Fourier transform to Laplace Transform

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1 Introduction

We know that for a given periodic function $x(t)$, we can find out its FT and the following two equations are used to get $X(\omega)$ from $x(t)$. The inverse formula is used to get $x(t)$ from $X(\omega)$.

$$\begin{aligned}
\text{Fourier transform of } x(t) : F(\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega\, t}\, dt \\
\text{Inverse Fourier transform of } X(\omega) : x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega
\end{aligned}$$

Note that $f(t)$ and $F(\omega)$ form a transform pair.

The existence of FT depends on the fact that the following integral converges.

$$\int_{-\infty}^{\infty} x(t)\, e^{-j\omega\, t}\, dt$$

We also note that the sufficient condition for existence of FT is that $x(t)$ must be an energy signal i.e.,

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt \text{ should be finite: also known as Perseval's condition}$$

The above condition is not necessary i.e., there exist signals which may not be energy signals but still have FT (example : unit step function $u(t)$

2 Fourier transform to Laplace transform

Generally for functions which grow monotonously with time , no FT can be found out. Let us consider such a function $x(t)$ whose FT can not be found out. Let us modify the function to $e^{-\sigma t}x(t)$ where $\sigma > 0.$, then this modified function will decay and we expect FT to exist for the function $e^{-\sigma t}x(t)$ - let us try to obtain FT of this modified function.

$$\begin{aligned}
\mathcal{F}[x(t)]e^{-\sigma t}] &= \int_{-\infty}^{\infty} \{x(t)e^{-\sigma t}\}\, e^{-j\omega\, t}\, dt \\
\text{or, } \mathcal{F}[x(t)]e^{-\sigma t}] &= \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma+j\omega)t}\, dt = F(\sigma + j\omega) \\
\text{where, } \sigma + j\omega & \quad \text{has dimension of frequency} \\
\text{define , } \sigma + j\omega &= s \text{ complex frequency} \\
\text{Thus, } \mathcal{F}[x(t)]e^{-\sigma t}] &= \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt
\end{aligned}$$

Since RHS of the last equation above will be function of s only, we can interpret it as follows:

$$\int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = X(s)$$

$X(s)$ is called **Laplace Transform** of $x(t)$.

The inverse formula

We know

$$\begin{aligned}
\mathcal{F}[x(t)]e^{-\sigma t}] &= \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma+j\omega)t}\, dt = F(\sigma+j\omega) \\
\therefore\ x(t)e^{-\sigma t} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma+j\omega)e^{j\omega t}\ d\omega \\
or,\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma+j\omega)e^{(\sigma+j\omega)t}\ d\omega \\
\text{Now let, } \sigma+j\omega &= s \text{ which means } d\omega = \frac{ds}{j} \\
\therefore\ x(t) &= \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)e^{st}\ ds
\end{aligned}$$

Summarizing the Laplace transform formulas for a function $x(t)$ are:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)e^{st}\ ds$$

The above formulas are called **Bilinear Laplace transform**. However for causal signals $x(t)u(t)$, the integration limits naturally changes to : from 0 to ∞ and this is called **Unilateral Laplace transform** and the equations of interest will be:

$$X(s) = \int_{0^-}^{\infty} x(t)\, e^{-st}\, dt \ \ \text{Lower limit } 0^- \text{ is used to include } \delta(t)$$

$$x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)e^{st}\ ds$$

As we shall see LT is a very popular and efficient tool to solve circuit problems where solution for switching transient as well as steady state solution can be obtained in one go.

LT of $u(t)$ and its ROC

$$\begin{aligned}
U(s) &= \int_0^\infty u(t)\, e^{-st}\, dt = \int_0^\infty e^{-st}\, dt \\
\text{or, } U(s) &= \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s} \\
\text{The above result is true provided } \lim_{t\to\infty} \frac{e^{-st}}{-s} &= 0 \\
\text{Let us see what does it mean?} & \\
\text{now, } \lim_{t\to\infty} \frac{e^{-st}}{-s} &= \lim_{t\to\infty} \frac{e^{-(\sigma+j\omega)t}}{-s} = \lim_{t\to\infty} \frac{e^{-\sigma t}}{-s} \times \frac{e^{-j\omega t}}{-s} \\
\text{Note } \left|e^{-j\omega t}\right| &\leq 1 \;\; \text{no matter what is the value of } t \\
\therefore \lim_{t\to\infty} \frac{e^{-(\sigma+j\omega)t}}{-s} &= 0 \text{ only when } \sigma > 0
\end{aligned}$$

This condition $\sigma > 0$ gives us the region of convergence for the function $u(t)$. In other words entire right half of the complex (or s) plane is the **Region of convergence or ROC** for $u(t)$. Also from another angle $\sigma > 0$, only will make $e^{-\sigma t}u(t)$ convergent and make it an energy signal. Thus for any other function $x(t)$, different ROC will exist and can be found out if we wish. In general we will belief *sigma* value has been picked up from ROC associated with $x(t)$.

3 Laplace transform of some useful functions

$$\mathcal{L}[x(t)u(t)] = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt\, dt$$

Using the above LT of various functions can be found out.

1. $\mathcal{L}[\delta(t)] = \delta(s) = 1$
2. $\mathcal{L}[u(t)] = U(s) = \frac{1}{s}$
3. $\mathcal{L}[r(t)] = \mathcal{L}[tu(t)] = \frac{1}{s^2}$
4. $\mathcal{L}[e^{at}\, u(t)] = \frac{1}{s-a}$
5. $\mathcal{L}[\sin\omega t\ u(t)] = \frac{\omega}{s^2+\omega^2}$
6. $\mathcal{L}[\cos\omega t\ u(t)] = \frac{s}{s^2+\omega^2}$

Also if $\mathcal{L}[x(t)] = X(s)$, then following useful results can be easily established.

1. $\mathcal{L}[x(t-a)u(t)] = e^{-as}X(s)$
2. $\mathcal{L}[e^{at}x(t)u(t)] = X(s-a)$
3. It follows then: $\mathcal{L}[e^{at}\sin\omega t\ u(t)] = \frac{\omega}{(s-a)^2+\omega^2}$
4. $\mathcal{L}[e^{at}\cos\omega t\ u(t)] = \frac{(s-a)}{(s-a)^2+\omega^2}$
5. $\mathcal{L}[\frac{dx}{dt}u(t)] = sX(s) - x(0)$
6. $\mathcal{L}[\frac{d^2x}{dt^2}u(t)] = s^2X(s) - s\frac{dx}{dt}(0) - x(0)$
7. $\mathcal{L}[\frac{d^3x}{dt^3}u(t)] = s^3X(s) - s^2\frac{d^2x}{dt^2}(0) - s\frac{dx}{dt}(0) - x(0)$
8. $\mathcal{L}[\int_0^t x(\tau)\,d\tau]u(t) = \frac{X(s)}{s}$
9. $\mathcal{L}[tx(t)u(t)] = -\frac{dX(s)}{ds}$
10. If $\mathcal{L}[x_1(t)u(t)] = X_1(s)$ and $\mathcal{L}[x_2(t)u(t)] = X_2(s)$ then $\mathcal{L}[x_1(t) * x_2(t)] = X_1(s)X_2(s)$

If you find any mistake in the above please bring it to my notice.

Circuit Elements : a deeper look

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Contents

1 Introduction

An electrical circuit or network will consist of sources and three circuit elements R, L and C. Inductor and capacitor are energy storing elements while a resistor is an energy dissipative element. In this note $v - i$ characteristics of these three circuit elements are discussed. The present value of the current in a resistor depends on the present value of the voltage only. In contrast present value of the current in an inductor is not only decided by the present value of the voltage across it but also depends upon the previous voltage applied earlier across it. Similarly present value of the voltage in a capacitor is not only decided by the present value of the current flowing through it but also depends upon the previous value of the current, the capacitor carried earlier. For this reason, inductor and capacitor are called elements with memory and a resistor is called a memory less element. It may be noted that to build up current in an inductor we have to apply voltage across it while we have to pump current in to capacitor to build up voltage across it.

If the excitation (voltage or current) are reasonably good (!) functions of time, then current through an inductor and voltage across a capacitor will be continuous in time. This conditions will fail or will be different if we allow excitation to be an impulse ($\delta(t)$). How to handle situations with $\delta(t)$ functions is also discussed.

2 v vs i relations of R, L and C

The current value of the current in a resistor depends solely upon the current value of the voltage. Hence it is called a memory less element. In a circuit if energy storing elements (inductor & capacitor) are present along with resistance and the D.C sources, response of the current due to switching of the circuit causes time varying current to flow before a steady state is reached. The voltage-current relationship of a resistor, inductor and a capacitor are shown in Figure 1.

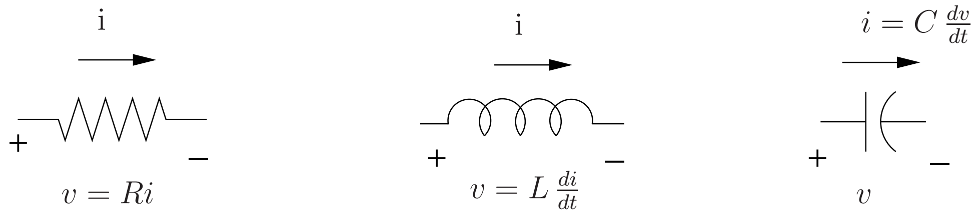

Figure 1:

$$\text{For resistance: } v = Ri$$

$$\text{For inductance: } v = L\frac{di}{dt}$$

$$\text{For capacitance: } i = C\frac{dv}{dt}$$

One should carefully note the direction of the current through the element and the polarity of the voltage across the element. The following important boundary conditions will be satisfied for the three elements when a switching is executed say at $t = 0$.

1. In a resistance since $v = Ri$, current can change in steps i.e., can change instantaneously. A resistor is an energy dissipating elements. In fact the current value of the current in a resistor depends solely upon the current value of the voltage. Hence it is called a memory less element.

2. In an inductor since, $v = L\,\frac{di}{dt}$, current can not change instantaneously as such a situation will call for voltage to be infinitely large ($\because\ \frac{di}{dt} \to \infty$). This essentially means that $i(0_-) = i(0_+)$. 0_- is the time immediately before switching is done and 0_+ is the time immediately after switching is done. The current value of the current in an inductor depends not only upon the current value of the voltage but also upon the previous voltage applied across it.. Hence it is called a element with memory.

3. When an inductor carries current, it stores energy which is equal to $\frac{1}{2}\,Li^2$

4. In a capacitor, since, $i = C\,\frac{dv}{dt}$, voltage can not change instantaneously as such a situation will call for current to be infinitely large ($\because\ \frac{dv}{dt} \to \infty$). This essentially means that $v(0_-) = v(0_+)$. 0_- is the time immediately before switching is done and 0_+ is the time immediately after switching is done. The current value of the voltage across a capacitor depends not only upon the present value of the current but also upon the currents passed through it previously. Hence it is called an element with memory.

5. When there is a voltage across a capacitor, it stores energy which is equal to $\frac{1}{2}\,Cv^2$

2.1 Deeper look at inductor & capacitor

About inductor

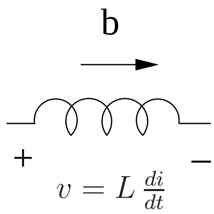

Figure 2:

Let us try to write current i in an inductor in terms of voltage:

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt$$

Thus the present value of the inductor current depends on the past history of the applied voltage. Suppose in a circuit at $t = 0$ some switching is executed, then the RHS of the above equation can be broken up into various pieces as below. Note $t = 0^-$ represent time immediately before $t = 0$ and $t = 0^+$ represents time immediately after $t = 0$.

$$i(t) = \frac{1}{L}\int_{-\infty}^{0^-} v(t)\,dt + \frac{1}{L}\int_{0^-}^{0^+} v(t)\,dt + \frac{1}{L}\int_{0^+}^{t} v(t)\,dt$$

$$
\begin{aligned}
\text{Now, } \frac{1}{L}\int_{0^-}^{0^+} v(t)\,dt &= 0 \text{ provided } v(t) \text{ is not an impulse} \\
\therefore\ i(t) &= \frac{1}{L}\int_{-\infty}^{0^-} v(t)\,dt + \frac{1}{L}\int_{0^+}^{t} v(t)\,dt \\
\therefore\ i(t) &= i(0^-) + \frac{1}{L}\int_{0^+}^{t} v(t)\,dt \\
\text{Now putting } t = 0^+,\ i(0^+) &= i(0^-) + \frac{1}{L}\int_{0^+}^{0^+} v(t)\,dt \\
\text{Thus } i(0^+) &= i(0^-) \\
\text{and } i(t) &= i(0^-) + \frac{1}{L}\int_{0^+}^{t} v(t)\,dt = i(0^-) + \frac{1}{L}\int_{0}^{t} v(t)\,dt
\end{aligned}
$$

Conclusion is that "in a circuit if some switching is carried out (at $t = 0$, current through an inductor can not change instantaneously".

Recall:

$$i(t) = i(0^-) + \frac{1}{L}\int_{0}^{t} v(t)\,dt$$

This equation suggests that an inductor with initial current $i(0^-)$ is equivalent to an inductor with no initial current and in parallel with a current source of strength $i(0^-)$ as depicted in Figure 3.

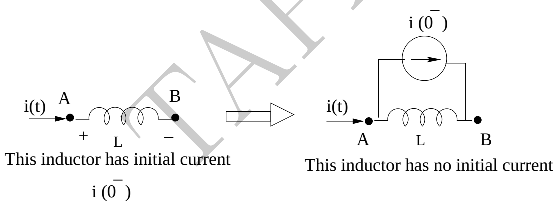

Figure 3:

If somebody loves to see inductor without any initial current $i_(0_-)$ and a capacitor without any initial voltage $v_(0_-)$ then he has to introduce additional current source of constant magnitude $i_(0_-)$ across the inductor and additional voltage source of constant magnitude $v_(0_-)$ in series the capacitor and pretend that initial conditions of the inductor and capacitor to be relaxed.

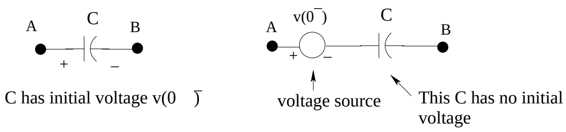

Figure 4:

About capacitor

Let us try to write down voltage $v(t)$ across a capacitor in terms of current:

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$$

Thus the present value of the capacitor voltage depends on the past history of the injected current. Suppose in a circuit at $t = 0$ some switching is executed, then the RHS of the above

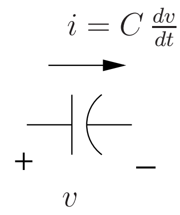

Figure 5:

equation can be broken up into various pieces as below. Note $t = 0^-$ represent time immediately before $t = 0$ and $t = 0^+$ represents time immediately after $t = 0$.

$$
\begin{aligned}
v(t) &= \frac{1}{C}\int_{-\infty}^{0^-} i(t)\,dt + \frac{1}{C}\int_{0^-}^{0^+} i(t)\,dt + \frac{1}{C}\int_{0^+}^{t} i(t)\,dt \\
\text{Now, } \frac{1}{C}\int_{0^-}^{0^+} i(t)\,dt &= 0, \text{ provided } i(t) \text{ is not an impulse} \\
v(t) &= \frac{1}{C}\int_{-\infty}^{0^-} i(t)\,dt + \frac{1}{C}\int_{0^+}^{t} i(t)\,dt \\
\therefore\ v(t) &= v(0^-) + \frac{1}{C}\int_{0^+}^{t} i(t)\,dt \\
\text{Now putting } t = 0^+,\ v(0^+) &= v(0^-) + \frac{1}{C}\int_{0^+}^{0^+} i(t)\,dt \\
\text{Thus, } v(0^+) &= v(0^-) \\
\text{and at any time } t,\ v(t) &= v(0^-) + \frac{1}{C}\int_{0^+}^{t} i(t)\,dt = \frac{1}{C}\int_{0}^{t} i(t)\,dt
\end{aligned}
$$

Conclusion is that "in a circuit if some switching is carried out (at $t = 0$, voltage across the capacitor can not change instantaneously".

Recall:

$$v(t) = v(0^-) + \frac{1}{C}\int_{0}^{t} i(t)\,dt$$

This equation suggests that a capacitor with initial voltage $v(0^-)$ is equivalent to an capacitor with no initial voltage and in series with a voltage source of strength $v(0^-)$ as depicted in Figure 6.

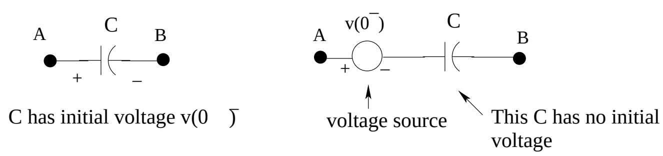

Figure 6:

3 Solution of differential equation when forcing function is an Impulse

Suppose we want to solve the following linear differential equation with a unit impulse forcing function.

$$\frac{dx}{dt} + 3x = \delta(t)$$

The solution requires a different approach altogether because of the because of the tricky nature of the impulse function as recapitulated below.

Impulse or Delta function : $\delta(t)$

In figure 7(a), a unit impulse function is shown. The function is shown as thin rectangular pulse,

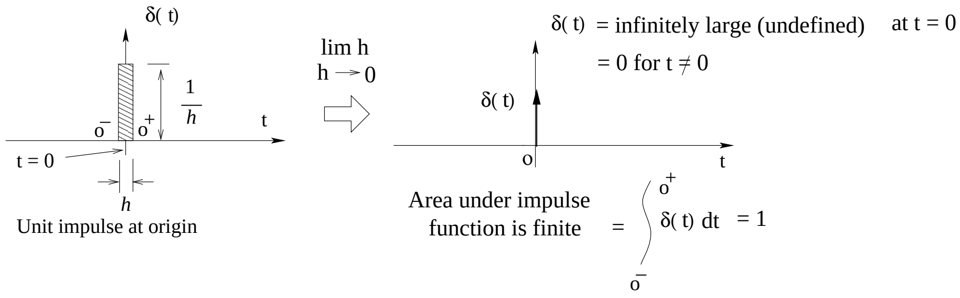

Figure 7: Impulse function

centered around the the origin. The width of the pulse is shown to be h and it's height to be $1/h$. The area enclosed by the pulse is unity, no matter whatever is the value of h you choose. This general rectangular pulse will become an unit impulse function as $h \to 0$ i.e., $h = 0^+ - 0^-$, where 0^- and 0^+ are small (as small as you can think of) perturbations in time in the negative and positive direction of time axis around $t = 0$. So the the height of the pulse tends to infinity and width tends to zero. Thus no point in talking about the functional value of this function at $t = 0$, since it approaches ∞. However, area under the impulse is finite and equal to unity and this can be mathematically described as:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = \int_{0^-}^{0^+} \delta(t)\, dt = 1 \text{ since elsewhere, } \delta(t) = 0$$

Coming back to the problem of solving the equation $\frac{dx}{dt} + 3x = \delta(t)$. Since $\delta(t)$ has no finite functional value at $t = 0$. It will be prudent to transform the RHS to the form $\int_{0^-}^{0^+} \delta(t)\, dt$ since this integral has a finite value of 1.

$$\begin{aligned}
\frac{dx}{dt} + 3x &= \delta(t) \\
\text{B.C is: } x(0^-) &= 0 \\
\text{Multiply both sides by} & \quad dt \text{ \& integrating from } 0^- \text{ to } 0^+, \\
\int_{0^-}^{0+} \frac{dx}{dt}\, dt + 3\int_{0^-}^{0+} x\, dt &= \int_{0^-}^{0^+} \delta(t)\, dt \\
\text{or, } \int_{0^-}^{0+} dx + 3\int_{0^-}^{0+} x\, dt &= 1 \\
\text{or, } x(0^+) - x(0^-) + 3 \times 0 &= 1 \\
\therefore\ x(0^+) &= 1
\end{aligned}$$

Note for $t > 0^+$, $\delta(t) = 0$, so we can reformulate the original problem as: solve $\frac{dx}{dt} + 3x = 0$ with the boundary condition $x(0^+) = 1$. The solution can now be written by inspection: $x(t) = e^{-3t}$ for $t > 0^+$. Note that there is a step jump of x from $t = 0^-$ to $t = 0^+$ and the amount by which this jump takes place is equal to 1 in this case.

For a second order linear differential, it can be shown that there will be a step jump of $\frac{dx}{dt}$ from $t = 0^-$ to $t = 0^+$ and the amount by which this jump takes place is equal to the strength of the impulse. To summaries:

$$\begin{aligned}
\text{For a 1st order system jump will be in} &\quad x \\
\text{For a 2nd order system jump will be in} &\quad \frac{dx}{dt} \\
\text{For a 3rd order system jump will be in} &\quad \frac{d^2x}{dt^2} \text{ and so on} \\
\text{In each case value of the jump will be} &= \text{Strength of the impulse}
\end{aligned}$$

Thus from the initial conditions given at $t = 0^-$, new initial conditions at $t = 0^+$ are to be found out and the differential equation is to be solved in the usual way assuming forcing function to be zero.

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Circuit Analysis with Laplace transform

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Contents

1 Introduction

To solve a circuit using Laplace transform is always advantageous because

1. The differential equations are transformed to algebraic equations after taking Laplace transform.
2. Finally to get the time domain expression of the variable, one has to take Laplace inverse.
3. Both steady state and transient solutions are obtained in one stroke.

1.1 R, L & C in s-domain

The voltage-current relationship of a resistance R is given by

$$\begin{aligned}\text{In time domain } v(t) &= Ri(t)\\ \text{Taking LT of both sides } V(s) &= RI(s)\\ \text{or, } R &= \frac{V(s)}{I(s)}\end{aligned}$$

The voltage-current relationship of an inductor L **having no initial current** is given by

$$\begin{aligned}\text{In time domain } v(t) &= L\frac{di(t)}{dt}\\ \text{Taking LT of both sides } V(s) &= sLI(s)\\ \text{or, } sL &= \frac{V(s)}{I(s)}\end{aligned}$$

Thus an initially relaxed inductor has an impedance of sL in s-domain.

The voltage-current relationship of a capacitor C **having no initial voltage** is given by

$$\begin{aligned}\text{In time domain } i(t) &= C\frac{dv(t)}{dt}\\ \text{Taking LT of both sides } I(s) &= sCV(s)\\ \text{or, } \frac{1}{sC} &= \frac{V(s)}{I(s)}\end{aligned}$$

Thus an initially relaxed capacitor may be considered to have an impedance of $\frac{1}{sC}$ in s-domain.

Therefore if a circuit has R, L & C, in the transformed domain s one can redraw the circuit replacing L & C respectively sL and $\frac{1}{sC}$ provided the inductor had no initial current and capacitor had no initial voltage.

1.2 Initially charged inductor and capacitor in s-domain

Recall that an inductor L, with initial current $i(0)$ can be represented as parallel combination of an uncharged inductor L with a constant current source $i(0)$ in time domain. Also recall that a capacitor C, with initial voltage $v(0)$ can be represented as series combination of an uncharged capacitor c with a constant voltage source $v(0)$ in time domain. Note LT of $i(0)$ is $\frac{i(0)}{s}$ in case of inductor and LT of $v(0)$ is $\frac{v(0)}{s}$.

The above discussion is summarized in figure 1.

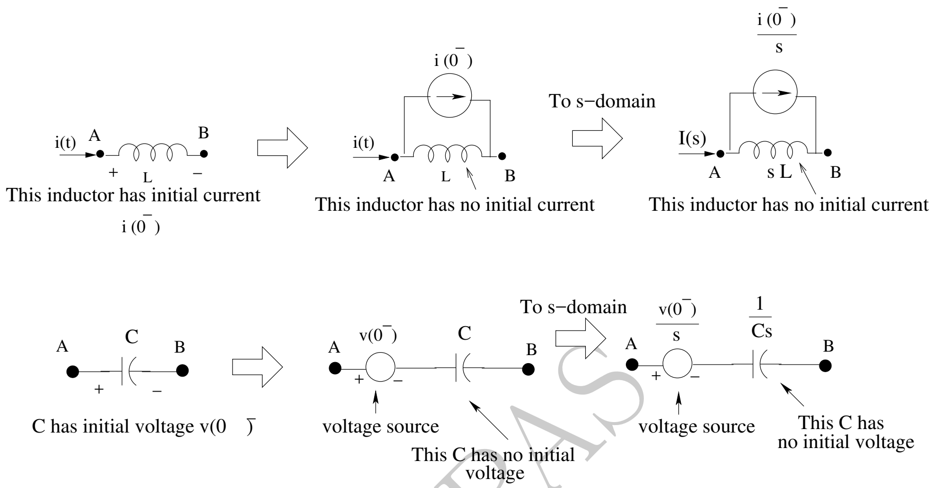

Figure 1:

The advantages we get by redrawing the circuit in s-domain are as follows:

1. Now **it is not necessary to write the differential equation in time domain** then take Laplace transform.

2. All voltages now will be of the form $V(s)$ and all currents will be $I(s)$.

3. Ratio of $V(s)$ and $I(s)$ of an element gives the impedance of the element $Z(s)$.

4. In s-domain also, you can adopt any method (mesh analysis, nodal method etc.) to solve for $I(s)$ in any branch.

1.2.1 Example-1

In the circuit shown in figure 2(a),it is given that capacitor voltage $v(0_-) = -\frac{1}{2}$ Volt. (i) Redraw the circuit in s-domain showing the initial condition. (ii) Calculate $i(t)$ for $t \geq 0$ when the input voltage $v_{in} = t$ for $t \geq 0$.

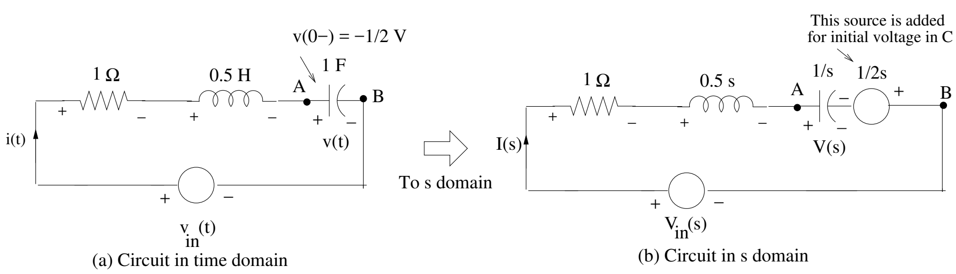

Figure 2:

1.2.2 Solution

We first redraw the circuit in s-domain as shown in figure 2(b). Since capacitor had initial voltage, its representation between A and B will be an uncharged capacitor and the initial voltage 1/2 V in series - note carefully the polarity of the voltage.

$$
\begin{aligned}
I(s) &= \frac{V_{in}(s) + \frac{1}{2s}}{1 + 0.5s + \frac{1}{s}} \\
\text{Now, } v_{in} &= t\,u(t) \\
\therefore\; V_{in}(s) &= \frac{1}{s^2} \\
\text{So, } I(s) &= \frac{\frac{1}{s^2} + \frac{1}{2s}}{1 + 0.5s + \frac{1}{s}} \\
&= \frac{s+2}{s^2+2s+2} \\
\therefore\; I(s) &= \frac{s+2}{s(s^2+2s+2)} = \frac{K}{s} + \frac{As+B}{s^2+2s+2} \\
K &= \left.\frac{s+2}{(s^2+2s+2)}\right|_{s=0} = 1 \\
\therefore\; I(s) &= \frac{s+2}{s(s^2+2s+2)} = \frac{1}{s} + \frac{As+B}{s^2+2s+2}
\end{aligned}
$$

To know A and B equate the numerators of both the sides.

$$
\begin{aligned}
(s^2+2s+2) + s(As+B) &= s+2 \\
\text{so, } A+1 &= 0 \text{ or, } A = -1 \\
B+2 &= 1 \text{ or, } B = -1 \\
\therefore\; I(s) &= \frac{1}{s} - \frac{s+1}{s^2+2s+2} \\
\text{or,} I(s) &= \frac{1}{s} - \frac{s+1}{(s+1)^2+1} \\
\text{Taking Laplace inverse: } i(t) &= \left(1 - e^{-t}\cos t\right) u(t)
\end{aligned}
$$

Example-2

Once we have drawn the circuit in s-domain showing the initial conditions for inductor current and capacitor voltage, we can adopt any known method of solving the circuit. For example, consider a circuit given in s-domain with some initial current in the inductor as shown in figure 3(a). The capacitor had no initial voltage.

Nodal method in s-domain

Let us apply nodal method to calculate the the node voltages V_{AO} and V_{BO} in s-domain. So **by inspection** we get the following two equations.

$$
\begin{aligned}
\left(\frac{1}{R_1} + \frac{1}{sL}\right) V_{AO}(s) - \frac{1}{sL} V_{BO}(s) &= \frac{V(s)}{R_1} - \frac{i(0)}{s} \\
-\frac{1}{sL} V_{AO}(s) + \left(\frac{1}{R_3} + \frac{1}{sL} + \frac{1}{R_2 + 1/sC}\right) V_{BO}(s) &= \frac{i(0)}{s}
\end{aligned}
$$

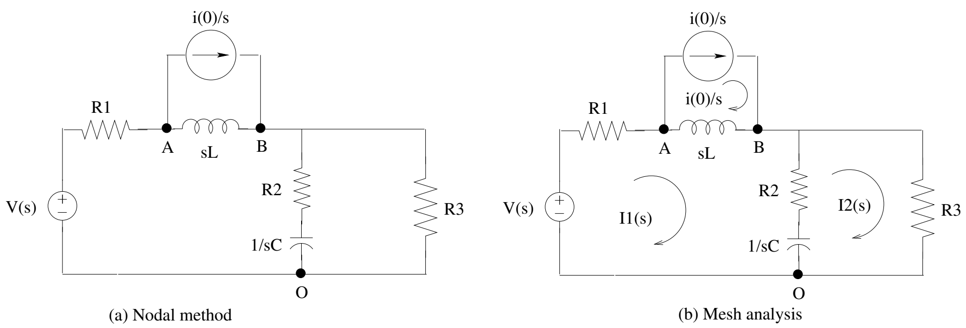

Figure 3:

Solving these two algebraic equations $V_{AO}(s)$ and $V_{BO}(s)$ can be obtained as function of s. Hence $v_{AO}(t)$ and $v_{BO}(t)$ by taking Laplace inverse. If we are also interested to find current say in the branch BO, then we first try to get $I_{BO}(s)$ and take its inverse as follows.

$$\begin{aligned} I_{BO}(s) &= \frac{V_{BO}(s)}{R_2 + 1/Cs} \\ i_{BO}(t) &= \text{Laplace inverse of } I_{BO}(s) \end{aligned}$$

Mesh analysis in s-domain

Let us now apply Mesh analysis to solve the above circuit. For mesh analysis refer to figure 3(b) where there are three meshes of which the current in the top mesh is known as $\frac{i(0)}{s}$. Assuming the other mesh currents as $I_1(s)$ and $I_2(s)$, KVL in the two meshes can be written **by inspection** as follows.

$$\begin{aligned} (R_1 + R_2 + sL + 1/sC)\, I_1(s) - (R_2 + 1/sC)\, I_2(s) - sL\frac{i(0)}{s} &= V(s) \\ -\,(R_2 + 1/sC)\, I_1(s) + (R_2 + R_3 + 1/sC)\, I_2(s) &= 0 \end{aligned}$$

Solving these two algebraic equations $I_1(s)$ and $I_2(s)$ can be obtained as function of s. Hence $i_1(t)$ and $i_2(t)$ by taking Laplace inverse. If we are also interested to find current say in the branch BO, then we first try to get $I_{BO}(s)$ and take its inverse as follows.

$$\begin{aligned} I_{BO}(s) &= I_1(s) - I_2(s) \\ i_{BO}(t) &= \text{Laplace inverse of } I_{BO}(s) \end{aligned}$$

Thevenin theorem in s-domain

For the same circuit figure 3(a), we want to find out the current through the resistance R_3. by applying Thevenin theorem. So to calculate Z_{th} refer to circuit shown in figure 4(a), where voltage source is shorted and current source is shown open circuited. Obviously:

$$Z_{th} = \frac{(R_1 + sL)(R_2 + 1/sC)}{R_1 + sL + R_2 + 1/sC}$$

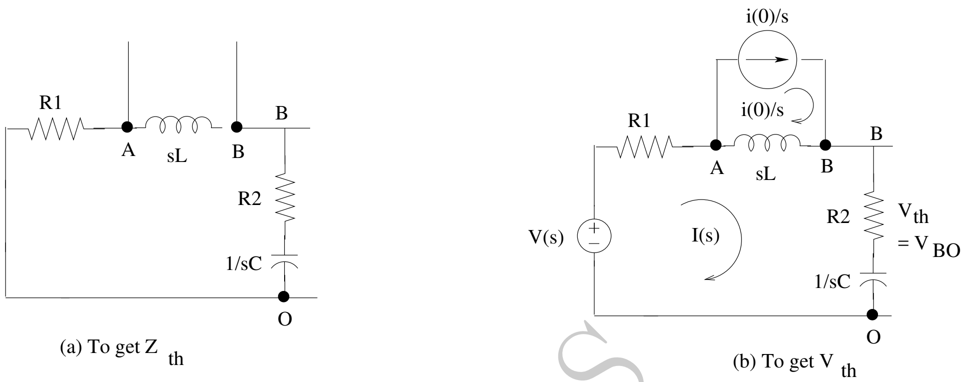

Figure 4:

To obtain Thevenin voltage we have to keep all the sources in the circuit as shown in figure 4(b) and calculate $V_{BO}(s)$ as follows.

$$\begin{aligned} V_{th}(s) = V_{BO}(s) &= I(s)(R_2 + 1/sC) \\ I(s) \text{ can be obtained from:} & \\ (R_1 + sL + R_2 + 1/sC)\, I(s) - sL\frac{i(0)}{s} &= V(s) \\ \therefore \text{ current through } R_3 &= \frac{V_{th}}{Z_{th}(s) + R_3} \end{aligned}$$

2 Conclusions

1. It is suggested that *better avoid writing differential equations in time domain first and take Laplace transform.*

2. By doing so, you can save lot of time.

3. You redraw the given time domain circuit, in s-domain replacing L by sL, C by $1/sC$. Note R remains R - unchanged.

4. Connect current source $\frac{i(0)}{s}$ across sL (with correct direction) if inductor had initial current.

5. Connect voltage source $\frac{v(0)}{s}$ in series with $1/sC$ (with correct polarity)if capacitor had initial voltage.

6. Now be in s-domain circuit, to find current in any branch $I_k(s)$ or voltage $V_k(s)$ across any element by any method you like.

7. Finally, take Laplace inverse of $I_k(s)$ or $V_k(s)$ to get $i_k(t)$ or $v_k(t)$.

8. Laplace transform (and corresponding inverse transform) of some standard and useful function such $u(t)$, sin ωt, cos ωt etc. and frequently used properties of LT must be at your finger tips to solve a circuit problem efficiently and with pleasure.

9. Also revisit your knowledge in partial fraction.

Table 1: **Properties of the Laplace Transform**

Property	**Signal**	**Transform**	**ROC**
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R [i.e., s is in the ROC if $(s - s_0)$ is in R]
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	"Scaled" ROC (i.e., s is in the ROC if (s/a) is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Initial- and Final Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Table 2: **Laplace Transforms of Elementary Functions**

Signal	**Transform**	**ROC**
1. $\delta(t)$	1	All s
2. $u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3. $-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4. $\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5. $-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6. $e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\Re e\{\alpha\}$
7. $-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\Re e\{\alpha\}$
8. $\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\Re e\{\alpha\}$
9. $-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\Re e\{\alpha\}$
10. $\delta(t-T)$	e^{-sT}	All s
11. $[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12. $[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13. $[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\Re e\{\alpha\}$
14. $[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\Re e\{\alpha\}$
15. $u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16. $u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$