Linear algebra for AI & ML

Eigenvalues / eigenvectors:

$A \in \underline{\underline{\mathbb{R}}}^{n \times n}$; $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ Linear transformation. (Scalar $\mathbb{R}$)

$x \mapsto Ax$

We are trying to see if there are any vectors $x \in \mathbb{R}^n$; $x \neq 0$ which upon action of A, (Ax) do not "<u>change the direction</u>". That means, x and Ax have <u>same direction</u>, (x and Ax are linear multiples of each other).

Such a vector $x \in \mathbb{R}^n$ $(x \neq 0)$ is called as an eigenvector and $Ax = \lambda x$ where λ (scalar multiple) is called as eigenvalue correspoind x.

Given $A \in \mathbb{R}^{n \times n}$, if there exists $x \neq 0$ in $\mathbb{R}^n$ such that $Ax = \lambda x$ for som $\lambda \in \mathbb{R}$, then x is called an eigenvector and λ is its corresponding eigenvalue.

$$Ax = \lambda x \qquad x \neq 0$$

$$\iff (A - \lambda I)\, x = 0$$

Note $(A - \lambda I)$ is a matrix in $\mathbb{R}^{n \times n}$ for some value of $\underline{\underline{\lambda}}$.

We will find $\lambda \in \mathbb{R}$ s.t. $(A - \lambda I)$ is <u>not</u> invertible or $(A - \lambda I)$ is <u>not</u> full rank.

$\Rightarrow$ for this λ, $\exists$ $x \neq 0$ in the null space of $(A - \lambda I)$.

$\Rightarrow \quad (A - \lambda I)x = 0$

$\Rightarrow \quad Ax = \lambda x$

$\Rightarrow$ λ and x is an eigenvalue, eigenvector pair (eigen pair).

<u>Notice</u>: $A - \lambda I$ is not invertible.

$$\Leftrightarrow \underbrace{\det(A - \lambda I)} = 0$$

<u>Examples:</u> $S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$\det(S - \lambda I) = \det\left[\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]$$

$$= \det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$\Rightarrow (2-\lambda)^2 - 1 = 0$

$\Rightarrow (2-\lambda-1)(2-\lambda+1) = 0$

$\lambda = 1, 3$ are eigenvectors.

$$(S - \lambda I) = S - 1I = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Let $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $y = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$; $z = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}$ for $\alpha \neq 0 \in \mathbb{R}$

$Sx = x$ $\quad$ x = eigenvector corresponding to the eigenvalue $\underline{\underline{\lambda = 1}}$

Similarly for $\lambda = 3$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector.

$N(S-\lambda I) = \{ x \in \mathbb{R}^n \mid (S-\lambda I)x = 0 \}$

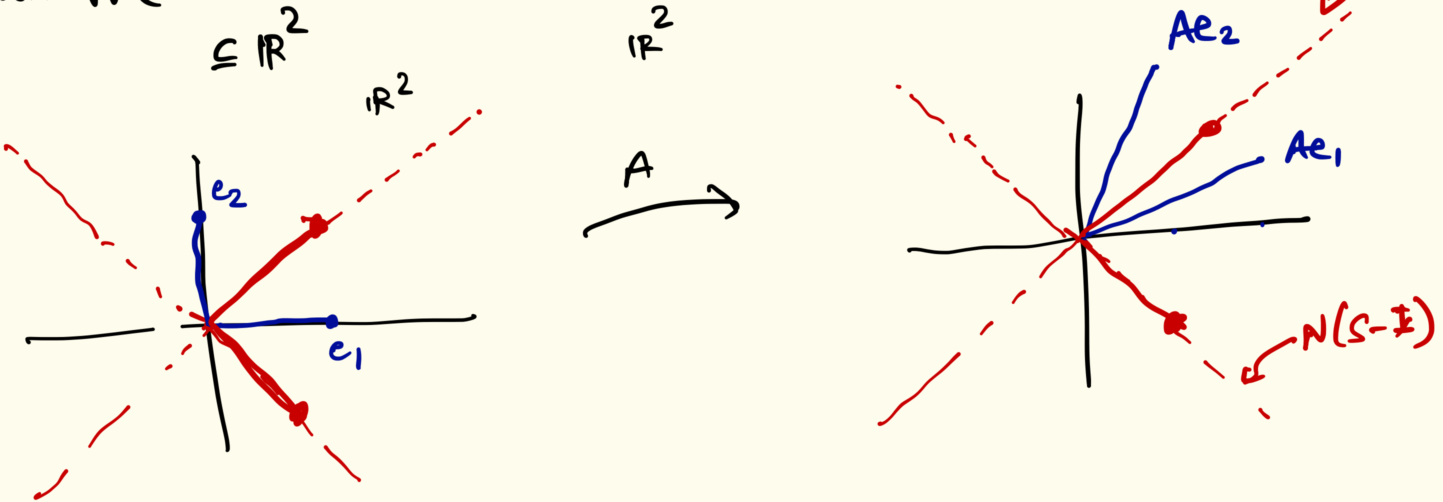
for every vector $x \in N(S-\lambda I)$

non-zero

$(S-\lambda I)x = 0 \Rightarrow$ $\underline{\underline{Sx = \lambda x}}$

$\dim N(S-\lambda I) = 1$ for $\boxed{\lambda = 1, 3}$

$\subseteq \mathbb{R}^2$



Ex:

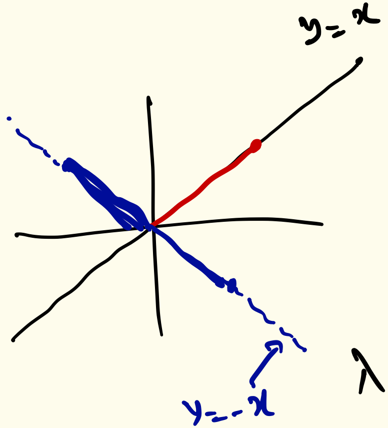
$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad : \quad \text{rotator matrix.}$$

$$\det(Q - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$$

$$= \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i \qquad \text{where } i = \sqrt{-1}$$

Ex:



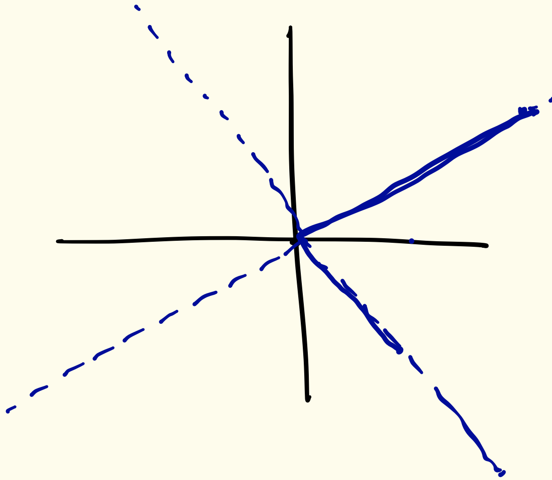
Q = Reflection through y=x line.

$$Q = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\boxed{\begin{aligned} Qv &= v \\ Qu &= -u \end{aligned}}$$

$$\lambda = 1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda = -1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Ex: $A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$



$$\det(A - \lambda I) = \det \begin{bmatrix} 8-\lambda & 3 \\ 2 & 7-\lambda \end{bmatrix}$$

$$= (8-\lambda)(7-\lambda) - 6)$$

$$= \lambda^2 - 15\lambda + 50$$

$$= (\lambda - 10)(\lambda - 5)$$

$$= 0$$

$\Rightarrow \lambda = 5, 10 \leftarrow$ eigenvalues.

$\lambda_1 = 5, \; x_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$; $\lambda_2 = 10, \; x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

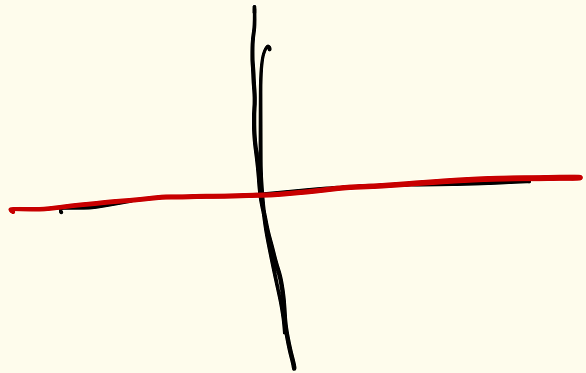
Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Eigenvalues are : 1, 1

eigenvectors:

$$(A - I) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

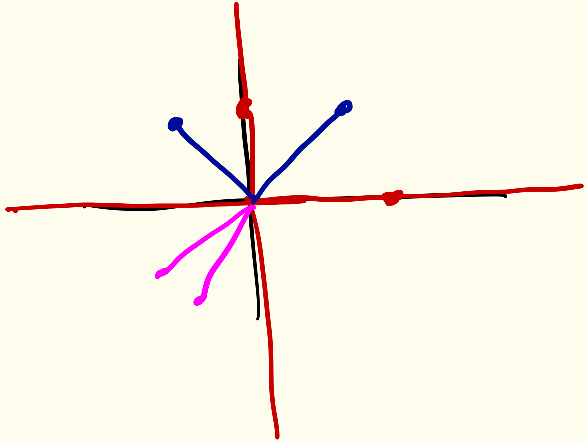


Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda_1 = 1, \quad \lambda_2 = 1$

$\downarrow \qquad \downarrow$

$e_1 \qquad e_2$



$\mathcal{N}(A - \lambda I) = \mathcal{N}\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right] =$ 2-dimensional.

span$\{e_1, e_2\}$

Observations :

i) $\text{Trace}(A) =$ sum of diagonal entries

$= \lambda_1 + \lambda_2$

$=$ sum of eigenvalues

ii) $\det(A) = \lambda_1 \lambda_2$

$\det(A - \lambda I) = 0$: quadratic polynomial.

Suppose λ_1 & λ_2 are roots of this polynomial.

$$\lambda^2 + a\lambda + b = 0$$

Note: Given $A \in \mathbb{R}^{n \times n}$

→ $\det(A - \lambda I) = 0$ (*)

→ get a monic polynomial of degree 'n'. (*)

→ compute the roots of this polynomial.

→ compute the corresponding eigenvectors.