Linear algebra for AI and ML

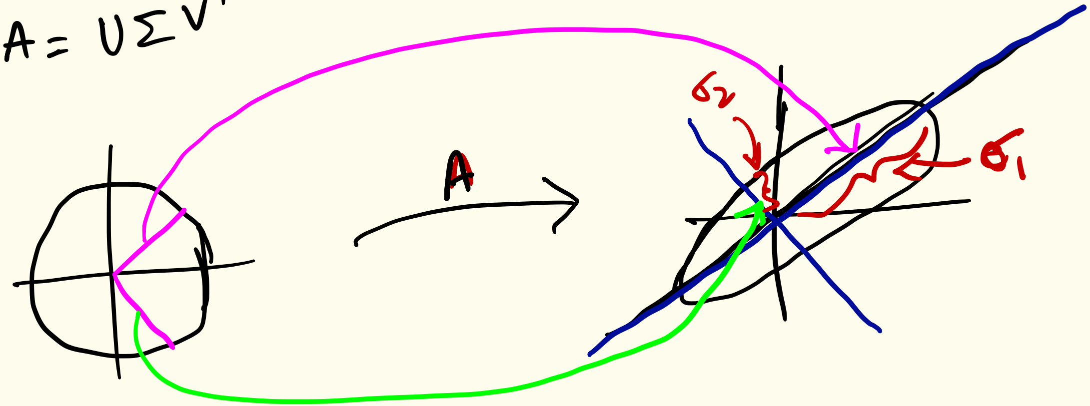
(October- 29)

Low Rank Approximations.

$A = \underline{\underline{U}} \Sigma \underline{\underline{V}}^T$: svd

$A \in \mathbb{R}^{2 \times 2}$

$A = U \Sigma V^T$



$A = U \Sigma V^T \Rightarrow AV = U\Sigma \Rightarrow$ $Av_1 = \sigma_1 \underline{u_1}$ ← direction of max. mag.

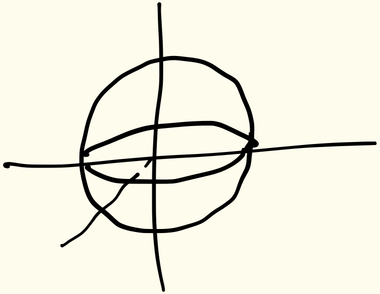
$Av_2 = \sigma_2 \underline{u_2}$ ← direction of min mag.

$v_2 \in \ker(A)$, $\sigma_2 = 0$

Generalise this:

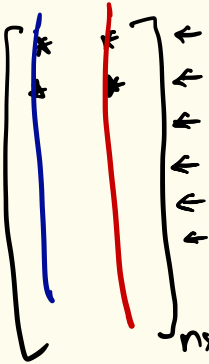
$n = m = 3$

$A \in \mathbb{R}^{3\times 3}$



$\xrightarrow{A}$ Ellipsoid

$A = \mathbb{R}^{n\times 2}$



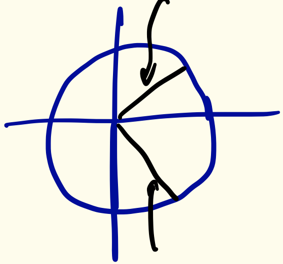
$n \times 2$

Height Weight

Weight $= \alpha$ Height $+ \underbrace{\varepsilon}_{0}$

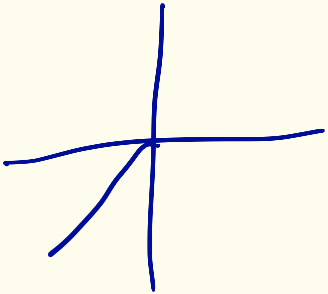
$A \in \mathbb{R}^{n \times 2}$

$A: \underline{\mathbb{R}^2} \longrightarrow \underline{\mathbb{R}^n}$



$\xrightarrow{A}$

ellipse in an n-dimensional space.



2-d subspace.

$A = U \Sigma V^T$

$AV = U\Sigma$

$Av_1 = \sigma_1 u_1 \quad ; \quad Av_2 = \sigma_2 u_2$

$= \text{span}\{u_1, u_2\}$

" 2-d subspace of $\mathbb{R}^n$

Why these LRAs are important??

Consider $\underline{A \in \mathbb{R}^{n \times n}}$

$$\underbrace{A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \cancel{\sigma_n u_n v_n^T}}_{\text{SVD of } A.}$$ be the

$$U = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}; \quad \Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix};$$

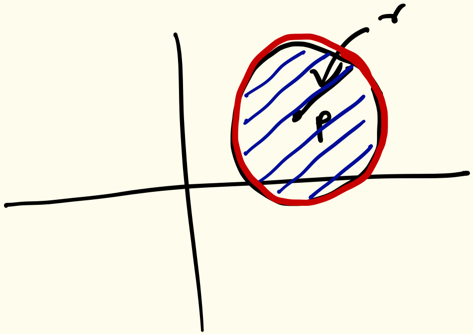
$$V^T = \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ & \vdots & \\ - & v_n^T & - \end{bmatrix}$$

$$A = U \Sigma V^T$$

Q: What is the "<u>nearest</u>" rank $(n-1)$ matrix to A??

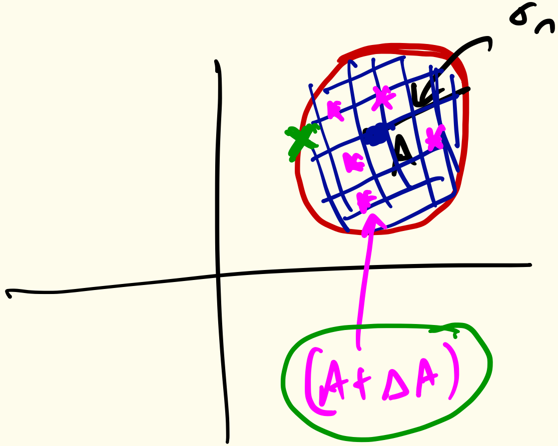
$$A_{n-1} = \sigma_1 u_1 v_1^T + \cdots + \sigma_{n-1} u_{n-1} v_{n-1}^T$$

$$\|A - A_{n-1}\|_2 = \sigma_n$$



extrapolate in the space of matrices.

Hypothetically [set of all $n \times n$ matrices



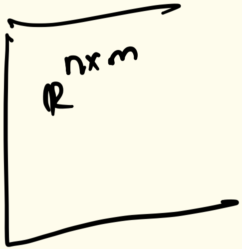
$\|\Delta A\|_2 < \sigma_n$

$Ax = b$

$\underbrace{(A+\Delta A)}\tilde{x} = (b+\Delta b)$

Observation:

Consider a randomly generated $n \times n$ matrix.



space : $\mathbb{R}^{n \times n} = \mathbb{R}^{n^2}$

Consider case $n = 2$

$\mathbb{R}^{2 \times 2}$: space of all 2×2 matrices.

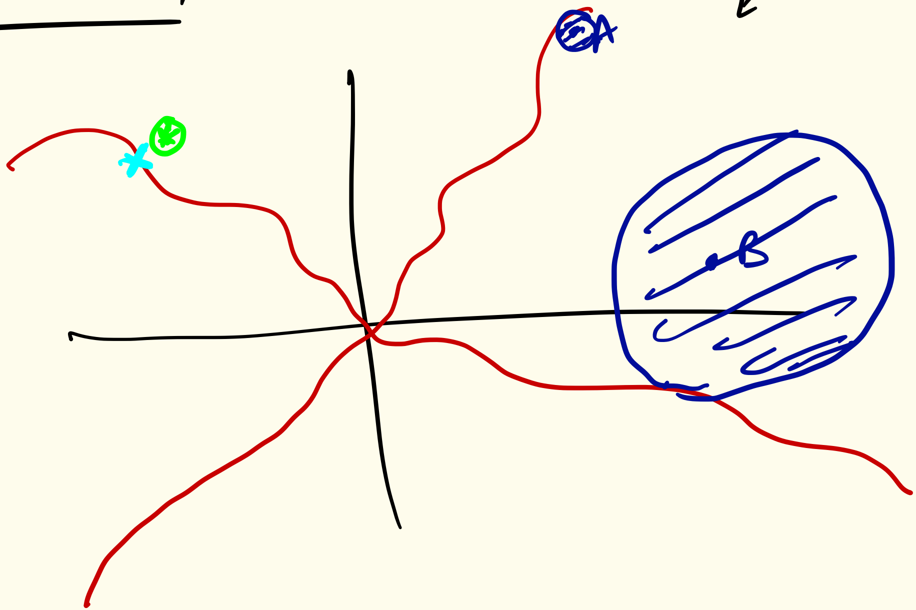
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = 0 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a singular, non-invertible matrix.

In $\mathbb{R}^4$ $\begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix}$

Hypothetically:

space of $n\times n$ matrices.



Property of invertibility (in case $n\times n$ matrices) or property of full column rank (columns being linearly independent) in case of $n\times m$, $n\geq m$

Generic properties.

$A \in \mathbb{R}^{n \times 2}$

$X = 2Y$

$$\begin{bmatrix} * & * \\ & \end{bmatrix} \leftarrow A$$

$$\tilde{X} \quad \tilde{Y}$$

$(A + \Delta A)$

$\mathbb{R}^{2 \times 2}$: $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$\det \begin{bmatrix} x & y \\ z & w \end{bmatrix} = 0$

$\Rightarrow \boxed{xw - yz = 0}$