Signals Basics

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1 Introduction

We shall in this section consider signals which vary with time. Value of the signal at different time may be shown graphically or if we are lucky the signal can be represented mathematically in terms of an equation.

1.1 Time Reversal: To get $x(-t)$ from $x(t)$

If we know, $x(t)$ then mathematically $x(-t)$ can be obtained by replacing t with $-t$ in $x(t)$. Naturally, both $x(t)$ and $x(-t)$ can be plotted. Essentially $x(-t)$ is nothing but the mirror image of $x(t)$ about the vertical axis. Also it can be interpreted as folding $x(t)$ about the vertical axis to get $x(-t)$. The idea is illustrated in self explained figure 1(a) and (b).

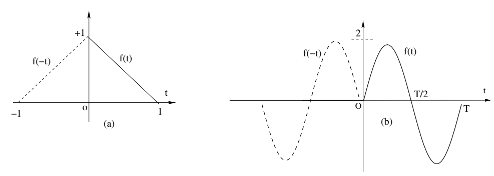

Figure 1: Getting $x(-t)$ from $x(t)$

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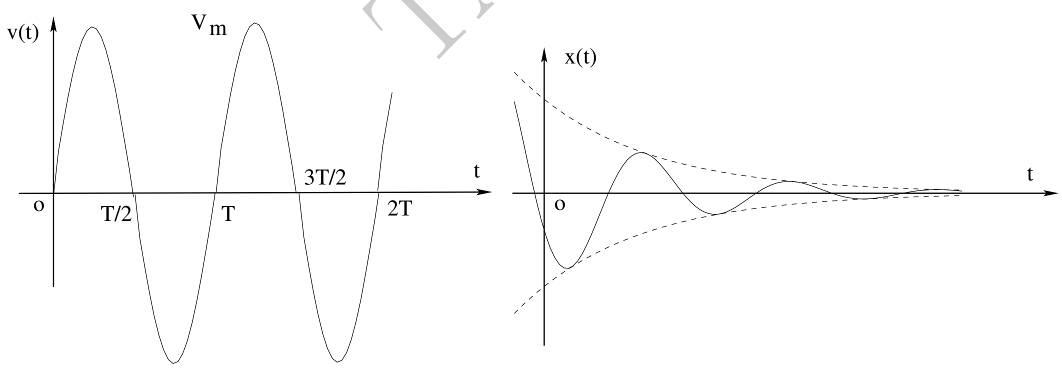
1.2 Types of signals

Usually these type of signals can be nicely expressed in terms of equations as shown in figure 2



(a) Sinusoidal

(b) Exponentially decaying sinusoid

Figure 2: Continuous time signals

Some more functions or signals, called the singularity functions having discontinuities at certain points are shown in figure 3.

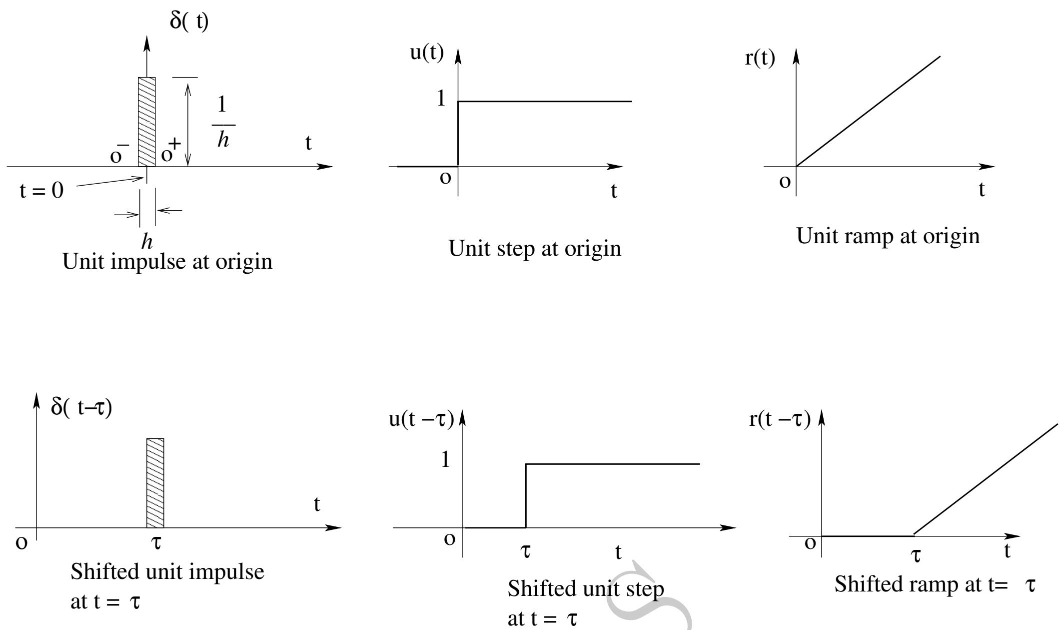


Figure 3: Singularity functions

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1.3 Properties of signals

2 Singularity functions

In this section we shall take up most useful singularity functions one at a time and discuss about its properties.

2.1 Impulse or Delta function

In figure 4(a) & (b), an unit impulse function is shown. The function is shown as thin rectangular pulse, centered around the the origin. The width of the pulse is shown to be h and it's height to be $1/h$. The area enclosed by the pulse is unity, no matter whatever is the value of h you choose. This general rectangular pulse will become an unit impulse function as $h \to 0$ i.e., $h = 0^+ - 0^-$, where 0^+ and 0^- are small perturbations in time in the negative and positive direction of time axis around $t = 0$. So the the height of the pulse tends to infinity and width tends to zero. Thus no

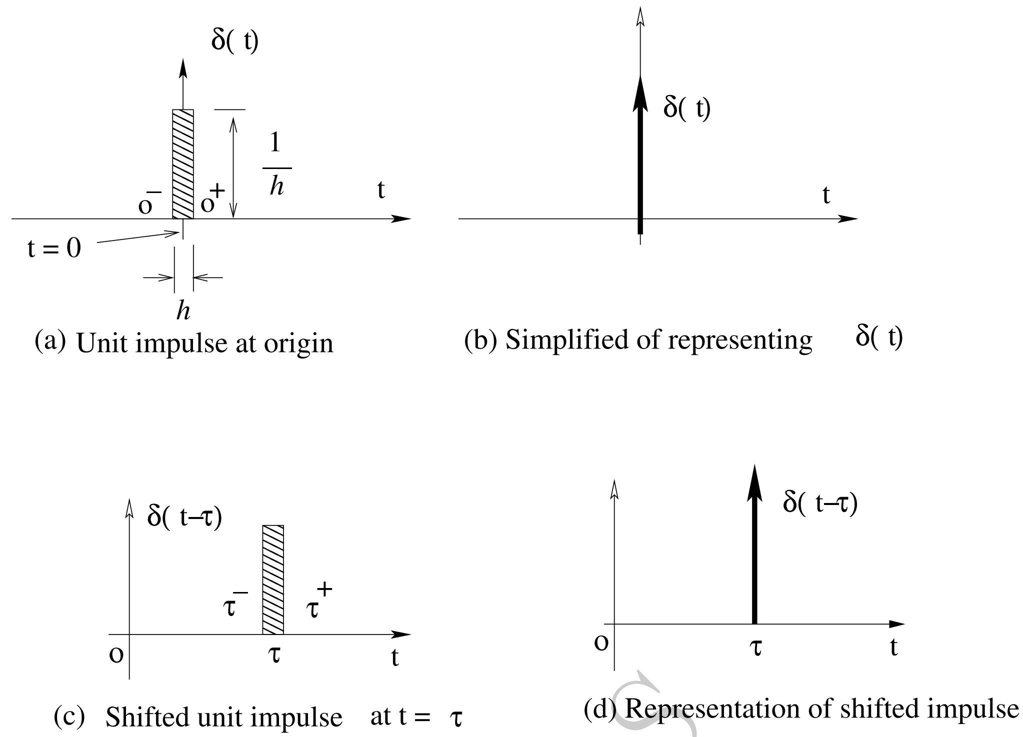

Figure 4: Impulse function

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point in talking about the functional value of this function at $t = 0$ as it is infinitely large. However area under the impulse is finite and equal to unity and this can be mathematically described as:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = \int_{0^-}^{0^+} \delta(t)\, dt = 1 \text{ since elsewhere, } \delta(t) = 0$$

The area enclosed is the strength of the impulse. For example $2\delta(t)$ represents an impulse of strength 2. Similarly for a shifted delta function $\delta(t-\tau)$ as shown in figure 4(c) & (d), following relation holds good.

$$\int_{-\infty}^{\infty} \delta(t-\tau)\, dt = \int_{\tau^-}^{\tau^+} \delta(t-\tau)\, dt = 1 \;\; \text{since, } \delta(t-\tau) = 0 \text{ for, } t \neq \tau$$

The consequence of this is that when you multiply a continuous function $x(t)$ with $\delta(t)$, following relations are true.

$$x(t)\delta(t) = x(0)\delta(t) \text{ and } x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau)$$

We can generate any continuous arbitrary function $x(t)$, using delta function as follows:

$$\begin{aligned}
\text{Consider the integral} \qquad & \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \\
= & \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,d\tau \\
= & \; x(t)\int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau \\
= & \; x(t)
\end{aligned}$$

Thus finally,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

Representation of unit impulse function is shown with a arrowed line as depicted in figure 4(c)

2.1.1 Time scaling of impulse function : $\delta(at)$

It will be shown here that $\delta(at) = \frac{1}{|a|}\delta(t)$ where a could be +ve or -ve real numbers. Let us calculate the integral:

$$I = \int_{-\infty}^{+\infty} x(t)\delta(at)\,dt$$

Now,

$$\begin{aligned}
I &= \int_{-\infty}^{+\infty} x(t)\delta(at)\,dt \\
\text{substitute } at = \tau & \qquad \text{and assume } a > 0 \\
\text{or, } I &= \int_{-\infty}^{+\infty} x(\tau/a)\,\frac{\delta(\tau)}{a}d\tau = \int_{-\infty}^{+\infty} x(0)\,\frac{\delta(\tau)}{a}d\tau \\
\text{thus, } I = \int_{-\infty}^{+\infty} x(t)\delta(at)\,dt &= \int_{-\infty}^{+\infty} x(\tau)\,\frac{\delta(\tau)}{a}d\tau \\
\text{Comparing we get, } \delta(at) &= \frac{1}{a}\delta(t)
\end{aligned}$$

We can repeat the above steps when a is -ve and will arrive at $\delta(at) = \frac{1}{|a|}\delta(t)$ which will be true for both +ve and -ve values of a.

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2.1.2 What about : $\delta(at+b)$?

Now

$$\delta(at+b) = \delta\left[a\left(t+\frac{b}{a}\right)\right] = \frac{1}{|a|}\delta\left(t+\frac{b}{a}\right)$$

This can be easily proved starting from basic integral as follows:

$$\begin{aligned}
I &= \int_{-\infty}^{+\infty} x(t)\delta(at+b)\,dt = \int_{-\infty}^{+\infty} x(t)\delta\left[a\left(t+\frac{b}{a}\right)\right]\,dt \\
\text{Substitute } t+\frac{b}{a} &= \tau \text{ and proceed to get the result shown below} \\
\delta(at+b) &= \delta\left[a\left(t+\frac{b}{a}\right)\right] = \frac{1}{|a|}\delta\left(t+\frac{b}{a}\right) \\
\text{which naturally means, } \delta\left[a(t+c)\right] &= \frac{1}{|a|}\delta\left(t+c\right)
\end{aligned}$$

2.1.3 What about : $x(t)\frac{d\{\delta(t)\}}{dt}$?

It will be shown here that $x(t)\frac{d\{\delta(t)\}}{dt} = -\frac{dx}{dt}(0)\,\delta(t) = -\dot{x}(0)\delta(t)$ We start with the integral :

$$I = \int_{-\infty}^{+\infty} x(t)\,\frac{[d(\delta(t)]}{dt}dt$$

Integrating the above by parts considering the first function to be $x(t)$ and the second function to be $\frac{d\{\delta(t)\}}{dt}$.

$$\begin{aligned}
I &= \int_{-\infty}^{+\infty} x(t)\,\frac{\{d(\delta(t)\}}{dt}dt \\
\text{or, } I &= x(t)\delta(t)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\frac{dx}{dt}\delta(t)\,dt = -\int_{-\infty}^{+\infty}\frac{dx}{dt}(0)\delta(t)\,dt \\
\therefore \int_{-\infty}^{+\infty} x(t)\,\frac{\{d(\delta(t)\}}{dt}dt &= -\int_{-\infty}^{+\infty}\frac{dx}{dt}(0)\delta(t)\,dt = -\frac{dx}{dt}(0)
\end{aligned}$$

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2.1.4 What about : $\int_{-\infty}^{+\infty} x(t)\delta\{f(t)\}\,dt$?

If an impulse function is described as $\delta\{f(t)\}$ where $f(t)$ is a function having several real roots, say a_1, a_2 $\cdots$ a_i etc. Therefore, it is expected we shall have a number of impulses occurring at those roots. However the strengths of those impulses may not be the same.

Consider a general root a_i which makes $f(a_i) = 0$. For this general root a_i, we shall first show that

$$\int_{-\infty}^{+\infty} x(t)\delta\{f(t)\}\,dt = \int_{-\infty}^{+\infty} x(t)\frac{\delta(t-a_i)}{|\dot{f}(a_i)|} = \frac{x(a_i)}{|\dot{f}(a_i)|}$$

Which essentially means that

$$\delta\{f(t)\} = \frac{\delta(t-a_i)}{|\dot{f}(a_i)|}$$

Now the value of the function $f(t)$, very close to a_i i.e., $(t-a_i) \to 0$ can be obtained from Taylor series expansion as follows:

$$\begin{aligned} f(t) &\approx f(a_i) + \dot{f}(a_i)(t-a_i) \text{ neglecting higher order terms} \\ \text{or, } f(t) &= \dot{f}(a_i)(t-a_i) \;\text{ since } f(a_i) = 0 \end{aligned}$$

Therefore for the i^{th} root we can write

$$\begin{aligned} \int_{-\infty}^{+\infty} x(t)\delta\{f(t)\}\,dt &= \int_{-\infty}^{+\infty} x(t)\delta\{\dot{f}(a_i)(t-a_i)\}\,dt \\ \text{Recall that } \delta\{a(t-b)\} &= \frac{\delta(t-b)}{|b|} \text{ use this in above equation to get} \\ \int_{-\infty}^{+\infty} x(t)\delta\{\dot{f}(a_i)(t-a_i)\}\,dt &= \int_{-\infty}^{+\infty} x(t)\frac{\delta(t-a_i)}{|\dot{f}(a_i)|}\,dt = \frac{x(a_i)}{|\dot{f}(a_i)|} \text{ for a single root } a_i \\ \text{For all the roots } \int_{-\infty}^{+\infty} x(t)\delta\{f(t)\}\,dt &= \sum_i \frac{x(a_i)}{|\dot{f}(a_i)|} \end{aligned}$$

Thus $\delta\{f(t)\}$ is nothing but a collection of as many impulses as the number of real roots with different strengths. That is

$$\delta\{f(t)\} = \sum_i \frac{\delta(t-a_i)}{|\dot{f}(a_i)|}$$

2.2 Unit step function

Unit step function shown in figure 5(a), can be described mathematically as follows:

$$u(t) = \begin{cases} 0, & \text{if } t < 0; \\ 1, & \text{if } t > 0; \end{cases}$$

Note that $u(t)$ is not defined at $t = 0$. Similarly mathematical description of the shifted unit step

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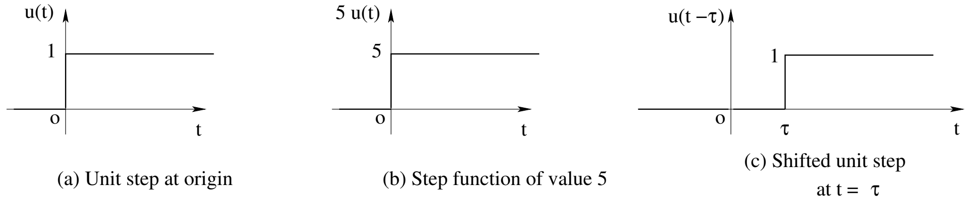

Figure 5: Step function

function shown in figure 5(c) is as follows:

$$u(t-\tau) = \begin{cases} 0, & \text{if } t < \tau; \\ 1, & \text{if } t > \tau; \end{cases}$$

Here also the step function is not defined at $t = \tau$. Figure 5(b), shows a step function of strength 5, i.e., it shows the plot of $5u(t)$.

2.3 Relationship between impulse & step function

Here we shall show that by integrating a delta function we shall get the step function and by differentiating a step function, delta function can be obtained.

$$\begin{aligned} \text{consider the integral} \quad & \int_{-\infty}^{t} \delta(\tau)\, d\tau \\ \text{and note} \int_{-\infty}^{t} \delta(\tau)\, d\tau &= 0 \text{ if } t < 0 \\ &= 1 \text{ if } t > 0 \\ \therefore \int_{-\infty}^{t} \delta(\tau)\, d\tau &= u(t) \\ \text{We can also conclude from this, } \frac{du}{dt} &= \delta(t) \end{aligned}$$

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2.4 Ramp function

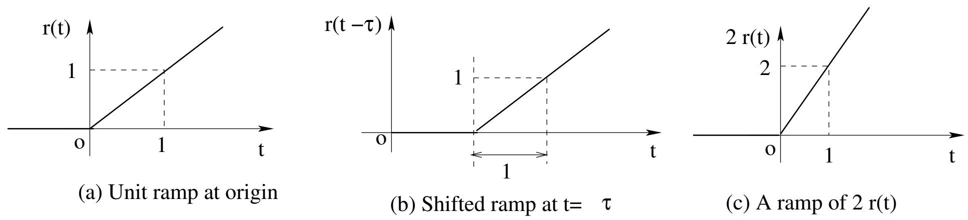

Figure 6: Ramp functions

An unit ramp function is shown in figure 6(a) and it is mathematically described as :

$$r(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ t, & \text{if } t \geq 0; \end{cases}$$

In the same way, a shifted unit ramp, shown in figure 6(b) can be described as follows:

$$r(t) = \begin{cases} 0, & \text{if } t \leq \tau; \\ t-\tau, & \text{if } t \geq \tau; \end{cases}$$

Finally a ramp with a higher slope $2r(t)$ is shown in figure 6(c) and it's mathematical description is:

$$r(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 2t, & \text{if } t \geq 0; \end{cases}$$

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2.5 Relationship between step & ramp function

The relationships are obvious and mentioned below. If an unit step function is integrated, unit ramp will be obtained and if a ramp is differentiated, step function will be obtained.

$$\begin{aligned} r(t) &= \int_{-\infty}^{t} u(t)\, dt \\ u(t) &= \frac{dr}{dt} \\ \text{also earlier, we have already got } \int_{-\infty}^{t} \delta(\tau)\, d\tau &= u(t) \\ \frac{du}{dt} &= \delta(t) \end{aligned}$$

The above relations can be visually more appealing as shown in figure 7

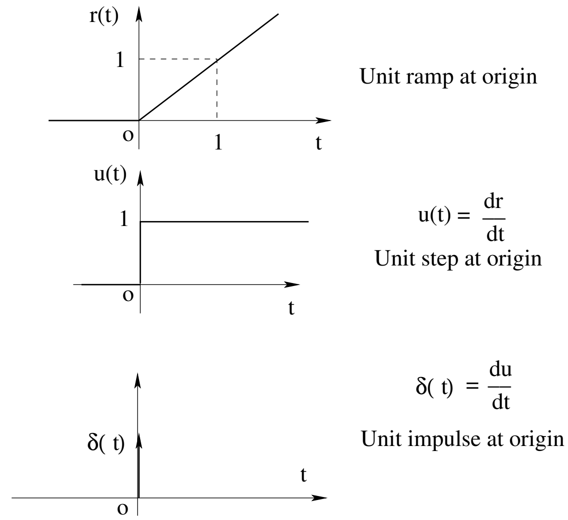

Figure 7: Relationship among Ramp,unit & impulse functions

2.6 Expressing a given signal in terms of singularity functions

Suppose a given signal $x(t)$ is multiplied with $u(t)$, then it means a signal whose value will be zero for $t < 0$, since $u(t) = 0$ for $t < 0$ and it will be $x(t)$ for $t > 0$ as $u(t) = 1$ for $t > 0$. In figure 8(a), we have plotted $x(t) \sin \omega t$ and its range is $-\infty < t < \infty$. If now this x(t) is multiplied by $u(t)$, the resulting function will become as shown in figure 8(c). The usefulness of the step and ramp functions are shown in the following example.

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Example

Write down the description of $x_1(t)$ and $x_2(t)$ shown in figure 9(a) and (b).

The expressions are written by inspection of the functions as:

$$x_1(t) = u(t) + u(t-1) - u(t-2) - u(t-3)$$

$$x_2(t) = r(t) - r(t-1) - 2u(t-3) + \frac{1}{2}r(t-3)$$

It should be noted that both the functions $x_1(t)$ and $x_2(t)$ can be expressed, if we like, elaborately

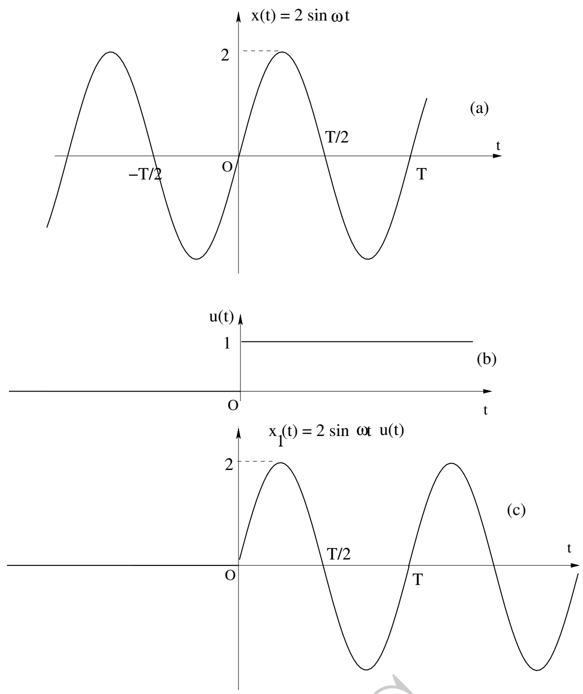

Figure 8: Sinusoidal signal multiplied with step function

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in different time zones. For $f_1(t)$, it will look like:

$$x_1(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } 0 < t < 1; \\ 2, & \text{if } 1 < t < 2; \\ 1, & \text{if } 2 < t < 3; \\ 0, & \text{if } t > 3; \end{cases}$$

We can easily see the advantage of using step and ramp functions for expressing a given function $x(t)$.

3 Even & odd functions

Any function may be either even or odd or neither even or odd. If $x(t) = x(-t)$, then the function is said to be even and if $x(t) = -x(-t)$, then the function is said to be odd. If function $x(t)$, does not satisfies any of the above condition, it is said to be neither odd nor even. Look at figure 10, where an even, odd and a general functions are shown.

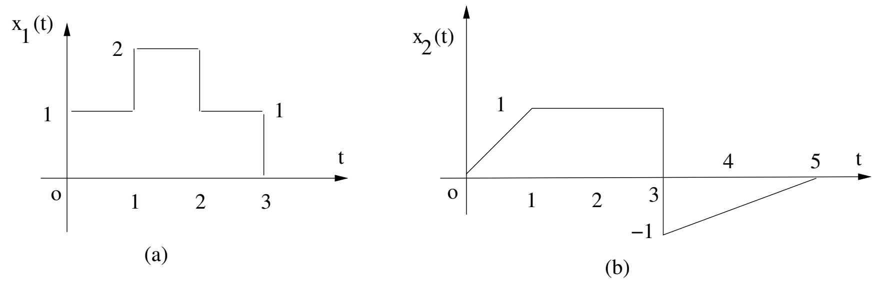

Figure 9: Examples with step & ramp functions.

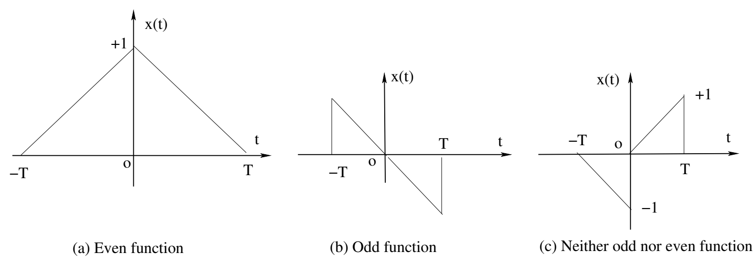

Figure 10: Even, odd & a general function

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One can easily verify that cosine function (cos t) is even while a sine function (sin t) is odd. It is interesting to note that a general function $x(t)$ can be shown to be a sum of an even function and an odd function. The even part and the odd part can be easily calculated as follows.

$$\begin{aligned}
\text{Let, } x(t) &= x_{\text{even}}(t) + x_{\text{odd}}(t) \\
\text{Then, } x(-t) &= x_{\text{even}}(-t) + x_{\text{odd}}(-t) \\
\text{Now applying properties of even \& odd functions:} & \\
\text{or, } x(-t) &= x_{\text{even}}(t) - x_{\text{odd}}(t) \\
\text{Now manipulating the first \& third equations:} & \\
x_{\text{even}}(t) &= \frac{x(t) + x(-t)}{2} \\
x_{\text{odd}}(t) &= \frac{x(t) - x(-t)}{2}
\end{aligned}$$

Example

Is the function $x(t) = 2t + 3$ is even, odd or neither (general)?, If the answer is neither, find out the even and odd part of the function.

Since $x(t) \neq x(-t)$ or $x(t) \neq -x(-t)$, we conclude that the function is general.

$$
\begin{aligned}
x_{\text{even}}(t) &= \frac{x(t)+x(-t)}{2} = \frac{1}{2}(2t+3-2t+3) = 3 \\
x_{\text{odd}}(t) &= \frac{x(t)-x(-t)}{2} = \frac{1}{2}(2t+3+2t-3) = 2t
\end{aligned}
$$

The problem just solved, a bit trivial in the sense that from the original function itself we could make out the answer.

3.1 Time scaling of a function

If a function $x(t)$ is known, then we shall be able to know $x(at)$ where a is a factor by which the time axis has been scaled. In essence, we have to replace t in $x(t)$, by at in order to get $x(at)$. Visual effect of time scaling of a signal $x(t)$ is shown in 11.

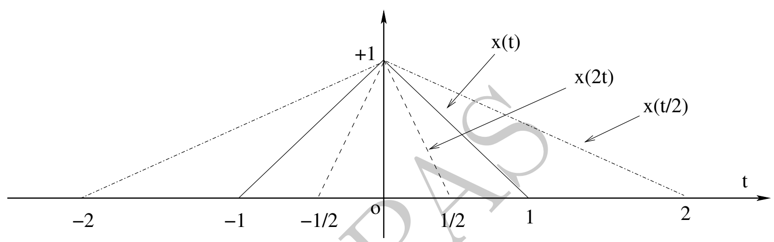

Figure 11: Time scaling of a signal

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The triangular pulse $x(t)$ is shown by firm line and its mathematical expression is

$$
\begin{aligned}
x(t) &= t+1 \text{ for, } \; -1 \leq t \leq 0 \\
x(t) &= 1-t \text{ for, } \; 0 \leq t \leq 1
\end{aligned}
$$

Now let us now try to find out $x(2t)$. Replace t by $2t$ in the above equation.

$$
\begin{aligned}
x(2t) &= 2t+1 \text{ for, } \; -1/2 \leq t \leq 0 \\
x(t) &= 1-2t \text{ for, } \; 0 \leq t \leq 1/2
\end{aligned}
$$

The effect of time scaling with $a = 2$, is to make the triangular pulse $x(2t)$ compressed between -1/2 to +1/2 as shown in figure 11 by dashed line. In the same way, if we choose scaling factor to be $a = 1/2$, the resulting waveform of $x(t/2)$ will be expanded between -2 to +2 as shown by the chain-dotted line in figure 11.

Example-1

A function $x(t)$ is shown in figure 12(a), plot $x(-2t-2)$.
It may be noted that all the three operations namely time shift,time scaling and time reversal operations to be executed on $x(t)$. Follow the following steps to get $x(-2t-2)$ from $x(t)$.

1. Do right shift operation to get $x(t-2)$ from $x(t)$ as shown in figure 12(b).
2. Now do time time scaling on $x(t-2)$ to get $x(2t-2)$ as shown in figure 12(c).
3. Finally do time reversal (or folding) operation on $x(2t-2)$ to get $x(-2t-2)$ as shown in 12(d)

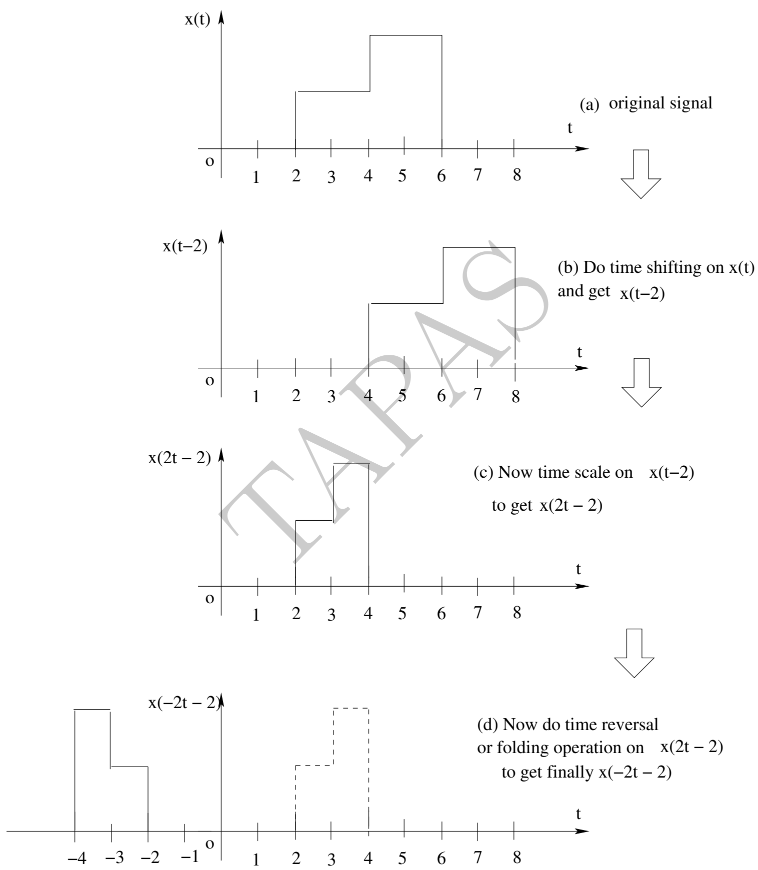

Figure 12: Time scaling of a signal

As discussed in the class, we could also get $x(-2t-2)$ from $x(t)$ following some other rules. End result will of course be same. I prefer the rule : **Shift** $\rightarrow$ **Scale** $\rightarrow$ **Reversal**.

4 Energy and Power of a signal

A signal may be also classified as *energy signal* or *power signal* or *neither energy or power signal.* Here we shall discuss how the energy and power associated with a given signal are calculated.

4.1 Energy of a signal

For a signal $x(t)$, total energy of the signal is defined as:

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt$$

If E is found to be finite and non-zero, then the signal is said to be energy signal.

4.2 Power of a signal

For a signal $x(t)$, average power of the signal is defined as:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

If P is found to be finite and non-zero, then the signal is said to be power signal.
Example-1 Is the unit step function $u(t)$ an energy or a power signal?
Solution: Let us first calculate energy associated with $u(t)$.

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$$\begin{aligned}
E &= \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt \\
&= \lim_{T\to\infty} \int_{-T}^{T} |u(t)|^2 \, dt \\
&= \lim_{T\to\infty} \int_{-T}^{T} dt \\
\lim_{T\to\infty} E &= (2T) \\
\therefore\ E &\rightarrow \infty
\end{aligned}$$

so, $u(t)$ is not an energy signal

Now let us calculate power associated with $u(t)$.

$$\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |u(t)|^2\,dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt \\
\lim_{T\to\infty} P &= \frac{1}{2T}(2T) = 1 \\
\therefore\ P \text{ is finite} & \\
\text{so, } u(t) \text{ is a power signal} &
\end{aligned}$$

Example-2 Is the function $\sin 5t$ an energy or a power signal?
Solution: Let us first calculate energy associated with $\sin\omega t$.

$$\begin{aligned}
E &= \lim_{T\to\infty}\int_{-T}^{T} \sin^2 5t\,dt \\
&= \lim_{T\to\infty}\int_{-T}^{T} (1-\cos 10t)\,dt \\
&= \lim_{T\to\infty}\int_{-T}^{T} dt \\
&= \lim_{T\to\infty}(2T) \\
\therefore\ \lim_{T\to\infty}\int_{-T}^{T} \sin^2 5t\,dt &\to \infty \\
\text{so, } \sin 5t \text{ is not an energy signal} &
\end{aligned}$$

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Let us now calculate the power associated with $\sin 5t$.

$$
\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sin^2 5t\, dt \\
&= \lim_{T\to\infty} \frac{1}{4T}\int_{-T}^{T} (1-\cos 10t)\, dt \\
&= \lim_{T\to\infty} \frac{1}{4T}\int_{-T}^{T} dt \\
&= \lim_{T\to\infty} \frac{1}{4T}(2T) = \frac{1}{2} \text{ finite} \\
\therefore \lim_{T\to\infty} \int_{-T}^{T} \sin^2 5t\, dt &= \frac{1}{2}
\end{aligned}
$$

so, $\sin 5t$ is a power signal

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