Linear algebra for AI & ML

(October-6)

Let $A \in \mathbb{R}^{n \times n}$; let λ_1 and λ_2 be two <u>distinct</u> eigenvalues of A with corresponding eigenvectors v_1 and v_2. Then v_1 and v_2 are linearly independent.

$Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$

To prove: v_1 & v_2 are linearly independent.

Consider $\alpha_1 v_1 + \alpha_2 v_2 = 0$ we

then we want to prove $\alpha_1 = \alpha_2 = 0$.

$$\alpha_1 v_1 + \alpha_2 v_2 = 0 \quad \text{—(1)}$$

$$\Rightarrow A(\alpha_1 v_1 + \alpha_2 v_2) = A(0) = 0$$

$$\Rightarrow \alpha_1 A v_1 + \alpha_2 A v_2 = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 = 0 \quad \text{—(2)}$$

Multiply first eqn by λ_1 and subtract it from (2) we get $\alpha_2(\lambda_2 - \lambda_1) v_2 = 0$

Note $v_2 \neq 0$ (Because its an eigen vector)

further $\lambda_1 \neq \lambda_2$ ($\Rightarrow \lambda_2 - \lambda_1 \neq 0$)

$\underbrace{(\lambda_2 - \lambda_2) v_2}_{} \neq 0 \Rightarrow \alpha_2 = 0$

Similarly, we can show that $\alpha_1 = 0$.

$\Rightarrow v_1$ and v_2 are linearly independent.

$A \in \mathbb{R}^{n \times n}$; let $\lambda_1, \dots, \lambda_n$ be eigenvalues of A, all distinct. Then corresponding eigenvectors $v_1, v_2, \dots, v_n$ are linearly independent.

$$Av_i = \lambda_i v_i \quad \text{for } i = 1, 2, \dots, n$$

Then $\{v_1, \ldots, v_n\}$ will be a basis of $\mathbb{R}^n$.

$$T = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$$AT = T \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$ $\leftarrow$ from eigenvalue eigenvector relationship.

Note that $\operatorname{rank}(T) = n \Rightarrow T$ is invertible.

$$A = T \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} T^{-1}$$

$$\boxed{A = T \Lambda T^{-1}}$$ ... diagonalization of the matrix A.

$A \in \mathbb{R}^{n \times n}$, let $\lambda_1, \dots, \lambda_n$ be the distinct eigenvalues of A.

let $v_1, \dots, v_n$ be the corresponding eigenvectors.

For any vector $x \in \mathbb{R}^n$

$$\underset{\sim}{x} = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

$$Ax = A(\alpha_1 v_1 + \dots + \alpha_n v_n)$$

$$= \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \dots + \alpha_n \lambda_n v_n$$

$$A(Ax) = A^2 x = A(\alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \dots + \alpha_n \lambda_n v_n)$$

$$= \alpha_1 \lambda_1^2 v_1 + \alpha_2 \lambda_2^2 v_2 + \dots + \alpha_n \lambda_n^2 v_n$$

continuing this process,

$$A^n x = \alpha_1 \lambda_1^n v_1 + \alpha_2 \lambda_2^n v_2 + \cdots + \alpha_n \lambda_n^n v_n$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be such that

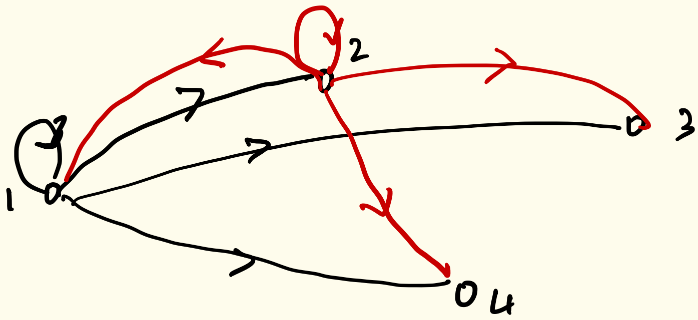
$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| \geq \cdots \geq |\lambda_n| \geq 0$$

$$\frac{A^n x}{\lambda_1^n} = \alpha_1 v_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n v_2 + \cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^n v_n$$

Let $n \to \infty$

Idea leads to power iteration / power method to compute dominant eigenvector and then corresponding eigenvalue.

Markov chains (with finite state space)



4×4 : matrix of transition probabilities

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{eigenvector}} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

eigenvalue

What we interested in is the eigenvectors of P^T.

(Check: Eigenvalues of A and A^T are same!!

$\det(A^T - \lambda I) = \det(A^T - \lambda I^T) = \det(A - \lambda I)^T = \det(A - \lambda I)$)

We, in particular, are interested in the eigenvector corresponding to eigenvalue 1 of P^T. (stationary distribution of the Markov chain)

$$P^T\left(P^T\left(P^T \begin{bmatrix} \cdot \\ \\ \\ \end{bmatrix}\right)\right) = \begin{bmatrix} \cdot \\ \\ \\ \end{bmatrix}$$