Contents

Section outline

1. **Handling numbers**
   - Radix number systems
   - Complementation
   - Conversion of bases
   - Binary to BCD
   - Binary codes
   - Error detecting code
   - Error correcting code
   - Mininum bits for 1-bit ECC
   - Mininum bits for 1-bit EDC

Radix number systems

- $N = a_m b^m + \ldots + a_1 b + a_0 + a_{-1} b^{-1} + \ldots + a_{-p} b^{-p}$
  $0 \leq a_i < b$, MSB: a_m, LSB: a_{-p}
- $123.45 = 1 \times 10^2 + 2 \times 10^1 + 3 \times 10^0 + 4 \times 10^{-1} + 5 \times 10^{-2}$
- Integer part: $a_m b^m + \ldots + a_1 b + a_0$
- Fractional part: $a_{-1} b^{-1} + \ldots + a_{-p} b^{-p}$
- Common bases: 10 – decimal, 2 – binary, 8 – octal, 16 – hexadecimal
- $1101.01 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} = 13.25$
- $31.1_4 = ?$
- $15.2_8 = ?$

Numbers in some bases

Base					
2	4	8	10	12	16
0000	0	0	0	0	0
0001	1	1	1	1	1
0010	2	2	2	2	2
0011	3	3	3	3	3
0100	10	4	4	4	4
0101	11	5	5	5	5
0110	12	6	6	6	6
0111	13	7	7	7	7
1000	20	10	8	8	8
1001	21	11	9	9	9
1010	22	12	10	α	A
1011	23	13	11	β	B
1100	30	14	12	10	C
1101	31	15	13	11	D
1110	32	16	14	12	E
1111	33	17	15	13	F

Complementation

- Complement of digit a, denoted a', in base b is ${a_b}' = (b-1)_b - a_b$
- Binary: $a'_2 = 1_2 - a_2$, $0' = 1$, $1' = 0$
- Decimal: $a'_{10} = 9_{10} - a_{10}$
  $0' = 9$, $1' = 8$, $2' = 7$, $3' = 6$, $4' = 5$, $5' = 4$, $6' = 3$, $7' = 2$, $8' = 1$, $9' = 0$
- Octal: $a'_8 = 7_8 - a_8$
  $0' = 7$, $1' = 6$, $2' = 5$, $3' = 4$, $4' = 3$, $5' = 2$, $6' = 1$, $7' = 0$
- For, $N = a_m b^m + \ldots + a_1 b + a_0$, let $M = a'_m b^m + \ldots + a'_1 b + a'_0$

$\therefore$ $M = (b-1-a_m)\, b^m + \ldots + (b-1-a_1)\, b + (b-1-a_0)$

$$\Rightarrow M = \sum_{i=1}^{m+1} b^i - \sum_{i=0}^{m} b^i - N = \left(b^{m+1} - 1\right) - N$$

- Diminished radix complement of N is $\left(b^{m+1} - 1\right) - N = M$
- Radix complement of N is $b^{m+1} - N = M + 1 = N'$
- $P - N = P + N' \mod b^m$ (for m digits)

Complementation (contd.)

Example (Decimal subtraction)

- $321 - 123 = 198$
- Ten's complement of 123: $876 + 1 = 877$
- $321 + 876 = 1198 = 198 \mod 10^3$

Example (Binary subtraction)

- $1\,0100\,0001 - 0\,0111\,1011 = 0\,1100\,0110$
- 2's complement of $0\,0111\,1011$: $1\,1000\,0100 + 1 = 1\,1000\,0101$
- $1\,0100\,0001 + 1\,1000\,0101 = 10\,1100\,0110 = 0\,1100\,0110 \mod 2^9$

Complementation (contd.)

	Num	twos'	two's
0	0000	1111	0000
1	0001	1110	1111
2	0010	1101	1110
3	0011	1100	1101
4	0100	1011	1100
5	0101	1010	1011
6	0110	1001	1010
7	0111	1000	1001
8	1000	0111	1000
9	1001	0110	0111

	Num	twos'	two's
0	0 0000	1 1111	0 0000
1	0 0001	1 1110	1 1111
2	0 0010	1 1101	1 1110
3	0 0011	1 1100	1 1101
4	0 0100	1 1011	1 1100
5	0 0101	1 1010	1 1011
6	0 0110	1 1001	1 1010
7	0 0111	1 1000	1 1001
8	0 1000	1 0111	1 1000
9	0 1001	1 0110	1 0111
10	0 1010	1 0101	1 0110
11	0 1011	1 0100	1 0101
12	0 1100	1 0011	1 0100
13	0 1101	1 0010	1 0011
14	0 1110	1 0001	1 0010
15	0 1111	1 0000	1 0001

Conversion of bases

- Number in base b_1 to be converted to base b_2
- If $b_1 < b_2$, use arithmetic of b_2
- $N = a_m b^m + \ldots + a_1 b + a_0 + a_{-1} b^{-1} + \ldots + a_{-p} b^{-p}$

Example (432.2_8 to decimal)

$432.2_8 = 4 \times 8^2 + 3 \times 8^1 + 2 \times 8^0 + 2 \times 8^{-1} = 282.25_{10}$

Example (1101.01_2 to decimal)

$1101.01_2 = 1 \times 23 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} = 13.25_{10}$

Conversion of bases

- Number in base b_1 to be converted to base b_2
- If $b_1 > b_2$, use arithmetic of b_1
- $N_{b_1} = \underbrace{a_m {b_2}^m + \ldots + a_1 b_2 + a_0}_{A} + \underbrace{a_{-1} {b_2}^{-1} + \ldots + a_{-p} {b_2}^{-p}}_{B}$
- $\frac{A}{b_2} = \underbrace{a_m {b_2}^{m-1} + \ldots + a_1}_{Q_0} + \frac{a_0}{b_2}$
- Least significant digit of A_{b_2} is the remainder of $\frac{a_0}{b_2}$
- If $Q_0 = 0$, terminate, otherwise, apply procedure recursively to Q_0

Conversion of bases (contd.

Example (548_{10} to octal (base 8))

Q_i	r_i	
68	4	a_0
8	4	a_1
1	0	a_2
	1	a_3

$548_{10} = 1044_8$

Example (345_{10} to base 6)

Q_i	r_i	
57	3	a_0
9	3	a_1
1	3	a_2
	1	a_3

$245_{10} = 1333_6$

Conversion of bases (contd.

- $b_2 B = a_{-1} + \underbrace{a_{-1} {b_2}^{-1} + \ldots + a_{-p} {b_2}^{1-p}}_{F}$
- The first digit of fractional part is the integer part of the product
- Continue recursively until F is non-zero

Example (0.3125_{10} to base 8)

- $0.3125 \times 8 = 2.5000$
- $0.5000 \times 8 = 4.0000$
- $a_{-1} = 2$, $a_{-2} = 4$
- $0.3125_{10} = 0.24_8$

Binary to BCD

$d = 0 - 9$: binary and BCD are identical

$d = 10 - 15$: 1 goes to the next higher place, $d - 10$ in current place

Alternately $d + 6 \mod 16$ in current place, if $d \geq 10$

$d = 12$: $d + 6 = 18 \mod 16 = 2$, 1 goes to next higher place

$12_{10} = 1100_2$, 1100 + 0110 = 1 0010

NB: LSB is unaffected, because LSB of $6_{10} = 0$

If bits are handled sequentially, 3 can be added (instead of 6) and then shifted left

110 + 011 = 1001 $\longrightarrow$ 1 0010

To be repeated until conversion is complete

Name Shift-and-add-3 or double-dabble

Binary of 48748 to BCD example

Op	B4	B3	B2	B1	B0	48748
L Sft	0000	0000	0000	0000	0001	1011111001101100
L Sft	0000	0000	0000	0000	0010	1011111001101100
L Sft	0000	0000	0000	0000	0101	1011111001101100
Add 3	0000	0000	0000	0000	1000	1011111001101100
L Sft	0000	0000	0000	0001	0001	1011111001101100
L Sft	0000	0000	0000	0010	0011	1011111001101100
L Sft	0000	0000	0000	0100	0111	1011111001101100
Add 3	0000	0000	0000	0100	1010	1011111001101100
L Sft	0000	0000	0000	1001	0101	1011111001101100
Add 3	0000	0000	0000	1100	1000	1011111001101100
L Sft	0000	0000	0001	1001	0000	1011111001101100
Add 3	0000	0000	0001	1100	0000	1011111001101100
L Sft	0000	0000	0011	1000	0000	1011111001101100

Binary of 48748 to BCD example

Op	B4	B3	B2	B1	B0	48748
Add 3	0000	0000	0011	1011	0000	1011111001101100
L Sft	0000	0000	0111	0110	0001	1011111001101100
Add 3	0000	0000	1010	1001	0001	1011111001101100
L Sft	0000	0001	0101	0010	0011	1011111001101100
Add 3	0000	0001	1000	0010	0011	1011111001101100
L Sft	0000	0011	0000	0100	0110	1011111001101100
Add 3	0000	0011	0000	0100	1001	1011111001101100
L Sft	0000	0110	0000	1001	0011	1011111001101100
Add 3	0000	1001	0000	1100	0011	1011111001101100
L Sft	0001	0010	0001	1000	0111	1011111001101100
Add 3	0001	0010	0001	1011	1010	1011111001101100
L Sft	0010	0100	0011	0111	0100	1011111001101100
Add 3	0010	0100	0011	1010	0100	1011111001101100
L Sft	0100	1000	0111	0100	1000	1011111001101100
End	4	8	7	4	8	

Correctness of binary to BCD conversion

- Given binary value is $B = b_{n-1}b_{n-2}\dots b_0$, $n = 15$ for the example
- Let D be the BCD number with digits $d_{m-1}...d_j...d_0$, $m \leq 4\frac{n}{3}$
- Let D_j be the value of the BCD number after the j^{th} shift
- $D_0 = 00\dots 0$ (D is initialised to 0)
- Initially, each BCD value y_i of digit d_i is valid (zero)
- Also, at least one bit of B is pending conversion
- On a left shift, each new BCD value of d_i' is $y_i' = 2y_i + m_{i-1}$ where m_{i-1} is the MSB of y_{i-1} if $i \geq 1$, otherwise the next input bit
- For the first three left shifts $D_j = 2D_{j-1} + b_{n-j}$ holds ($D_1 = 2D_0 + b_{15}$, $D_2 = 2D_1 + b_{14}$, $D_3 = 2D_2 + b_{13}$)
- If the bits are exhausted, then the conversion correctly terminates
- Otherwise, if any $y_i' \geq 5$, it's updated to $y_i' = 2y_i + m_{i-1} + 3$
- MSB of d_j is the carry to be shifted into d_{j+1}
- On the next left shift, $D_j = 2D_{j-1} + b_{n-j}$ again holds
- Conversion algorithm is reversible

BCD 48748 to Binary example

Op	B4	B3	B2	B1	B0	
Input	0100	1000	0111	0100	1000	
R Sft Sub 3	0010 0010	0100 0100	0011 0011	1010 0111	0100 0100	0000000000000000 0000000000000000
R Sft Sub 3	0001 0001	0010 0010	0001 0001	1011 1000	1010 0111	0000000000000000 0000000000000000
R Sft Sub 3	0000 0000	1001 0110	0000 0000	1100 1001	0011 0011	1000000000000000 1000000000000000
R Sft Sub 3	0000 0000	0011 0011	0000 0000	0100 0100	1001 0110	1100000000000000 1100000000000000
R Sft Sub 3	0000 0000	0001 0001	1000 0101	0010 0010	0011 0011	0110000000000000 0110000000000000
R Sft Sub 3	0000 0000	0000 0000	1010 0111	1001 0110	0001 0001	1011000000000000 1011000000000000
R Sft Sub 3	0000 0000	0000 0000	0011 0011	1011 1000	0000 0000	1101100000000000 1101100000000000

BCD 48748 to Binary example

R Sft Sub 3	0000 0000	0000 0000	0001 0001	1100 1001	0000 0000	0110110000000000 0110110000000000
R Sft Sub 3	0000 0000	0000 0000	0000 0000	1100 1001	1000 0101	0011011000000000 0011011000000000
R Sft Sub 3	0000 0000	0000 0000	0000 0000	0100 0100	1010 0111	1001101100000000 1001101100000000
R Sft	0000	0000	0000	0010	0011	1100110110000000
R Sft	0000	0000	0000	0001	0001	1110011011000000
R Sft Sub 3	0000 0000	0000 0000	0000 0000	0000 0000	1000 0101	1111001101100000 1111001101100000
R Sft	0000	0000	0000	0000	0010	1111100110110000
R Sft	0000	0000	0000	0000	0001	0111110011011000
R Sft	0000	0000	0000	0000	0000	1011111001101100
End						48748

Binary codes

- Binary coding scheme for decimal digits
- Sequence of bits $x_3x_2x_1x_0$ (say) for N is it's code word
- Each position i may have a weight w_i (**weighted code**); $N = \sum w_ix_i$
- For BCD $w_3 = 8, w_2 = 4, w_1 = 2, w_0 = 1$
- Sum of weights is 9 for self-complementing code

		weights													
N		8	4	2	1		2	4	2	1		6	4	2	-3
0		0	0	0	0		0	0	0	0		0	0	0	0
1		0	0	0	1		0	0	0	1		0	1	0	1
2		0	0	1	0		0	0	1	0		0	0	1	0
3		0	0	1	1		0	0	1	1		1	0	0	1
4		0	1	0	0		0	1	0	0		0	1	0	0
5		0	1	0	1		1	0	0	1		1	0	1	1
6		0	1	1	0		1	1	0	0		0	1	1	0
7		0	1	1	1		1	1	0	1		1	1	0	0
8		1	0	0	0		1	1	1	0		1	0	1	0
9		1	0	0	1		1	1	1	1		1	1	1	1

Binary codes

BCD					Excess-3					Cyclic					Gray			
0	0	0	0		0	0	1	1		0	0	0	0		0	0	0	0
0	0	0	1		0	1	0	0		0	0	0	1		0	0	0	1
0	0	1	0		0	1	0	1		0	0	1	1		0	0	1	1
0	0	1	1		0	1	1	0		0	0	1	0		0	0	1	0
0	1	0	0		0	1	1	1		0	1	1	0		0	1	1	0
0	1	0	1		1	0	0	0		1	1	1	0		0	1	1	1
0	1	1	0		1	0	0	1		1	0	1	0		0	1	0	1
0	1	1	1		1	0	1	0		1	0	0	0		0	1	0	0
1	0	0	0		1	0	1	1		1	1	0	0		1	1	0	0
1	0	0	1		1	1	0	0		0	1	0	0		1	1	0	1

- Excess-3, Cyclic and Gray codes are **unweighted** codes
- Excess-3 code is formed by adding 3 (0011) to the BCD value
- It's is self-complementing (n+3 $+$ (9-n)+3 $=$ 15)
- Adjacent code words of a cyclic code differ only in one place in the range 0..9, also, 0 and 9 are adjacent
- What if the codes are: 8, 4, -2, -1

Binary codes (contd.)

BCD				Excess-3				Cyclic				Gray			
0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	1
0	0	1	0	0	1	0	1	0	0	1	1	0	0	1	1
0	0	1	1	0	1	1	0	0	0	1	0	0	0	1	0
0	1	0	0	0	1	1	1	0	1	1	0	0	1	1	0
0	1	0	1	1	0	0	0	1	1	1	0	0	1	1	1
0	1	1	0	1	0	0	1	1	0	1	0	0	1	0	1
0	1	1	1	1	0	1	0	1	0	0	0	0	1	0	0
1	0	0	0	1	0	1	1	1	1	0	0	1	1	0	0
1	0	0	1	1	1	0	0	0	1	0	0	1	1	0	1

- Gray code is cyclic (in the range 0..15, 0 and 15 being adjacent for a 4-bit code) and also a reflected code – not cyclic in 0..9
- $g_i = b_i \oplus b_{i+1}$, $g_{n-1} = b_{n-1}$; $b_i = ?$
- $g_i \oplus b_{i+1} = b_i \oplus b_{i+1} \oplus b_{i+1} = b_i \oplus 0 = b_i$

Binary codes (contd.)

N	Binary				Gray			
0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	1
2	0	0	1	0	0	0	1	1
3	0	0	1	1	0	0	1	0
4	0	1	0	0	0	1	1	0
5	0	1	0	1	0	1	1	1
6	0	1	1	0	0	1	0	1
7	0	1	1	1	0	1	0	0
8	1	0	0	0	1	1	0	0
9	1	0	0	1	1	1	0	1
10	1	0	1	0	1	1	1	1
11	1	0	1	1	1	1	1	0
12	1	1	0	0	1	0	1	0
13	1	1	0	1	1	0	1	1
14	1	1	1	0	1	0	0	1
15	1	1	1	1	1	0	0	0

- $g_i = b_i \oplus b_{i+1}$, $g_{n-1} = b_{n-1}$
- n and it's bitwise complement $\tilde{n}$ are placed symmetrically about the middle of the table
- Their Gray codes should differ only in the MSB
- Let $n \equiv b_{n-1}b_{n-2}\dots b_0$ and it's Gray code be $g_{n-1}g_{n-2}\dots g_0$
- By the rule the gray code of $\tilde{n}$ is

$$\begin{array}{cccc}
\overline{b_{n-1}} & \overline{b_{n-2}} & \dots & \overline{b_0} \\
0 & \overline{b_{n-1}} & \dots & \overline{b_1} \\
\hline
\overline{b_{n-1}} & b_{n-2} \oplus b_{n-1} & \dots & b_0 \oplus b_1 \\
\hline
\overline{g_{n-1}} & g_{n-2} & \dots & g_0
\end{array}$$

- Thus the Gray codes of n and $\tilde{n}$ differ only in the MSB

Binary codes (contd.)

Is the Gray code weighted?

- Can we find weights such that $\sum_i w_i x_{i,j} = j$?
- Suppose it's weighted
- Utilise the property that adjacent codes differ in one place only
- $\forall i \exists j | (j+1) - j = \sum_i w_i \left(x_{i,j+1} - x_{i,j}\right) = \pm w_i = 1$ (why?)
- This precludes representation of 2^n values for a n-bit Gray code

Is the Excess-3 code weighted?

- Can we find weights such that $\sum_i w_i x_{i,j} = j$?
- $w_2 = 1$ [$1 \mapsto 4$ (0100)]
- $w_3 = 5$ [$5 \mapsto 8$ (1000)]
- $w_1 + w_0 = 0$ [$0 \mapsto 3$ (0011)]
- But, $w_2 + w_1 + w_0 = 5 \neq 4$ [$4 \mapsto 7$ (0111)] – inconsistent

Excess-3 arithmetic

Example (Excess-3 addition)

- $825 + 528 = 1353$
- Excess-3

	0	0	1	1	1	0	1	1	0	1	0	1	1	0	0	0
+	0	0	1	1	1	0	0	0	0	1	0	1	1	0	1	1
	0	1	1	1	10	0	1	1	1	0	1	1	10	0	1	1
	0	1	0	0	0	1	1	0	1	0	0	0	0	1	1	0

Example (Excess-3 subtraction)

- $825 - 528 = 297 \rightarrow 825 + 471 + 1 = 1297 = 297 \mod 1000$
- Excess-3

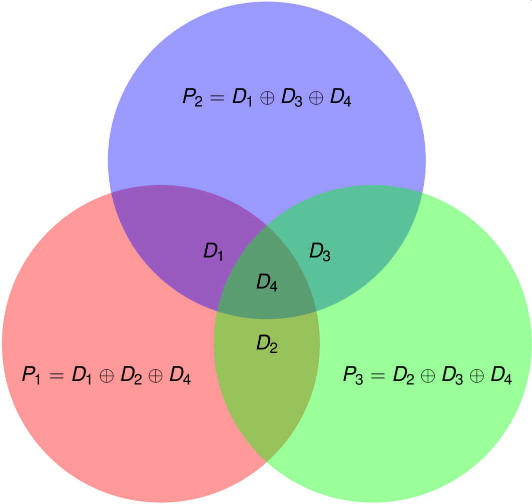
	0	0	1	1	1	0	1	1	0	1	0	1	1	0	0	0
+	0	0	1	1	0	1	1	1	1	0	1	0	0	1	0	10
	0	1	1	1	10	0	1	0	1	1	1	1	1	1	0	1
	0	1	0	0	0	1	0	1	1	1	0	0	1	0	1	0

Error detecting code

N		Even Parity BCD						2-out-of-5, $\binom{5}{2}=10$						63210 BCD				
		8	4	2	1	p		0	1	2	4	7		6	3	2	1	0
0		0	0	0	0	0		0	0	0	1	1		0	0	1	1	0
1		0	0	0	1	1		1	1	0	0	0		0	0	0	1	1
2		0	0	1	0	1		1	0	1	0	0		0	0	1	0	1
3		0	0	1	1	0		0	1	1	0	0		0	1	0	0	1
4		0	1	0	0	1		1	0	0	1	0		0	1	0	1	0
5		0	1	0	1	0		0	1	0	1	0		0	1	1	0	0
6		0	1	1	0	0		0	0	1	1	0		1	0	0	0	1
7		0	1	1	1	1		1	0	0	0	1		1	0	0	1	0
8		1	0	0	0	1		0	1	0	0	1		1	0	1	0	0
9		1	0	0	1	0		0	0	1	0	1		1	1	0	0	0

- Hamming distance: number of bits differing between two codes
- If minimum Hamming distance between any two code words is d then $d-1$ single bit errors can be detected

Error correcting code



Correction for single bit error

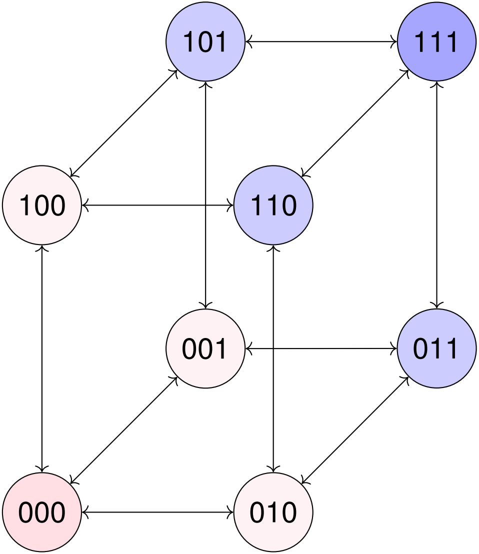
- D_1 P_1 and P_2 affected, P_3 unaffected
- D_2 P_1 and P_3 affected, P_2 unaffected
- D_3 P_2 and P_3 affected, P_1 unaffected
- D_4 P_1, P_2 and P_2 affected
- P_1 D_1, D_2, D_3, P_1 P_2 and P_2 unaffected, D_1, D_2, D_3
- P_2 D_1, D_2, D_3, P_1 P_2 and P_3 unaffected
- P_3 D_1, D_2, D_3, P_1 P_1 and P_2 unaffected

Relating data and parity bits

- Association of parity bits to the data bits may be done according to the table below

Bits indices	7	6	5	4	3	2	1
Binary	111	110	101	100	011	010	001
Data/parity	d_4	d_3	d_2	p_3	d_1	p_2	p_1
Association	p_3, p_2, p_1	p_3, p_2	p_3, p_1	p_3	p_2, p_1	p_2	p_1

- Bit at 2^i positions (1, 2, 4) are for parity, others for data
- p_1 covers data bit positions having 1 in LSB (1:p_1, 3:d_1, 5:d_2, 7:d_4)
- p_2 covers data bit positions having 1 in next higher bit position (2:p_2, 3:d_1, 6:d_3, 7:d_4)
- p_3 covers data bit positions having 1 in next higher bit position (4:p_3, 5:d_2, 6:d_3, 7:d_4)
- This scheme may be generalised



- Consider codes 000 and 111 and all possible single bit errors
- Any single bit error code can be tracked backed to 000 or 111
- Achive by maintaining Hamming distance of 3 between the code words
- If d is the minimum Hamming distance between code words, up to $\lfloor \frac{d-1}{2} \rfloor$-bit errors can be corrected

Mininum bits for 1-bit ECC

- Let there be m information bits in total of n bits; $m+p=n$
- n patterns for 1-bit error in a code word; 1 valid pattern
- Reserve $n+1$ patterns for each code
- $(n+1)2^m \leq 2^n$
- $n+1 \leq 2^{n-m} = 2^p$
- $m+p+1 \leq 2^p$
- For $m=4$ $p=?$
- Say $p=3$ then $2^p = 2^3 = 8 \geq 4+3+1 = 8$

Mininum bits for 1-bit EDC

- For single bit error, all codes at Hamming distance of 1 from a valid code are in error
- Since there is no recovery errorneous codes can be "shared" between valid codes
- Adjacent codes must have separate colours (valid:✓, error:✗)

	000	001	011	010	110	111	101	100
00	✓	✗	✓	✗	✓	✗	✓	✗
01	✗	✓	✗	✓	✗	✓	✗	✓
11	✓	✗	✓	✗	✓	✗	✓	✗
10	✗	✓	✗	✓	✗	✓	✗	✓

- For single bit error, at most half the codes are usable
- For m bits of data, $n = m + 1$ bits are needed for EDC
- BCD cannot be accommodated in 4-bits