1

The relation $\sim$ is not an equivalence relation because it is not reflexive.

$$1 \in \mathbb{Z} \text{ however } \gcd(1,1) = 1$$

$$(1,1) \notin \sim$$

$$\implies \sim \text{ is not reflexive}$$

$$\therefore \sim \text{ is not an equivalence relation}$$

2

Yes the relation (R) is a POSET as it is reflexive, antisymmetric and transitive

- Reflexive:
  - $(a,a) \in R$
  - $(b,b) \in R$
  - $(c,c) \in R$
  - $(d,d) \in R$

  $\therefore \forall x \in A, (x,x) \in R$, so R is reflexive
- Antisymmetric: $\nexists (x,y) \in R, x \neq y$ such that $(y,x) \in R, \therefore R$ is antisymmetric
- Transitive:
  - $(a,a) \in R, (a,d) \in R, (a,d) \in R$
  - $(b,b) \in R, (b,d) \in R, (b,d) \in R$
  - $(c,c) \in R, (c,d) \in R, (c,d) \in R$
  - $(a,d) \in R, (d,d) \in R, (a,d) \in R$
  - $(b,d) \in R, (d,d) \in R, (b,d) \in R$
  - $(c,d) \in R, (d,d) \in R, (c,d) \in R$

  $\therefore R$ is transitive via proof by exhaustion

3

R is not an equivalence relation as $(6,6) \notin R \therefore R$ is not reflexive.

4

$R_1 \cup R_2$ is not an equivalence relation, consider

- $A = \{x, y, z\}$
- $R_1 = \{(x, x), (y, y), (z, z), (x, y), (y, x)\}$
- $R_2 = \{(x, x), (y, y), (z, z), (y, z), (z, y)\}$
- $R_1 \cup R_2 = \{(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), (z, y)\}$

Now, $R_1 \cup R_2$ contains $(x, y), (y, z)$ however, $(x, z) \notin R_1 \cup R_2$

$$\therefore R_1 \cup R_2 \text{ is not transitive}$$

$$\therefore R_1 \cup R_2 \text{ is not an equivalence relation}$$

5

(a)
- Reflexive: $\forall (x, y) \in A, (x, y)R(x, y)$ as $xy = xy$
- Symmetric:

$$\begin{gathered} \forall (x_1, y_1), (x_2, y_2) \in A \\ (x_1, y_1)R(x_2, y_2) \implies (x_2, y_2)R(x_1, y_1) \\ (x_1y_1 = x_2y_2 \implies x_2y_2 = x_1y_1) \end{gathered}$$

- Transitive:

$$\begin{aligned} \forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in A, (x_1, y_1)R(x_2, y_2), (x_2, y_2)R(x_3, y_3) \\ (x_1, y_1)R(x_2, y_2) \implies x_1y_1 = x_2y_2 \\ (x_2, y_2)R(x_3, y_3) \implies x_2y_2 = x_3y_3 \\ \implies x_1y_1 = x_3y_3 \\ \implies (x_1, y_1)R(x_3, y_3) \end{aligned}$$

$\therefore$ R is transitive

R is reflexive, symmetric and transitive $\therefore R$ is an equivalence relation.

(b) All equivalence classes are of the form

$$[(1, i)], \forall i \in \mathbb{N}$$

$$[(1, i)] = \{(x, y) | xy = i, \text{ and } x, y \in \mathbb{N}\}$$

(c) $[(1, 2)]$ has two elements $(1, 2), (2, 1)$, infact all equivalence classes of the form $[(1, p)]$ have two elements where $p \in Primes$

(d) $[(1, 4)]$ has three elements $(1, 4), (2, 2), (4, 1)$, infact all equivalence classes of the form $[(1, p^2)]$ have two elements where $p \in Primes$

6

- Reflexive:
$$\forall x \in A, |x-x| = 0, 0 \text{ is even}$$
$$\therefore \forall x \in A, (x,x) \in R$$
- Symmetric:
$$|x-y| = |y-x|$$
$$\therefore (x,y) \in R \implies (y,x) \in R$$
- Transitive: Let $(x,y),(y,z) \in R$
$$\begin{aligned} |x-y| &= 2\lambda_1, \lambda_1 \in \mathbb{Z}^* \\ x-y &= 2\mu_1, \mu_1 \in \mathbb{Z} \\ |y-z| &= 2\lambda_2, \lambda_2 \in \mathbb{Z}^* \\ y-z &= 2\mu_2, \mu_2 \in \mathbb{Z} \\ \implies x-y+y-z &= 2(\mu_1+\mu_2) \\ \implies x-z &= 2\mu_3, \mu_3 \in \mathbb{Z} \\ \implies |x-z| &= 2\lambda_3(\text{even}), \lambda_3 \in \mathbb{Z}^* \\ &\implies (x-z) \in R \end{aligned}$$

  $\therefore R$ is transitive

R is reflexive, symmetric and transitive so R is an equivalence relation.

7

Any equivalence relation ρ on set A induces a partition of A. So we can count partitions instead of equivalence relations

Type	Counts
4	$\binom{4}{0} = 1$
3,1	$\binom{4}{3} = 4$
2,2	$\frac{\binom{4}{2}}{2} = 3$
2,1,1	$\binom{4}{2} = 6$
1,1,1,1	$\binom{4}{4} = 1$

$\implies$ Number of Equivalence Relations on A = 1+4+3+6+1=15

S.No	Type	Equivalence Clases
1	4	$\{1,2,3,4\}$
2	$3,1$	$\{1,2,3\},\{4\}$
3	$3,1$	$\{1,2,4\},\{3\}$
4	$3,1$	$\{1,4,3\},\{2\}$
5	$3,1$	$\{4,2,3\},\{1\}$
6	$2,2$	$\{1,2\},\{3,4\}$
7	$2,2$	$\{1,3\},\{2,4\}$
8	$2,2$	$\{1,4\},\{3,2\}$
9	$2,1,1$	$\{1,2\},\{3\},\{4\}$
10	$2,1,1$	$\{1,3\},\{2\},\{4\}$
11	$2,1,1$	$\{1,4\},\{2\},\{3\}$
12	$2,1,1$	$\{3,2\},\{1\},\{4\}$
13	$2,1,1$	$\{4,2\},\{1\},\{3\}$
14	$2,1,1$	$\{3,4\},\{1\},\{2\}$
15	$1,1,1,1$	$\{1\},\{2\},\{3\},\{4\}$

8

The statement is true.
Proof:

- $\implies$ (If) We can prove this by its contrapositive
  If R is not antisymmetric,
  $$\exists (x,y),(y,x) \in R | x \neq y$$
  However, any closure of R would still contain $(x,y),(y,x)$ and would continue to remain antisymmetric $\implies$ no antisymmetric closure of R can exist

- $\impliedby$ (Only-If) R is antisymmetric $\implies$ the antisymmetric closure of R is itself, which exists

Total number of antisymmetric relations on a finite set of size n is given by $2^n \times 3^{\binom{n}{2}}$.
Proof:

- A relation R on a set A is antisymmetric if $\forall x,y \in A, (x,y),(y,x) \in R \implies x = y$.

- CASE 1: First we look at all pairs $(x,y)|x = y$. The number of such pairs is n, one for each element in A. We may have $(x,x) \in R$ or $(x,x) \notin R$. There are n such pairs, and 2 possibilities for each, so the total relations in this case are 2^n.

- CASE 2: Now we look at all pairs $(x,y)|x \neq y$. The number of such pairs is $\binom{n}{2}$, the number of ways of selecting 2 objects from a set of n objects. We may have
  - $(x,y) \in R, (y,x) \notin R$
  - $(y,x) \in R, (x,y) \notin R$
  - $(x,y),(y,x) \notin R$.

There are $\binom{n}{2}$ such pairs, and 3 possibilities for each, so the total relations in this case are $3^{\binom{n}{2}}$.

- CASE 1 and CASE 2 exhaust all possible pairs of elements. Using the multiplication rule of counting on the results of the two cases, the total number of antisymmetric relations on a finite set of size n are thus $2^n \times 3^{\binom{n}{2}}$.

9

Let $S = x, y$

$$R_1 = (x, x), (y, y), (x, y)$$
$$R_2 = (x, x), (y, y), (y, x)$$
$$R_1 \cup R_2 = (x, x), (y, y), (x, y), (y, x)$$

This is not antisymmetric $(x, y), (y, x) \in R_1 \cup R_2, x \neq y$
$\therefore R_1 \cup R_2$ is not a POSET on S

10

(a) Not a POSET,

$$(5, 1) \preccurlyeq (5, 2)$$
$$(5, 2) \preccurlyeq (5, 1)$$
$$(5, 2) \neq (5, 1)$$

$\therefore$ not antisymmetric, so not a POSET

(b) Is a POSET,

- Reflexive:

$$\forall (x, y) \in \mathbb{N} \times \mathbb{N}, (x, y) \preccurlyeq (x, y), x \leq x, y \geq y$$

  $\therefore$ it is reflexive.

- Antisymmetric:

$$\text{Let } \exists (x_1, y_1), (x_2, y_2) \in \mathbb{N} \times \mathbb{N} : (x_1, y_1) \preccurlyeq (x_2, y_2), (x_2, y_2) \preccurlyeq (x_1, y_1)$$
$$\implies x_1 \leq x_2, x_2 \leq x_1, y_1 \geq y_2, y_2 \geq y_1$$
$$\implies x_1 = x_2, y_1 = y_2$$
$$\implies (x_1, y_1) = (x_2, y_2)$$

  $\therefore$ it is antisymmetric.

- Transitive:

$$\text{Let } \exists (x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{N} \times \mathbb{N} : (x_1, y_1) \preccurlyeq (x_2, y_2), (x_2, y_2) \preccurlyeq (x_3, y_3)$$

$$\implies x_1 \leq x_2, x_2 \leq x_3, y_3 \geq y_2, y_2 \geq y_1$$

$$\implies x_1 \leq x_3, y_3 \geq y_1$$

$$\implies (x_1, y_1) \preccurlyeq (x_3, y_3)$$

$\therefore$ it is transitive.

11

No this is not a POSET on $P(S)$

$$\text{Let } S = \{a_1, a_2, a_3\}$$

$$a_1 \preccurlyeq a_2 \ (|a_1| = 1 \leq |a_2| = 1)$$

$$a_2 \preccurlyeq a_1 \ (|a_2| = 1 \leq |a_1| = 1)$$

$$a_1 \neq a_2$$

$\therefore (P(S), \preccurlyeq)$ is not antisymmetric

12

a We know that $x \vee 1 = 1, x \vee 0 = x$, Let $M_R \vee I_n = S$

$$\therefore S[i][j] = \begin{cases} 1 & i = j \\ M_R[i][j] & i \neq j \end{cases}$$

$$\implies \forall i \, 0 \leq i < n, S[i][i] = 1$$

$\therefore$ the relation holds for all (x,x) in the set

b We know that $M_R[i][j] = M_R^t[j][i]$, Let $M_R \vee M_R^t = S$

$$S[i][j] = M_R^t[i][j] \vee M_R[i][j]$$

$$= M_R[j][i] \vee M_R[i][j]$$

$$= M_R[j][i] \vee M_R^t[j][i]$$

$$= S[j][i]$$

$$\therefore S[i][j] = S[j][i]$$

So if (x,y) $\in$ the relation $\iff$ (y,x) $\in$ the relation

13

(a) Let A be the set of all bit strings of length three or more.

- Reflexive: for some string y all bits of y agree, $\therefore (y, y) \in R$ thus R is reflexive
- Symmetric: Trivial to see that if all bits execpt the first three agree for some x and y, then both (x,y) and (y,x) would belong to R. Thus R is symmetric
- Transitive: Let $(x, y) \in R, (y, z) \in R$, so all bit after the third position agree for x and y, and for y and z, so they would agree for x and z, $\therefore (x, z) \in R$. Thus R is transitive

R is reflexive, symmetric,transitive so it is an equivalence relation

(b) Not true, $(0101, 0000) \in R, (0000, 0101) \in R$, but $0101 \neq 0000 \therefore$ R is not antisymmetric, so R is not a POSET on the given set.

14

For a finite totally ordered set, by definition all subsets(except empty) would have a least element, therefore a finite totally ordered set would be well ordered, so there exists no such set.

15

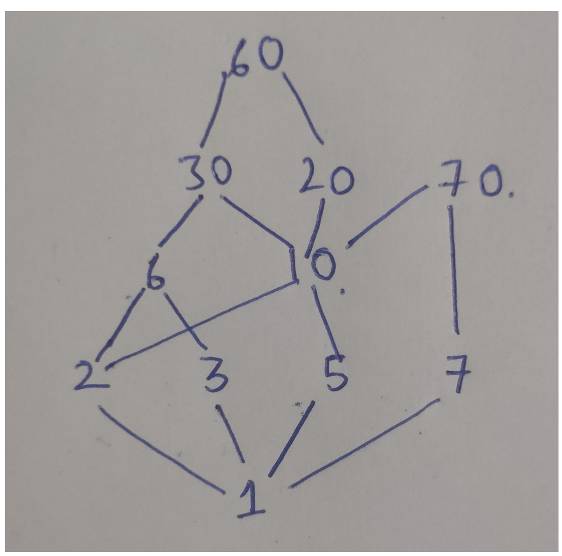

Figure 1: Hasse diagram

(a)

(b) Maximal elements : 60,70 [maximal elements have no sucessor]

(c) Minimal element : 1 [minimal elements have no predecessor]

(d) For greatest element(M) to exist $\forall x \in S, x \leq M$, where M is the greatest element.
There exist no element $M \in S$ that satisfies the above condition, hence the poset has no greatest element
Greatest element has all other elements as it's predecessors(direct/indirect)

(e) For least element(m) to exist $\forall x \in S, m \leq x$, where m is the least element.
The given poset has 1 as it's least element as it satisfies the above condition
Least element has all other elements as it's sucessors(direct/indirect)

(f) Upper bound of $\{2, 5\}$: $10, 20, 30, 60, 70$

(g) LUB of 2,5 : 10

(h) Lower bounds of 6,10 : 1,2

(i) GLB of 2,5 : 1

(j) This Poset is not a Lattice as many subsets have non-existent joins(LUB)
$\{20, 70\}, \{30, 70\}, \{60, 70\}$: LUB Does Not Exist (only some subsets listed)
$\therefore$ This Poset is not a lattice

16

$[A], [B], [C]$ are all lattices

17

Property	(1)	(2)	(3)	(4)	(5)
Distributive	NO	NO	YES	YES	YES
Complemented	YES	YES	NO	YES	NO

DISTRIBUTIVE LATTICE CHECK :

- Every lattice element has atmost 1 complement

COMPLEMENTED LATTICE CHECK :

- Every lattice element has atleast 1 complement

(1)

x	$\bar{x}$
b	c,d
c	b
d	b

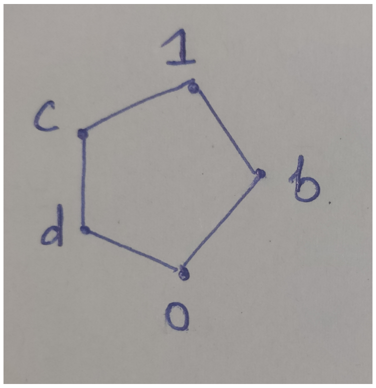

Figure 2: (q17 (1))

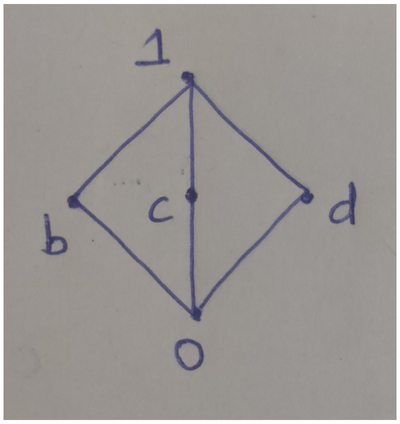

Figure 3: (q17 (2))

(2)

x	$\bar{x}$
b	c,d
c	b,d
d	b,c

(4)

x	$\bar{x}$
b	c
c	b

18

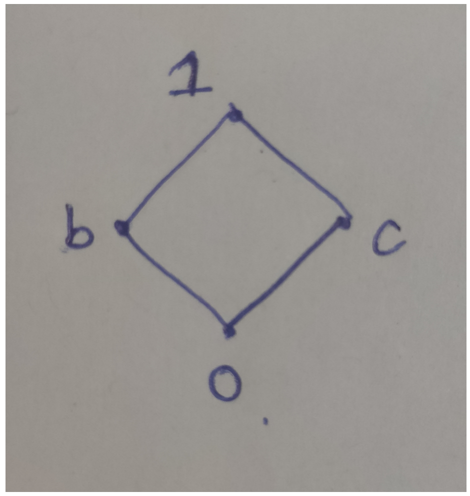

Figure 4: (q17 (4))

SUBSET	MAXIMAL(M)	MINIMAL(m)	GREATEST	LEAST	UB	LB	LUB	GLB
$\{d, k, f\}$	$\{k\}$	$\{d, f\}$	$\{k\}$	DNE	$\{k, l, m\}$	DNE	$\{k\}$	DNE
$\{b, h, f\}$	$\{h, f\}$	$\{b, f\}$	DNE	DNE	$\{l, m\}$	DNE	$\{k\}$	DNE
$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d, h, i, j, k, l, m\}$	$\{a, b, d\}$	$\{d\}$	$\{d\}$
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	DNE	DNE	$\{k, l, m\}$	DNE	$\{k\}$	DNE
$\{l, m\}$	$\{l, m\}$	$\{l, m\}$	DNE	DNE	DNE	$\{a, b, c, d, e, f, g, h, k\}$	DNE	$\{k\}$

Table 1: Answers of 18

19

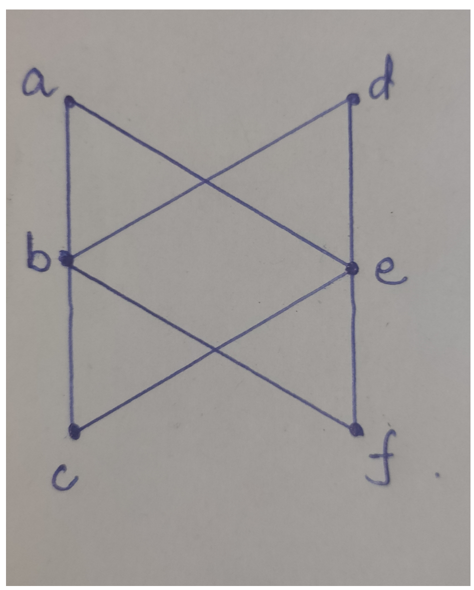

Figure 5: Poset P1 for (a),(d),(e)

- In the poset P1(figure), consider the subset $\{a, b, c, d, e, f\}$

It has :
2 maximal elements : a,d
2 minimal elements : c,f

- consider the poset $(\mathbb{Z}, \preccurlyeq)$, where $x \preccurlyeq y \iff x \leq y$
  and take the subset $(-\infty, 4]$
  Maximal element : 4
  Minimal element : Does Not exist

- Yes, as shown in the above example

- In the poset P1(figure), consider the subset $\{b, e\}$
  It has :
  Lower Bound : $\{c, f\}$
  GLB : Does Not exist

- In the poset P1(figure), consider the subset $\{b, e\}$
  It has :
  Upper Bound : $\{a, d\}$
  LUB : Does Not exist

20

(a) Let S be the set of divisors of 60. The given poset is a lattice as

$$\forall x, y \in S, x \vee y, x \wedge y \in S$$

(i.e) the meet and join exist and belong to the set, for all pairs of elements in S

Meet : $x \vee y \equiv \text{LCM}(x, y)$
Join : $x \wedge y \equiv \text{GCD}(x, y)$

(b) Let S be the power set of $\{0, 1, 2\}$. The given poset is a lattice as

$$\forall x, y \in S, x \vee y, x \wedge y \in S$$

(i.e) the meet and join exist and belong to the set, for all pairs of elements in S

Meet : $x \vee y \equiv x \cup y$
Join : $x \wedge y \equiv x \cap y$

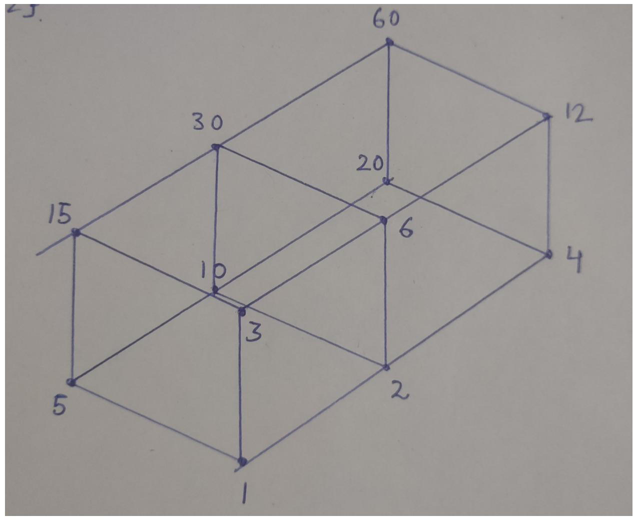

Figure 6: Hasse diagram for divisors of 60

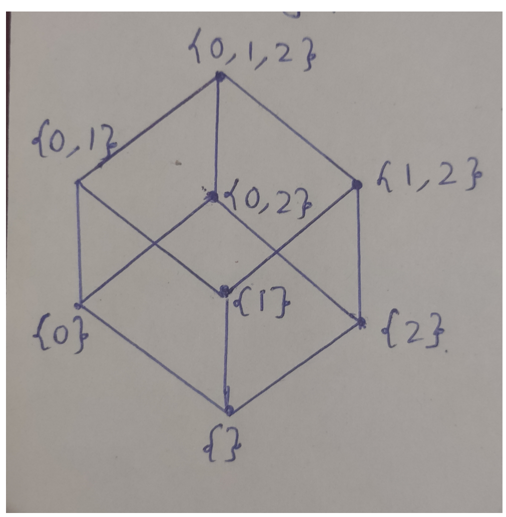

Figure 7: Hasse diagram for subsets of 0,1,2