Differential Operator Method of Finding A Particular Solution to An Ordinary Nonhomogeneous Linear Differential Equation with Constant Coefficients

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We systematically introduce the idea of applying differential operator method to find a particular solution of an ordinary nonhomogeneous linear differential equation with constant coefficients when the nonhomogeneous term is a polynomial function, exponential function, sine function, cosine function or any possible product of these functions. In particular, different from the differential operator method introduced in literature, we propose and highlight utilizing the definition of the inverse of differential operator to determine a particular solution. We suggest that this method should be introduced in textbooks and widely used for determining a particular solution of an ordinary nonhomogeneous linear differential equation with constant coefficients in parallel to the method of undetermined coefficients.

I. INTRODUCTION

When we determine a particular solution of an ordinary nonhomogeneous linear differential equation of constant coefficients with nonhomogeneous terms being a polynomial function, an exponential function, a sine function, a cosine function or any possible products of these functions, usually only the method of undetermined coefficients is introduced in undergraduate textbooks [1–3]. The basic idea of this method is first assuming the general form of a particular solution with the coefficients undetermined, based on the nonhomogeneous term, and then substituting the assumed solution to determine the coefficients. The calculation process of this method is very lengthy and time-consuming. In particular, in the case that the assumed particular solution duplicates a solution to the associated homogeneous equation, one must multiply a certain power function to erase the duplication, and students usually get confused and frustrated in applying this method to find a particular solution of the equation.

Nevertheless, differential operator method provide a convenient and effective method of finding a particular solution of an ordinary nonhomogeneous linear differential equation of constant coefficients with the nonhomogeneous terms being a polynomial function, an exponential function, a sine function, a cosine function or any possible products of these functions. The efficiency of this method in determining a particular solution is based on the following facts: exponential function is an eigenfunction of $D \equiv d/dx$; sine and cosine functions are eigenfunctions of second-order differential operator D^2; There exists an exponential shift theorem when a polynomial of differential operator acts on a product of exponential function with another continuous function. Especially, if we apply the inverse of differential operator acting on a differentiable function, which is an integration over the function, this method can greatly reduce the complication and arduousness of the calculation.

However, the introduction to this method only scatters partially in some online lecture notes of differential equations [4]. Hence, here we give a systematic introduction to the method including both rigorous mathematical principles and detailed calculation techniques. In Sect.II, we introduce the definitions of differential operator and its higher order versions as well their actions on differentiable functions. We emphasize the fact that exponential functions are eigenfunctions of differential operator and both sine and cosine functions are eigenfunctions of differential operators of even orders. Further, we define the polynomial of differential operator and list its properties when it acts on differentiable functions. We highlight two remarkable properties including eigenvalue substitution rule and exponential shift rule, which will play important roles in solving differential equations.

In addition, we introduce the kernel of the polynomial operator and point out it is the solution to the associated linear homogeneous differential equations. In Sect. III, based on the Fundamental Theorem of Calculus, we define the inverses of differential operator and its higher order versions. Then according to the definition of a function of differential operator given by the Taylor series we define the inverse of a polynomial of differential operator and show some of its properties when it acts on differentiable functions. Sect. IV contains the main content of the article. We use a large number of examples to show how the differential operator method works efficiently in determining a particular solution for a nonhomogeneous linear differential equation with constant coefficients when the nonhomogeneous term is a polynomial function, an exponential function, a sine, a cosine function or any possible product of these functions. Sect V is a a brief summary on the application of differential operator method in solving nonhomogeneous linear ordinary differential equations with constant coefficients.

II. DIFFERENTIAL OPERATOR AND ACTION ON A DIFFERENTIABLE FUNCTION

A. Definition of differential operator

A differential operator D acting a differentiable function $y = f(x)$ on R takes the form

$$D = \frac{d}{dx} \tag{1}$$

The action of D and its higher order versions D^n on a at least n-times differentiable function $y = f(x)$ is just to take the derivatives of the function:

$$Df(x) = \frac{df(x)}{dx}, \tag{2}$$

$$D^2 f(x) = D[Df(x)] = \frac{d}{dx}\left[\frac{df(x)}{dx}\right] = \frac{d^2 f(x)}{dx^2},$$

$$\vdots$$

$$D^n f(x) = D[D^{n-1}f(x)] = \frac{d}{dx}\left[\frac{d^{n-1}f(x)}{dx^{n-1}}\right] = \frac{d^n f(x)}{dx^n} \tag{3}$$

For example, let

$$y = x^3 + x + 3e^{2x} - 5\sin(3x) \tag{4}$$

then

$$Dy = \frac{d}{dx}\left[x^3 + x + 3e^{2x} - 5\sin(3x)\right] = 3x^2 + 1 + 6e^{2x} + 15\cos(3x) \tag{5}$$

B. Action of differential operator on elementary functions and eigenvalue of differential operator

The actions of differential operator on elementary functions including exponential function, sine and cosine function, and polynomial functions are listed as follows:

1. **Exponential function**

$$De^{\lambda x} = \frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x} \tag{6}$$

It shows $e^{\lambda x}$ is an eigenfunction of D with eigenvalue λ in the representation space composed of differentiable function on R.

Straightforwardly, for the higher order version D^n, when $n \geq 2$ is a positive integer, there exists

$$D^n e^{\lambda x} = \frac{d^n}{dx^n}e^{\lambda x} = \frac{d}{dx}\left[\frac{d}{dx}\left(\cdots \frac{d}{dx}e^{\lambda x}\right)\right] = \lambda^n e^{\lambda x} \tag{7}$$

2. **Action of D^2 on sine and cosine functions**

$$D^2 \sin(\beta x) = -\beta^2 \sin(\beta x),$$
$$D^2 \cos(\beta x) = -\beta^2 \cos(\beta x) \tag{8}$$

This shows both $\sin(\beta x)$ and $\cos(\beta x)$ are eigenfunctions of D^2 with eigenvalue $-\beta^2$. Further, consider the higher order version $D^{2n} = \left(D^2\right)^n$, we have

$$D^{2n}\sin(\beta x) = \left(D^2\right)^n \sin(\beta x) = D^2\left[D^2\left(\cdots D^2 \sin \beta x\right)\right] = \left(-\beta^2\right)^n \sin(\beta x),$$
$$D^{2n}\cos(\beta x) = \left(D^2\right)^n \cos(\beta x) = D^2\left[D^2\left(\cdots D^2 \cos \beta x\right)\right] = \left(-\beta^2\right)^n \cos(\beta x) \tag{9}$$

3. **Action of D on power function**

$$Dx^k = \frac{d}{dx}x^k = kx^{k-1},$$
$$D^n x^k = \frac{d^n}{dx^n}x^k = k(k-1)\cdots(k-n+1)x^{k-n} \tag{10}$$

where $k \geq 1$ is a positive integer. Especially,

$$D^n x^k = 0 \quad \text{for } k < n \tag{11}$$

For example,

$$D^5 x^3 = \frac{d^5}{dx^5} x^3 = 0 \tag{12}$$

C. Polynomial of Differential Operator D and Property

Definition 1: Let

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{13}$$

be a polynomial of real variable x of degree n, where a_0, a_1, $\cdots$, a_n are real constants. Then

$$P_n(D) = a_n D^n + a_{n-1} D^{n-1} + a_1 D + \cdots + a_0 \tag{14}$$

is called a polynomial of differential operator D of degree n. In the representation space composed of at least n times differentiable functions $f(x)$ on R,

$$P_n(D) = P_n\left(\frac{d}{dx}\right) = a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 D + a_0 \tag{15}$$

The properties of a polynomial of differential operator D are listed as follows:

1. **Linearity**

    Theorem 1: Let $P_n(D)$ be a polynomial of differential operator D of degree n defined in (14). Then there exists

    $$P_n(D)\left[af(x) + bg(x)\right] = a P_n(D) f(x) + b P_n(D) g(x) \tag{16}$$

    where a and b are constants, and both $f(x)$ and $g(x)$ are differentiable functions of at least n-times.

2. **Sum rule and product rule**

    Theorem 2: Let $P_n(D)$ and $Q_m(D)$ be polynomials of differential operator D of degree n and m, respectively. Then there exist

    (1). Sum rule

    $$\left[P_n(D) + Q_m(D)\right] f(x) = \left[Q_m(D) + P_n(D)\right] f(x) = P_n(D) f(x) + Q_m(D) f(x) \tag{17}$$

where $f(x)$ is a differentiable function of at least $\max(n, m)$ times.

(2). Product rule

$$[P_n(D)Q_m(D)]\, g(x) = [Q_m(D)P_n(D)]\, g(x) = P_n(D)\, [Q_m(D)g(x)]$$
$$= Q_m(D)\, [P_n(D)g(x)] \qquad (18)$$

where $f(x)$ is a differentiable function of at least $\max(n, m)$ times.

Theorems 1 and **2** can be proved straightforwardly with the explicit representation forms of $P_n(D)$ and $Q_m(D)$ on the space of differentiable functions.

3. **Eigenvalue substitution rule**

Theorem 3: Let $P_n(D)$ be a polynomial of differential operator D of degree n. Then there exist

$$(1). \quad P_n(D)e^{\lambda x} = P_n(\lambda)e^{\lambda x} = e^{\lambda x}\, P_n(\lambda) \qquad (19)$$

$$(2). \quad P_n(D^2)\sin\beta x = P_n(-\beta^2)\sin\beta x = \sin\beta x\, P_n(-\beta^2),$$

$$P_n(D^2)\cos\beta x = P_n(-\beta^2)\cos\beta x = \cos\beta x\, P_n(-\beta^2) \qquad (20)$$

where λ and β are real constants.

Proof: (1). First we have

$$P_n(D)e^{\lambda x} = P_n\left(\frac{d}{dx}\right)e^{\lambda x} = \left(a_n\frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1D + a_0\right)e^{\lambda x}$$
$$= \left(a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0\right)e^{\lambda x} = P_n(\lambda)e^{\lambda x} = e^{\lambda x}P_n(\lambda) \qquad (21)$$

(2). First applying Eq.(8), we obtain

$$P_n(D^2)\sin\beta x = \left[a_n(D^2)^n + a_{n-1}(D^2)^{n-1} + \cdots + a_1D^2 + a_0\right]\sin\beta x$$
$$= \left[a_n(-\beta^2)^n + a_{n-1}(-\beta^2)^{n-1} + \cdots + a_1(-\beta^2) + a_0\right]\sin\beta x$$
$$= P_n(-\beta^2)\sin\beta x = \sin\beta x\, P_n(-\beta^2) \qquad (22)$$

Similarly there is the result for $P_n(D^2)\cos\beta x$.

4. **Exponential shift rule**

Theorem 4: Let $P_n(D)$ be a polynomial of differential operator D of degree n. Then there exists

$$D^n \left[e^{\lambda x} f(x) \right] = e^{\lambda x} (D + \lambda)^n f(x), \tag{23}$$

$$P_n(D) \left[e^{\lambda x} f(x) \right] = e^{\lambda x} P_n (D + \lambda) f(x) \tag{24}$$

where $f(x)$ is a differentiable function of at least n times on R.

Proof: (1). We use the method of mathematical induction to prove the result (23).

First, the result arises for $n = 1$:

$$D \left[e^{\lambda x} f(x) \right] = \frac{d}{dx} \left[e^{\lambda x} f(x) \right] = e^{\lambda x} \left(\lambda + \frac{d}{dx} \right) f(x) = e^{\lambda x} (D + \lambda) f(x) \tag{25}$$

Second, assume the result holds for $n = k$:

$$D^k \left[e^{\lambda x} f(x) \right] = e^{\lambda x} (D + \lambda)^k f(x) \tag{26}$$

Then

$$\begin{aligned}
D^{k+1} \left[e^{\lambda x} f(x) \right] &= D \left[D^k \left(e^{\lambda x} f(x) \right) \right] = D \left[e^{\lambda x} (D + \lambda)^k f(x) \right] \\
&= e^{\lambda x}(D + \lambda) \left[(D + \lambda)^k f(x) \right] = e^{\lambda x} (D + \lambda)^{k+1} f(x)
\end{aligned} \tag{27}$$

Therefore, we have the general result (23).

(2). The result (24) follows straightforwardly from (23):

$$\begin{aligned}
P_n(D) \left[e^{\lambda x} f(x) \right] &= \left[a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 \right] \left[e^{\lambda x} f(x) \right] \\
&= e^{\lambda x} \left[a_n (D + \lambda)^n + a_{n-1}(D + \lambda)^{n-1} + \cdots + a_1(D + \lambda) + a_0 \right] f(x) \\
&= e^{\lambda x} P_n(D + \lambda) f(x)
\end{aligned} \tag{28}$$

D.   Kernel of Differential Operator

According to the general definition on the kernel of an operator, the kernel of differential operator D^k, $k = 1, 2, \cdots$, in the space of differentiable real functions is defined as follows:

Definition 3:

$$\ker D^k = \left\{ f(x) \middle| D^k f(x) = 0 \right\} \tag{29}$$

Due to Eq.(11), $\ker D^k$ is the set of polynomial functions of degree at most $k-1$,

$$\ker D^k = \operatorname{span} \left\{ 1, x, \cdots, x^{k-1} \right\}$$
$$= \left\{ c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \cdots + c_1 x + c_0 \middle| c_0, c_1, \cdots, c_k \text{ are constants} \right\} \tag{30}$$

Further, we can write down the kernel of the polynomial $P_n(D)$ of differential operator D, which is the solution space of a linear n-th order ordinary homogeneous differential equation with constant coefficients,

$$P_n(D)y(x) = a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0 \tag{31}$$

The kernel $\ker P_n(D)$ takes different forms depending on the form of $P_n(D)$. Applying the exponential shift rule (23) and eigenvalue substitution rule, we can easily find the following results:

Theorem 5:

(1). If $P_n(D)$ is product of n distinct linear factors (for simplicity, we take $a_n = 1$ for now and later),

$$P_n(D) = (D - r_1)(D - r_2) \cdots (D - r_n) = \prod_{i=1}^{n}(D - r_i),$$
$$r_1 \neq r_2 \neq \cdots \neq r_n \tag{32}$$

then

$$\ker P_n(D) = \operatorname{span} \left\{ e^{r_1 x}, e^{r_2 x}, \cdots, e^{r_n x} \right\}$$
$$= \left\{ \sum_{i=1}^{n} c_i e^{r_i x} \middle| c_1, c_2, \cdots, c_n \text{ are constants} \right\} \tag{33}$$

(2). If $P_n(D)$ is product of k distinct repeated linear factors,

$$P_n(D) = (D - r_1)^{n_1}(D - r_2)^{n_2} \cdots (D - r_k)^{n_k} = \prod_{i=1}^{k}(D - r_i)^{n_i},$$
$$r_1 \neq r_2 \neq \cdots \neq r_k \geq 1, \quad \sum_{i=1}^{k} n_i = n \tag{34}$$

then

$$\ker P_n(D) = \operatorname{span} \left\{ xe^{r_1 x}, x^2 e^{r_1 x}, \cdots, x^{n_1-1}e^{r_1 x}; \cdots; xe^{r_k x}, x^2 e^{r_k x}, \cdots, x^{n_k-1}e^{r_k x} \right\} \tag{35}$$

(3). If $P_n(D)$ is product of k distinct quadratic factors, and in this case there must be $n = 2k$,

$$P_n(D) = \left[(D - \alpha_1)^2 + \beta_1^2\right]\left[(D - \alpha_2)^2 + \beta_2^2\right]\cdots\left[(D - \alpha_k)^2 + \beta_k^2\right]$$
$$= \prod_{i=1}^{k}\left[(D - \alpha_i)^2 + \beta_i^2\right], \quad \alpha_i \neq \alpha_j, \text{ or } \beta_i \neq \beta_j \text{ for } i \neq j \tag{36}$$

then

$$\ker P_n(D) = \text{span}\left\{e^{\alpha_1 x}\cos(\beta_1 x), e^{\alpha_1 x}\sin(\beta_1 x); \cdots; e^{\alpha_k x}\cos(\beta_k x), e^{\alpha_k x}\sin(\beta_k x)\right\} \tag{37}$$

(4). If $P_n(D)$ is product of p repeated distinct quadratic factors,

$$P_n(D) = \left[(D - \alpha_1)^2 + \beta_1^2\right]^{n_1}\left[(D - \alpha_2)^2 + \beta_2^2\right]^{n_2}\cdots\left[(D - \alpha_p)^2 + \beta_p^2\right]^{n_p}$$
$$= \prod_{i=1}^{p}\left[(D - \alpha_i)^2 + \beta_i^2\right]^{n_i}, \quad \alpha_i \neq \alpha_j, \text{ or } \beta_i \neq \beta_j \text{ for } i \neq j \tag{38}$$

then

$$\ker P_n(D) = \text{span}\left\{e^{\alpha_1 x}\cos(\beta_1 x), e^{\alpha_1 x}\sin(\beta_1 x), xe^{\alpha_1 x}\cos(\beta_1 x), xe^{\alpha_1 x}\sin(\beta_1 x),\right.$$
$$\cdots, x^{n_1-1}e^{\alpha_1 x}\cos(\beta_1 x), x^{n_1-1}e^{\alpha_1 x}\sin(\beta_1 x);$$
$$e^{\alpha_2 x}\cos(\beta_2 x), e^{\alpha_2 x}\sin(\beta_2 x), xe^{\alpha_2 x}\cos(\beta_2 x), xe^{\alpha_2 x}\sin(\beta_2 x),$$
$$\cdots, x^{n_2-1}e^{\alpha_2 x}\cos(\beta_2 x), x^{n_2-1}e^{\alpha_2 x}\sin(\beta_2 x);$$
$$\cdots; e^{\alpha_p x}\cos(\beta_p x), e^{\alpha_p x}\sin(\beta_p x), xe^{\alpha_p x}\cos(\beta_p x), xe^{\alpha_p x}\sin(\beta_p x),$$
$$\left.\cdots, x^{n_p-1}e^{\alpha_p x}\cos(\beta_p x), x^{n_p-1}e^{\alpha_p x}\sin(\beta_p x)\right\} \tag{39}$$

If $P_n(D)$ is a mixture of the above four cases, so are the corresponding terms from each of the kernels. For example, let

$$P_{12}(D) = (D - 2)(D - 5)^3\left[(D + 3)^2 + 4\right]\left[(D - 7)^2 + 16\right]^4 \tag{40}$$

then

$$\ker P_{12}(D) = \text{span}\left\{e^{2x}; e^{5x}, xe^{5x}, x^2e^{5x}; e^{-3x}\cos(2x), e^{-3x}\sin(2x); e^{7x}\cos(4x), e^{7x}\sin(4x),\right.$$
$$\left. xe^{7x}\cos(4x), xe^{7x}\sin(4x), x^2e^{7x}\cos(4x), x^2e^{7x}\sin(4x), x^3e^{7x}\cos(4x), x^3e^{7x}\sin(4x)\right\}$$
$$= a_0 e^{2x} + \left(b_0 + b_1 x + b_2 x^2\right)e^{5x} + (c_1\cos 2x + c_2\sin 2x)e^{-3x}$$
$$+ \left(d_0 + d_1 x + d_2 x^2 + d_3 x^3\right)e^{7x}\cos 4x + \left(e_0 + e_1 x + e_2 x^2 + e_3 x^3\right)e^{7x}\sin 4x \tag{41}$$

where a_0, b_0, b_1, b_2, c_1, c_2, d_0, $\cdots$, d_3 and e_0, $\cdots$, e_3 are real numbers.

III.   INVERSE OF DIFFERENTIA OPERATOR AND ACTION ON A CONTINUOUS FUNCTION

A.   Definition of the Inverse of Differential Operator

The inverse of differential operator D can be defined according to the Fundamental Theorem of Calculus. Because there exists

$$\frac{d}{dx} \int_{x_0}^{x} f(x_1)dx_1 = D \int_{x_0}^{x} f(x_1)dx_1 = f(x) \tag{42}$$

where $f(x)$ is a continuous function on a finite interval $[a, b]$ and x_0 is an arbitrary constant on $[a, b]$, therefore, the action of the inverse D^{-1} of the differential operator D on a continuous function $f(x)$ can be defined in terms of the following definite integral:

$$D^{-1}f(x) = \int_{x_0}^{x} f(x_1)dx_1 \tag{43}$$

Further, we can successively define the action of higher order inverse differential operator on a continuous function on a finite interval:

$$D^{-2}f(x) = \left(D^{-1}\right)^2 f(x) = D^{-1}\left[D^{-1}f(x)\right] = \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} f(x_2)dx_2, \tag{44}$$

$$D^{-3}f(x) = \left(D^{-1}\right)^3 f(x) = \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2 \int_{x_0}^{x_2} f(x_3)dx_3, \tag{45}$$

$$\vdots$$

$$D^{-n}f(x) = \left(D^{-1}\right)^n f(x) = \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2 \cdots \int_{x_0}^{x_{n-1}} f(x_n)dx_n \tag{46}$$

The justification of these definitions can be demonstrated by the following examples:

1. $D^{-1}\cos(3x) = \int_{x_0}^{x} \cos(3x_1)dx_1 = \frac{1}{3}\left[\sin(3x) - \sin(3x_0)\right]$

$$= \frac{1}{3}\sin(3x) + c = \frac{1}{3}\sin(3x) + \ker D \tag{47}$$

2. $D^{-2}x = \int_{x_0}^{x} \int_{x_0}^{x_1} dx_2 x_2 dx_2 = \int_{x_0}^{x} \frac{1}{2}\left(x_1^2 - x_0^2\right) = \frac{1}{6}x^3 - \frac{1}{2}x_0^2 x + \frac{1}{3}x_0^3$

$$= \frac{1}{6}x^3 + c_1 x + c_2 = \frac{1}{6}x^3 + \ker D^2 \tag{48}$$

B.   Function of Differential Operator

As a generalization of the polynomial $P_n(D)$, a function of the differential operator D can be defined in a way similar to the definition of a square matrix [5].

Definition 2: Let $f(x)$ be a uniformly convergent function on R. Then a function of $f(D)$ of the differential D is defined in terms of the Taylor series expansion of $f(x)$ about $x = 0$:

$$f(D) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) D^n = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f(x)}{dx^n}\Big|_{x=0} D^n = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f(x)}{dx^n}\Big|_{x=0} \frac{d^n}{dx^n} \tag{49}$$

For examples, two typical functions of D are listed as follows:

$$(1). \quad \frac{1}{1-D} = \sum_{n=0}^{n} D^n = 1 + D + D^2 + \cdots + D^n + \cdots \tag{50}$$

$$(2). \quad e^D = \sum_{n=0}^{\infty} \frac{1}{n!} D^n = 1 + D + \frac{1}{2!} D^2 + \cdots + \frac{1}{n!} D^n + \cdots \tag{51}$$

C.   Inverse of Polynomial of Differential Operator

Using the Taylor series representation (49) on a function of differential operator D, the product rule (18) of polynomials of differential operator D, and the definitions (43), (44), (45), (46) of the action of the inverse operator D^{-k} on a continuous function, we can define the inverse of a polynomial operator $P_n(D)$ as follows:

Definition 4: Let

$$P_n(D)f(x) = \left(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 \right) f(x) = g(x) \tag{52}$$

where both $f(x)$ and $g(x)$ are differentiable functions.

(1). If $a_0 \neq 0$, then

$$f(x) = [P_n(D)]^{-1} g(x) = \frac{1}{P_n(D)} g(x) = \frac{1}{a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0} g(x)$$
$$= \sum_{p=0}^{\infty} (-1)^p \frac{1}{a_0} \left(\frac{a_n}{a_0} D^n + \cdots + \frac{a_1}{a_0} D \right)^p g(x) \tag{53}$$

(2). If all $a_0 = a_1 = \cdots a_{k-1} = 0$ $(1 \leq k \leq n)$ and $a_k \neq 0$, then

$$f(x) = [P_n(D)]^{-1} g(x) = \frac{1}{P_n(D)} g(x) = \frac{1}{a_n D^n + a_{n-1} D^{n-1} + \cdots + a_k D^k} g(x)$$
$$= \frac{1}{a_n D^n + a_{n-1} D^{n-1} + \cdots + a_k D^k} g(x)$$
$$= \frac{1}{a_n D^{n-k} + a_{n-1} D^{n-k-1} + \cdots + a_k} D^{-k} g(x) \tag{54}$$

The definition of $[P_n(D)]^{-1}$ and the product rule (18) yields the following property

Corollary 1: Let $P_n(D)$ and $Q_m(D)$ be two polynomials of D of degree n and m, respectively. Then

$$\frac{1}{P_n(D)Q_m(D)}f(x) = \frac{1}{Q_m(D)P_n(D)}f(x)$$
$$= \frac{1}{P_n(D)}\left[\frac{1}{Q_m(D)}\right]f(x) = \frac{1}{Q_m(D)}\left[\frac{1}{P_n(D)}\right]f(x) \tag{55}$$

where $f(x)$ is a differentiable function on a certain interval.

Corollary 2: Let $P_n(D)$ and $Q_m(D)$ be two polynomials of D of degree n and m, respectively. Then

$$\frac{1}{P_n(D)}f(x) = \frac{1}{P_n(D)Q_m(D)}[Q_m(D)f(x)] = Q_m(D)\left[\frac{1}{P_n(D)Q_m(D)}f(x)\right] \tag{56}$$

where $f(x)$ is a differentiable function on a certain interval.

IV.   DIFFERENTIAL OPERATOR METHOD OF SOLVING NONHOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

The general form of an n-th order nonhomogeneous linear ordinary differential equation with constant coefficients takes the following form:

$$a_n\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = g(x) \tag{57}$$

where a_0, a_1, $\cdots$, a_n are constants. Expressed in terms of differential operator, the equation reads

$$\left(a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0\right)y = P_n(D)y = g(x) \tag{58}$$

Then formally a particular solution of the equation is

$$Y(x) = \frac{1}{P_n(D)}g(x) \tag{59}$$

Due to the exponential shift rule (24) and the eigenvalue substitution rule (19), (20) as well as the feature (11), this approach, like the method of undetermined coefficients, works only when $g(x)$ is a polynomial function $P_k(x)$ of degree k, an exponential function $e^{\alpha x}$, $\sin(\beta x)$, $\cos(\beta x)$ or any possible product of these functions. In the following we discuss each case of $g(x)$

A. The case $g(x)$ is an exponential function

We first give the following results for $g(x) = Ae^{\alpha x}$ in order to show how the differential operator approach works in this case, where A is a constant.

Theorem 6: A particular solution to a differential equation

$$P_n(D)y(x) = Ae^{\alpha x} \tag{60}$$

reads as follows:

1. If $P_n(\alpha) \neq 0$, then

$$Y(x) = \frac{1}{P_n(D)} (Ae^{\alpha x}) = \frac{1}{P_n(\alpha)} (Ae^{\alpha x}) \tag{61}$$

2. If $P_n(\alpha) = 0$, i.e., $P_n(D) = (D - \alpha)^k P_{n-k}(D)$, $1 \leq k \leq n$ then

$$Y(x) = \frac{1}{P_n(D)} (Ae^{\alpha x}) = \frac{A}{P_{n-k}(\alpha)} \left(\frac{1}{k!} x^k + \ker D^k \right) e^{\alpha x} \tag{62}$$

Proof: According to the rule of eigenvalue substitution (19), we have

$$P_n(D)Ae^{\alpha x} = AP_n(\alpha)e^{\alpha x} \tag{63}$$

(1). Since $P_n(\alpha) \neq 0$, we divide the equation (63) by $P_n(\alpha)$, then

$$\frac{1}{P_n(\alpha)} [P_n(D) (Ae^{\alpha x})] = P_n(D) \left(\frac{Ae^{\alpha x}}{P_n(\alpha)} \right) = Ae^{\alpha x} \tag{64}$$

In comparison with the differential equation (60), it shows $Y(x) = Ae^{\alpha x}/P_n(\alpha)$.

(2). Since $P_n(\alpha) = 0$, the polynomial $P_n(D)$ must take the following form for a certain $1 \leq k \leq n$,

$$P_n(D) = P_{n-k}(D) (D - \alpha)^k, \quad P_{n-k}(\alpha) \neq 0, \tag{65}$$

and the equation becomes

$$P_n(D)y(x) = (D - \alpha)^k P_{n-k}(D)y(x) = Ae^{\alpha x} \tag{66}$$

Because

$$(D - \alpha)^k P_{n-k}(D) \left[\left(\frac{1}{k!} x^k + \ker D^k \right) e^{\alpha x} \right]$$

$$= P_{n-k}(\alpha)(D - \alpha)^k \left[e^{\alpha x} \left(\frac{1}{k!} x^k + \ker D^k \right) \right]$$

$$= P_{n-k}(\alpha) e^{\alpha x} D^k \left(\frac{1}{k!} x^k + \ker D^k \right) = P_{n-k}(\alpha) e^{\alpha x} \tag{67}$$

therefore, $y(x)$ given in (62) is the solution to the differential equation.

Theorem 6 shows a formal method of finding a particular solution of differential equation when the nonhomogeneous term is an exponential function,

$$Y(x) = \frac{1}{P_n(D)} \left(A e^{\alpha x} \right) = \frac{1}{P_{n-k}(D) \left(D - \alpha \right)^k)} \left(A e^{\alpha x} \right)$$

$$= \frac{A e^{\alpha x}}{P_{n-k}(\alpha)} D^{-k} 1 = \frac{A x^k e^{\alpha x}}{k! P_{n-k}(\alpha)} \tag{68}$$

The following two examples shows how this method works.

(1). A particular solution to the differential equation

$$3y'' - 2y' + 6y = 5e^{3x} \tag{69}$$

is

$$Y(x) = \frac{1}{3D^2 - 2D + 8} \left(5e^{3x} \right) = \frac{5e^{3x}}{3 \cdot 3^2 - 2 \cdot 3 + 8} = \frac{5}{29} e^{3x} \tag{70}$$

(2). A particular solution to the differential equation

$$(D - 1)(D + 5)(D - 2)^3 y(x) = 3e^{2x} \tag{71}$$

is

$$Y(x) = \frac{1}{(D - 1)(D + 5)(D - 2)^3} \left(3e^{2x} \right) = \frac{3e^{2x}}{(2 - 1)(2 + 5)} D^{-3} 1$$

$$= \frac{3e^{2x}}{7} \frac{1}{3!} x^3 = \frac{1}{14} x^3 e^{2x} \tag{72}$$

A straightforward result comes from **Theorem 6**, which is called the Exponential Input Theorem.

Corollary 3 (Exponential Input Theorem):

Let $P_n(D)$ be a polynomial of D of degree n. Then the nonhomogeneous linear differential equation (60) has a particular solution of the following form:

$$Y(x) = \begin{cases} \dfrac{Ae^{\alpha x}}{P_n(\alpha)}, & \text{if } P_n(\alpha) \neq 0 \\[2mm] \dfrac{Axe^{\alpha x}}{P_n'(\alpha)}, & \text{if } P_n(\alpha) = 0 \text{ but } P_n'(\alpha) \neq 0 \\[2mm] \dfrac{Ax^2 e^{\alpha x}}{P_n''(\alpha)}, & \text{if } P_n(\alpha) = P_n'(\alpha) = 0 \text{ but } P_n''(\alpha) \neq 0 \\[2mm] \vdots \\[2mm] \dfrac{Ax^k e^{\alpha x}}{P_n^{(k)}(\alpha)}, & \text{if } P_n(\alpha) = P_n'(\alpha) = \cdots = P_n^{(k-1)}(\alpha) = 0 \text{ but } P_n^{(k)}(\alpha) \neq 0 \end{cases} \tag{73}$$

Proof: We still start from Eq.(63).

(1). For the case $P_n(\alpha) \neq 0$, the proof is the same as that for **Theorem 6**.

(2). If $P_n(\alpha) = 0$, while $P_n'(\alpha) \neq 0$ we first take the derivative on the the equation (63) with respect to α, then

$$\frac{d}{d\alpha}\left[P_n(D)\left(Ae^{\alpha x}\right)\right] = \frac{d}{d\alpha}\left[AP_n(\alpha)e^{\alpha x}\right],$$
$$P_n(D)\left(Axe^{\alpha x}\right) = A\left[P_n'(\alpha)e^{\alpha x} + P_n(\alpha)\left(xe^{\alpha x}\right)\right] = AP_n'(\alpha)e^{\alpha x} \tag{74}$$

We divide the above equation by $P_n'(\alpha)$ and obtain

$$\frac{1}{P_n'(\alpha)}\left[P_n(D)\left(Axe^{\alpha x}\right)\right] = P_n(D)\left[\frac{Axe^{\alpha x}}{P_n'(\alpha)}\right] = Ae^{\alpha x} \tag{75}$$

Therefore, a particular solution of the differential equation (60) is $y(x) = Axe^{\alpha x}/P_n'(\alpha)$.

(3). If $P_n(\alpha) = P_n'(\alpha) = 0$, while $P_n''(\alpha) \neq 0$, we take the derivative the equation (63) twice with respect to α,

$$\frac{d^2}{d\alpha^2}\left[P_n(D)\left(Ae^{\alpha x}\right)\right] = \frac{d^2}{d\alpha^2}\left[AP_n(\alpha)e^{\alpha x}\right],$$
$$P_n(D)\left(Ax^2 e^{\alpha x}\right) = A\left[P_n''(\alpha)e^{\alpha x} + 2P_n'(\alpha)xe^{\alpha x} + P_n'(\alpha)x^2 e^{\alpha x}\right] = AP_n''(\alpha)e^{\alpha x} \tag{76}$$

Then dividing the above equation by $P_n'(\alpha)$, we have

$$P_n(D)\left[\frac{Ax2e^{\alpha x}}{P_n''(\alpha)}\right] = Ae^{\alpha x} \tag{77}$$

Hence in this case the solution to the differential equation is $y(x) = Ax^2 e^{\alpha x}/P_n''(\alpha)$ (60).

(4). For the general case, $P_n(\alpha) = P'_n(\alpha) = \cdots = P_n^{(k-1)}(\alpha) = 0$, while $P_n^{(k)}(\alpha) \neq 0$, we take the derivative on the equation (63) k times with respect to α,

$$\frac{d^k}{d\alpha^k} [P_n(D)(Ae^{\alpha x})] = \frac{d^k}{d\alpha^k} [AP_n(\alpha)e^{\alpha x}] \tag{78}$$

With an application of the product rule for higher order derivative,

$$\frac{d^k}{dx^k} [f(x)g(x)] = \sum_{p=0}^{k} \binom{k}{p} f^{(k-p)}(x)g^{(p)}(x), \tag{79}$$

it yields

$$P_n(D)\left(Ax^k e^{\alpha x}\right) = A\sum_{p=0}^{k} \binom{k}{p} P_n^{(k-p)}(\alpha) x^p e^{\alpha x}$$
$$= A\left[P_n^{(k)}(\alpha)e^{\alpha x} + kP_n^{(k-1)}(\alpha)xe^{\alpha x} + \frac{k(k-1)}{2}P_n^{(k-2)}(\alpha)x^2 e^{\alpha x} + \cdots + P_n(\alpha)x^k e^{\alpha x}\right]$$
$$= AP_n^{(k)}(\alpha)e^{\alpha x} \tag{80}$$

Therefore, a particular solution of the equation in this case is $y(x) = Ax^k e^{\alpha x}/P_n^{(k)}(\alpha)$.

One can also apply **Corollary 3** to find a particular solution of a differential equation. For example, determine a particular solution of the equation

$$(D-2)(D-4)^3 y = 5e^{4t} \tag{81}$$

Since

$$P'(D) = (D-4)^2(4D-10), \;\; P''(D) = (D-4)(12D-36), \;\; P^{(3)}(D) = 12(D-7),$$
$$P(4) = P'(4) = P''(4) = 0, \;\; P^{(3)}(4) \neq 0 \tag{82}$$

so a particular solution of the equation is

$$Y(x) = \frac{5x^3 e^{4x}}{P^{(3)}(4)} = -\frac{5}{36}x^3 e^{4x} \tag{83}$$

B. The case $g(x)$ is a polynomial function $P_k(x)$

In this case, due to the fact (11) it is most convenient to use **Definition 4** (see Eqs.(53) and (53)) to find a particular solution of a differential equation. We use the following two examples to illustrate the idea.

Example 1: Determine a particular solution of the equation

$$y''' - 5y'' + 3y' + 2y = x^2 + 3x - 2 \tag{84}$$

The differential operator form of the equation is

$$\left(D^3 - 5D^2 + 3D + 2\right)y = 2x^3 + 4x^2 - 6x + 5 \tag{85}$$

and consequently, a particular solution of the equation is

$$
\begin{aligned}
Y(x) &= \frac{1}{D^3 - 5D^2 + 3D + 2}\left(2x^3 + 4x^2 - 6x + 5\right) \\
&= \frac{1}{2}\frac{1}{1 + (D^3 - 5D^2 + 3D)/2}\left(2x^3 + 4x^2 - 6x + 5\right) \\
&= \frac{1}{2}\left[1 - \frac{1}{2}(D^3 - 5D^2 + 3D) + \frac{1}{4}(-5D^2 + 3D)^2 - \frac{1}{8}(3D)^3\right]\left(2x^3 + 4x^2 - 6x + 5\right) \\
&= \frac{1}{2}\left(1 - \frac{3}{2}D + \frac{19}{4}D^2 - \frac{91}{8}D^3\right)\left(2x^3 + 4x^2 - 6x + 5\right) \\
&= x^3 - \frac{5}{2}x^2 + \frac{39}{2}x - \frac{169}{4}
\end{aligned} \tag{86}
$$

Example 2: Determine a particular solution of the equation

$$y''' - 3y'' + 2y' = x^3 - 2x^2 \tag{87}$$

A particular solution of the above equation is

$$
\begin{aligned}
Y(x) &= \frac{1}{D^3 - 3D^2 + 2D}\left(x^3 - 2x^2\right) = \frac{1}{D(1 - D)(2 - D)}\left(x^3 - 2x^2\right) \\
&= \frac{1}{D(2 - D)}\left(1 + D + D^2 + D^3\right)\left(x^3 - 2x^2\right) \\
&= \frac{1}{D}\left[\frac{1}{2}\left(1 + \frac{1}{2}D + \frac{1}{4}D^2 + \frac{1}{8}D^3\right)\right](x^3 + x^2 + 2x + 2) \\
&= \frac{1}{2}D^{-1}\left(x^3 + \frac{5}{2}x^2 + \frac{9}{2}x + \frac{17}{4}\right) = \frac{1}{8}x^4 + \frac{5}{12}x^3 + \frac{9}{8}x^2 + \frac{17}{8}x
\end{aligned} \tag{88}
$$

C.   The case $g(x)$ is either a sine or cosine function

For $g(x) = A\sin\beta x$ or $g(x) = A\cos\beta x$, the solution is formally given by Eq.(59) as previous cases:

$$Y(x) = \frac{1}{P_n(D)}\left(A\sin\beta x + B\sin\beta x\right) \tag{89}$$

There are two cases:

(1). $P_n(D)\sin\beta x \neq 0$ and $P_n(D)\cos\beta x \neq 0$

In this case, we first employ the result of **Corollary 2** and the simple algebraic operation $(a+b)(a-b) = a^2 - b^2$ to make the denominator become a function of D^2, and then apply the eigenvalue substitution rule (20). We use the following example to illustrate this method.

Example: Determine a particular solution of the equation

$$2y''' + y'' - 5y' + 3y = 3\sin 2x \tag{90}$$

A particular solution of the equation is

$$
\begin{aligned}
Y(x) &= \frac{1}{2D^3 + D^2 - 5D + 3}(3\sin 2x) = \frac{(2D^3 - 5D) - (D^2 + 3)}{(2D^3 - 5D)^2 - (D^2 + 3)^2}(3\sin 2x)\\
&= \frac{2D^3 - D^2 - 5D - 3}{D^2(2D^2 - 5)^2 - (D^2 + 3)^2}(3\sin 2x)\\
&= \frac{3}{(-4)\left[2\cdot(-4) - 5\right]^2 - (-4+3)^2}\left(2\frac{d^3}{dx^3} - \frac{d^2}{dx^2} - 5\frac{d}{dx} - 3\right)\sin 2x\\
&= \frac{3}{677}(26\cos 2x - \sin 2x) \tag{91}
\end{aligned}
$$

(2). $P_n(D)\sin\beta x = 0$ and $P_n(D)\cos\beta x = 0$

In this case, due to $\left(D^2 + \beta^2\right)\sin\beta x = 0$ and $\left(D^2 + \beta^2\right)\cos\beta x = 0$, $P_n(D)$ must takes the following form,

$$P_n(D) = P_{n-2k}(D)\left(D^2 + \beta^2\right)^k, \quad k \geq 1 \tag{92}$$

To determine a particular solution in this case, we first introduce the following result.

Theorem 7: There exist

$$
\begin{aligned}
\frac{1}{(D^2 + \beta^2)^{2p}}\cos\beta x &= \frac{(-1)^p}{(2p)!(2\beta)^{2p}}x^{2p}\cos\beta x,\\
\frac{1}{(D^2 + \beta^2)^{2p}}\sin\beta x &= \frac{(-1)^p}{(2p)!(2\beta)^{2p}}x^{2p}\sin\beta x, \quad p = 1, 2, \cdots \tag{93}\\
\frac{1}{(D^2 + \beta^2)^{2p+1}}\cos\beta x &= \frac{(-1)^p}{(2p+1)!(2\beta)^{2p+1}}x^{2p+1}\sin\beta x,\\
\frac{1}{(D^2 + \beta^2)^{2p+1}}\sin\beta x &= \frac{(-1)^{p+1}}{(2p+1)!(2\beta)^{2p+1}}x^{2p+1}\cos\beta x, \quad p = 0, 1, 2, \cdots \tag{94}
\end{aligned}
$$

Proof: Applying the exponential shift rule, we have

$$\left(D^2 + \beta^2\right)^k \left(x^k e^{i\beta x}\right) = e^{i\beta x}\left[(D+i\beta)^2 + \beta^2\right]^k x^k$$
$$= e^{i\beta x}\left(D + 2i\beta\right)^k D^k x^k = e^{i\beta x} i^k (2\beta)^k k! \tag{95}$$

For $k = 2p$, $p = 1, 2, \cdots$, using the Euler formula, we obtain

$$\left(D^2 + \beta^2\right)^{2p} \left(x^{2p}\cos\beta x + ix^{2p}\sin\beta x\right) = (2p)!(2\beta)^{2p}(-1)^p \left(\cos\beta x + i\sin\beta x\right) \tag{96}$$

The real and imaginary parts yield

$$\left(D^2 + \beta^2\right)^{2p}\left(x^{2p}\cos\beta x\right) = (2p)!(2\beta)^{2p}(-1)^p \cos\beta x,$$
$$\left(D^2 + \beta^2\right)^{2p}\left(x^{2p}\sin\beta x\right) = (2p)!(2\beta)^{2p}(-1)^p \sin\beta x \tag{97}$$

and lead to the result (93).

Similarly, for $k = 2p + 1$, $p = 0, 1, \cdots$, there arises

$$\left(D^2 + \beta^2\right)^{2p+1}\left(x^{2p+1}\cos\beta x + ix^{2p+1}\sin\beta x\right)$$
$$= (2p+1)!(2\beta)^{2p}(-1)^p \left(i\cos\beta x - \sin\beta x\right) \tag{98}$$

The real and imaginary parts give the result (94).

We hence can determine a particular solution (89) for the case $P_n(D) = 0$ as follows:

$$Y(x) = \frac{1}{P_{n-2k}(D)\left(D^2+\beta^2\right)^k}\left(A\cos\beta x + B\sin\beta x\right)$$
$$= \frac{1}{\left(D^2+\beta^2\right)^k}\left[\frac{1}{P_{n-2k}(D)}\left(A\cos\beta x + B\sin\beta x\right)\right] \tag{99}$$

For example, we determine a particular solution of the differential equation

$$(D-1)^2(D-2)(D^2+4)^2 y(x) = 4\sin(2x) \tag{100}$$

Then

$$Y(x) = \frac{1}{(D-1)^2(D-2)(D^2+4)^2}\left(4\sin 2x\right) = \frac{4}{(D^2+4)^2}\left[\frac{1}{(D-1)^2(D-2)}\sin 2x\right]$$
$$= \frac{4}{(D^2+4)^2}\left[\frac{(D+1)^2(D+2)}{(D^2-1)^2(D^2-4)}\sin 2x\right]$$
$$= -\frac{1}{50}\frac{1}{(D^2+4)^2}\left(\frac{d^3}{dx^3} + 4\frac{d^2}{dx^2} + 5\frac{d}{dx} + 2\right)\sin 2x$$
$$= -\frac{1}{50}\frac{1}{(D^2+4)^2}\left(2\cos 2x - 14\sin 2x\right) = \frac{1}{800}x^2\left(\cos 2x - 7\sin 2x\right) \tag{101}$$

An alternative method for solving the case $g(x) = A\cos\beta(x) + B\sin\beta x$ is first using the Euler formula to express $\cos\beta x$ and $\sin\beta x$ in terms of complex exponential functions,

$$\cos\beta x = \frac{1}{2}\left(e^{i\beta x} + e^{-i\beta x}\right), \quad \sin\beta x = \frac{1}{2}\left(e^{i\beta x} - e^{-i\beta x}\right) \tag{102}$$

and then determining a particular solution in the same way as $g(x)$ being a complex exponential function. For example, the particular solutions of the equations (90) and (100) can be worked out as follows:

(1). For the equation (90), we have

$$\begin{aligned}
Y(x) &= \frac{1}{2D^3 + D^2 - 5D + 3}(3\sin 2x) \\
&= \frac{1}{2D^3 + D^2 - 5D + 3}\left[\frac{3}{2i}\left(e^{2ix} - e^{-2ix}\right)\right] \\
&= \frac{3}{2i}\left[e^{2ix}\left(\frac{1}{2(D+2i)^3 + (D+2i)^2 - 5(D+2i) + 3}1\right)\right. \\
&\quad\left. -e^{-2ix}\left(\frac{1}{2(D-2i)^3 + (D-2i)^2 - 5(D-2i) + 3}1\right)\right] \\
&= \frac{3}{2i}\left(\frac{e^{2ix}}{-1 - 26i} - \frac{e^{-2ix}}{-1 + 26i}\right) \\
&= \frac{3}{2i}\frac{1}{677}\left[(-1+26i)(\cos 2x + i\sin 2x) - (-1-26i)(\cos 2x - i\sin 2x)\right] \\
&= \frac{3}{677}(26\cos 2x - \sin 2x) \tag{103}
\end{aligned}$$

(2). For the equation (100), we have

$$\begin{aligned}
Y(x) &= \frac{1}{(D-1)^2(D-2)(D^2+4)^2}(4\sin 2x) \\
&= \frac{1}{(D-1)^2(D-2)(D^2+4)^2}\left[\frac{4}{2i}\left(e^{2ix} - e^{-2ix}\right)\right] \\
&= \frac{2}{i}\left[e^{2ix}\left(\frac{1}{(D+2i-1)^2(D+2i-2)[(D+2i)^2+4]^2}1\right)\right. \\
&\quad\left. -e^{-2ix}\left(\frac{1}{(D-2i-1)^2(D-2i-2)[(D-2i)^2+4]^2}1\right)\right] \\
&= \frac{2}{i}\left[\frac{e^{2ix}}{(2i-1)^2(2i-2)}\left(\frac{1}{D^2(D+4i)^2}1\right) - \frac{e^{-2ix}}{(-2i-1)^2(-2i-2)}\left(\frac{1}{D^2(D-4i)^2}1\right)\right] \\
&= \frac{1}{16i}\left[\frac{e^{2ix}}{(1-2i)^2(1-i)} + \frac{e^{-2ix}}{(1+2i)^2(1+i)}\right]D^{-2}1 \\
&= \frac{1}{800i}\left[(-3+4i)(1+i)(\cos 2x + i\sin 2x) + (-3-4i)(1-i)(\cos 2x - i\sin 2x)\right]\frac{1}{2}x^2 \\
&= \frac{1}{800}x^2(\cos 2x - 7\sin 2x) \tag{104}
\end{aligned}$$

D. The case $g(x)$ is a product of exponential function and polynomial function

For $g(x) = P_m(x)e^{\alpha x}$, one should first apply the exponential shift rule to move out of the exponential function, and then proceed to determine a particular solution as the case $g(x)$ being a polynomial function:

$$Y(x) = \frac{1}{P_n(D)}\left(P_m(x)e^{\alpha x}\right) = e^{\alpha x}\left[\frac{1}{P_n(D+\alpha)}P_m(x)\right] \tag{105}$$

For examples,

(1). A particular solution of the equation

$$(D-3)^2(D^2-2D+5)(D+2)y(x) = (x^2-3x+1)e^{2x} \tag{106}$$

is

$$\begin{aligned}
Y(x) &= \frac{1}{(D-3)^2[(D-1)^2+4](D+2)}\left[(x^2-3x+1)e^{2x}\right]\\
&= e^{2x}\frac{1}{(D-1)^2[(D+1)^2+4](D+4)}(x^2-3x+1)\\
&= e^{2x}\frac{1}{20-27D+2D^3+4D^4+D^5}(x^2-3x+1)\\
&= e^{2x}\frac{1}{20-27D}(x^2-3x+1) = e^{2x}\frac{1}{20}\left[1+\frac{27}{20}\frac{d}{dx}+\left(\frac{27}{20}\right)^2\frac{d^2}{dx^2}\right](x^2-3x+1)\\
&= \frac{1}{20}\left(x^2-\frac{3}{10}x+\frac{119}{200}\right)e^{2x} \tag{107}
\end{aligned}$$

(2). A particular solution of the equation

$$(D-3)(D-2)^2(D+1)y(x) = (4x-2)e^{2x} \tag{108}$$

is

$$\begin{aligned}
Y(x) &= \frac{1}{(D-2)^2(D-3)(D+1)}\left[(4x-2)e^{2x}\right] = e^{2x}\frac{1}{(D-1)(D+3)D^2}(4x-2)\\
&= e^{2x}\frac{1}{D^2(D^2+2D-3)}(4x-2) = e^{2x}\frac{1}{D^2}\left[\frac{1}{2D-3}(4x-2)\right]\\
&= e^{2x}\frac{1}{D^2}\left[-\frac{1}{3}\left(1+\frac{2}{3}\frac{d}{dx}\right)(4x-2)\right] = -\frac{1}{3}e^{2x}D^{-2}\left(4x+\frac{2}{3}\right)\\
&= -\frac{1}{9}x^2(2x+1)e^{2x} \tag{109}
\end{aligned}$$

E.   The case $g(x)$ is a product of polynomial function and sine or cosine functions

In this case, $g(x) = P_m(x)(A\cos\beta x + B\sin\beta x)$. The most convenient way of determining a particular solution is to first express sine and cosine function in terms of complex exponential functions, and then use the method of dealing with the product of a polynomial function and exponential function to find a particular solution:

$$Y(x) = \frac{1}{P_n(D)}\left[P_m(x)\left(A\cos\beta x + B\sin\beta x\right)\right]$$

$$= \frac{1}{2}\left(A - iB\right)e^{i\beta x}\left[\frac{1}{P_n(D + i\beta)}P_m(x)\right] + \frac{1}{2}\left(A + iB\right)e^{-i\beta x}\left[\frac{1}{P_n(D - i\beta)}P_m(x)\right] \qquad (110)$$

For examples,

(1). A particular solution to the differential equation

$$y'' - 4y = (x^2 - 3)\sin 2x \qquad (111)$$

is

$$Y(x) = \frac{1}{D^2 - 4}\left[\left(x^2 - 3\right)\sin 2x\right] = \frac{1}{D^2 - 4}\left[\left(x^2 - 3\right)\frac{1}{2i}\left(e^{2ix} - e^{-2ix}\right)\right]\left(x^2 - 3\right)$$

$$= \frac{1}{2i}\left[e^{2ix}\frac{1}{(D + 2i)^2 - 4} - e^{-2ix}\frac{1}{(D - 2i)^2 - 4}\right]\left(x^2 - 3\right)$$

$$= \frac{1}{2i}\left[e^{2ix}\frac{1}{D^2 + 4iD - 8} - e^{-2ix}\frac{1}{D^2 - 4iD - 8}\right]\left(x^2 - 3\right)$$

$$= \frac{i}{16}\left[e^{2ix}\left(1 + \frac{1}{2}iD - \frac{1}{8}D^2\right) - e^{-2ix}\left(1 - \frac{1}{2}iD - \frac{1}{8}D^2\right)\right]\left(x^2 - 3\right)$$

$$= -\frac{1}{32}\left[\left(4x^2 - 13\right)\sin 2x + 4x\cos 2x\right] \qquad (112)$$

(2). A particular solution to the equation

$$y'' + 4y = 4x^2\cos 2x \qquad (113)$$

is

$$Y(x) = \frac{1}{D^2 + 4}\left(4x^2\cos 2x\right) = \frac{1}{D^2 + 4}\left[2x^2\left(e^{2ix} + e^{-2ix}\right)\right]$$

$$= 2\left[e^{2ix}\frac{1}{(D + 2i)^2 + 4}x^2 + e^{-2ix}\frac{1}{(D - 2i)^2 + 4}x^2\right]$$

$$= 2\left[e^{2ix}\frac{1}{D(D + 4i)}x^2 + e^{-2ix}\frac{1}{D(D - 4i)}x^2\right]$$

$$= \frac{1}{4i}\left[e^{2ix}D^{-1}\left(2x^2 + ix - \frac{1}{4}\right) - e^{-2ix}D^{-1}\left(2x^2 - ix - \frac{1}{4}\right)\right]$$

$$= \frac{1}{24}\left[6x^2\cos 2x + x(8x^2 - 3)\sin 2x\right] \qquad (114)$$

F. The case $g(x)$ is a product of exponential function and sine or cosine functions

For $g(x) = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$, as other cases involving sine or cosine functions, the best way of determining a particular solution for an equation is to first express sine and cosine function in terms of complex exponential functions. There are two possibilities.

(1). If $P_n(\alpha \pm i\beta) \neq 0$, then one can directly apply the eigenvalue substitution rule to get a particular solution:

$$
\begin{aligned}
Y(x) &= \frac{1}{P_n(D)} \left[e^{\alpha x} (A \cos \beta x + B \sin \beta x) \right] \\
&= \frac{A - iB}{2P_n(\alpha + i\beta)} e^{(\alpha + i\beta)x} + \frac{A + iB}{2P_n(\alpha - i\beta)} e^{(\alpha - i\beta)x}
\end{aligned}
\tag{115}
$$

For example, a particular solution of the equation

$$
y'' - 2y' + 2y = e^{2x} (2 \cos x - 6 \sin x)
\tag{116}
$$

is

$$
\begin{aligned}
Y(x) &= \frac{1}{D^2 - 2D + 2} \left[e^{2x} (2 \cos x - 6 \sin x) \right] \\
&= \frac{1}{D^2 - 2D + 2} \left[(1 + 3i)e^{(2+i)x} + (1 - 3i)e^{(2-i)x} \right] \\
&= (1 + 3i)\frac{e^{(2+i)x}}{(2+i)^2 - 2(2+i) + 2} + (1 - 3i)\frac{e^{(2-i)x}}{(2-i)^2 - 2(2-i) + 2} \\
&= \frac{2}{5} e^{2x} (7 \cos x - \sin x)
\end{aligned}
\tag{117}
$$

An alternative way is to first utilize the exponential rule to move out $e^{\alpha x}$ and then use the method of dealing with sine or cosine function to determine a particular solution. For example, a particular solution of the above equation in this approach is obtained as follows:

$$
\begin{aligned}
Y(x) &= e^{2x} \frac{1}{(D+2)^2 - 2(D+2) + 2} (2 \cos x - 6 \sin x) = e^{2x} \frac{1}{D^2 + 2D + 2} (2 \cos x - 6 \sin x) \\
&= e^{2x} \frac{D^2 + 2 - 2D}{(D^2+2)^2 - 4D^2} (2 \cos x - 6 \sin x) = \frac{1}{5} e^{2x} \left(\frac{d^2}{dx^2} - 2\frac{d}{dx} + 2 \right) (2 \cos x - 6 \sin x) \\
&= \frac{2}{5} e^{2x} (7 \cos x - \sin x)
\end{aligned}
\tag{118}
$$

(2). If $P_n(\alpha \pm i\beta) = 0$, one should first apply the exponential shift rule and then the definition for the inverse of differential operator. For example, we determine a particular solution of the equation

$$(D-1)(D^2 - 2D + 5)y = 4e^{3x}\cos 2x \qquad (119)$$

Then

$$
\begin{aligned}
Y(x) &= \frac{1}{(D-1)(D^2 - 6D + 13)}\left(4e^{3x}\cos 2x\right) \\
&= \frac{1}{[(D-3)^2 + 4](D-1)}\left\{2\left[e^{(3+2i)x} + e^{(3-2i)x}\right]\right\} \\
&= \frac{1}{(D-3)^2 + 4}\left[\frac{2}{2+2i}e^{(3+2i)x}\right] + \frac{1}{(D-3)^2 + 4}\left[\frac{2}{2-2i}e^{(3-2i)x}\right] \\
&= \frac{1}{2}(1-i)e^{(3+2i)x}\frac{1}{D(D+4i)}1 + \frac{1}{2}(1+i)e^{(3-2i)x}\frac{1}{D(D-4i)}1 \\
&= \frac{1}{8i}xe^{3x}(1-i)(\cos 2x + i\sin 2x) - \frac{1}{8i}xe^{3x}(1+i)(\cos 2x + i\sin 2x) \\
&= \frac{1}{4}xe^{3x}(\sin 2x - \cos 2x) \qquad (120)
\end{aligned}
$$

One can also work first take out $e^{\alpha x}$ with the exponential shift rule and then apply the results (93) and (94) of **Theorem 7**. The particular solution of the above example in this approach reads

$$
\begin{aligned}
Y(x) &= \frac{1}{(D-1)[(D-3)^2 + 4])}\left(4e^{3x}\cos 2x\right) = 4e^{3x}\frac{1}{(D+2)(D^2 + 4)}\cos 2x \\
&= 4e^{3x}\frac{1}{D^2 + 4}\left(\frac{D-2}{D^2 - 4}\cos 2x\right) = e^{3x}\frac{1}{D^2 + 4}(\sin 2x + \cos 2x) \\
&= \frac{1}{4}xe^{3x}(\sin 2x - \cos 2x) \qquad (121)
\end{aligned}
$$

G. The case $g(x)$ is a product of polynomial function, exponential function and sine or cosine functions

Finally, we consider the most general case, $g(x) = P_m(x)e^{\alpha x}(A\cos\beta x + B\sin\beta x)$. The method of getting a particular is the same as the previous case that $g(x)$ is a product of a polynomial function and an exponential function:

$$
\begin{aligned}
Y(x) &= \frac{1}{P_n(D)}\left[P_m(x)e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)\right] \\
&= \frac{1}{2}(A - iB)\frac{1}{P_n(D + \alpha + i\beta)}P_m(x) + \frac{1}{2}(A + iB)\frac{1}{P_n(D + \alpha - i\beta)}P_m(x) \qquad (122)
\end{aligned}
$$

For examples,

(1). A particular solution to the differential equation

$$y'' - 5y' + 6y = e^x \cos 2x + e^{2x} (3x + 4) \sin x \tag{123}$$

is

$$Y(x) = \frac{1}{D^2 - 5D + 6} \left[e^x \cos 2x + e^{2x} (3x + 4) \sin x \right] = Y_1(x) + Y_2(x), \tag{124}$$

$$Y_1(x) = \frac{1}{D^2 - 5D + 6} (e^x \cos 2x) = \frac{1}{(D - 2)(D - 3)} (e^x \cos 2x)$$

$$= e^x \frac{1}{(D - 1)(D - 2)} \cos 2x = e^x \frac{(D + 1)(D + 2)}{(D^2 - 1)(D^2 - 4)} \cos 2x$$

$$= \frac{1}{40} e^x \left(\frac{d}{dx} + 1 \right) \left(\frac{d}{dx} + 2 \right) \cos 2x = -\frac{1}{20} e^x (\cos 2x + 3 \sin 2x), \tag{125}$$

$$Y_2(x) = \frac{1}{D^2 - 5D + 6} \left[(3x + 4) e^{2x} \cos x \right]$$

$$= \frac{1}{2} \frac{1}{(D - 2)(D - 3)} \left[(3x + 4) \left(e^{(2+i)x} + e^{(2-i)x} \right) \right]$$

$$= \frac{1}{2} e^{(2+i)x} \frac{1}{(D + i)(D - 1 + i)} (3x + 4)$$

$$+ \frac{1}{2} e^{(2-i)x} \frac{1}{(D - i)(D - 1 - i)} (3x + 4)$$

$$= \frac{1}{2i} e^{(2+i)x} \frac{1}{D - 1 + i} (3x + 4 + 3i) - \frac{1}{2i} e^{(2-i)x} \frac{1}{D - 1 - i} (3x + 4 - 3i)$$

$$= -\frac{1}{2} e^{2x} \left[(3x + 10) \cos x + (3x + 1) \sin x \right] \tag{126}$$

(2). A particular solution of the equation

$$y'' + 2y' + 2y = e^{-x} \left(3 + 2 \sin x + 4x^2 \cos x \right) \tag{127}$$

is

$$Y(x) = \frac{1}{D^2 + 2D + 2} \left[e^{-x} \left(3 + 2 \sin x + 4x^2 \cos x \right) \right] = Y_1(x) + Y_2(x) + Y_3(x) \tag{128}$$

where

$$Y_1(x) = \frac{1}{D^2 + 2D + 2} (3e^{-x}) = \frac{3e^{-x}}{(-1)^2 - 2 + 2} = 3e^{-x},$$

$$Y_2(x) = \frac{1}{D^2 + 2D + 2} (2e^{-x} \sin x) = \frac{1}{(D + 1)^2 + 1} (2e^{-x} \sin x)$$

$$= 2e^{-x} \frac{1}{D^2 + 1} \sin x = -xe^{-x} \cos x, \tag{129}$$

$$Y_3(x) = \frac{1}{D^2 + 2D + 2} \left(4x^2 e^{-x} \cos x \right) = \frac{1}{(D + 1)^2 + 1} \left(4x^2 e^{-x} \cos x \right)$$

$$= \frac{1}{(D+1)^2+1}\left[2x^2\left(e^{(-1+i)x}+e^{(-1-i)x}\right)\right]$$

$$= 2e^{(-1+i)x}\frac{1}{D(D+2i)}x^2 + 2e^{(-1-i)x}\frac{1}{D(D-2i)}x^2$$

$$= -ie^{(-1+i)x}D^{-1}\left(x^2+ix-\frac{1}{2}\right) + ie^{(-1+i)x}D^{-1}\left(x^2-ix-\frac{1}{2}\right)$$

$$= \frac{1}{3}e^{-x}\left[3x^2\cos x + x(2x^2-3)\sin x\right] \tag{130}$$

V.   SUMMARY

We have introduced both the mathematical principle and computing technique of using differential operator method to find a particular solution of an ordinary nonhomogeneous linear differential equation with constant coefficients when the nonhomogeneous term is a polynomial function, exponential function, sine function, cosine function or any possible product of these functions. We have reviewed some rules in the differential operations including the eigenvalue substitution rule and the exponential shift rule. Furthermore, we have used a number of examples to illustrate the practical application of this approach. In particular, comparing with the introductions on this differential operator method scattered in some lecture notes, we propose and highlight the application of the inverse of differential operator in determining a particular solution, and overcome the difficulty when a polynomial of differential operator in a differential equation is singular. The differential operator method has great advantages over the well-known method of undetermined coefficients introduced in textbooks in determining a particular solution. Therefore, it is meaningful to make a systematic introduction and put an emphasis on this method. We hope that this method can be systematically introduced in textbooks and widely used for determining a particular solution of an ordinary nonhomogeneous linear differential equation with constant coefficients, in parallel to the method of undetermined coefficients.

Rusjan for various discussions and encouragements.

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[4] See for example, "math.mit.edu/ jorloff/suppnotes/suppnotes03/o.pdf"; "http://web.mit.edu/jorloff/www/18.03-esg/notes/topic6.pdf"

[5] See for example, D. Poole, *Linear Algeba*, 3rd edition, Brooks/Cole, 2011.