In proof by cases, should the implications for all the cases be true or is it sufficient if even one of the implications is true?

$(p \vee q \vee r) \to s$ ✓ $(p \to s) \wedge (q \to s) \wedge (r \to s)$

$\equiv \neg(p \vee q \vee r) \vee s$ ✗ $(p \to s) \vee (q \to s) \vee (r \to s)$

$\equiv (\neg p \wedge \neg q \wedge \neg r) \vee s$

$\equiv (\neg p \vee s) \wedge (\neg q \vee s) \wedge (\neg r \vee s)$

$\equiv (p \to s) \wedge (q \to s) \wedge (r \to s)$

How to prove the following statement by induction?

$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n \text{ for all } m \geq 1 \text{ and } n \geq 0.$$

① Induction on m (with n arbitrary)

② Induction on n (with m arbitrary)

③ Induction on $m+n$

② Base cases:

$n = 0$ LHS $= F_m$

RHS $= F_m F_1 + F_{m-1} F_0 = F_m$

$n = 1$ LHS $= F_{m+1}$

RHS $= F_m F_2 + F_{m-1} F_1 = F_m + F_{m-1}$

Induction

$n \geq 2$

$$\text{hyp:}\begin{cases} F_{m+n-1} = F_m F_n + F_{m-1} F_{n-1} \\ F_{m+n-2} = F_m F_{n-1} + F_{m-1} F_{n-2} \end{cases}$$

$$F_{m+n} = F_m(F_n + F_{n-1}) + F_{m-1}(F_{n-1} + F_{n-2})$$

$$= F_m F_{n+1} + F_{m-1} F_n$$

③ Induction on $m+n$

<u>Base</u> $m+n=1 \rightarrow m=1, n=0$

$m+n=2 \rightarrow m=1, n=1$

or $m=2, n=0$

<u>Induction</u> Take $m+n \geq 3$

True for $m+n-1$, $m+n-2$

Case 1: $m \geq 3$

$m+n-1 = (m-1)+n$

$m+n-2 = (m-2)+n$

Case 2: $n \geq 2$

$m+n-1 = m+(n-1)$

$m+n-2 = m+(n-2)$

Case 3: $m < 3$ and $n < 2$

$m \leq 2$ and $n \leq 1$

$m + n \geq 3$

$m = 2$ and $n = 1$

Prove that: $\gcd(F_{n+1}, F_n) = 1$ for all $n \geq 0$.

① $p \mid F_{n+1}$ and $p \mid F_n$

$F_{n-1} = F_{n+1} - F_n$ is divisible by p

$F_{n-2} = F_n - F_{n-1}$ is divisible by p

...

F_1 is divisible by p

$p \mid 1$ ↙

② By induction on n.

Prove that:

$$\gcd(F_m, F_n) = F_{\gcd(m,n)} \text{ for all } m, n \text{ (not both zero).}$$

$m \geq n$ $\qquad m = qn + r$

$\gcd(F_m, F_n) = \gcd(F_n, F_r)$

$m = n + k$ $\qquad F_m = F_n F_{k+1} + F_{n-1} F_k$

$\gcd(F_m, F_n) = \gcd(F_{n-1} F_k, F_n)$

$= \gcd(F_k, F_n)$

$= \gcd(F_{m-n}, F_n) = \gcd(F_{m-2n}, F_n)$

$= \cdots = \gcd(F_{m-qn}, F_n)$

$= \gcd(F_r, F_n)$

$= \gcd(F_n, F_r)$

$\gcd(c + kb, b)$

$= \gcd(c, b)$

How to prove the following statement?

$$\forall a, b, c \in \mathbb{N}\left[\left(\gcd(a,b)=1\right) \Rightarrow \exists x \in \mathbb{N}\left(\gcd(a+bx, c)=1\right)\right]$$

$a = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ $\quad e_i > 0$

$b = q_1^{f_1} q_2^{f_2} \cdots q_s^{f_s}$ $\quad f_i > 0$

x

$c = p_1^{u_1} p_2^{u_2} \cdots p_r^{u_r} q_1^{v_1} q_2^{v_2} \cdots q_s^{v_s} \pi_1^{w_1} \cdots \pi_t^{w_t}$

$u_i \geq 0, \; v_j \geq 0, \; w_k > 0$

Consider the following function with m and n non-negative integers.

```
int g ( int m, int n)
{
  if ((m == 0) || (n == 0)) return 1;
  return g(m,n-1) + g(m-1,n);
}
```

Express the return value of g(2,n) as a function of n.

$$g(0,n) = 1 \quad \forall\, n \geq 0$$

$$g(1,n) = g(1,n-1) + g(0,n)$$

$$= g(1,n-1) + 1$$

$$= g(1,n-2) + 2$$

$$\ldots$$

$$= g(1,0) + n = n+1$$

$$g(2,n) = g(2,n-1) + g(1,n) = g(2,n-1) + n+1$$

$$= g(2,n-2) + (n) + (n+1) = \ldots = g(2,0) + 2 + \ldots + n+1$$

$$= 1 + 2 + \ldots + n+1 = \frac{(n+1)(n+2)}{2}$$

```
for (i=1; i<=n; ++i) L[i] = 0;

for (i=1; i<=n; ++i)
    for (j=i; j<=n; j+=i)
        L[j] = 1 - L[j];
```

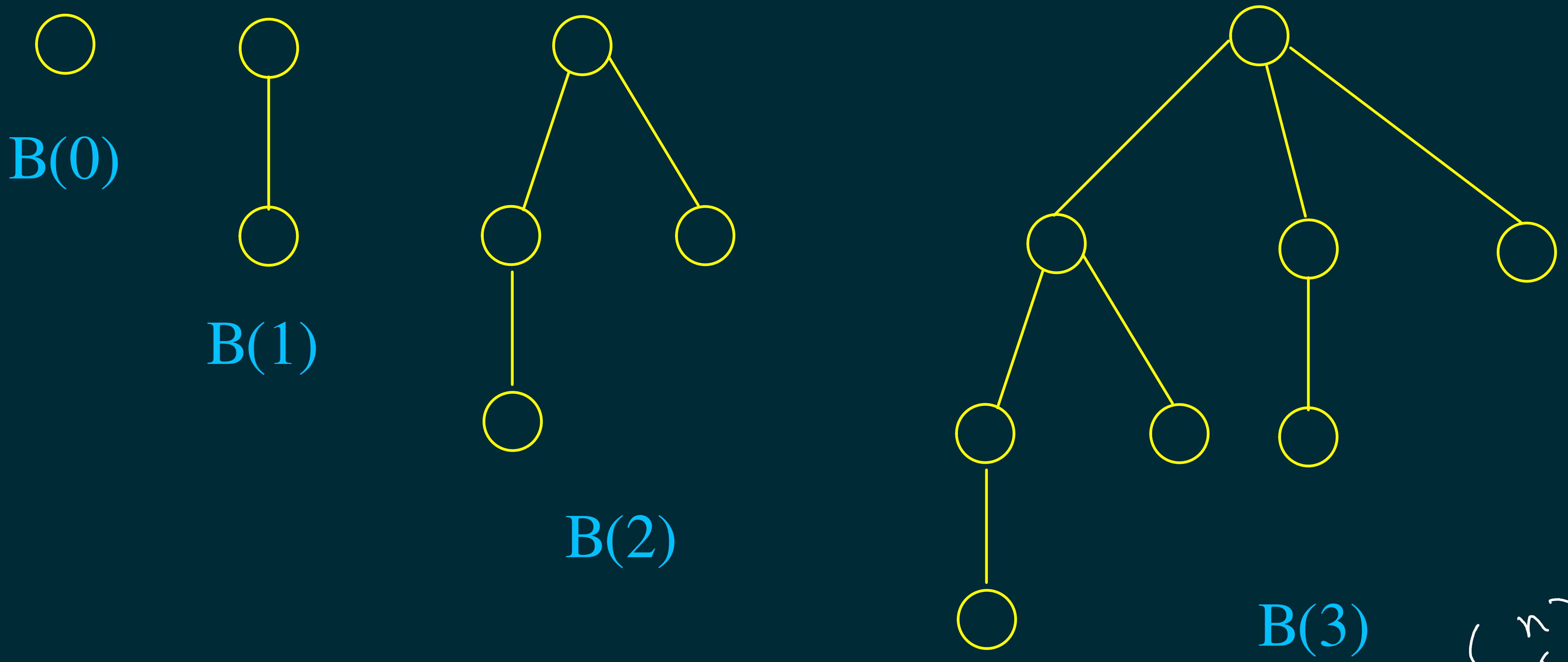
After this, L[i] = 1 for which values of i?

$$i = p_1^{e_1} \cdots p_r^{e_r}$$

# of factors of i

$$= (e_1+1)(e_2+1)\cdots(e_r+1)$$

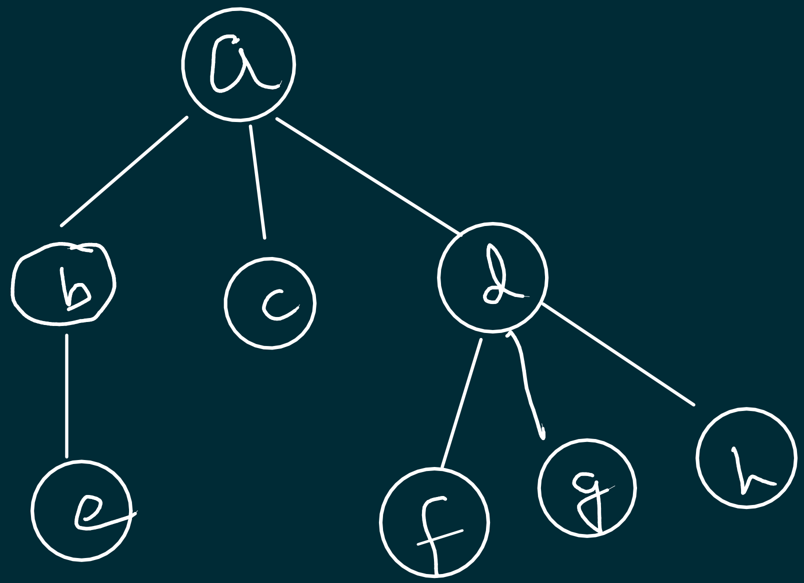
Binomial Trees



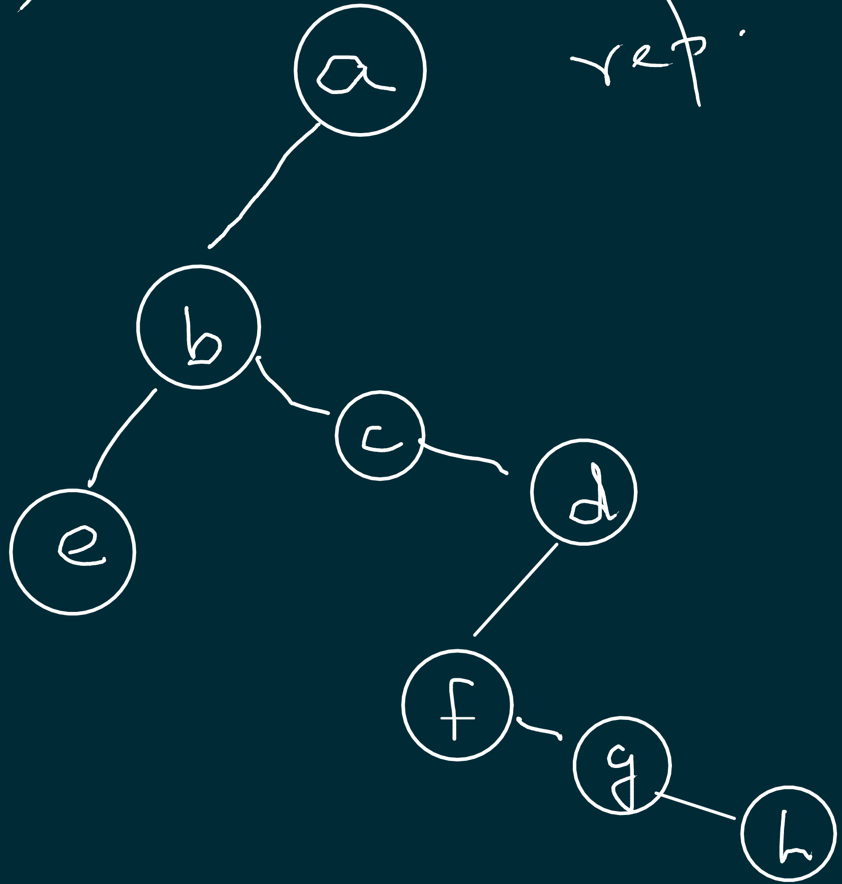
$\binom{n}{i} = 0$ if $n < i$

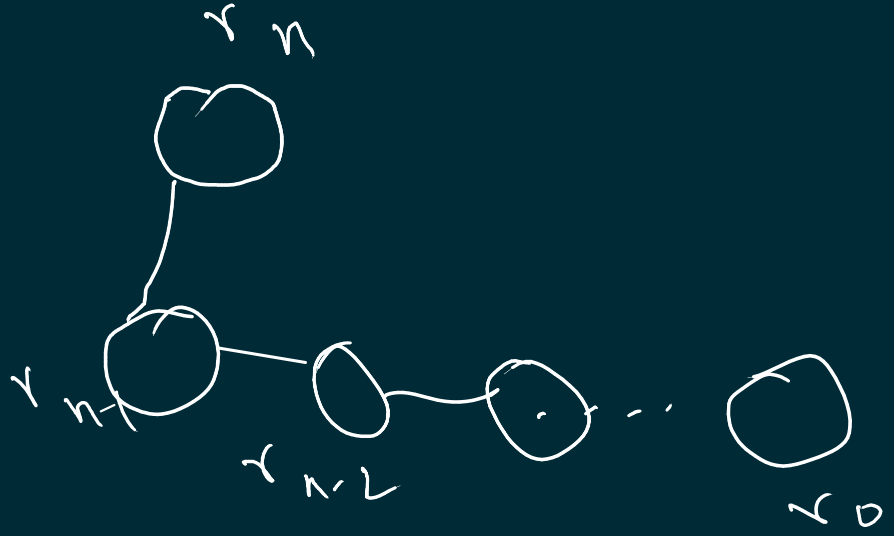
Prove that there are $\binom{n}{i}$ nodes in B(n) at level i.

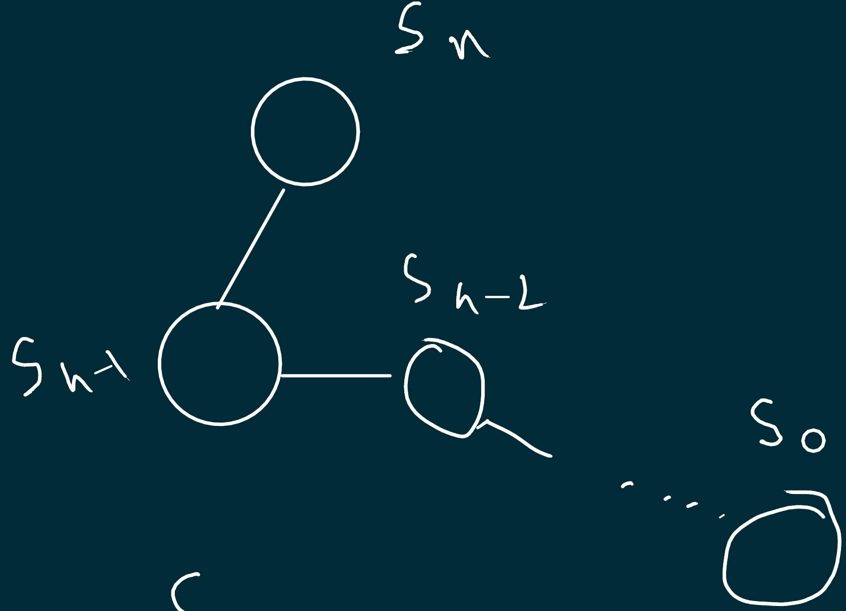
B(n) contains 2^n nodes (easy to prove by strong induction).
You are given two disjoint copies of B(n). How can you efficiently make a single copy of B(n+1)?



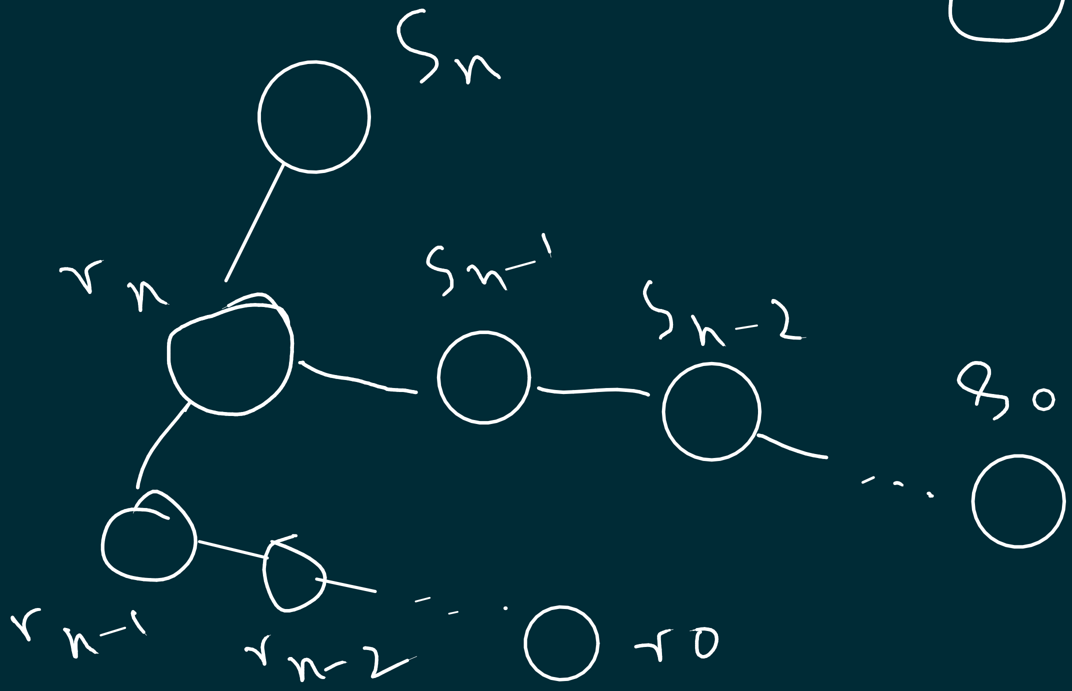
First-child-next-sibling rep.







$O(1)$ time



Binomial heaps