1. Let G be a context-free grammar. Prove that the problem whether $\mathscr{L}(G) = \mathscr{L}(G)\,\mathscr{L}(G)$ is undecidable.

$$\overline{HP} \leq \underbrace{\{ G \mid \mathcal{L}(G) = \Delta^* \}}_{L}$$

$$M \# x \longmapsto G \quad \text{s.t.} \quad \mathcal{L}(G) = \overline{\text{VALCOMP}(M, x)}$$

If M does not halt on x, then $L = \Delta^*$

If $\underline{M \text{ halts on } x}$, then $L \neq \Delta^*$

For this exercise, the same reduction works.

$$\Delta^* = \Delta^* \Delta^*$$

a valid comp history $\longrightarrow$ $\# C_0 \# C_1 \# C_2 \# \ldots \# C_n \#$

$\in L$ $\quad\quad$ $\in L$

not the start config (head position)

2. For a Turing machine M and an input w for M, define

$$\text{VALCOMP-ALT}(M,w) = \{\ \# C_0 \# C_1^R \# C_2 \# C_3^R \# \cdots \# \mid C_0C_1C_2C_3\cdots \text{ is a valid computation history of } M \text{ on } w\ \}$$

$\boxed{w \# w^R}$

Prove the following assertions.

(a) The complement of VALCOMP-ALT(M,w) is context-free.

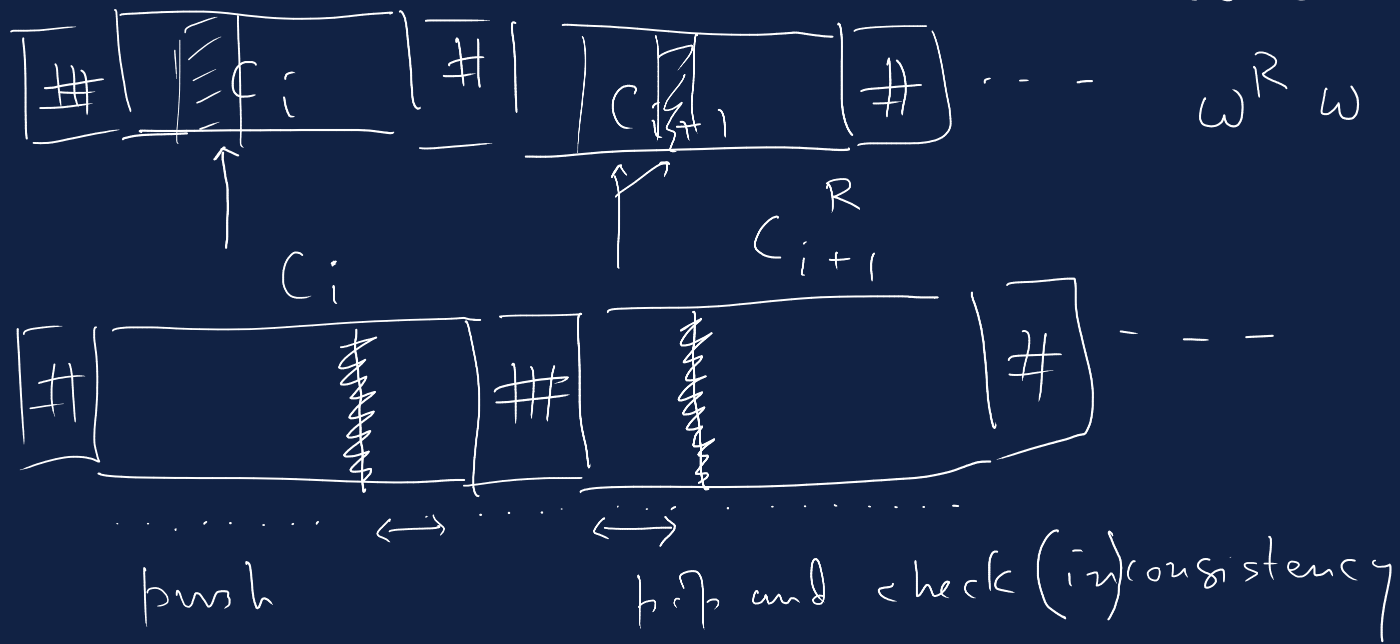
$\sim \{ \underline{ww} \}$

$\underline{w \# w^R \# w}$

ww^R

$w^R w$

Cannot be done by a DPDA



can be done by a DPDA

(b) VALCOMP-ALT(M,w) is the intersection of two DCFLs.

$$VCAE = \{ \# C_0 \# C_1^R \# C_2 \# C_3^R \# \cdots \# \mid C_i \text{ is consistent with } C_{i+1} \text{ for all even } i \}$$

$$VCAO = \{ \cdots \mid \text{for all odd } i \}$$

$$VCA(M, w) = VCAE \cap VCAO.$$

3. Prove that it is undecidable whether the intersections of two CFLs is empty.

$$\overline{HP} \leq \{ G_1 \# G_2 \mid \mathcal{L}(G_1) \cap \mathcal{L}(G_2) = \emptyset \}$$

$$M \# x \longmapsto G_1, G_2$$

G_1: a CFG for VCAE

G_2: a CFG for VCAO

M does not halt on x $\Rightarrow$ there are no valid comp histories of M on x

$\Rightarrow VCA = \emptyset$

$\Rightarrow VCAE \cap VCAO = \emptyset$.

Similly, if M halts on x

$\Rightarrow VCAE \cap VCAO \neq \emptyset$