{U}| = 11$.

a) If $A = \{1, 2, 3, 4\}$, then $|A| = 4$ and here we have

i) $A \subseteq \mathcal{U}$;	**ii)** $A \subset \mathcal{U}$;	**iii)** $A \in \mathcal{U}$;
iv) $\{A\} \subseteq \mathcal{U}$;	**v)** $\{A\} \subset \mathcal{U}$; but	**vi)** $\{A\} \notin \mathcal{U}$.

$\boxed{\{\{1, 2, 3, 4\}\}}$

How can we claim that, $\mathcal{A} \in \mathcal{P}(\mathcal{A})$?

$A = \{1, 2\}$ $\quad P(A) = \{\phi, \{1\}, \{2\}, \overbrace{\{1, 2\}}^{A}\}$

$\underline{A \in P(A)}$

Example Relation: $\rho = \{(x, y) \mid y = x + 1 \ and \ x, y \in \mathbb{Z}\}$

NOT Reflexive, NOT Symmetric, NOT Transitive, BUT Anti-symmetric ?

✓

(x,y) & (y,x) $\Rightarrow x = y$

$\left.\begin{array}{l} x = y+1 \\ y = x+1 \end{array}\right\} x = y$

$\left.\begin{array}{l} y = x+1 \\ x = y+1 \end{array}\right\}$ together ✗

Let $f : \mathcal{A} \to \mathcal{B}$, with $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$.

Now, if $\mathcal{A}_1 \subset \mathcal{A}_2$, then $f(\mathcal{A}_1) \subseteq f(\mathcal{A}_2)$ – Will the equality also hold?

✓ Yes

Index Set and Partitions

Index Set

Definition: Let $\mathcal{I} \neq \phi$ and $\forall i \in \mathcal{I}$, let $\mathcal{A}_i \subseteq \mathcal{U}$ (universal set). Then, $\mathcal{I}$ is called an *index set*, and each $i \in \mathcal{I}$ is an index.

Set Operations: (Union) $\bigcup_{i \in \mathcal{I}} \mathcal{A}_i = \{x \mid \exists i \in \mathcal{I},\ x \in \mathcal{A}_i\}$

(Intersection) $\bigcap_{i \in \mathcal{I}} \mathcal{A}_i = \{x \mid \forall i \in \mathcal{I},\ x \in \mathcal{A}_i\}$

Generalized DeMorgan's Law: $\overline{\bigcup_{i \in \mathcal{I}} \mathcal{A}_i} = \bigcap_{i \in \mathcal{I}} \overline{\mathcal{A}_i}$ and $\overline{\bigcap_{i \in \mathcal{I}} \mathcal{A}_i} = \bigcup_{i \in \mathcal{I}} \overline{\mathcal{A}_i}$

Partition of a Set

Definition: Let $\mathcal{S}$ be a non-empty set. A family of non-empty subsets, $\{\mathcal{S}_i \mid i \in \mathcal{I}\}$ ($\mathcal{I}$ being the index set) is said to form a partition of $\mathcal{S}$ if the following two condition holds:

- $\bigcup_{i \in \mathcal{I}} \mathcal{S}_i = \mathcal{S}$ (Complete Set Cover), and
- $\mathcal{S}_i \cap \mathcal{S}_j = \phi, \forall i, j \in \mathcal{I}$ and $i \neq j$ (Pairwise Disjoint).

Example: Let $\mathcal{Z}_0 = \{3m \mid m \text{ is an integer}\} = \{0, \pm 3, \pm 6, \ldots\}$,
$\mathcal{Z}_1 = \{3m + 1 \mid m \text{ is an integer}\} = \{\ldots, -8, -5, -2, +1, +4, +7, \ldots\}$
$\mathcal{Z}_2 = \{3m + 2 \mid m \text{ is an integer}\} = \{\ldots, -7, -4, -1, +2, +5, +8, \ldots\}$
Now, $\mathcal{Z}_0 \cup \mathcal{Z}_1 \cup \mathcal{Z}_2 = \mathbb{Z}$ and $\mathcal{Z}_0 \cap \mathcal{Z}_1 = \mathcal{Z}_1 \cap \mathcal{Z}_2 = \mathcal{Z}_2 \cap \mathcal{Z}_0 = \phi$

Definitions: Poset, Maximal/Minimal Elements and Greatest/Least Elements?

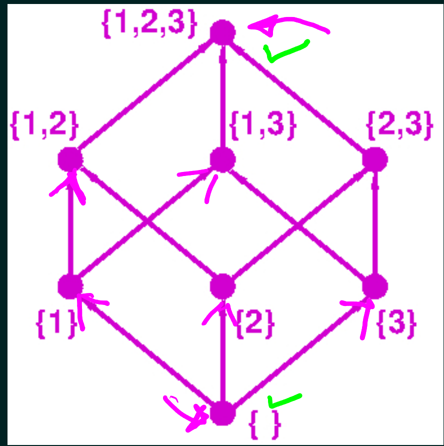
Example: $\mathcal{S} = \{1, 2, 3\}$ with (*i*) $(\mathcal{P}(\mathcal{S}), \subseteq)$, and (*ii*) $(\mathcal{P}(\mathcal{S}) - (\{\phi\} \cup \mathcal{S}), \subseteq)$

$\boxed{\text{Poset}}$

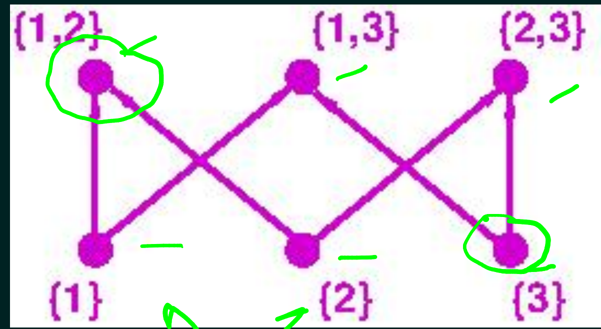
↳ Reflexive

↳ Antisym

↳ Transitive



cover



glb → 0

0 x

$A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

$\boxed{A \in P(A)}$ ✓

$\{A\} \subset P(A)$ ✓

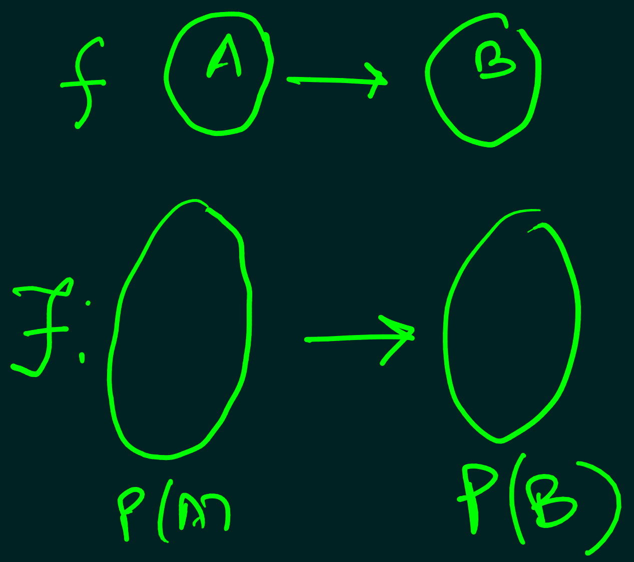
Tutorial Problems

Let $A, B, C \in \mathcal{U}$ are three arbitrary sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Prove that, $B = C$.

$$B = B \cap (A \cup B)$$
$$= B \cap (A \cup C) = (B \cap A) \cup (B \cap C)$$
$$= (A \cap C) \cup (B \cap C) = (A \cup B) \cap C$$
$$= (A \cup C) \cap C$$
$$= C$$

$$B = B \cup (A \cap B) = \cdots\cdots = C$$

For a function $f : A \to B$, define a function $\mathscr{F} : \mathscr{P}(A) \to \mathscr{P}(B)$ as $\mathscr{F}(S) = f(S)$ for all $S \subseteq A$. Prove that:
(a) $\mathscr{F}$ is injective if and only if f is injective. **(b)** $\mathscr{F}$ is surjective if and only if f is surjective.



(a) $\Leftarrow$ f is injective $\left\{ \begin{array}{l} A_1, A_2 \in A \\ f(A_1) = f(A_2) \\ \Rightarrow A_1 = A_2 \end{array} \right.$

$\mathcal{F}(S_1) = \mathcal{F}(S_2)$

$\Rightarrow f(S_1) = f(S_2)$

$\Rightarrow S_1 = S_2$

$\Rightarrow \mathcal{F}$ is injective

?

(b) $\underline{f}$ is onto) surjective

for any $b \in B$

we have $a \in A$, s.t. $f(a) = b$

any $y \in \mathcal{P}(B) \longrightarrow x \in \mathcal{P}(A)$ $\mathcal{F}(x) = y$?

$f(S) = \{ f(s) \mid s \in S \}$

Let $f : A \to B$ be a function and σ an equivalence relation on B. Define a relation ρ on A as: $a\,\rho\,a'$ if and only if $f(a)\,\sigma\,f(a')$.

(a) Prove that ρ is an equivalence relation on A.

(b) Define a map $\bar{f} : A/\rho \to B/\sigma$ as $[a]_\rho \mapsto [f(a)]_\sigma$. Prove that $\overline{f}$ is well-defined.

(a) Ref: $a \rho a$ iff $f(a) \sigma f(a)$ ✓

Sym: if $a \rho a'$ then $a' \rho a$?

$\leftrightarrow f(a) \underline{\sigma} f(a') \Rightarrow f(a') \sigma f(a)$

Tran: .. — . . . $a \underline{\rho a'}$ and $a' \underline{\rho} a'' \Rightarrow a \rho a''$

(b) $\underline{A}/\underline{\rho} = \{ [a_1]_\rho, [a_2]_\rho, [a_3]_\rho \dots \}$

x, y

$[x]_\rho = [y]_\rho$

$f : A \to B$ well-defined

$[a_1 = a_2 \quad , a_1, a_2 \in A$

? $[a_1]_\rho = [a_2]_\rho$ then

$[f(a_1)]_\sigma = [f(a_2)]_\sigma \Rightarrow f(a_1) = f(a_2)]$

$a_1 \rho a_2 \leftrightarrow f(a_1) \sigma f(a_2)$

Let $f : A \to B$ be a function and σ an equivalence relation on B. Define a relation ρ on A as: $a \,\rho\, a'$ if and only if $f(a) \,\sigma\, f(a')$. Define a map $\bar{f} : A/\rho \to B/\sigma$ as $[a]_\rho \mapsto [f(a)]_\sigma$.

(c) Prove that $\bar{f}$ is injective. ✓

(d) Prove or disprove: If f is a bijection, then so also is $\bar{f}$. ✓ Yes

(e) Prove or disprove: If $\bar{f}$ is a bijection, then so also is f. No

(c) $[f(a_1)]_\sigma = [f(a_2)]_\sigma \Rightarrow f(a_1) \,\sigma\, f(a_2) \Rightarrow a_1 \rho a_2$

$\Rightarrow [a_1]_\rho = [a_2]_\rho$

$\therefore \bar{f}$ is injective

(d) f is bijection any $b \in B \longrightarrow a \in A$ $f(a)=b$

$\Rightarrow \bar{f}$ is onto any $[b]_\sigma = [f(a)]_\sigma$

$\in B/\sigma \Rightarrow [a]_\rho \longmapsto [b]_\sigma$

(e) $A = \{x, y, z\}$ $B = \{1, 2\}$

$f(x) = f(y) = 1$ $f(z) = 2$

$\rho = \{(x,x), (y,y), (x,y), (y,x), (z,z)\}$

$\sigma = \{(1,1), (2,2)\}$

$A/\rho = \{[x,y], [z]\}$

$B/\sigma = \{[1], [2]\}$

$\bar{f}([x,y]) = 1$ $\bar{f}([z]) = 2$

Let ρ be a total order on A. We call ρ a *well-ordering* of A if every non-empty subset of A contains a least element. In this exercise, we plan to construct a well-ordering of $A = \mathbb{N} \times \mathbb{N}$.

(a) Define a relation ρ on A as $(a,b)\ \rho\ (c,d)$ if and only if $a \leqslant c$ or $b \leqslant d$.

(b) Define a relation σ on A as $(a,b)\ \sigma\ (c,d)$ if and only if $a \leqslant c$ and $b \leqslant d$.

(c) Define a relation $\leqslant_L$ on A as $(a,b) \leqslant_L (c,d)$ if either (i) $a < c$ or (ii) $a = c$ and $b \leqslant d$.

Prove or disprove: ρ, σ, $\leqslant_L$ is a well-ordering of A.

partial order ✓ → total order ✓

(a) Is this a P.O.? ✗

$(a,b)\ \rho\ (a,b)$ ✓ Ref.

$(1,2)\ \rho\ (2,1)$

$(2,1)\ \rho\ (1,2)$

Antisym ✗ $\Rightarrow (1,2) \neq (2,1)$

(b) - Is it a P.O.? ✓

Ref. ✓ Trans ✓

Antisym ✓

- Is it a T.O.? ✗

$(1,2) \not\sigma (2,1) \leftarrow (\mathbb{N} \times \mathbb{N}) = A$

$(\neg$ well-order$)$

(c) P.O.? ✓

T.O.? ✓

[Lexicographic order]

well ordering? $\rightarrow \{(x, b) \mid b \text{ any}\} \Rightarrow \{(x, y) \mid y \text{ least}\}$

least

least

[*Genesis of rational numbers*] Define a relation ρ on $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ as $(a,b)\ \rho\ (c,d)$ if and only if $ad = bc$. Prove that ρ is an equivalence relation. Argue that A/ρ is essentially the set $\mathbb{Q}$ of rational numbers. In abstract algebra, we say that $\mathbb{Q}$ is the *field of fractions* of the integral domain $\mathbb{Z}$.

$ad = bc \Rightarrow \frac{a}{b} = \frac{c}{d}$

$(a,b)\ \rho\ (a,b)$ Ref ✓

$(a,b)\ \rho\ (c,d) \Rightarrow (c,d)\ \rho\ (a,b)$ ✓

$ad = bc$ and $cf = ed$

$(c,d)\ \rho\ (e,f)$

$\frac{a}{b} = \frac{c}{d}$ $(a,b)\ \rho\ (e,f)$ $\frac{c}{d} = \frac{e}{f}$

Trans ✓

$\left[\frac{1}{2}\right] = \left[\frac{2}{4}\right] = \left[\frac{3}{6}\right] = \cdots$

$\left[\frac{1}{3}\right] = \left[\frac{2}{6}\right] = \left[\frac{3}{9}\right] = \cdots$

Rationals

$\left\{ \left[\frac{1}{2}\right], \left[\frac{1}{3}\right], \left[\frac{1}{4}\right], \cdots \right\}$

Let A be the set of all functions $\mathbb{N}_0 \to \mathbb{R}^+$.

(a) Define a relation Θ on A as $f \, \Theta \, g$ if and only if $f = \Theta(g)$. Prove that Θ is an equivalence relation.

(b) Define a relation O on A as $f \, \mathrm{O} \, g$ if and only if $f = \mathrm{O}(g)$. Argue that O is not a partial order.

— DO YOURSELF —

Let A be the set of all functions $\mathbb{N}_0 \to \mathbb{R}^+$.
Define a relation O on A/Θ as $[f]\ \mathrm{O}\ [g]$ if and only if $f = \mathrm{O}(g)$.

(c) Establish that the relation O is well-defined.

(d) Prove that O is a partial order on A/Θ.

(e) Prove or disprove: O is a total order on A/Θ.

(f) Prove or disprove: A/Θ is a lattice under O.

— DO YOURSELF —