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(2) For $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$
↳ invertible

We have

→ maximum magnification of A

$$\text{maxmag}(A) = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2=1} \|Ax\|_2$$

→ minimum magnification of A

$$\text{minmag}(A) = \min_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \min_{\|x\|_2=1} \|Ax\|_2$$

→ condition number of A

$$\text{cond}(A) = \|A\|_2 \|A^{-1}\|_2$$

A is invertible. Hence for any $Ax = y$ we can write $x = A^{-1}y$

So $$\text{maxmag}(A) = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{y\neq 0} \frac{\|y\|_2}{\|A^{-1}y\|_2}$$

$$\Rightarrow \quad \text{maxmag}(A) = \max_{y \neq 0} \frac{1}{\dfrac{\|A^{-1}y\|_2}{\|y\|_2}}$$

$$= \frac{1}{\min\limits_{y \neq 0} \dfrac{\|A^{-1}y\|_2}{\|y\|_2}}$$

$$= \frac{1}{\text{minmag}(A^{-1})} \quad - (1)$$

Hence, proved

Now, by definition we have

$$\|A\|_2 = \text{maxmag}(A)$$

$$\text{cond}(A) = \|A\|_2 \|A^{-1}\|_2 = \text{maxmag}(A)\,\text{maxmag}(A^{-1})$$

$$= \frac{\text{maxmag}(A)}{\text{minmag}((A^{-1})^{-1})}$$

[Using (1)]

$$= \frac{\text{maxmag}(A)}{\text{minmag}(A)}$$

Hence, proved.