Fourier Series-I

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August 5, 2019

Contents

1 Fourier Series of Periodic Signals

1.1 Periodic signals

Any periodic signal $f(t)$ of period T, satisfies the following condition.

$$f(t) = f(t + T) = f(t - T)$$

Some periodic signals are shown in figure 1.

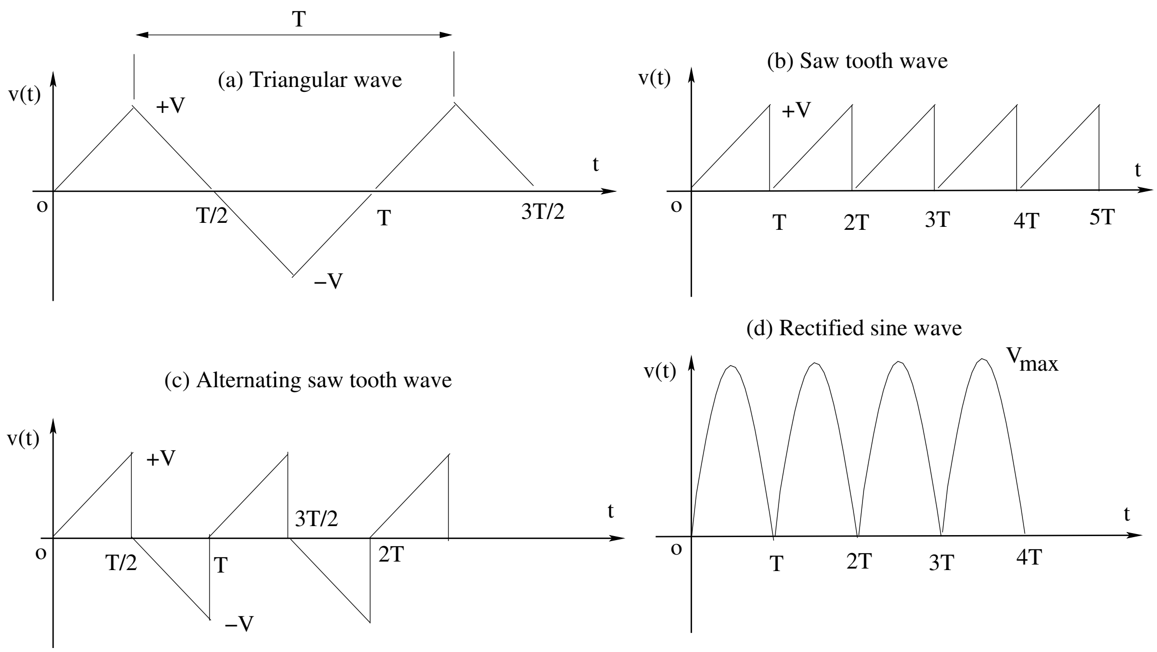


Figure 1: Square wave

Any periodic signal can be expressed as sum of infinite series of cosine and sine terms along with a constant value.

$$f(t) = a_0 + \sum_{n=0}^{\infty} a_n \cos n\omega t + \sum_{n=0}^{\infty} b_n \sin n\omega t$$

If the periodic function is described in terms of θ where $\theta = \omega\, t$ and $\omega = 2\pi/T$ ω is known as the angular frequency in radian per second. ω and frequency are related by $\omega = 2\pi f$.

1.2 How to get a_0

To get a_0, let us multiply both sides of the above equation by dt and integrate both sides over a period (i.e., 0 to T)

$$f(t)\,dt = a_0\,dt + \sum_{n=0}^{\infty} a_n \cos n\omega t\,dt + \sum_{n=0}^{\infty} b_n \sin n\omega t\,dt$$

$$\text{or} \int_0^T f(t)\,dt = \int_0^T a_0\,dt + \int_0^T \sum_{n=0}^{\infty} a_n \cos n\omega t\,dt + \int_0^T \sum_{n=0}^{\infty} b_n \sin n\omega t\,dt$$

Since a_0, a_n and b_n are not functions of time, integration sign can be put inside and summation sign as shown below.

$$\text{or} \int_0^T f(t)\,dt = \int_0^T a_0\,dt + \sum_{n=0}^{\infty} a_n \int_0^T \cos n\omega t\,dt + \sum_{n=0}^{\infty} b_n \int_0^T \sin n\omega t\,dt suwaveeven$$

Integrations of the second and third terms are all zero as we are integrating sine or cosine terms over a complete period.

$$\text{or} \int_0^T f(t)\,dt = \int_0^T a_0\,dt = Ta_0$$

$$\therefore\; a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

Thus the value of a_0 is nothing but the *average* value of the periodic signal over a cycle.

1.3 How to get a_n and b_n

Suppose we want to get an expression for a_m where m is a fixed integer and n however can assume integer values from 0 to ∞

Now we multiply both sides by cos $m\omega t$ and integrate both sides over a period

$$f(t)\cos m\omega t\,dt = a_0 \cos m\omega t\,dt + \cos m\omega t \sum_{n=0}^{\infty} a_n \cos n\omega t\,dt + \cos m\omega t \sum_{n=0}^{\infty} b_n \sin n\omega t\,dt$$

Since m is a fixed integer, it can be pushed inside the summation side.

$$\int_0^T f(t)\cos m\omega t\,dt = a_0 \int_0^T \cos m\omega t\,dt + \sum_{n=0}^{\infty} a_n \int_0^T \cos m\omega t \cos n\omega t\,dt + \sum_{n=0}^{\infty} b_n \int_0^T \cos m\omega t \sin n\omega t\,dt$$

Now recall the following results.

$$\begin{aligned}
\int_0^T \cos m\omega t \cos n\omega t \, dt &= 0 \text{ if } m \neq n \\
\int_0^T \cos m\omega t \cos n\omega t \, dt &= \frac{T}{2} \text{ if } m = n \\
\int_0^T \cos m\omega t \sin n\omega t \, dt &= 0 \text{ for all } m \text{ and } n
\end{aligned}$$

Therefore applying the above results we get,

$$\begin{aligned}
\int_0^T f(t) \cos m\omega t \, dt &= a_m \frac{T}{2} \\
a_m &= \frac{2}{T} \int_0^T f(t) \cos m\omega t \, dt \\
\therefore \; a_n &= \frac{2}{T} \int_0^T f(t) \cos n\omega t \, dt
\end{aligned}$$

In the same way it can be shown that,

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega t \, dt$$

To summarize we can say that given a periodic function $f(t)$ of period T, we can express it as follows:

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=0}^{\infty} a_n \cos n\omega t + \sum_{n=0}^{\infty} b_n \sin n\omega t \\
\text{where } a_0 &= \frac{1}{T} \int_0^T f(t) \, dt \\
a_n &= \frac{2}{T} \int_0^T f(t) \cos n\omega t \, dt \\
b_n &= \frac{2}{T} \int_0^T f(t) \sin n\omega t \, dt
\end{aligned}$$

1.4 Periodic function expressed in terms of angle

Some times the periodic functions are expressed in terms of θ where $\theta = \omega t$. In that case Fourier series for a function will look like :

$$f(\theta) = a_0 + \sum_{n=0}^{\infty} a_n \cos n\theta + \sum_{n=0}^{\infty} b_n \sin n\theta$$

To calculate a_o, a_n and b_n, we put $\theta = \omega t$ which means that $d\theta = \omega\, dt$. Let us change variable t to θ in formulas for a_o, a_n and b_n noting that $t = 0$ corresponds to $\theta = 0$ and $t = T$ corresponds to $\theta = \omega T = 2\pi$.

$$\begin{aligned} f(\theta) &= a_0 + \sum_{n=0}^{\infty} a_n \cos n\theta + \sum_{n=0}^{\infty} b_n \sin n\theta \\ \text{where } a_0 &= \frac{1}{T}\int_0^{2\pi} f(\theta)\, d\theta/\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta \end{aligned}$$

In the same way,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta) \cos n\theta\, d\theta \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta) \sin n\theta\, d\theta \end{aligned}$$

2 Some special functions

The periodic function $f(t)$, whose Fourier expansion is needed may fall under one of the following categories.

1. The function may be *even* which means $f(t) = f(-t)$.
   If the function$ft)$, is even, note that $f(t) \cos n\omega t$ is also even and $f(t) \sin n\omega t$ is odd. Then,

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(t) \cos n\omega t\, dt = \frac{4}{T}\int_0^{T/2} f(t) \cos n\omega t\, dt \\ b_n &= \frac{2}{T}\int_0^T f(t) \sin n\omega t\, dt = 0 \end{aligned}$$

   Conclusion is for even function no sine terms will be present since $b_n = 0$

2. The function may be *odd* which means $f(t) = -f(-t)$. If the function$ft)$, is odd, note that $f(t) \cos n\omega t$ is also odd and $f(t) \sin n\omega t$ is even. Then,

$$\begin{aligned} a_n &= \frac{2}{T} \int_0^T f(t) \cos n\omega t \, dt = 0 \\ b_n &= \frac{2}{T} \int_0^T f(t) \sin n\omega t \, dt = \frac{4}{T} \int_0^{T/2} f(t) \sin n\omega t \, dt \end{aligned}$$

   Conclusion is, for odd function no cosine terms will be present since $a_n = 0$

3. The function may be neither even nor odd. In this case all the terms are expected to be present.

4. The function may have *half wave* symmetry which means $f(t) = -f(t+T/2) = -f(t-T/2)$. If the function has half wave symmetry, we note the following:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_0^T f(t) \, dt \\ &= \frac{1}{T} \int_0^{T/2} f(t) \, dt + \frac{1}{T} \int_{T/2}^T f(t) \, dt \\ &= \frac{1}{T} \int_0^{T/2} f(t) \, dt - \frac{1}{T} \int_{T/2}^T f(t - T/2) \, dt \\ \text{Now put } t &= x + T/2 \\ a_0 &= \frac{1}{T} \int_0^{T/2} f(t) \, dt - \frac{1}{T} \int_0^T f(x) \, dt = 0 \end{aligned}$$

Let us get a_n:

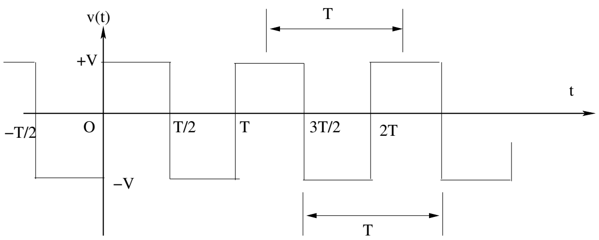
$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt \\
&= \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt + \frac{2}{T}\int_{T/2}^{T} f(t)\cos n\omega t\,dt \\
&= \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt - \frac{2}{T}\int_{T/2}^{T} f(t-T/2)\cos n\omega t\,dt \\
\text{Now put } t &= x + T/2 \\
&= \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt - \frac{2}{T}\int_0^{T/2} f(x)\cos(n\omega x + n\omega T/2)\,dx \\
\text{but, } \omega T &= 2\pi \\
\text{so } a_n &= \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt - \frac{2}{T}\int_0^{T/2} f(x)\cos(n\omega x + n\pi)\,dx \\
\text{obviously, if } n \text{ is even, } a_n &= \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt - \frac{2}{T}\int_0^{T/2} f(x)\cos n\omega x\,dx = 0 \\
\text{if } n \text{ is odd , } a_n &= \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt - \frac{2}{T}\int_0^{T/2} f(x)\cos(n\omega x + \pi)\,dx \\
&= \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt + \frac{2}{T}\int_0^{T/2} f(x)\cos n\omega x\,dx \\
\text{finally if } n \text{ is odd,} a_n &= \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt
\end{aligned}$$

5. The function may have *quarter wave* symmetry. If the following two conditions are satisfied then the periodic signal become quarter wave symmetry.

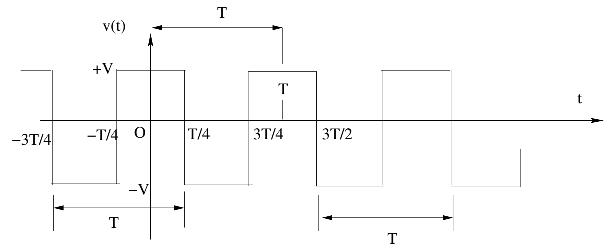
   - The periodic function must be either even or odd.
   - The periodic function must be half wave symmetric.

We already know that for a periodic function with half wave symmetry only odd harmonics are present. In addition if it is an even function, then only cosine odd harmonic terms will be present and if it is an odd function then only sine odd harmonic terms will be present.

It will be shown below that to find out the Fourier coefficients integration is to be carried out only from 0 to $T/4$.



(a) Square wave as odd function



(b) Square wave as even function

Figure 2: Square wave Odd & Even Representation

Case-1: Assume the function $f(t)$ to be even.

$$\begin{aligned}\text{because of half wave symmetry } a_n &= \frac{2}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt \\ \text{then } f(t)\cos n\omega t \text{ is also even} & \\ \therefore\ a_n &= \frac{4}{T}\int_0^{T/4} f(t)\cos n\omega t\,dt\end{aligned}$$

Case-2: Assume the function $f(t)$ to be odd.

$$\begin{aligned}\text{because of half wave symmetry } b_n &= \frac{2}{T}\int_0^{T/2} f(t)\sin n\omega t\,dt \\ \text{then } f(t)\cos n\omega t \text{ is even} & \\ \therefore\ b_n &= \frac{4}{T}\int_0^{T/4} f(t)\sin n\omega t\,dt\end{aligned}$$

3 Some examples

3.1 Example-1:Periodic Square Wave (odd)

Find out the Fourier expansion of the periodic square wave shown in figure 2a

Solution

Before we start solving the problem, let us inspect the waveform and conclude the following:

1. The periodic function is odd.

2. Therefore only sine terms will be present. So $a_n = 0$ for all $n = 1, 2, 3 \cdots \infty$

3. b_n will be present and $a_0 = 0$

4. Is the waveform is quarter wave symmetric? Yes it is. Since $v(t)$ is odd and the wave form is half wave symmetric. Thus there can not be any even harmonic present.

5. Finally we conclude $v(t)$ will have only sine terms with only odd harmonics. In other words, $b_n = 0$ for $n = 2, 4, 6, \ldots \infty$

Point to be noted is without any calculation we can know many important things about $v(t)$ just by looking at it. Now over the period T, $v(t)$ can be described as,

$$v(t) = \begin{cases} +V, & \text{if } 0 < t < T/2; \\ -V, & \text{if } T/2 < t < T; \end{cases}$$

Therefore only b_n is to be calculated.

$$\begin{aligned} b_n &= \frac{2}{T}\int_0^T v(t)\sin n\omega t\, dt \\ &= \frac{4}{T}\int_0^{T/2} V \sin n\omega t\, dt \\ &= \frac{4V}{T}\int_0^{T/2} \sin n\omega t\, dt \\ &= \frac{4V}{T}\left.\frac{\cos n\omega t}{n\omega}\right|_0^{T/2} \\ &= \frac{4V}{n\omega T}\left(\cos n\omega T/2 - 1\right) \\ &= \frac{4V}{n\omega T}\left(\cos n\pi - 1\right) \\ &= \frac{4V}{n2\pi}\left[(-1)^n - 1\right] \\ &= \frac{2V}{n\pi}\left[(-1)^n - 1\right] \end{aligned}$$

$\therefore\ b_n = \frac{4V}{n\pi}$ when n is odd and $b_n = 0$ if n is even and the periodic function can be written as sum of infinite sine terms as follows.

$$v(t) = \sum_{n=0}^{\infty} \frac{4V}{n\pi} \sin n\omega t = \frac{4V}{\pi} \sin \omega t + \frac{4V}{3\pi} \sin 3\omega t + \frac{4V}{5\pi} \sin 5\omega t + \cdots \text{to } \infty$$

3.2 Example-2: Periodic Square Wave (even)

Find out the Fourier expansion of the periodic square wave shown in figure 2b

Solution

The function can be described as:

$$v(t) = \begin{cases} +V, & \text{if } 0 < t < T/2; \\ -V, & \text{if } T/2 < t < T; \end{cases}$$

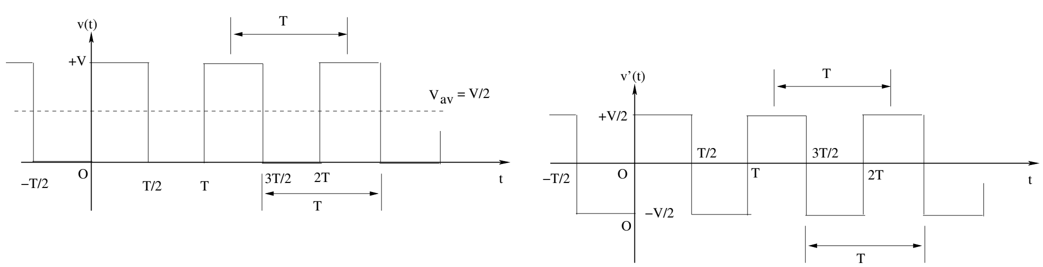
This wave form of figure 2b is same as waveform shown in 2a except that $t = 0$ (origin) is shifted. It is noted that it is an even function and has quarter wave symmetry. Fourier expansion will lead to only cosine terms having odd harmonics. So $a_0 = 0$ and $b_n = 0$ and a_n is given by:

$\therefore\ a_n = \frac{4V}{n\pi}$ when n is odd and $a_n = 0$ if n is even and the periodic function can be written as sum of infinite sine terms as follows.

$$v(t) = \sum_{n=0}^{\infty} \frac{4V}{n\pi} \cos n\omega t = \frac{4V}{\pi} \cos \omega t + \frac{4V}{3\pi} \cos 3\omega t + \frac{4V}{5\pi} \cos 5\omega t + \cdots \text{to } \infty$$

3.3 Example-3: Periodic Square Wave

Find out the Fourier expansion of the periodic square wave shown in figure 3 (a).



(a) Square wave with swing between 0 to V

(b) If V/2 is subtracted, swing is between −V/2 to +V/2

Figure 3: Square wave

Solution

A careful look of $v(t)$ shows that, it is neither even nor odd. However if we subtract the average value $V/2$ of $v(t)$ from $v(t)$ we get $v'(t)$ as shown figure 3(b). $v'(t)$ is periodic square wave swinging between $-V/2$ to $+V/2$. This problem we have already solved in example-3 (V is to be replaced by $v/2$). Thus,

$$\begin{aligned} v'(t) = v(t) - \frac{V}{2} &= \sum_{n=0}^{\infty} \frac{4V/2}{n\pi} \sin n\omega t \\ v(t) &= \frac{V}{2} + \sum_{n=0}^{\infty} \frac{2V}{n\pi} \sin n\omega t \\ v(t) &= \frac{V}{2} + \frac{2V}{\pi} \sin \omega t + \frac{2V}{3\pi} \sin 3\omega t + \frac{2V}{5\pi} \sin 5\omega t + \cdots \text{to } \infty \end{aligned}$$

3.4 Example-4: Half wave rectified wave

Find out the Fourier expansion of the periodic square wave shown in figure 4.

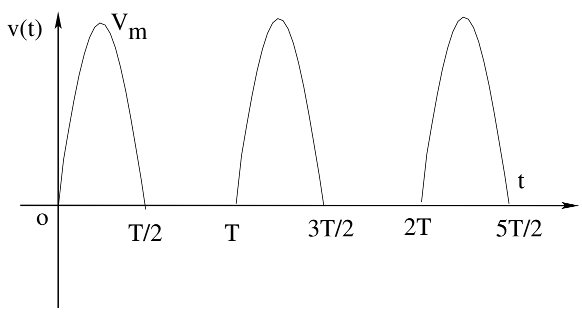


Figure 4: Half wave rectified wave

Solution

The function can be mathematically described as:

$$v(t) = \begin{cases} V_m \sin \omega t & \text{if } 0 < t < T/2 \\ 0 & \text{if } T/2 < t < T \end{cases}$$

The function is neither odd nor even and has no half wave symmetric. Let us calculate the Fourier coefficients.

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_0^T v(t)\,dt \\
&= \frac{1}{T}\int_0^T V_m \sin\omega t\,dt \\
\therefore\; a_0 &= \frac{V_m}{\pi}
\end{aligned}$$

Now a_n is calculated.

$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt \\
&= \frac{2}{T}\int_0^{T/2} V_m \sin\omega t\,\cos n\omega t\,dt \\
&= \frac{V_m}{T}\int_0^T [\sin(n+1)\omega t - \sin(n-1)\omega t]\,dt \\
&= \frac{V_m}{T}\left[\left.\frac{\cos(n+1)\omega t}{(n+1)\omega}\right|_{T/2}^{0} + \left.\frac{\cos(n-1)\omega t}{(n-1)\omega}\right|_{0}^{T/2}\right] \text{ for } n \neq 1 \\
\text{After simplifying, we get } a_n &= -\frac{V_m}{\pi}\frac{1+(-1)^n}{n^2-1}
\end{aligned}$$

a_1 however is to be calculated directly by putting $n = 1$ in the formula as follows:

$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^{T/2} V_m \sin\omega t\,\cos n\omega t\,dt \\
a_1 &= \frac{2}{T}\int_0^{T/2} V_m \sin\omega t\,\cos\omega t\,dt = 0
\end{aligned}$$

Thus we can see that if n is, $a_n = 0$ i.e., $a_1 = a_3 = a_5 = a_7 \cdots = 0$.

However for even n:

$$a_n = -\frac{2V_m}{\pi(n^2-1)} \text{ for } n = 2, 4, 6 \cdots$$

Now let us calculate b_n:

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_0^{T/2} V_m \sin\omega t \sin n\omega t\, dt \\
&= \frac{V_m}{T}\int_0^{T/2} [\cos(n-1)\omega t - \cos(n+1)\omega t\, dt] \\
b_n &= 0 \text{ for all } n > 0 \\
\text{and } b_1 &= \frac{2}{T}\int_0^{T/2} V_m \sin\omega t \sin \omega t\, dt \\
&= \frac{1}{T}\int_0^{T/2} V_m\,(1-\cos 2\omega t)\, dt \\
\therefore\ b_1 &= \frac{V_m}{2} \\
\text{Finally, } v(t) &= \frac{V_m}{\pi} - \sum_{n=\text{even}}^{\infty} \frac{2V_m}{\pi(n^2-1)} \cos n\omega t + \frac{V_m}{2}\sin\omega t
\end{aligned}
$$

3.5 Example-5: Full wave rectified wave

Find out the Fourier expansion of the periodic square wave shown in figure 5.

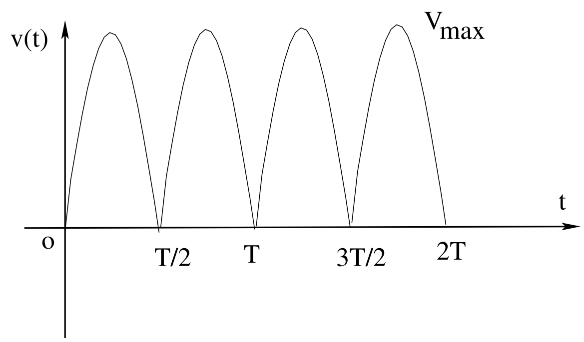


Figure 5: Full wave rectified wave

Solution

The function can be mathematically described as:

$$v(t) = V_m \sin\omega t \text{ for } 0 < t < T$$

The Fourier coefficients can be found out in the usual ways using the appropriate formulas for a_0, a_n and b_n. However here we shall try to find out the Fourier constants by using the results we have obtained for half wave rectified wave in the previous problem. Now the full wave rectified wave $v(t)$, can be thought of as sum of two half wave rectified waves $v_1(t)$ and $v_2(t)$ where $v_2(t) = v_1(t - T/2)$. Therefore,

$$\begin{aligned}
v_1(t) &= \frac{V_m}{\pi} - \sum_{n=\text{even}}^{\infty} \frac{2V_m}{\pi(n^2-1)} \cos n\omega t + \frac{V_m}{2} \sin \omega t \\
v_2(t) &= \frac{V_m}{\pi} - \sum_{n=\text{even}}^{\infty} \frac{2V_m}{\pi(n^2-1)} \cos n\omega(t - T/2) + \frac{V_m}{2} \sin \omega(t - T/2) \\
&= \frac{V_m}{\pi} - \sum_{n=\text{even}}^{\infty} \frac{2V_m}{\pi(n^2-1)} \cos (n\omega t - n\pi) + \frac{V_m}{2} \sin (\omega t - \pi) \\
v_2(t) &= \frac{V_m}{\pi} - \sum_{n=\text{even}}^{\infty} \frac{2V_m}{\pi(n^2-1)} \cos n\omega t - \frac{V_m}{2} \sin \omega t \\
\therefore\ v(t) &= v_1(t) + v_2(t) \\
\text{Thus } v(t) &= \frac{2V_m}{\pi} - \sum_{n=\text{even}}^{\infty} \frac{4V_m}{\pi(n^2-1)} \cos n\omega t
\end{aligned}$$

3.5.1 To summarize:

1. If the periodic function is neither even nor odd and without half wave symmetry, then all the terms (sine, cosine & average value) and all the harmonics (even & odd) will be present.

2. If the periodic function has *half wave symmetry*, then average value, c_0 will be zero and only odd harmonics will be present.

3. A periodic wave is said to have *quarter wave symmetry* when it is having half wave symmetry and as well it is either even or odd function.

   (a) Suppose the periodic function is even function and has half wave symmetry. It has then quarter wave symmetry. In this case average value will be zero ($c_0 = 0$), only cosine terms will be present ($a_k \neq 0$) and ($b_k = 0$). Also only odd harmonics (k = odd) will be present.

   (b) Suppose the periodic function is odd function and has half wave symmetry. It has then quarter wave symmetry. In this case, average value will be zero ($c_0 = 0$), only sine terms will be present ($a_k = 0$) and ($b_k \neq 0$). Also only odd harmonics (k = odd) will be present.

4 Alternative Expression of Fourier Series

We have seen earlier that a periodic function $f(t)$ can expanded as follows:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

The sine and cosine terms can be combined either into a sine or cosine term as worked out below. In stead of the two constants a_n and b_n, the Fourier expansion will now have magnitude $\bar{c}_n$ and phase constants ϕ_n for different harmonics.

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \\
&= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \\
&= \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left[\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos n\omega t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin n\omega t \right] \\
\text{Let, } \bar{c}_n &= \sqrt{a_n^2 + b_n^2} \\
\sin \phi_n &= \frac{a_n}{\sqrt{a_n^2 + b_n^2}} \\
\text{and, } \cos \phi_n &= \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \ \therefore \ \tan \phi_n = \frac{a_n}{b_n} \\
\text{Finally, } f(t) &= a_0 + \sum_{n=1}^{\infty} \bar{c}_n \sin(n\omega t + \phi_n)
\end{aligned}$$

If we define:

$$\begin{aligned}
\cos \phi_n &= \frac{a_n}{\sqrt{a_n^2 + b_n^2}} \\
\sin \phi_n &= \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \\
\text{Then, } \tan \phi_n &= \frac{b_n}{a_n} \\
\text{We get, } f(t) &= a_0 + \sum_{n=1}^{\infty} \bar{c}_n \cos(n\omega t - \phi_n)
\end{aligned}$$

Signals and Networks : Fourier Series-II

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Contents

1 Periodic impulse function

Find out the Fourier expansion of the periodic impulse function shown in figure 1. The periodic unit impulse function is an even function. Average value and cosine terms will be present. Fundamental angular frequency is $2\pi/T$. The property of impulse function is:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Which means:

$$\int_{-T/2}^{T/2} \delta(t)\, dt = 1$$

Now we calculate a_0 and a_n below.

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, dt \tag{1}$$

$$= \frac{1}{T} \tag{2}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \delta(t)\, \cos n\,\omega t dt = \frac{2}{T}\int_{-T/2}^{T/2} \delta(t)\, dt \tag{3}$$

$$= \frac{2}{T} \tag{4}$$

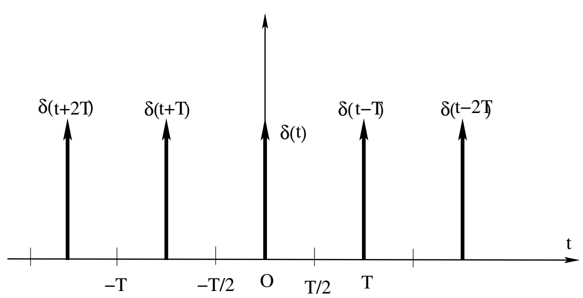

Figure 1: Periodic impulse function

Thus the Fourier expansion is:

$$\delta(t) = \frac{1}{T} + \frac{2}{T}\cos \omega t + \frac{2}{T}\cos 2\omega t + \frac{2}{T}\cos 3\omega t + \cdots \infty$$

It is interesting to note that all harmonics are present with equal strength of value $2/T$ and the average value is $1/T$

2 Complex form of Fourier Series

In this section we shall show that, expansion of a periodic function into a Fourier series can be done in a very compact way. Instead of representing function in terms of the constants a_0, a_n and b_n or a_0, $\bar{c}_n$ and ϕ_n, the function may be represented by a single complex constant c_n as shown below.

$$
\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \\
\text{Now by Euler's theorem, } f(t) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \frac{e^{jn\omega t} + e^{-jn\omega t}}{2} + b_n \frac{e^{jn\omega t} - e^{-jn\omega t}}{2j} \right) \\
f(t) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \frac{e^{jn\omega t} + e^{-jn\omega t}}{2} - jb_n \frac{e^{jn\omega t} - e^{-jn\omega t}}{2} \right) \\
f(t) &= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - jb_n}{2} e^{jn\omega t} + \frac{a_n + jb_n}{2} e^{-jn\omega t} \right) \\
\text{Now let, } \frac{a_n - jb_n}{2} &= c_n \\
\text{Then, } \frac{a_n + jb_n}{2} &= c_n^{\star} = \text{Complex conjugate of } c_n \\
f(t) &= a_0 + \sum_{n=1}^{\infty} \left(c_n e^{jn\omega t} + c_n^{\star} e^{-jn\omega t} \right) \\
\text{Recall, } c_n^{\star} &= c_{-n} \text{ then the above can be written as:} \\
f(t) &= a_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega t} \\
&= a_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=-1}^{-\infty} c_n e^{jn\omega t} \text{ 3rd term manipulated intelligently} \\
&= \sum_{n=-1}^{-\infty} c_n e^{jn\omega t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} \text{ Note, } c_0 = a_0 \\
&= \sum_{n=-\infty}^{-1} c_n e^{jn\omega t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} \\
\text{Finally, } f(t) &= \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega t} \text{ Is it not very compact way of writing } f(t)?
\end{aligned}
$$

If now the compact equation of Fourier series is expanded, we get

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega t} \\ f(t) &= \cdots + c_{-3}\, e^{-j3\omega t} + c_{-2}\, e^{-j2\omega t} + c_{-1}\, e^{-j1\omega t} + c_0 + c_1\, e^{j1\omega t} + c_2\, e^{j2\omega t} + c_3\, e^{j3\omega t} + \cdots \end{aligned}$$

Note that each term of the right hand side is complex, but when all the terms are summed up it must return a real function. This is true because we can pair the negative frequency term with the corresponding positive frequency term, for example $c_{-3}\, e^{-j3\omega t} + c_3\, e^{j3\omega t}$ will yield always real number.

2.1 How to get c_n ?

An expression for c_n can be easily found out by using the a_n and b_n as follows:

$$\begin{aligned} c_n &= \frac{a_n - jb_n}{2} \\ &= \frac{2}{T}\frac{1}{2}\left(\int_0^T f(t)\cos\, n\omega t\, dt - j\frac{2}{T}\int_0^T f(t)\sin\, n\omega t\, dt\right) \\ &= \frac{1}{T}\int_0^T f(t)\left(\cos\, n\omega t - j\sin\, n\omega t\right)\, dt \\ \therefore\; c_n &= \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega t}\, dt \end{aligned}$$

2.2 An useful property of Fourier Series

We know that for a given periodic function $f(t)$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega t}$$

$$_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega t}\, dt$$

If we differentiate $f(t)$ we get ,

$$\frac{df}{dt} = \sum_{n=-\infty}^{\infty} jn\omega C_n\, e^{jn\omega t} = \sum_{n=-\infty}^{\infty} C'_n\, e^{jn\omega t} \quad \text{where,} C'_n = jn\omega C_n$$

Thus $\frac{df}{dt}$ will be a periodic function of coefficients $C'_n = jn\omega C_n$.

If $\frac{df}{dt}$ happens to be periodic delta function, then C'_n calculation becomes much simpler and the C_n of the original $f(t)$ will be simply, $C_n = \frac{C'_n}{jn\omega}$. Refer to figure 2 to get the idea. Now, over a

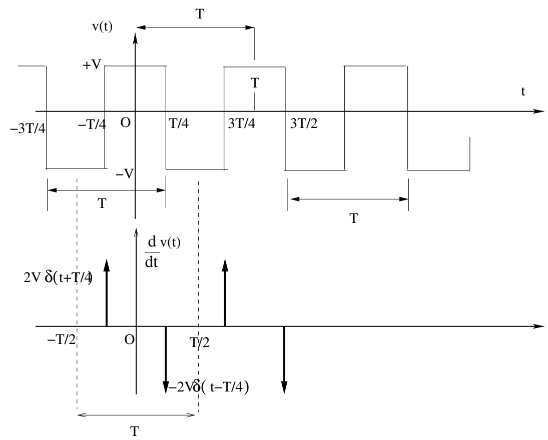

Figure 2: Sqare wave & its derivative

chosen period T,

$$\frac{df}{dt} = 2V\delta(t + T/4) - 2V\delta(t - T/4)$$

therefore,C'_n can be easily calculated as follows

$$C'_n = \frac{1}{T}\int_{-T/2}^{T/2} (2V\delta(t + T/4) - 2V\delta(t - T/4))\, e^{-jn\omega t}\, dt$$

Finally C_n of the $f(t)$ will be simply $\frac{C'_n}{jn\omega}$.

C'_n

3 Some examples

3.1 Triangular periodic signal

Let us consider the triangular even periodic signal $v(t)$ shown in figure 3. For this signal $a_0 = 0$, $b_n = 0$ and only a_n is to be evaluated. Since it is a quarter wave symmetric even function.

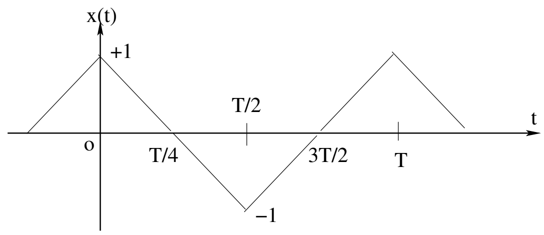

Figure 3: Triangular wave

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_0^T v(t)\,\cos\, n\omega t\,dt \\
\text{or, } v(t) &= \frac{4}{T}\int_0^{T/2} v(t)\,\cos\, n\omega t\,dt \;\text{ since the integrand is even} \\
\text{now, } v(t) &= 1-\frac{4}{T}\,t \;\text{ for } 0<t<\pi \\
\therefore\; a_n &= \frac{4}{T}\int_0^{T/2}\left(1-\frac{4}{T}\,t\right)\cos n\omega t\,dt \\
&= -\frac{16}{T^2}\int_0^{T/2} t\,\cos n\omega t\,dt \\
&= -\frac{16}{T^2}\left[t\,\frac{\sin\, n\omega t}{n\omega}\bigg|_0^{T/2}-\int_0^{T/2}\frac{\sin\, n\omega t}{n\omega}\,dt\right] \\
&= \frac{16}{T^2}\int_0^{T/2}\frac{\sin\, n\omega t}{n\omega}\,dt \\
&= -\frac{16}{T^2}\frac{\cos\, n\omega t}{n^2\omega^2}\bigg|_0^{T/2} = \frac{4}{\pi^2}\frac{1-(-1)^n}{n^2} \\
v(t) &= \frac{8}{\pi^2 n^2}\;\text{ where } n \text{ is odd, } n=1,3,5\cdots\infty \\
v(t) &= \frac{8}{\pi^2}\sum_{n=\text{odd}}^{\infty}\frac{\cos\, n\omega t}{n^2}
\end{aligned}
$$

One interesting side result is shown below from the above result.

$$
\begin{aligned}
\therefore\; v(t=0) &= \frac{8}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots\infty\right) \\
\text{But from the graph, }, v(t=0) &= 1 \\
\therefore\; 1 &= \frac{8}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots\infty\right) \\
\text{or, } \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots\infty &= \frac{\pi^2}{8}
\end{aligned}
$$

1. Students are advised to work out the same problem after calculating complex F.S coefficients C_k and eventually get a_n.

2. Also differentiate twice the triangular function to get periodic impulse function. Calculate C'_k of this periodic impulse function; finally get C_k of the original triangular function.

Exercise on parabolic periodic function

Look at the periodic parabolic signal shown in figure 4. Over a period the function is defined as:

$$v(t) = t^2 \text{ for, } -T/2 < t < T/2$$

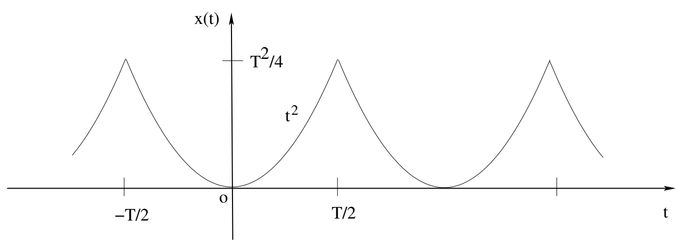

Figure 4: Parabolic wave

Find out F.S coefficients a_n, b_n and c_n by any method you like.
Try to establish

$$\sum_{n=\text{even}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}$$

Note that from the previous worked out problem we know that

$$\sum_{n=\text{odd}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

4 Addition of periodic signals

Suppose there are two periodic signals $f_1(t)$ and $f_2(t)$ with periods T_1 and T_2 respectively. If we add these two signals, will it remain periodic and if so what will be the period of the added signals?

$$
\begin{aligned}
\because\ f_1(t) \text{ is periodic } f_1(t) &= f_1(t+mT_1)\ m \text{ an integer} \\
\text{Also, } f_2(t) \text{ is periodic } f_2(t) &= f_2(t+nT_2)\ n \text{ an integer} \\
f(t) = f_1(t) + f_2(t) &= f_1(t+mT_1) + f_2(t+nT_2) \\
\text{If } f(t) \text{ is to be periodic, } & \text{ with fundamental period } T \\
\text{then, } f(t) &= f(t+T) = f_1(t+T) + f_2(t+T) \\
\text{Thus, } f(t+T) &= f_1(t+mT_1) + f_2(t+nT_2) \\
\text{or, } f_1(t+T) + f_2(t+T) &= f_1(t+mT_1) + f_2(t+nT_2) \\
\text{which means that, } mT_1 &= T \\
\text{and, } nT_2 &= T \\
\therefore\ \frac{T_1}{T_2} &= \frac{n}{m}
\end{aligned}
$$

1. So for the given signals $f_1(t)$ and $f_2(t)$, we shall calculate T_1 and T_2
2. find out the ratio $\frac{T_1}{T_2}$. If the ratio is a rational number (since m and n are integers), then $f(t)$, $(= f_1(t) + f_2(t))$ must be periodic. If the ratio happens to be an irrational number then $f(t)$ is not periodic.
3. If $f(t)$ is periodic, then the fundamental period of $f(t)$ will be $T = mT_1$ or $T = nT_2$

4.1 Examples

Example-1
Check whether the signal $(\sin 2t + \cos 5t)$ is periodic. If it is periodic, find out its period.
Solution

$$\begin{aligned}
\text{Now, } T_1 &= \frac{2\pi}{2} \\
\text{and } T_2 &= \frac{2\pi}{5} \\
\text{ratio: } \frac{T_1}{T_2} &= \frac{5}{2} \\
\text{The ratio is rational,} & \quad \text{so the signal is periodic} \\
\text{Fundamental period of the signal } T &= 2T_1 \text{ or, } 5T_2 \\
\therefore\ T &= 2 \times \frac{2\pi}{2} = 2\pi
\end{aligned}$$

Example-2

Check whether the signal $(\sin\sqrt{2}t + \cos 2t)$ is periodic. If it is periodic, find out its period.

Solution

$$\begin{aligned}
\text{Now, } T_1 &= \frac{2\pi}{\sqrt{2}} \\
\text{and } T_2 &= \frac{2\pi}{2} \\
\text{ratio: } \frac{T_1}{T_2} &= \frac{2}{\sqrt{2}} = \sqrt{2} \\
\text{The ratio is irrational,} & \quad \text{so the signal is not periodic.} \\
\text{Question of fundamental period } T \text{ of the signal} & \quad \text{does not arise.}
\end{aligned}$$

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