NISARG UPADHYAYA 19CS30031

A5) $\Rightarrow$ Matrix multiplication is associative.

$$\text{Let } A \in \mathbb{R}^{m \times n}, \quad B \in \mathbb{R}^{n \times p}, \quad C \in \mathbb{R}^{p \times q}$$

$$R = AB \in \mathbb{R}^{m \times p}, \quad S = BC \in \mathbb{R}^{n \times q}$$

$$\text{Let } T = (AB)C = RC \Rightarrow t_{ij} = \sum_{k=1}^{p} r_{ik} \cdot c_{kj}$$

$$\Rightarrow r_{ik} = \sum_{l=1}^{n} a_{il} \cdot b_{lk}$$

$$\Rightarrow t_{ij} = \sum_{k=1}^{p} \left(\sum_{l=1}^{n} a_{il} \cdot b_{lk} \right) \cdot c_{kj}$$

$$= \sum_{k=1}^{p} \sum_{l=1}^{n} a_{il} \cdot b_{lk} \cdot c_{kj}$$

[Multiplication can be distributed over addition]

$$\text{Let } T' = A(BC) = AS \Rightarrow t'_{ij} = \sum_{k=1}^{n} a_{ik} s_{kj}$$

$$\Rightarrow s_{kj} = \sum_{l=1}^{p} b_{kl} \cdot c_{lj}$$

$$\Rightarrow t'_{ij} = \sum_{k=1}^{n} \left(\sum_{l=1}^{p} b_{kl} \cdot c_{lj} \right) \cdot a_{ik}$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{p} a_{ik} \cdot b_{kl} \cdot c_{lj}$$ [Multiplication can be distributed over addition]

$$= \sum_{l=1}^{n} \sum_{k=1}^{p} a_{il} \cdot b_{lk} \cdot c_{kj}$$ [Change of variable $l \rightarrow k$, $k \rightarrow l$]

Hence $t_{ij} = t'_{ij} \; \forall \; i, j \Rightarrow T = T' \Rightarrow (AB)C = A(BC)$ ✓

$\Rightarrow$ Matrix multiplication is not commutative.

Consider the following counterexample.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 4 \\ 7 & 3 \end{bmatrix} \neq BA = \begin{bmatrix} 8 & 7 \\ -1 & -4 \end{bmatrix}$$

$\Rightarrow$ When multiplying two matrices of dimensions $A(p \times q)$ and $B(q \times r)$ the ij^{th} element of multiplication is given by

$$\sum_{k=1}^{q} a_{ik} \cdot b_{kj}$$

$$\Rightarrow \text{Total steps} = \underbrace{q}_{\text{multiplications}} + \overbrace{(q-1)}^{\text{additions}}$$

$$\approx 2q$$

This is done for $1 \le i \le p$,
$1 \le j \le r$

$\Rightarrow p * r$ times.

Total complexity of multiplication

$$\approx 2pqr$$
$$= O(pqr).$$

MATRIX	DIMENSIONS
A	$p \times q$
B	$q \times r$
C	$r \times t$
AB	$p \times r$
BC	$q \times t$

$\Rightarrow$ Time complexity for $(AB)C$

$$= (pqr) + (prt)$$

$\Rightarrow$ Time complexity for $A(BC)$

$$= (qrt) + (pqt)$$

We want $pqr + prt < qrt + pqt$

$$\Rightarrow \frac{1}{t} + \frac{1}{q} < \frac{1}{p} + \frac{1}{r}$$