1. Let $L_k = \{ w \in \{0,1\}^* \mid \text{the k-th last symbol of w is } 1 \}$. Prove that:

(a) Any DFA for L_k contains at least 2^k states.

(b) Any NFA for L_k contains at least $k+1$ states.

Suppose that some NFA N with $\leq k$ states accepts L_k.

At most 2^k subsets of Q.

$\hat{\Delta}(s, 00\ldots0) \subseteq Q$

$\hat{\Delta}(s, 00\ldots1) \subseteq Q$

$\hat{\Delta}(s, 00\ldots10) \subseteq Q$

$\vdots$

$\hat{\Delta}(s, \underbrace{11\ldots1}_{\text{all k-length strings}}) \subseteq Q$

<u>Case 1</u>: $\hat{\Delta}(s, \alpha) = \emptyset$ $\times$

— gets <u>stuck</u> at some point

N also gets stuck on $\alpha 1 \beta$ for any β with $|\beta| = k-1$

<u>Case 2</u>: All non-empty

$$\alpha_i \; \underbrace{[\;\ldots\;0\;\ldots\;]}_{\leftarrow \ell-1 \rightarrow}\underbrace{[\;\gamma\;]}_{\leftarrow k-1 \rightarrow}$$

$$\alpha_j \; [\;\ldots\;1\;\ldots\;][\;\gamma\;]$$

$P = \hat{\Delta}(s, \alpha_i)$

$= \hat{\Delta}(s, \alpha_j)$

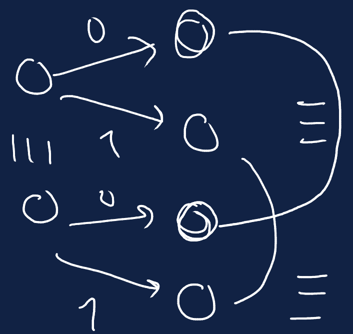
$\hat{\Delta}(P, \gamma) = R$

2. Consider the language

$A_k = \{w \in \{0,1\}^* \mid \text{The last } k \text{ positions in } w \text{ have at least one } 1\}$.

(a) Find the minimum number of states that a DFA accepting A_k must have.

(b) Design the minimal DFA for A_k.



Remember last k symbols read.

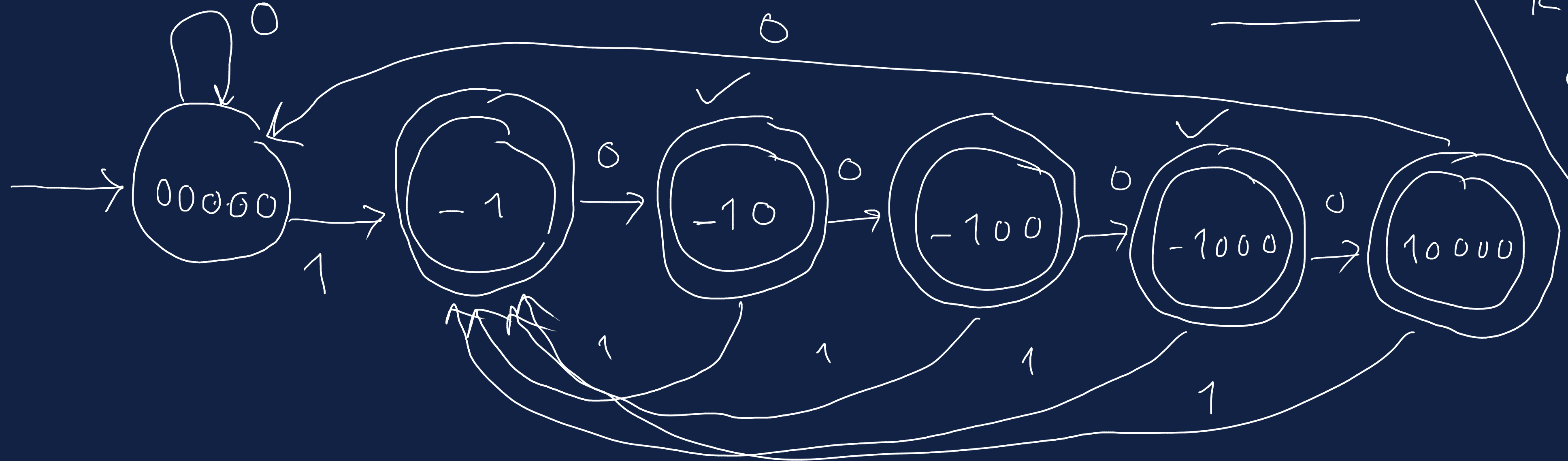
$\Rightarrow$ A DFA with 2^k states.

(a) It suffices to remember when the last 1 is read.

$k = 5$

$k+1$ states suffice.

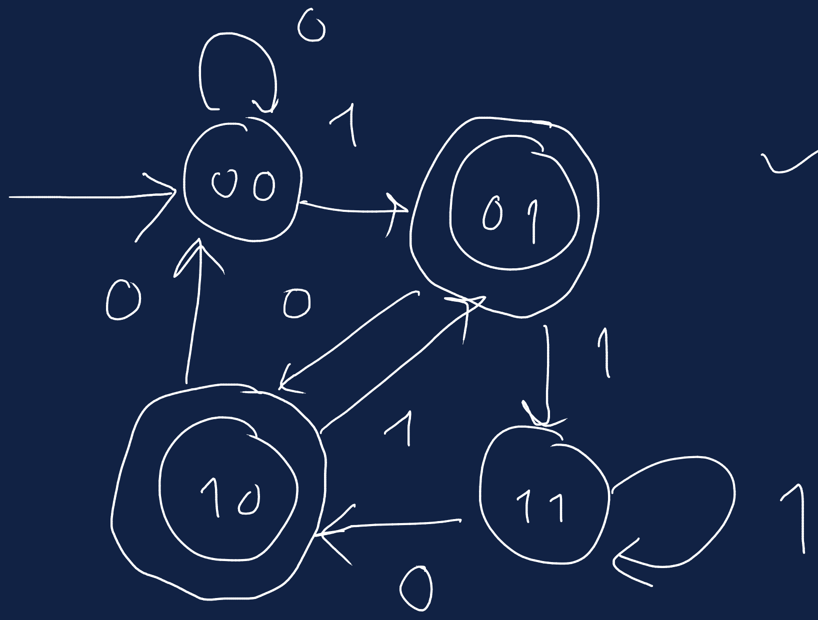
(b)



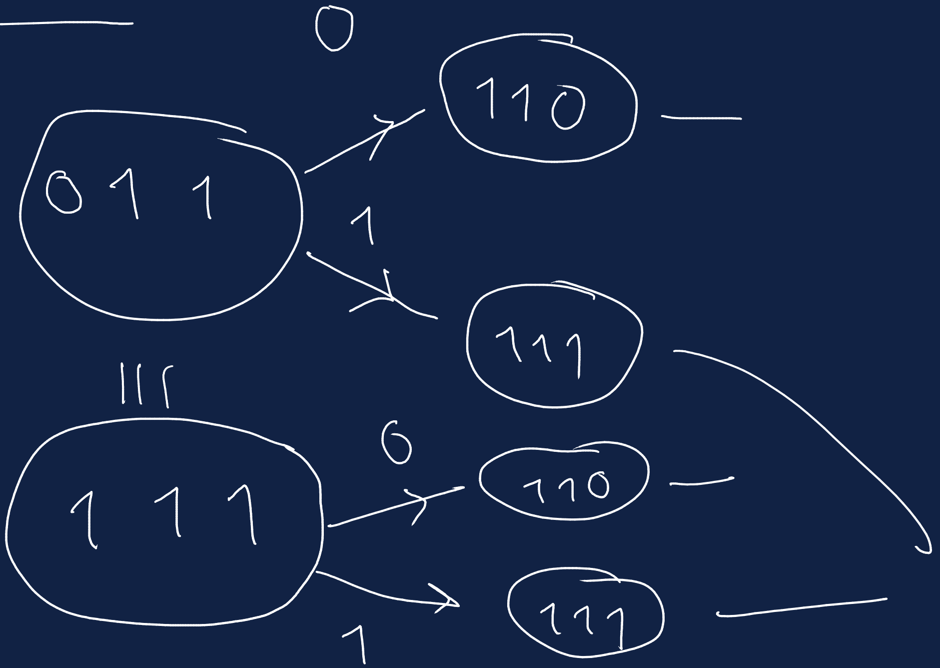
3. Let E_k = { $w \in \{0,1\}^*$ | the last k positions of w contain exactly one 1 }.
Prove/Disprove: Any DFA for E_k contains at least 2^k states.

$k = 1$

$k = 2$



$k = 3$



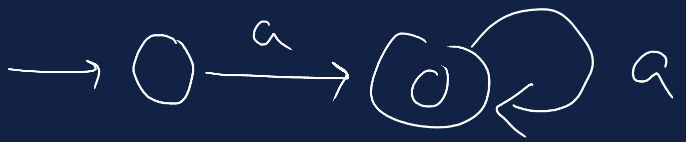
2^k states are not needed for $k = 3$.

Ans: $\binom{k+1}{2} + 1$ Why?

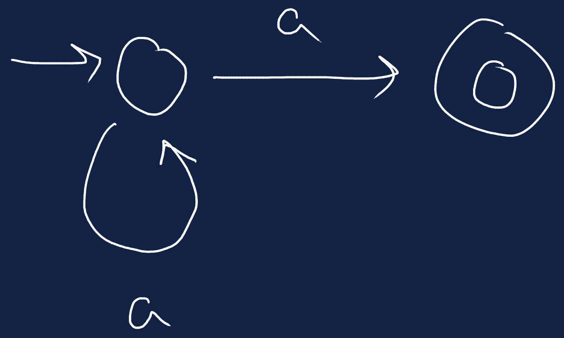
4. Prove that the minimal NFA for a regular language is not necessarily unique.

$$L = \mathcal{L}(a^+)$$

aa^*



a^*a



Show that no one-state NFA can accept L.

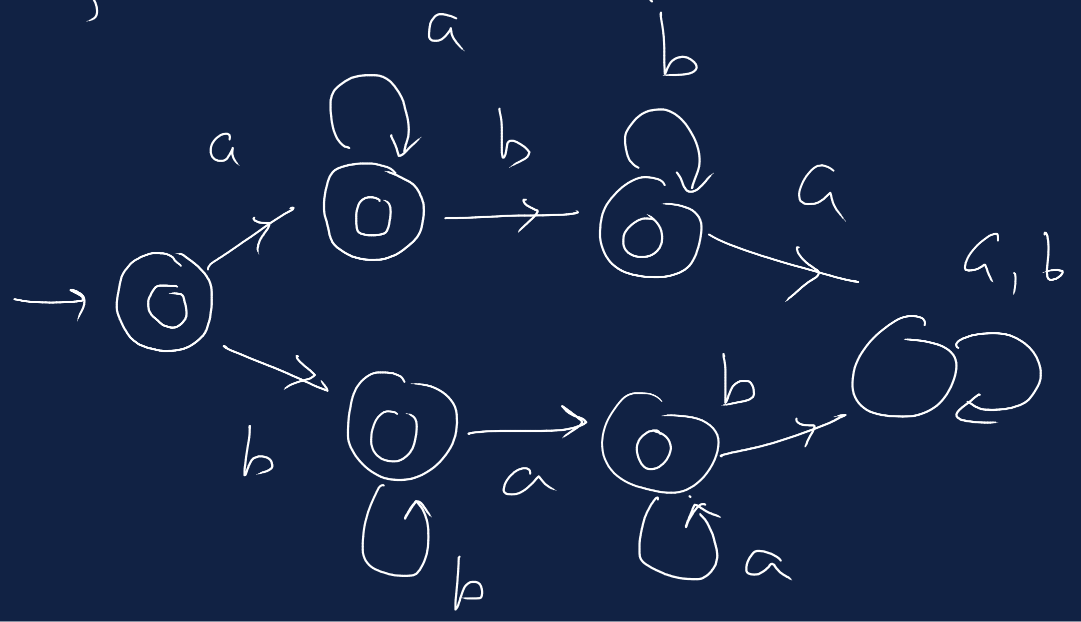
5. Consider the DFA with $\Sigma = \{0,1\}$, $Q = \{0,1,2,3,4\}$, $s = 0$, $F = \{0\}$, and $\delta(q,a) = (q^2+a) \bmod 5$. Minimize the DFA.

6. Consider the regular language R = L(a*b*+b*a*) over {a,b}. Find the equivalence classes of $\equiv_R$.

$\{\epsilon\}$, a^+, b^+, a^+b^+, b^+a^+, {all strings containing both ab and ba}

No string in b^+ can be equivalent to any string in a^+b^+.

Take $z = a$.



7. Use the Myhill–Nerode theorem to prove that the language

$$EQ = \{\ w \in \{a,b\}^* \mid \#a(w) = \#b(w)\ \}$$

is not regular.

To show that $\equiv_{EQ}$ has infinite index.

What do the equivalence classes look like?

$$[w] = \left\{ x \in \{a,b\}^* \;\middle|\; \underbrace{\#a(x) - \#b(x)}_{s(x)\,=\,s(w)} = \#a(w) - \#b(w) \right\}$$

$s(x)$: surplus of a's over b's.

L $\quad\quad x \equiv_L y$

$$\Leftrightarrow \forall z \in \Sigma^* \left(xz \in L \Leftrightarrow yz \in L \right)$$

$L = EQ$

$$S(xz) = S(x) + S(z)$$

$$S(yz) = S(y) + S(z)$$

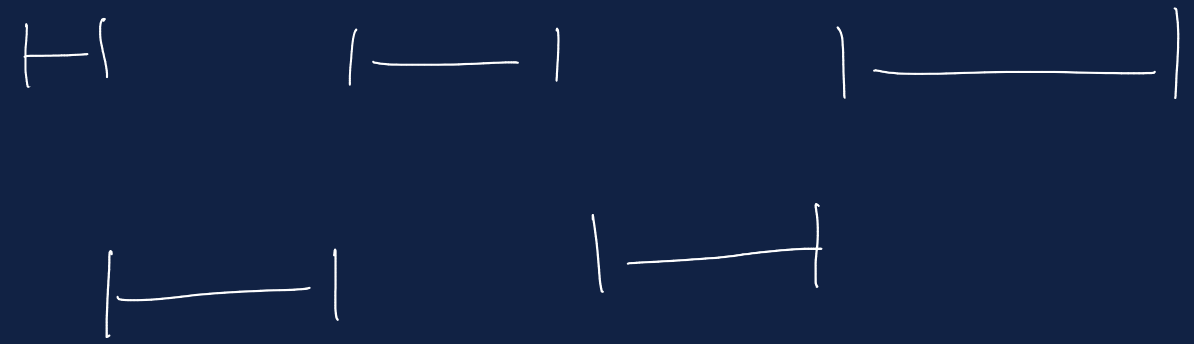
$$S(x) = S(y) \Leftrightarrow \forall z \left[S(xz) = S(yz) \right]$$

Example of $L \subseteq \{a, b\}^*$ s.t. neither L nor $\sim L$ contains an infinite regular language.

— A is regular $\Rightarrow$ length(A) is u.p.

L

$\sim L$



$$[2^n, 2^{n+1})$$