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A6)

A matrix X that satisfies $XA = I$ is called left inverse of A.

where if $A \in \mathbb{R}^{m \times n}$

then $X \in \mathbb{R}^{n \times m}$

$I \in \mathbb{R}^{n \times n}$

↳ identity matrix.

For existence of left inverse, columns of A should be linearly independent.

(a) $A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_1 = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Inverse Exists $\longleftrightarrow$ This is only possible when $x_1 = 0$. Hence, linearly independent.

Let x be some left inverse of A.

Let Y be the set of all y s.t. $yA = 0$.

Then the set of all left inverses is given by $\{x + y \mid y \in Y\}$.

$$(x+y)A = xA + yA = I + O = I.$$

PROOF

Let x be any arbitrary matrix $[a \ b \ c \ d \ e]$

s.t. $xA = I \Rightarrow a + d = 1 \Rightarrow d = 1 - a$

So the set of all matrices x s.t. $xA = I$ can be defined as $X = \left\{ \begin{bmatrix} a \\ b \\ c \\ 1-a \\ e \end{bmatrix} \middle| \ a, b, c, e \in \mathbb{R} \right\}$

Let y be any arbitrary matrix $[a \ b \ c \ d \ e]$

s.t. $y.A = O \Rightarrow a + d = 0 \Rightarrow d = -a$

So the set of all matrices y s.t. $yA = O$ can be defined as $Y = \left\{ \begin{bmatrix} a \\ b \\ c \\ -a \\ e \end{bmatrix} \middle| \ a, b, c, e \in \mathbb{R} \right\}$

We want to prove that given some left inverse $x \in X$ we can characterise the complete set X using set Y.

Let the given inverse be $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1-x_1 \\ x_5 \end{bmatrix}$. Then any

arbitrary $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ 1-z_1 \\ z_5 \end{bmatrix}$ can be generated as $x + y$

where $y = \begin{bmatrix} z_1 - x_1 \\ z_2 - x_2 \\ z_3 - x_3 \\ x_1 - z_1 \\ z_5 - x_5 \end{bmatrix} \in Y$.

Hence, proved.

(b) $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \\ 3 & 3 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} x_2 = \begin{bmatrix} 2x_1 \\ -2x_2 \\ 3x_1+3x_2 \end{bmatrix}$

Inverse exists $\longleftrightarrow$ This is 0 only when $x_1 = x_2 = 0$. Hence, linearly independent.

Similar to part (a) if x is some left inverse of A and Y be the set of all y s.t. $yA = 0$ Then set of all left inverses is given by

$$\{x + y \mid y \in Y\}.$$

In this case, following the same notations of part (a)

$$X = \left\{ \begin{bmatrix} a & \frac{1-2a}{2} & \frac{1-2a}{3} \\ b & \frac{-2b-1}{2} & \frac{-2b}{3} \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

$$Y = \left\{ \begin{bmatrix} a & -a & \frac{-2a}{3} \\ b & -b & \frac{-2b}{3} \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

Let the given inverse $x = \begin{bmatrix} x_1 & \frac{1-2x_1}{2} & \frac{1-2x_1}{3} \\ x_2 & \frac{-2x_2-1}{2} & -\frac{2x_2}{3} \end{bmatrix}$. Then any arbitrary

$z = \begin{bmatrix} z_1 & \frac{1-2z_1}{2} & \frac{1-2z_1}{3} \\ z_2 & \frac{-2z_2-1}{2} & -\frac{2z_2}{3} \end{bmatrix}$ can be generated as $x+y$

where $y = \begin{bmatrix} z_1-x_1 & x_1-z_1 & \frac{2(x_1-z_1)}{3} \\ z_2-x_2 & x_2-z_2 & \frac{2(x_2-z_2)}{3} \end{bmatrix} \in Y$