Damping coefficient: ζ, Natural frequency:ω_n & Damped natural frequency: ω_d

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This is to be read as continuation to the previous lecture on the second order system.

1 R-L-C series circuit switched on to a DC voltage

Let the the circuit shown in figure is initially relaxed. At $t = 0$, DC voltage V is switched on. We want to find out voltage across the capacitor $v(t)$ and current $i(t)$ in the circuit for $t \geq 0$.

$$
\begin{aligned}
\text{KVL equation: } L\frac{di}{dt} + Ri + v &= V \\
\text{Now, } i &= C\frac{dv}{dt} \\
\text{so, } LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v &= V \\
\text{or, } \frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v &= \frac{1}{LC}V
\end{aligned}
$$

We shall replace the parameter values of the above equation by two new variables ω_n and ζ as follows:

$$
\begin{aligned}
\omega_n &= \frac{1}{\sqrt{LC}} \\
\text{and } \frac{R}{L} &= 2\zeta\omega_n \\
\text{New equation: } \frac{d^2v}{dt^2} + 2\zeta\omega_n\frac{dv}{dt} + \omega_n^2 v &= V
\end{aligned}
$$

It will be shown that the physical interpretation of the response can now be made in terms of ζ (called damping coefficient) and ω_n (called the undamped natural frequency of the system).

$$
\begin{aligned}
\text{characteristic equation: } m^2 + 2\zeta\omega_n m + \omega_n^2 &= 0 \\
\text{characteristic roots: } m_{1,2} &= -\zeta\omega_n \pm \left(\sqrt{\zeta^2 - 1}\right)\omega_n
\end{aligned}
$$

1.1 Case-1: when roots are complex

Roots will be complex if $\zeta < 1$ and can be written as:

$$\begin{aligned} m_{1,2} &= -\zeta\omega_n \pm j\left(\sqrt{1-\zeta^2}\right)\omega_n = -\zeta\omega_n \pm j\omega_d \\ \text{where, } \omega_d &= \left(\sqrt{1-\zeta^2}\right)\omega_n \\ \text{It may be noted: } \omega_d^2 + (\zeta\omega_n)^2 &= \omega_n^2 \\ \text{Defining } \tan\theta &= \frac{\omega_d}{\zeta\omega_n} \end{aligned}$$

The figure 1 is very handy to remember relationship among ω_n, $\zeta\omega_n$ and ω_d and the angle θ.

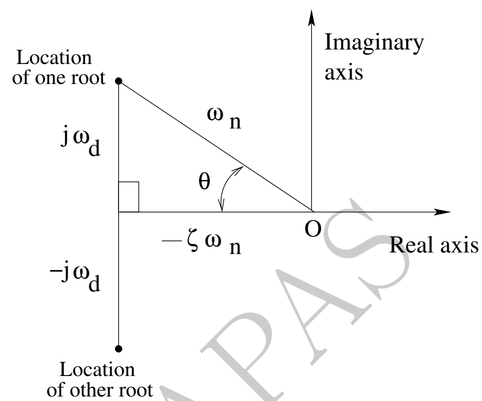

Figure 1

$$\begin{aligned} \therefore \text{ Natural response: } v_n(t) &= Ae^{(-\zeta\omega_n + j\omega_d)t} + Be^{(-\zeta\omega_n - j\omega_d)} \\ \text{forced response: } v_f(t) &= V \\ \therefore\ v(t) &= Ae^{(-\zeta\omega_n + j\omega_d)t} + Be^{(-\zeta\omega_n - j\omega_d)} + V \\ \text{and } i(t) = C\frac{dv}{dt} &= CA(-\zeta\omega_n + j\omega_d)e^{(-\zeta\omega_n + j\omega_d)t} + CB(-\zeta\omega_n - j\omega_d)e^{(-\zeta\omega_n - j\omega_d)t} \end{aligned}$$

Now apply the initial conditions: $v(0^-) = v(0^+) = 0$ and $i(0^-) = i(0^+) = 0$ to generate the following two equations for getting constants A and B.

$$
\begin{aligned}
A + B &= -V \\
(-\zeta\omega_n + j\omega_d)A + (-\zeta\omega_n - j\omega_d)B &= 0 \\
\text{solving we get:} & \\
A &= \frac{j(\zeta\omega_n + j\omega_d)}{2\omega_d} V = Re^{j\theta} \text{ say} \\
\text{where } R &= \frac{\sqrt{(\zeta\omega_n)^2 + \omega_d^2}}{2\omega_d} V \text{ and } \theta = \tan^{-1}\frac{\omega_d}{\zeta\omega_n} + \frac{\pi}{2} \\
\text{and } B &= \frac{-j(\zeta\omega_n - j\omega_d)}{2\omega_d} V = A^\star = Re^{-j\theta} \\
v(t) &= Re^{j\theta} e^{(-\zeta\omega_n + j\omega_d)t} + Re^{-j\theta} e^{(-\zeta\omega_n - j\omega_d)t} + V \\
\text{or, } v(t) &= Re^{-\zeta\omega_n t}\left[e^{j(\omega_d t + \theta)} + e^{-j(\omega_d t + \theta)}\right] V + V \\
\text{or, } v(t) &= 2Re^{-\zeta\omega_n t}\cos(\omega_d t + \theta) + V
\end{aligned}
$$

Putting values of R and θ from above:

$$
\begin{aligned}
\text{or, } v(t) &= \frac{\sqrt{(\zeta\omega_n)^2 + \omega_d^2}}{\omega_d} V e^{-\zeta\omega_n t} \cos\left(\omega_d t + \tan^{-1}\frac{\omega_d}{\zeta\omega_n} + \frac{\pi}{2}\right) + V \\
\text{or, } v(t) &= V - \frac{\sqrt{(\zeta\omega_n)^2 + \omega_d^2}}{\omega_d} V e^{-\zeta\omega_n t} \sin\left(\omega_d t + \tan^{-1}\frac{\omega_d}{\zeta\omega_n}\right) \\
\text{but } \omega_d^2 + (\zeta\omega_n)^2 &= \omega_n^2 \\
\therefore\ v(t) &= V - \frac{\omega_n}{\omega_d} V e^{-\zeta\omega_n t} \sin\left(\omega_d t + \tan^{-1}\frac{\omega_d}{\zeta\omega_n}\right) \\
\therefore\ v(t) &= V - \left(\frac{V}{\sqrt{1-\zeta^2}}\right) e^{-\zeta\omega_n t} \sin\left(\omega_d t + \tan^{-1}\frac{\omega_d}{\zeta\omega_n}\right) \\
\therefore\ v(t) &= V - V_{\text{peak}}\, e^{-\zeta\omega_n t} \sin\left(\omega_d t + \tan^{-1}\frac{\omega_d}{\zeta\omega_n}\right) \\
\text{where: } V_{\text{peak}} &= \left(\frac{V}{\sqrt{1-\zeta^2}}\right)
\end{aligned}
$$

The exponentially decaying sinusoidal terms will ultimately vanish and and final voltage across the capacitor will be V. The amplitude of the sinusoid is being modulated by $e^{-\zeta\omega_n t}$. Figures 2 and 3 show respectively the step response of a second order system for $\zeta = 0.5$ and $\zeta = 0.3$. With lesser value of ζ, the system becomes more oscillatory with higher value of overshoot.

$$\begin{aligned} &\text{For } \omega_n = 2 \;\&\; \zeta = 0.5 \\ &\qquad v(t) \;=\; 5 - 5.77e^{-t}\sin\left(1.732t + 60^\circ\right) \\ &\text{For } \omega_n = 2 \;\&\; \zeta = 0.3 \\ &\qquad v(t) \;=\; 5 - 5.24e^{-0.6t}\sin\left(1.91t + 72.54^\circ\right) \end{aligned}$$

$v(t) = 5 - 5.77e^{-t}\sin\left(1.732t + 60^\circ\right)$

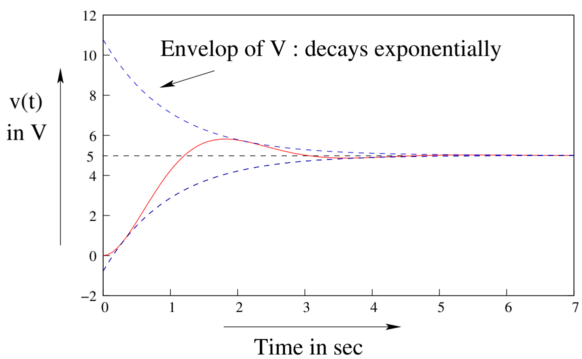

Figure 2: With $\zeta = 0.5$ & $\omega_n = 2$

$v(t) = 5 - 5.24e^{-0.6t}\sin\left(1.91t + 72.54^\circ\right)$

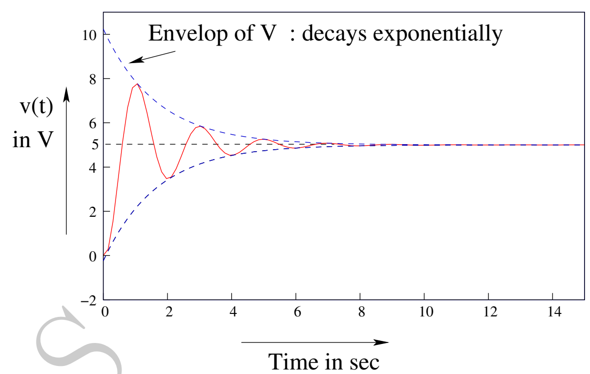

Figure 3: With $\zeta = 0.3$ & $\omega_n = 2$

1.2 Case-2: when roots are real and equal

We have:

$$\begin{aligned} \text{characteristic equation: } m^2 + 2\zeta\omega_n m + \omega_n^2 &= 0 \\ \text{characteristic roots: } m_{1,2} &= -\zeta\omega_n \pm \left(\sqrt{\zeta^2 - 1}\right)\omega_n \\ \text{roots are equal if : } \zeta &= 1 \\ \text{characteristic roots: } m_1 = m_2 &= -\zeta\,\omega_n = m \end{aligned}$$

Therefore solution for $v(t)$ in this case will be:

$$\begin{aligned}
v(t) &= (A + Bt)e^{-\zeta\,\omega_n t} + V \\
\text{initial conditions: } v(0) &= V \text{ and } i(0) = 0 \text{ give} \\
A + V &= 0 \text{ or } A = -V \\
\text{and } -\zeta\,\omega_n A + B &= 0 \text{ or} \\
\text{or } B &= -\zeta\,\omega_n V \\
\therefore\ v(t) &= V - Ve^{-\zeta\,\omega_n t} - \zeta\,\omega_n V t e^{-\zeta\,\omega_n t} \\
\text{current can be obtained from: } i(t) &= C\frac{dv}{dt}
\end{aligned}$$

Here v(t) will reach the final value of V without suffering any oscillation. Under this condition the system is said to be *critically damped*.

1.3 Case-3: when roots are real and distinct

We have:

$$\begin{aligned}
\text{characteristic equation: } m^2 + 2\zeta\omega_n m + \omega_n^2 &= 0 \\
\text{characteristic roots: } m_{1,2} &= -\zeta\omega_n \pm \left(\sqrt{\zeta^2 - 1}\right)\omega_n \\
\text{roots are real and distinct if : } \zeta &> 1 \\
\text{characteristic roots: } m_1 &= -\zeta\,\omega_n + \left(\sqrt{\zeta^2 - 1}\right)\omega_n \\
\text{and } m_2 &= -\zeta\,\omega_n - \left(\sqrt{\zeta^2 - 1}\right)\omega_n \\
\text{Note: } m_1 &> m_2 \\
\text{and } m_1 - m_2 &= 2\sqrt{\zeta^2 - 1}
\end{aligned}$$

Therefore solution for $v(t)$ in this case, will be:

$$
\begin{aligned}
v(t) &= Ae^{m_1 t} + Be^{m_2} + V \\
\text{initial conditions: } v(0) &= V \text{ and } i(0) = 0 \text{ give} \\
A + B + V &= 0 \text{ or } A + B = -V \\
\text{and } m_1 A + m_2 B &= 0 \\
\text{Solving, we get: } A &= \frac{m_2 V}{(m_1 - m_2)} \\
\text{and } B &= -\frac{m_1 V}{(m_1 - m_2)} \\
\therefore\ v(t) &= \frac{m_2 V}{(m_1 - m_2)} e^{m_1 t} - \frac{m_1 V}{(m_1 - m_2)} e^{m_2 t} + V \\
\text{or, } v(t) &= V + \frac{-\zeta\,\omega_n - \left(\sqrt{\zeta^2 - 1}\right) V}{2\sqrt{\zeta^2 - 1}} e^{\left(-\zeta\,\omega_n + \sqrt{\zeta^2-1}\,\omega_n\right)t} \\
&\quad - \frac{-\zeta\,\omega_n + \left(\sqrt{\zeta^2 - 1}\right)\,\omega_n V}{2\sqrt{\zeta^2 - 1}} e^{\left(-\zeta\,\omega_n - \sqrt{\zeta^2-1}\,\omega_n\right)t} \\
\text{current can be obtained from: } i(t) &= C\frac{dv}{dt}
\end{aligned}
$$