Problem	Languages
decidable	recursive
semidecidable	recursively enumerable r.e.
co-semidecidable	co-r.e.
not semidecidable	non-r.e.

HP, MP, $\overline{HP}$, $\overline{MP}$

1. Prove that the problem whether a Turing machine M on a given input x reenters its start state is undecidable.

Prove $L = \{N \# x \mid N$ reenters its start state on input $x\}$ is not recursive.

$$HP \leq_m L$$

$$M \# x \longmapsto N \# x$$

($N \# y$, but here we take $y = x$)

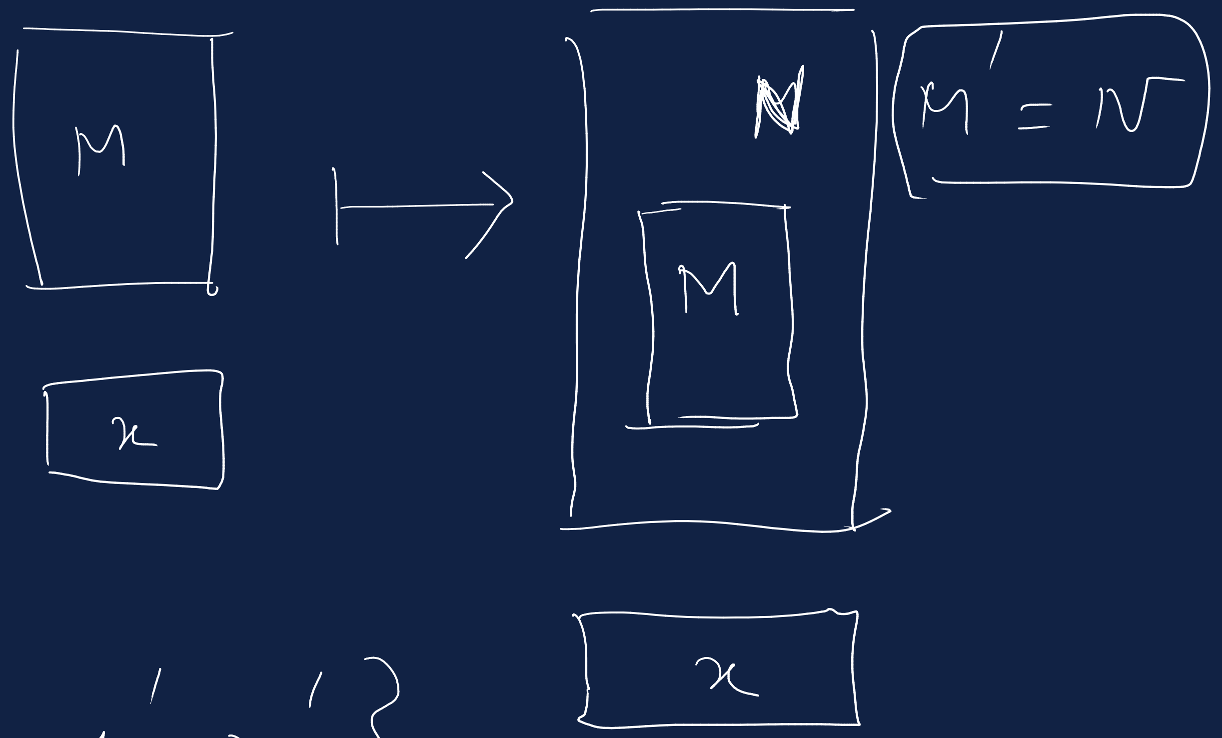
M halts on $x \Rightarrow$ N reenters ---

M does not halt on $x \Rightarrow$ N does not reenter ---

$$M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, s, t, r)$$

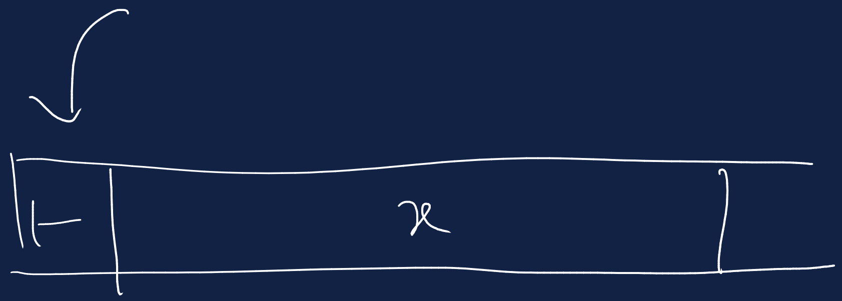
$$M' = (Q', \Sigma, \Gamma', \vdash, \sqcup, \delta', s', t', r')$$

N



M'

$$Q' = Q \cup \{s', t', r'\}$$



$$\delta'(s', \vdash) = (s, \vdash, R)$$

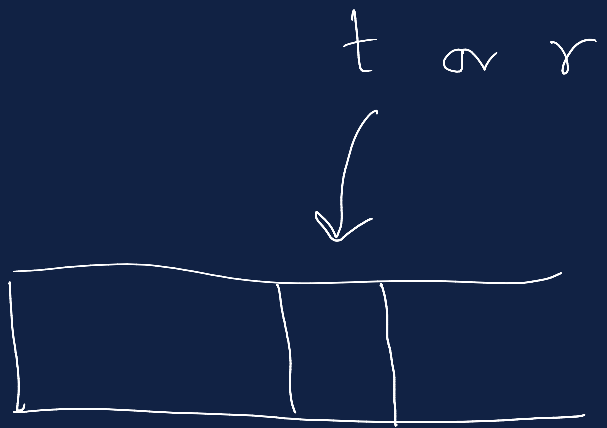
$$\rightarrow \delta'(s', a) = (s, a, L)$$

$$\forall a \in \Gamma \setminus \{\vdash\}.$$

M enters t or r

$$\delta'(t, \vdash) = (t, \vdash, R)$$

$$\delta'(r, \vdash) = (r, \vdash, R)$$



$$\delta'(t, a) = (s', \checkmark, R)$$

$$\delta'(r, a) = (s', X, R)$$

$$\Gamma' = \Gamma \cup \{\checkmark, X\}$$

$$\delta'(s, \checkmark) = (t', \checkmark, R)$$

$$\delta'(s, X) = (r', X, R)$$

2. Consider the language

AL_{2021} = { M | M is a Turing machine which accepts at least 2021 input strings }.

(a) Prove that AL_{2021} is recursively enumerable.

<u>Simulate</u> M on <u>all strings</u> on a <u>time-sharing basis</u>.

$x_0, x_1, x_2, x_3, \ldots\ldots$

If M accepts 2021 (or more) strings, this will be eventually discovered.

If M accepts < 2021 strings, the simulations will not stop.

(b) Prove that AL_{2021} is not recursive.

$$HP \leq AL_{2021}$$

$$M\#x \longmapsto N$$

(The i/p for N is y → ind of M and x)

M halts on x $\Leftrightarrow$ N accepts at least 2021 strings.

N, on input y, does the following:

1. Simulate M on x.
2. If the simulation halts, accept (y).

M halts on $x \Rightarrow \mathcal{L}(N) = \Sigma^*$

M does not halt on $x \Rightarrow \mathcal{L}(N) = \emptyset$

3. Consider the language

E_{2021} = { M | M is a Turing machine that accepts exactly 2021 input strings }.

Prove that E_{2021} is not recursively enumerable.

$\overline{HP} \leq E_{2021}$

$M\#x \longmapsto N$

N accepts exactly 2021 strings $\iff$ M does $\underline{\underline{\text{not}}}$ halt on x.

$y_1, y_2, \ldots, y_{2021}$ $\quad y_i = a^i$ where $a \in \Sigma$.

N, on input y, does the following:

1. If $y = y_i$ for some i, accept (and halt).
2. Simulate M on x.
3. Accept y.

M does not halt on x $\Rightarrow L(N) = \{y_1, y_2, \ldots, y_{2021}\}$

M halts on x $\Rightarrow L(N) = \Sigma^*$

4. Consider the language

$EQ = \{ M \# N \mid M$ and N are Turing machines with $\mathscr{L}(M) = \mathscr{L}(N) \}$.

Prove that EQ is not recursive.

Supply a reduction from HP.

5. Is EQ recursively enumerable?

Either construct a (non-total) Turing machine for the language,
or propose a reduction from the complement of HP.