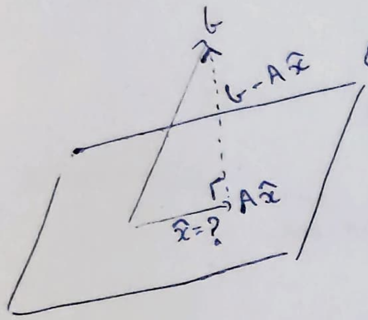
NISARG UPADHYAYA 19CS30031

(4) Let A be the given matrix with columns as $a_1, a_2, \dots a_n$

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$



Plane representing subspace which is the column space of A.

Any vector in this plane can be represented as a linear combination of columns of A.

Let $\hat{x}$ be the coefficient vector of that linear combination which gives the closest vector to b in colspace (A).

From the figure its clear that $b - A\hat{x} \perp$ plane

Hence any vector in the plane is perpendicular to $b - A\hat{x}$. $a_1, a_2 \dots a_n$ are all in this plane. So we have

$$a_1^T(b - A\hat{x}) = 0, \quad a_2^T(b - A\hat{x}) = 0, \quad \dots \quad a_n^T(b - A\hat{x}) = 0$$

Hence, the name "normal equations"

Together, we can write them as $A^T(b - A\hat{x}) = 0$

$$\Rightarrow \quad A^T b = A^T A \hat{x}$$

If the columns of A are linearly dependent we can have infinite solutions.

Because columns are dependent we have $Ay = 0$ for some $y \neq 0$.

If $\hat{x}$ is the least squares solution

$$\|A\hat{x} - b\|_2^2 = \min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2$$

then any solution of the form $\hat{x} + \alpha y$ is also a solution (α is any real number)

because $A(\hat{x} + \alpha y) - b$

$= A\hat{x} + \alpha Ay - b$

$= A\hat{x} - b \quad [\because, Ay = 0]$.