1

The relation $\sim$ is not an equivalence relation because it is not reflexive.

$$1 \in \mathbb{Z} \text{ however } \gcd(1,1) = 1$$

$$(1,1) \notin \sim$$

$$\implies \sim \text{ is not reflexive}$$

$$\therefore \sim \text{ is not an equivalence relation}$$

2

Yes the relation (R) is a POSET as it is reflexive, antisymmetric and transitive

- Reflexive:
  - $(a,a) \in R$
  - $(b,b) \in R$
  - $(c,c) \in R$
  - $(d,d) \in R$

  $\therefore \forall x \in A, (x,x) \in R$, so R is reflexive
- Antisymmetric: $\nexists (x,y) \in R, x \neq y$ such that $(y,x) \in R, \therefore R$ is antisymmetric
- Transitive:
  - $(a,a) \in R, (a,d) \in R, (a,d) \in R$
  - $(b,b) \in R, (b,d) \in R, (b,d) \in R$
  - $(c,c) \in R, (c,d) \in R, (c,d) \in R$
  - $(a,d) \in R, (d,d) \in R, (a,d) \in R$
  - $(b,d) \in R, (d,d) \in R, (b,d) \in R$
  - $(c,d) \in R, (d,d) \in R, (c,d) \in R$

  $\therefore R$ is transitive via proof by exhaustion

3

R is not an equivalence relation as $(6,6) \notin R \therefore R$ is not reflexive.

4

$R_1 \cup R_2$ is not an equivalence relation, consider

- $A = \{x, y, z\}$
- $R_1 = \{(x, x), (y, y), (z, z), (x, y), (y, x)\}$
- $R_2 = \{(x, x), (y, y), (z, z), (y, z), (z, y)\}$
- $R_1 \cup R_2 = \{(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), (z, y)\}$

Now, $R_1 \cup R_2$ contains $(x, y), (y, z)$ however, $(x, z) \notin R_1 \cup R_2$

$$\therefore R_1 \cup R_2 \text{ is not transitive}$$

$$\therefore R_1 \cup R_2 \text{ is not an equivalence relation}$$

5

(a)
- Reflexive: $\forall (x, y) \in A, (x, y)R(x, y)$ as $xy = xy$
- Symmetric:

$$\forall (x_1, y_1), (x_2, y_2) \in A$$
$$(x_1, y_1)R(x_2, y_2) \implies (x_2, y_2)R(x_1, y_1)$$
$$(x_1y_1 = x_2y_2 \implies x_2y_2 = x_1y_1)$$

- Transitive:

$$\forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in A, (x_1, y_1)R(x_2, y_2), (x_2, y_2)R(x_3, y_3)$$
$$\begin{aligned} (x_1, y_1)R(x_2, y_2) &\implies x_1y_1 = x_2y_2 \\ (x_2, y_2)R(x_3, y_3) &\implies x_2y_2 = x_3y_3 \\ &\implies x_1y_1 = x_3y_3 \\ &\implies (x_1, y_1)R(x_3, y_3) \end{aligned}$$

$\therefore$ R is transitive

R is reflexive, symmetric and transitive $\therefore$ R is an equivalence relation.

(b) All equivalence classes are of the form

$$[(1, i)], \forall i \in \mathbb{N}$$

$$[(1, i)] = \{(x, y) | xy = i, x, y \in \mathbb{N}\}$$

(c) $[(1, 2)]$ has two elements $(1, 2), (2, 1)$, infact all equivalence classes of the form $[(1, p)]$ have two elements where $p \in Primes$

(d) $[(1, 4)]$ has three elements $(1, 4), (2, 2), (4, 1)$, infact all equivalence classes of the form $[(1, p^2)]$ have two elements where $p \in Primes$

6

- Reflexive:
$$\forall x \in A, |x-x| = 0, 0 \text{ is even}$$
$$\therefore \forall x \in A, (x,x) \in R$$

- Symmetric:
$$|x-y| = |y-x|$$
$$\therefore (x,y) \in R \implies (y,x) \in R$$

- Transitive: Let $(x,y),(y,z) \in R$
$$\begin{aligned}
|x-y| &= 2\lambda_1, \lambda_1 \in \mathbb{Z}^* \\
x-y &= 2\mu_1, \mu_1 \in \mathbb{Z} \\
|y-z| &= 2\lambda_2, \lambda_2 \in \mathbb{Z}^* \\
y-z &= 2\mu_2, \mu_2 \in \mathbb{Z} \\
\implies x-y+y-z &= 2(\mu_1+\mu_2) \\
\implies x-z &= 2\mu_3, \mu_3 \in \mathbb{Z} \\
\implies |x-z| &= 2\lambda_3(\text{even}), \lambda_3 \in \mathbb{Z}^* \\
&\implies (x-z) \in R
\end{aligned}$$

  $\therefore R$ is transitive

R is reflexive, symmetric and transitive so R is an equivalence relation.

7

Any equivalence relation ρ on set A induces a partition of A. So we can count partitions instead of equivalence relations

Type	Counts
4	$\binom{4}{0} = 1$
3,1	$\binom{4}{3} = 4$
2,2	$\frac{\binom{4}{2}}{2} = 3$
2,1,1	$\binom{4}{2} = 6$
1,1,1,1	$\binom{4}{4} = 1$

$\implies$ Number of Equivalence Relations on A = 1+4+3+6+1=15

S.No	Type	Equivalence Clases
1	4	$\{1,2,3,4\}$
2	$3,1$	$\{1,2,3\},\{4\}$
3	$3,1$	$\{1,2,4\},\{3\}$
4	$3,1$	$\{1,4,3\},\{2\}$
5	$3,1$	$\{4,2,3\},\{1\}$
6	$2,2$	$\{1,2\},\{3,4\}$
7	$2,2$	$\{1,3\},\{2,4\}$
8	$2,2$	$\{1,4\},\{3,2\}$
9	$2,1,1$	$\{1,2\},\{3\},\{4\}$
10	$2,1,1$	$\{1,3\},\{2\},\{4\}$
11	$2,1,1$	$\{1,4\},\{2\},\{3\}$
12	$2,1,1$	$\{3,2\},\{1\},\{4\}$
13	$2,1,1$	$\{4,2\},\{1\},\{3\}$
14	$2,1,1$	$\{3,4\},\{1\},\{2\}$
15	$1,1,1,1$	$\{1\},\{2\},\{3\},\{4\}$

8

The statement is true.
Proof:

- $\implies$ (If) We can prove this by its contrapositive
  If R is not antisymmetric,
  $$\exists(x,y),(y,x) \in R | x \neq y$$
  However, any closure of R would still contain $(x,y),(y,x)$ and would continue to remain antisymmetric $\implies$ no antisymmetric closure of R can exist
- $\impliedby$ (Only-If) R is antisymmetric $\implies$ the antisymmetric closure of R is itself, which exists

Total number of antisymmetric relations on a finite set of size n is given by $2^n \times 3^{\binom{n}{2}}$.
Proof:

- A relation R on a set A is antisymmetric if $\forall x,y \in A, (x,y),(y,x) \in R \implies x = y$.
- CASE 1: First we look at all pairs $(x,y)|x = y$. The number of such pairs is n, one for each element in A. We may have $(x,x) \in R$ or $(x,x) \notin R$. There are n such pairs, and 2 possibilities for each, so the total relations in this case are 2^n.
- CASE 2: Now we look at all pairs $(x,y)|x \neq y$. The number of such pairs is $\binom{n}{2}$, the number of ways of selecting 2 objects from a set of n objects. We may have
  - $(x,y) \in R, (y,x) \notin R$
  - $(y,x) \in R, (x,y) \notin R$
  - $(x,y),(y,x) \notin R$.

There are $\binom{n}{2}$ such pairs, and 3 possibilities for each, so the total relations in this case are $3^{\binom{n}{2}}$.

- CASE 1 and CASE 2 exhaust all possible pairs of elements. Using the multiplication rule of counting on the results of the two cases, the total number of antisymmetric relations on a finite set of size n are thus $2^n \times 3^{\binom{n}{2}}$.

9

Let $S = x, y$

$$R_1 = (x,x),(y,y),(x,y)$$

$$R_2 = (x,x),(y,y),(y,x)$$

$$R_1 \cup R_2 = (x,x),(y,y),(x,y),(y,x)$$

This is not antisymmetric $(x,y),(y,x) \in R_1 \cup R_2, x \neq y$
$\therefore R_1 \cup R_2$ is not a POSET on S

10

(a) Not a POSET,

$$(5,1) \preccurlyeq (5,2)$$

$$(5,2) \preccurlyeq (5,1)$$

$$(5,2) \neq (5,1)$$

$\therefore$ not antisymmetric, so not a POSET

(b) Is a POSET,

- Reflexive:

$$\forall (x,y) \in \mathbb{N} \times \mathbb{N}, (x,y) \preccurlyeq (x,y), x \leq x, y \geq y$$

$\therefore$ it is reflexive.

- Antisymmetric:

$$\text{Let } \exists (x_1,y_1),(x_2,y_2) \in \mathbb{N} \times \mathbb{N} : (x_1,y_1) \preccurlyeq (x_2,y_2),(x_2,y_2) \preccurlyeq (x_1,y_1)$$

$$\implies x_1 \leq x_2, x_2 \leq x_1, y_1 \geq y_2, y_2 \geq y_1$$

$$\implies x_1 = x_2, y_1 = y_2$$

$$\implies (x_1,y_1) = (x_2,y_2)$$

$\therefore$ it is antisymmetric.

- Transitive:

$$\text{Let } \exists (x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{N} \times \mathbb{N} : (x_1, y_1) \preccurlyeq (x_2, y_2), (x_2, y_2) \preccurlyeq (x_3, y_3)$$

$$\implies x_1 \le x_2, x_2 \le x_3, y_3 \ge y_2, y_2 \ge y_1$$
$$\implies x_1 \le x_3, y_3 \ge y_1$$
$$\implies (x_1, y_1) \preccurlyeq (x_3, y_3)$$

$\therefore$ it is transitive.

11

No this is not a POSET on $P(S)$

$$\text{Let } S = \{a_1, a_2, a_3\}$$

$$a_1 \preccurlyeq a_2 \ (|a_1| = 1 \le |a_2| = 1)$$
$$a_2 \preccurlyeq a_1 \ (|a_2| = 1 \le |a_1| = 1)$$
$$a_1 \ne a_2$$

$\therefore (P(S), \preccurlyeq)$ is not antisymmetric

12

a We know that $x \vee 1 = 1, x \vee 0 = x$, Let $M_R \vee I_n = S$

$$\therefore S[i][j] = \begin{cases} 1 & i = j \\ M_R[i][j] & i \ne j \end{cases}$$

$$\implies \forall i\, 0 \le i < n, S[i][i] = 1$$

$\therefore$ the relation holds for all (x,x) in the set

b We know that $M_R[i][j] = M_R^t[j][i]$, Let $M_R \vee M_R^t = S$

$$S[i][j] = M_R^t[i][j] \vee M_R[i][j]$$

$$= M_R[j][i] \vee M_R[i][j]$$
$$= M_R[j][i] \vee M_R^t[j][i]$$
$$= S[j][i]$$
$$\therefore S[i][j] = S[j][i]$$

So if (x,y) $\in$ the relation $\iff$ (y,x) $\in$ the relation

13

(a) Let A be the set of all bit strings of length three or more. Then the relation R on A

14

For a finite totally ordered set, by definition all subsets(except empty) would have a least element, therefore a finite totally ordered set would be well ordered, so there exists no such set.

15

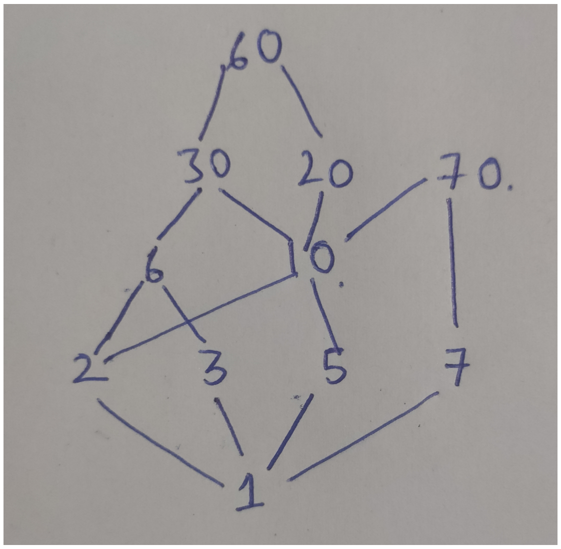

Figure 1: Hasse diagram

(a)

(b) Maximal elements : 60,70 [maximal elements have no sucessor]

(c) Minimal element : 1 [minimal elements have no predecessor]

(d) For greatest element(M) to exist $\forall x \in S, x \leq M$, where M is the greatest element.
There exist no element $M \in S$ that satisfies the above condition, hence the poset has no greatest element
Greatest element has all other elements as it's predecessors(direct/indirect)

(e) For least element(m) to exist $\forall x \in S, m \leq x$, where m is the least element.
The given poset has 1 as it's least element as it satisfies the above condition
Least element has all other elements as it's sucessors(direct/indirect)

(f) Upper bound of $\{2, 5\}$: $10, 20, 30, 60, 70$

(g) LUB of 2,5 : 10

(h) Lower bounds of 6,10 : 1,2

(i) GLB of 2,5 : 1

(j) This Poset is not a Lattice as many subsets have non-existent meets(LUB) $\{20, 70\}, \{30, 70\}, \{60, 70\}$: LUB Does Not Exist (only some subsets listed)
$\therefore$ This Poset is not a lattice

16

$[A], [B], [C]$ are all lattices

17

Property	(1)	(2)	(3)	(4)	(5)
Distributive	NO	NO	YES	YES	YES
Complemented	YES	YES	NO	YES	NO

DISTRIBUTIVE LATTICE CHECK :

- Every lattice element has atmost 1 complement

COMPLEMENTED LATTICE CHECK :

- Every lattice element has atleast 1 complement

(1)

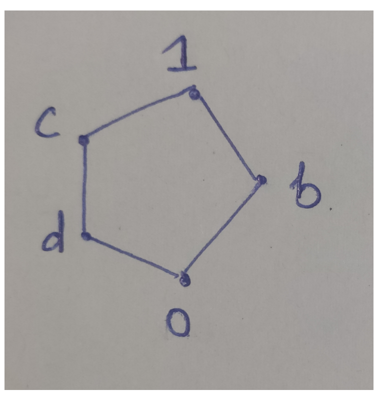

Figure 2

x	x
b	c,d
c	b
d	b

(2)

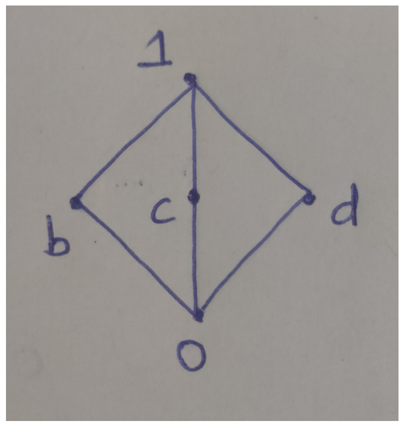

Figure 3

x	x
b	c,d
c	b
d	b

(4)

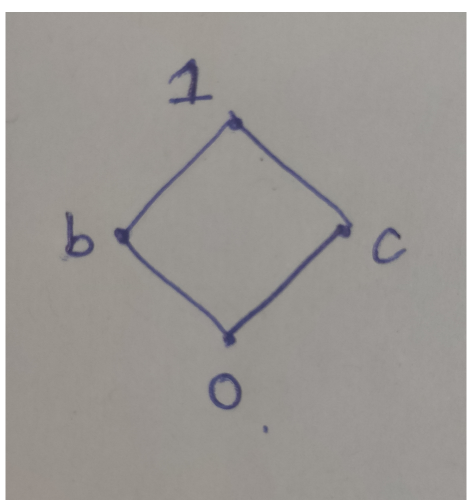

Figure 4

x	x
b	c,d
c	b
d	b

18

SUBSET	MAXIMAL(M)	MINIMAL(m)	GREATEST	LEAST	UB	LB	LUB	GLB
$\{d,k,f\}$	$\{k\}$	$\{d,f\}$	$\{k\}$	DNE	$\{k,l,m\}$	DNE	$\{k\}$	DNE
$\{b,h,f\}$	$\{h,f\}$	$\{b,f\}$	DNE	DNE	$\{l,m\}$	DNE	$\{k\}$	DNE
$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d,h,i,j,k,l,m\}$	$\{a,b,d\}$	$\{d\}$	$\{d\}$
$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	DNE	DNE	$\{k,l,m\}$	DNE	$\{k\}$	DNE
$\{l,m\}$	$\{l,m\}$	$\{l,m\}$	DNE	DNE	DNE	$\{a,b,c,d,e,f,g,h,k\}$	DNE	$\{k\}$

19

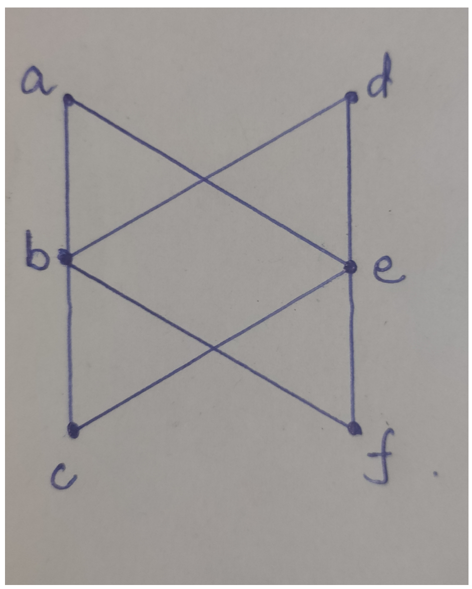

Figure 5: Poset P1 for (a),(d),(e)

- In the poset P1(figure), consider the subset $\{a,b,c,d,e,f\}$
  It has :
  2 maximal elements : a,d
  2 minimal elements : c,f
- consider the poset $(\mathbb{Z}, \preccurlyeq)$, where $x \preccurlyeq y \iff x \leq y$
  and take the subset $(-\infty, 4]$
  Maximal element : 4
  Minimal element : Does Not exist

- Yes, as shown in the above example
- In the poset P1(figure), consider the subset $\{b, e\}$
  It has :
  Lower Bound : $\{c, f\}$
  GLB : Does Not exist
- In the poset P1(figure), consider the subset $\{b, e\}$
  It has :
  Upper Bound : $\{a, d\}$
  LUB : Does Not exist

20

(a) Let S be the set of divisors of 60. The given poset is a lattice as

$$\forall x, y \in S, x \vee y, x \wedge y \in S$$

(i.e) the meet and join exist and belong to the set, for all pairs of elements in S

Meet : $x \vee y \equiv \text{LCM}(x, y)$
Join : $x \wedge y \equiv \text{GCD}(x, y)$

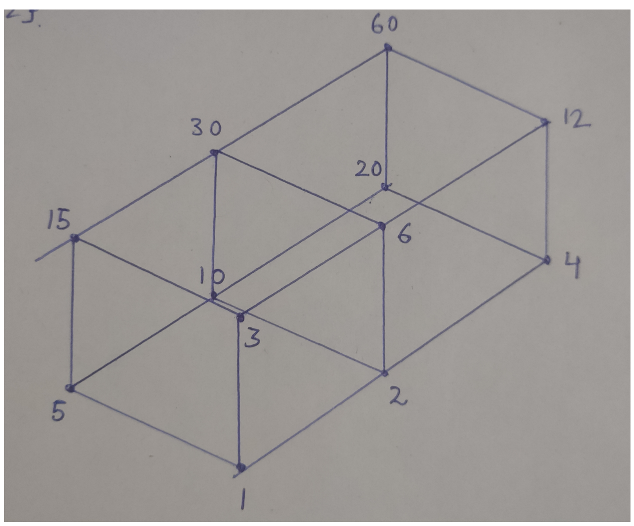

Figure 6: Hasse diagram for divisors of 60

(b) Let S be the power set of $\{0, 1, 2\}$. The given poset is a lattice as

$$\forall x, y \in S, x \vee y, x \wedge y \in S$$

(i.e) the meet and join exist and belong to the set, for all pairs of elements in S

Meet : $x \vee y \equiv x \cup y$
Join : $x \wedge y \equiv x \cap y$

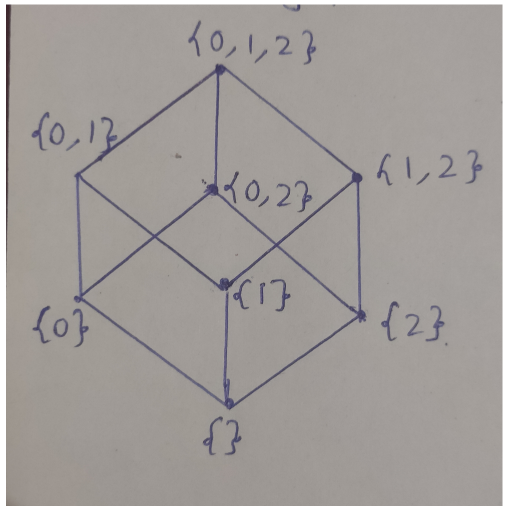

Figure 7: Hasse diagram for subsets of 0,1,2