NISARG UPADHYAYA 19CS30031

A3>

$$\|x\|_w = \sqrt{\sum_{i=1}^{n} w_i x_i^2}$$

① Non-negative homogenity. Let $\alpha \in \mathbb{R}$

$$\|\alpha x\|_w = \sqrt{\sum_{i=1}^{n} w_i (\alpha x_i)^2} = \sqrt{\sum_{i=1}^{n} \alpha^2 w_i x_i^2} = \sqrt{\alpha^2 \sum_{i=1}^{n} w_i x_i^2}$$

$$= |\alpha| \sqrt{\sum_{i=1}^{n} w_i x_i^2}$$

$$= |\alpha| \, \|x\|_w$$

② Non-negativity.

$$\|x\|_w = \sqrt{w_1 x_1^2 + w_2 x_2^2 + \cdots w_n x_n^2}$$

$$w_i > 0 \,,\; x_i^2 \geq 0 \implies \sum w_i x_i^2 \geq 0$$

$$\implies \sqrt{\sum w_i x_i^2} \geq 0$$

$$\implies \|x\|_w \geq 0$$

③ Definiteness.

$\Rightarrow$ $$\|x\|_w = \sqrt{\sum_{i=1}^{n} w_i x_i^2} = 0$$

Any $w_i x_i^2 \geq 0$ $[\because, w_i > 0 \ \& \ x_i^2 \geq 0]$

$$\implies w_1 x_1^2 = w_2 x_2^2 = \cdots w_n x_n^2 = 0$$

$$\implies x_1 = x_2 = \cdots x_n = 0 \quad [\because, w_i > 0 \implies w_i \neq 0]$$

$$\implies x = 0 \quad - (\mathrm{I})$$

$\Leftarrow$ $$x = 0 \implies x_1 = x_2 = \cdots x_n = 0$$

$$\implies w_1 x_1^2 = w_2 x_2^2 = \cdots w_n x_n^2 = 0$$

$$\implies \sqrt{\sum_{i=1}^{n} w_i x_i^2} = \sqrt{\sum 0} = \sqrt{0} = 0$$

$$\implies \|x\|_w = 0 \quad - (\mathrm{II})$$

From (I) & (II),

$\|x\|_w = 0 \iff x = 0$

④ Triangle inequality

Note that weighted norm of a ~~matrix~~ vector x with weights $w_1, w_2, \ldots w_n$ is the same as 2-norm of the vector $[\sqrt{w_1}\, x_1 \quad \sqrt{w_2}\, x_2 \; \cdots \; \sqrt{w_n}\, x_n]^T = \hat{x}$

$$\therefore \; \|\hat{x}\|_2 = \sqrt{(\sqrt{w_1}\, x_1)^2 + \cdots (\sqrt{w_n}\, x_n)^2} = \sqrt{w_1 x_1^2 + \cdots w_n x_n^2} = \|x\|_w .$$

Then using the triangle inequality of the 2-norm

$$\|x\|_w + \|y\|_w = \|\hat{x}\|_2 + \|\hat{y}\|_2$$

$$\geq \|\hat{x} + \hat{y}\|_2 \quad -①$$

But $\hat{x} + \hat{y} = [\sqrt{w_1}\, x_1 \quad \sqrt{w_2}\, x_2 \cdots \sqrt{w_n}\, x_n]^T$

$$+ [\sqrt{w_1}\, y_1 \quad \sqrt{w_2}\, y_2 \cdots \sqrt{w_n}\, y_n]^T$$

$$= [\sqrt{w_1}(x_1+y_1) \quad \sqrt{w_2}(x_2+y_2) \; \cdots \; \sqrt{w_n}(x_n+y_n)]^T$$

$$\|\hat{x}+\hat{y}\|_2 = \sqrt{(\sqrt{w_1}(x_1+y_1))^2 + \cdots (\sqrt{w_n}(x_n+y_n))^2}$$

$$= \sqrt{w_1 (x_1+y_1)^2 + \cdots w_n(x_n+y_n)^2}$$

$$= \|x+y\|_w \quad -⑪$$

From ① & ⑪

$$\|x\|_w + \|y\|_w \geq \|x+y\|_w$$

Hence, proved.