1. Find the generating function of the sequence

$$1,\ 2,\ 0,\ 3,\ 4,\ 0,\ 5,\ 6,\ 0,\ 7,\ 8,\ 0,\ \cdots.$$

1 2 3 4 5 6 7 8 9 10 11 12 ...

−3 −6 −9 −12 ...

−1 −1 −2 −2 −3 −3 ...

$$\frac{1}{(1-x)^2}$$

$$-\frac{3x^2}{(1-x^3)^2}$$

$$-\frac{(x+1)x^3}{(1-x^3)^2}$$

2. Let A(x) be the generating function of the sequence a0, a1, a2, ⋯. Express the generating function of the sequence

a0 + a1, a2 + a3, a4 + a5, a6 + a7, ⋯

in terms of A( ).

$$(a_0 + a_1) + (a_2 + a_3)x + (a_4 + a_5)x^2 + \cdots$$

$$= (a_0 + a_2 x + a_4 x^2 + \cdots)$$

$$+ (a_1 + a_3 x + a_5 x^2 + \cdots)$$

$$= [A(\sqrt{x}) + A(-\sqrt{x})]/2 + \frac{1}{2\sqrt{x}}[A(\sqrt{x}) - A(-\sqrt{x})]$$

3. Let $F_n$, $n \geqslant 0$, denote the Fibonacci sequence. Prove that $\sum_{n\in\mathbb{N}_0} \frac{F_n}{2^n} = 2$.

$$F(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\rho x)(1-\bar{\rho} x)}$$

$x = \frac{1}{2}$ (legal)

$$|\rho x| < 1$$

$$|\bar{\rho} x| < 1$$

$$F(1/2) = \frac{1/2}{1-\frac{1}{2}-\frac{1}{4}}$$

$$= 2$$

4. Let A(x) be the EGF of a0, a1, a2, $\cdots$. Express the EGF of the sequence

a1 – a0, a2 – a1, a3 – a2, $\cdots$

in terms of A( ).

$$(a_1 - a_0) + x(a_2 - a_1) + \frac{x^2}{2!}(a_3 - a_2) + \frac{x^3}{3!}(a_4 - a_3) + \cdots$$

$$= \left(a_1 + a_2 x + \frac{a_3}{2!}x^2 + \frac{a_4}{3!}x^3 + \cdots\right) - A(x)$$

$$= \left(a_1 + a_2 x + \frac{a_3}{2!}x^2 + \frac{a_4}{3!}x^3 + \cdots\right) - A(x)$$

$$= A'(x) - A(x).$$ $\leftarrow$ check

5. Let $A$ be a (real-valued) random variable, and $n \in \mathbb{N}_0$. The $n$-th moment of $A$ (about zero) is defined as $\mu_n = \mathrm{E}[A^n]$. The exponential generating function of the sequence $\mu_0, \mu_1, \mu_2, \mu_3, \dots$ is called the *moment generating function* $M_A(x)$ of $A$. Prove that $M_A(x) = \mathrm{E}[e^{xA}]$ (provided that this expectation exists).

discrete r.v. space for A is $a_i, i \in I$.

$$\mu_n = E[A^n] = \sum_{i \in I} p_i a_i^n \qquad \Pr[A = a_i] = p_i$$

$$M_A(x) = \mu_0 + \mu_1 x + \mu_2 \frac{x^2}{2!} + \mu_3 \frac{x^3}{3!} + \cdots$$

$$= \sum_{n \geq 0} \left( \sum_{i \in I} p_i a_i^n \right) \frac{x^n}{n!} = \sum_{i \in I} p_i \sum_{n \geq 0} a_i^n \frac{x^n}{n!}$$

$$= \sum_{i \in I} p_i e^{a_i x} = E[e^{Ax}].$$

6. Let $a_n$, $n \geqslant 0$, be the sequence satisfying

$$\begin{aligned} a_0 &= 1, \\ a_n &= 2 + 2a_0 + 2a_1 + 2a_2 + \cdots + 2a_{n-2} + a_{n-1} \quad \text{for } n \geqslant 1. \end{aligned}$$

Deduce that the generating function of this sequence is $\dfrac{1+x}{1-2x-x^2}$. Solve for $a_n$.

Use convolution.

7. The generating function $A(x)$ of a sequence $a_0, a_1, a_2, a_3, \ldots$ satisfies $A'(x) = 1 + A(x)$. Prove that $A(x) = (a_0+1)e^x - 1$.

$$1 + A(x) = (1+a_0) + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$A'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$a_1 = (1+a_0), \quad a_2 = \frac{1}{2} a_1 = \frac{1}{2}(1+a_0)$$

$$a_3 = \frac{1}{3} a_2 = \frac{1}{3!}(1+a_0) \quad \cdots$$

$$A(x) = -1 + (1+a_0)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= -1 + (1+a_0)e^x.$$