Solution of Linear Differential Equations

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Contents

1 Introduction

- Linear differential equations (LDE) describe the dynamics of many a physical system including circuits.
- One is bound to handle LDE, if the system consists of Energy Storage Elements like mass, spring in physical system or inductor and capacitor in a network.
- After writing the differential equation (using KVL, KCL in circuits, it is necessary to solve them.
- Laplace transform method is widely used to get the solution of LDE.
- Here we shall try to get the solution of LDE by classical method.
- Solution of LDE by classical method gives better insight into the problem.
- Our goal in this note will be to obtain the solution of LDE classically and quickly.

2 First order differential Equation

Consider the following first order LDE

$$\frac{dy}{dt} + ay = x(t)$$

Multiply both sides with e^{at}

$$\begin{aligned}
e^{at}\frac{dy}{dt} + ae^{at}y &= e^{at}x(t) \\
\text{or, } \frac{d(e^{at}y)}{dt} &= e^{at}x(t) \\
\int d(e^{at}y) &= A + \int e^{at}x(t)\,dt \ \text{ where } A = \text{integration constant} \\
\therefore\ y(t) &= Ae^{-at} + e^{-at}\int e^{at}x(t)\,dt = y_n(t) + y_f(t)
\end{aligned}$$

1. $y_n(t)$ is called *natural response* (also called complementary function) as it does not depend on the forcing function $x(t)$. This will exist if the there is initially stored energy in the system and the value of A is determined from the initial condition.

2. $y_f(t)$ is called the *response due to forcing function* (also known as particular integral) as it depends on $x(t)$.

We evaluate the integral $\int e^{at}x(t)\,dt$ to get $y_f(t)$

$$\begin{aligned}
\int e^{at}x(t)\,dt &= x\frac{e^{at}}{a} - \int \dot{x}\frac{e^{at}}{a}dt \text{ ; Integrating by parts} \\
&= \frac{1}{a}xe^{at} - \frac{1}{a^2}\dot{x}e^{at} + \int \ddot{x}\,\frac{e^{at}}{a^2}dt \text{ ; Once again Integrating by parts} \\
\therefore\ y_f(t) &= \frac{1}{a}x - \frac{1}{a^2}\dot{x} + \frac{1}{a^3}\ddot{x} + \cdots\infty
\end{aligned}$$

Told in language we can say that response due to forcing function $f(t)$ is nothing but **linear combination of the forcing function and its higher order derivatives up to infinite terms**

$$\begin{aligned}
y_f(t) &= k_o x + k_1\dot{x} + k_2\ddot{x} + \cdots\infty \\
\text{Hence total solution is } y(t) &= Ae^{-at} + (k_o x + k_1\dot{x} + k_2\ddot{x} + \cdots\infty)
\end{aligned}$$

Exception to this rule

That $y_f(t) = k_o x + k_1\dot{x} + k_2\ddot{x} + \cdots\infty$ is not true for $x(t) = Ae^{-at}$ where $-a$ is the characteristic root of the first order differential equation. Let us see, what will then be $y_f(t)$?

$$\begin{aligned}
\text{Given equation is: } \frac{dy}{dt} + ay &= x(t) = Ae^{-at} \\
e^{at}\frac{dy}{dt} + ae^{at}x &= A \text{ ;Multiply with the integrating factor} \\
\frac{d}{dt}(e^{at}y) &= A \\
\text{or, } d(e^{at}y) &= A\,dt \\
\text{Integrating, } e^{at}y &= A\,t + B \\
\text{or, } y &= (A\,t + B)e^{-at} \text{ ;Total solution} \\
\therefore\ y_n(t) &= Be^{-at} \text{ ;Natural response} \\
\text{and, } y_f(t) &= A\,te^{-at} \text{ ;Solution due to forcing function} \\
\therefore\ \text{Total solution is : } y(t) &= Be^{-at} + A\,te^{-at} = (At + B)e^{-at}
\end{aligned}$$

3 Second order differential equation

Let us consider a second order linear differential equation with forcing function $f(t)$.

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = x(t) \tag{1}$$

Solution of eq^n (1) is: $y(t) = y_n(t) + y_f(t)$
Where, $y_n(t)$ is the natural response with $x(t) = 0$ and $y_f(t)$ is the particular integral or solution due to forcing function. Let operator $D = \frac{d}{dt}$, hence the equation can be rewritten as

$$\begin{aligned} (D^2 + aD + b)y &= x(t) \\ \text{If roots are } m_1, m_2 & \\ \text{The equation can be written as, } (D - m_1)(D - m_2)y &= x(t) \\ \text{Let, } (D - m_2)y &= z(t) \\ \therefore\ (D - m_1)z &= x(t) \end{aligned}$$

We shall first solve for $z(t)$ from the last first order differential equation. After knowing $z(t)$, we shall solve for $y(t)$ from $(D - m_2)y = z(t)$, which is also a first order differential equation with $z(t)$ appearing as forcing function.

Case1: $m_1 \neq m_2$

In this case roots are not equal i.e., $m_1 \neq m_2$.

$$\begin{aligned} (D - m_1)z &= x(t) \\ \therefore\ z(t) &= Ae^{m_1 t} + (k_o x + k_1 \dot{x} + k_2 \ddot{x} + \cdots \infty) \end{aligned}$$

Now the first equation becomes:

$$\begin{aligned} (D - m_2)y &= z(t) \\ \text{or, } (D - m_2)y &= Ae^{m_1 t} + (k_o x + k_1 \dot{x} + k_2 \ddot{x} + \cdots \infty) \end{aligned}$$

The last equation is once again a first order differential equation and its forcing function is:

$$Ae^{m_1 t} + (k_o x + k_1 \dot{x} + k_2 \ddot{x} + \cdots \infty)$$

So the total solution will be:

$$y(t) = A_1 e^{m_1 t} + A_2 e^{m_2 t} + (K_o x + K_1 \dot{x} + K_2 \ddot{x} + \cdots \infty)$$

Case2: $m_1 = m_2 = m$

Here the roots are equal i.e., $m_1 = m_2 = m$.

$$\begin{aligned}
(D^2 + aD + b)y &= x(t) \\
\text{If roots are } m_1 = m_2 = m & \\
\text{The equation can be written as, } (D-m)(D-m)y &= x(t) \\
\text{Let, } \ (D-m)y &= z(t) \\
\therefore \ (D-m)z &= x(t)
\end{aligned}$$

We shall first solve for $z(t)$ from the last first order differential equation. After knowing $z(t)$, we shall solve for $y(t)$ from $(D-m)y = z(t)$, which is also a first order differential equation.

$$\begin{aligned}
(D-m)z &= x(t) \\
\therefore \ z(t) &= Ae^{mt} + (k_o x + k_1 \dot{x} + k_2 \ddot{x} + \cdots \infty)
\end{aligned}$$

Now the first equation becomes:

$$\begin{aligned}
(D-m)y &= z(t) \\
\text{or, } (D-m)y &= Ae^{mt} + (k_o x + k_1 \dot{x} + k_2 \ddot{x} + \cdots \infty)
\end{aligned}$$

The forcing function on RHS has a term Ae^{mt}. Therefore the total solution for $y(t)$ is given by:

$$y(t) = (A_1 + A_2 t)e^{mt} + (K_o x + K_1 \dot{x} + K_2 \ddot{x} + \cdots \infty)$$

3.1 n^{th} order differential equation

The above idea can easily be be extended to solve a general n^{th} order differential equation given below.

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} x + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_n = x(t)$$

In terms of operator D, the equation can be written as

$$\begin{aligned}
\left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + a_3 D^{n-3} + \cdots + a_n\right) y &= x(t) \\
\text{or, } (D - m_1)(D - m_2)(D - m_3)x \cdots (D - m_n)y &= x(t) \\
\therefore \text{ Roots of characteristic equation are } &: m_1, m_2, \cdots m_n
\end{aligned}$$

Case1: All roots are distinct

If all the roots are distinct i.e.,$m_1 \neq m_2 \neq m_3 \neq \cdots m_n$, The solution will be:

$$y(t) = A_1 e^{m_1 t} + A_2 e^{m_2 t} + A_3 e^{m_3 t} + \cdots A_n e^{m_n t} + (k_o x + k_1 \dot{x} + k_2 \ddot{x} + \cdots \infty)$$

Case2: When there are repeated roots

Let us assume that first two roots are equal ($m_1 = m_2 = m_1$ (say)) and next three roots are equal ($m_3 = m_4 = m_5 = m_3$ (say)), then the solution will be:

$$x(t) = (A_1 + B_1 t)e^{m_1 t} + (A_3 + B_3 t + C_3 t^2)e^{m_3 t} + A_4 e^{m_4 t} + \cdots A_n e^{m_n t} + (k_o f + k_1 \dot{x} + k_2 \ddot{x} + \cdots \infty)$$

Case3: When roots are complex

Complex roots will always appear as conjugate pair. If one root is $c_1 + jc_2 = c\, e^{j\theta}$, the other root has to be $c_1 - jc_2 = c\, e^{-j\theta}$. Since these roots can be considered to be distinct, we can apply the same rule established earlier for getting the solution.

3.2 General comments

1. Linear differential equation with boundary condition will be given.
2. Write down the characteristic equation and solve for its roots.
3. Examine whether there are repeated roots or not.
4. Then write down the expression for natural response $x_n(t)$ which will be sum of several exponential terms.
5. Solution $y_f(t)$, due to forcing function $x(t)$ will be linear combinations of the $x(t)$ and its higher order derivatives.
6. Total solution for $y(t)$ then will be sum of $y_n(t)$ and $y_f(t)$.

7. Now constants appearing in $y_f(t)$ and $x_n(t)$ are to be determined.

8. $y_f(t)$ has to satisfy the differential equation alone, for all time. The like terms of LHS & RHS can be equated to manufacture derived equations involving k's.

9. Now only constants appearing in $x_n(t)$ are to be determined. These are determined from the boundary conditions given. Use total solution when applying boundary conditions.

Example-1

Find out the solution of the following differential equation

$$\frac{dy}{dt} + 3y = t \quad \text{Boundary condition : } y(0) = 1$$

Solution

Characteristic equation is $m + 3 = 0$; hence $m = -3$. Also $x(t) = t$. So the solution is:

$$\begin{aligned}
\text{Natural response is } \; y_n(t) &= Ae^{-3t} \\
\text{Response due to forcing function is } \; y_f(t) &= k_o t + k_1 \\
\text{Total solution is } \; x(t) = Ae^{-3t} + k_o t + k_1 &
\end{aligned}$$

As we know $y_f(t) = k_o t + k_1$ alone satisfies the equation.

$$\begin{aligned}
\frac{d}{dt}(k_o t + k_1) + 3(k_o t + k_1) &= t \\
\text{or, } k_o + 3k_o t + 3k_1 &= t \\
\text{or, } k_o + 3k_1 + 3k_o t &= t \\
\text{Note that, } & \quad \text{this relation is to be true for all time } t \\
\text{hence, } k_o + 3k_1 &= 0 \\
\text{and, } 3k_o &= 1 \\
\therefore \; k_0 = \frac{1}{3} \text{ and } k_1 &= -\frac{1}{9} \\
\text{Thus, } y(t) &= Ae^{-3t} + \frac{1}{3}t - \frac{1}{9}
\end{aligned}$$

Now A is to be determined from the boundary condition $y(0) = 1$.

$$1 = A - \frac{1}{9} \;\; \therefore \;\; A = \frac{10}{9}$$

So the complete solution is:

$$y(t) = \frac{10}{9}e^{-3t} + \frac{1}{3}t - \frac{1}{9}$$

4 When forcing function is constant or exponential

In circuit analysis two kinds of excitations (forcing functions) are very common - D.C excitation and sinusoidal excitation. Mathematically $x(t) =$ constant or $x(t) = V_m \sin \omega t$. Is it possible to get the solution due to forcing function $y_f(t)$ rather easily when the forcing functions are either D.C or sinusoidally varying? Answer to this is yes.

A constant value or sinusoidally varying quantity can be expressed in terms of exponential function. Let us consider a forcing function $x(t) = Ae^{st}$, where s may be real or complex. For example if $s = 0$, $f(t) = A =$ constant. Also a sine or cosine function can be expressed in terms of exponential as follows:

$$\begin{aligned}
\sin \omega t &= \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \\
\cos \omega t &= \frac{e^{j\omega t} + e^{-j\omega t}}{2}
\end{aligned}$$

Thus we conclude that sinusoidal quantities can also be expressed as exponentials. Now consider a second order differential equation try to solve for $y_f(t)$ when $x(t) = Ae^{st}$.

$$\begin{aligned}
\text{Given equation: } \frac{d^2y}{dt^2} + a\frac{dy}{dt} + by &= x(t) \\
\text{Forcing function:} x(t) &= Ae^{st} \text{ where } s \text{ is a constant} \\
\text{or, } \frac{d^2y_f}{dt^2} + a\frac{dy_f}{dt} + by_f &= Ae^{st}
\end{aligned}$$

Since LHS must be equal to RHS for all t, nature of $y_f(t)$ must be of the following form

$$y_f(t) = Ye^{st} \text{ where } Y \text{ is a constant}$$

Putting this in the differential equation we get,

$$\begin{aligned} s^2Ye^{st} + asYe^{st} + bYe^{st} &= Ae^{st} \\ \therefore\ y_f(t) = Ye^{st} &= \frac{Ae^{st}}{(s^2+as+b)} \\ y_f(t) = Ye^{st} &= \frac{Ae^{st}}{(s^2+as+b)} \end{aligned}$$

For the equation $(D^2 + aD + b)y = Ae^{st}$ with forcing function of the form Ae^{st}, response $y_f(t)$ can be written as:

$$y_f(t) = \left.\frac{Ae^{st}}{(D^2+aD+b)}\right|_{D=s} = \frac{Ae^{st}}{(s^2+as+b)}$$

Let us apply this rules to solve the following examples.

Example-2

Solve the following differential equation

$$\frac{dy}{dt} + 2y = 10 \ \text{; Boundary condition } y(0) = 0$$

Let us first solve for $y_f(t)$.

$$\begin{aligned} \text{The given equation is: } (D+2)y_f(t) &= 10 \text{ ; Note } x(t) = 10 = 10e^{0t} \text{ constant} \\ \therefore\ y_f(t) &= \left.\frac{10}{(D+2)}\right|_{D=0} = 5 \\ \text{Since characteristic root } m &= -2 \\ \therefore\ y_n(t) &= Ae^{-2t} \\ \text{Total solution is: } y(t) &= y_f(t) + y_n(t) \\ \text{or, } y(t) &= 5 + Ae^{-2t} \\ \text{Boundary condition: } y(0) &= 0 \text{ gives } A = -5 \\ \therefore\ \text{Total solution is: } y(t) &= 5 - 5e^{-2t} \end{aligned}$$

Example-3

Solve the following differential equation

$$\frac{dy}{dt} + 2y = 10\cos 2t \ \text{; Boundary condition } y(0) = 0$$

Let us first solve for $y_f(t)$.

$$
\begin{aligned}
\text{The given equation is: } (D+2)y_f(t) &= 10\cos 2t \\
\text{Now, } 10\cos 2t &= 10\frac{(e^{j2t}+e^{-j2t})}{2} = 5e^{j2t}+5e^{-j2t} \\
(D+2)y_f(t) &= 5e^{j2t}+5e^{-j2t} \\
\text{or, } y_f(t) &= \left.\frac{5e^{j2t}}{(D+2)}\right|_{D=j2} + \left.\frac{5e^{-j2t}}{(D+2)}\right|_{D=-j2} \\
\text{or, } y_f(t) &= \frac{5e^{j2t}}{(j2+2)} + \frac{5e^{-j2t}}{(-j2+2)} \\
\text{or, } y_f(t) &= \frac{5(2+j2)e^{j2t}}{8} + \frac{5(2-j2)e^{-j2t}}{8} \\
&= \frac{10}{8}\left[(1+j1)e^{j2t}+(1-j1)e^{-j2t}\right] \\
&= \frac{5}{4}\left[(e^{j2t}+e^{-j2t})+j(e^{j2t}-1e^{-j2t})\right] \\
&= \frac{5}{4}\left[2\cos 2t - 2\sin 2t\right] \\
\therefore\ y_f(t) &= \frac{5}{2}\left[\cos 2t - \sin 2t\right] \\
\text{Since characteristic root } m &= -2 \\
\therefore\ x_n(t) &= Ae^{-2t} \\
\text{Total solution is: } y(t) &= y_f(t)+y_n(t) \\
\text{or, } y(t) &= \frac{5}{2}\left[\cos 2t - \sin 2t\right] + Ae^{-2t} \\
\text{Boundary condition: } y(0) &= 0 \text{ gives } A = -\frac{5}{2} \\
\therefore\ \text{Total solution is: } y(t) &= \frac{5}{2}\left[\cos 2t - \sin 2t\right] - \frac{5}{2}e^{-2t}
\end{aligned}
$$