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A1) (a) $P_n(\mathbb{R}) = \{a_0 + a_1x + \dots a_nx^n \mid a_0, a_1, \dots a_n \in \mathbb{R}\}$

$\Rightarrow$ VECTOR ADDITION

1. Closure

$a = a_0 + a_1x + \dots a_nx^n \quad , a_0, a_1, \dots a_n \in \mathbb{R} \quad - (\text{I})$

$b = b_0 + b_1x + \dots b_nx^n \quad , b_0, b_1, \dots b_n \in \mathbb{R} \quad - (\text{II})$

$\Rightarrow a+b = (a_0+b_0) + (a_1+b_1)x + \dots (a_n+b_n)x^n$

From (I) & (II) $\quad a_0+b_0, a_1+b_1, \dots a_n+b_n \in \mathbb{R}$

$\Rightarrow a+b \in P_n(\mathbb{R})$

2. Commutative

$a+b = (a_0+b_0) + (a_1+b_1)x + \dots (a_n+b_n)x^n$

$= (b_0+a_0) + (b_1+a_1)x + \dots (b_n+a_n)x^n$

$= b+a$

3. Associative

$a+(b+c) = a_0 + a_1x + \dots a_nx^n + (b_0+c_0) + (b_1+c_1)x + \dots (b_n+c_n)x^n$

$= (a_0+b_0+c_0) + (a_1+b_1+c_1)x + \dots (a_n+b_n+c_n)x^n$

$= (a_0+b_0) + (a_1+b_1)x + \dots (a_n+b_n)x^n + c_0 + c_1x + \dots c_nx^n$

$= (a+b) + c$

4. Additive identity

$0 + 0x + \dots 0x^n$

$a+0 = (a_0+0) + (a_1+0)x + \dots (a_n+0)x^n$

$= (0+a_0) + (0+a_1)x + \dots (0+a_n)x^n$

$= 0 + a$

$= a$

5. Additive inverse

For $a_0, a_1, \dots a_n \in \mathbb{R}$ we have $-a_0, -a_1, \dots -a_n \in \mathbb{R}$

$\Rightarrow a = a_0 + a_1x + \dots a_nx^n, \; -a = (-a_0) + (-a_1x) + \dots (-a_n)x^n \in P_n(\mathbb{R})$

$a+(-a) = (a_0-a_0) + (a_1-a_1)x + \dots (a_n-a_n)x^n = 0 + 0x + \dots 0x^n = 0$

=) SCALAR MULTIPLICATION

$$\beta, \alpha \in \mathbb{R}$$
$$a = a_0 + a_1 x + \cdots a_n x^n \quad , a_0, a_1, \ldots a_n \in \mathbb{R}$$
$$b = b_0 + b_1 x + \cdots b_n x^n \quad , b_0, b_1, \ldots b_n \in \mathbb{R}$$

1. Closure

$\alpha a = (\alpha a_0) + (\alpha a_1) x + \cdots (\alpha a_n) x^n \in P_n(\mathbb{R})$ as $\alpha a_0, \alpha a_1, \ldots \alpha a_n \in \mathbb{R}$

2. Associative

$$\alpha(\beta a) = \alpha\left[(\beta a_0) + (\beta a_1)x + \cdots (\beta a_n)x^n\right]$$
$$= (\alpha\beta a_0) + (\alpha\beta a_1)x + \cdots (\alpha\beta a_n)x^n$$
$$= (\alpha\beta)[a_0 + a_1 x + \cdots a_n x^n]$$
$$= (\alpha\beta) a$$

3. Distributive

$$\Rightarrow (\alpha+\beta) a = (\alpha+\beta)a_0 + (\alpha+\beta)a_1 x + \cdots (\alpha+\beta) a_n x^n$$
$$= (\alpha a_0 + \alpha a_1 x + \cdots \alpha a_n x^n) + (\beta a_0 + \beta a_1 x + \cdots \beta a_n x^n)$$
$$= \alpha a + \beta a$$

$$\Rightarrow \alpha(a+b) = \alpha(a_0+b_0) + \alpha(a_1+b_1)x + \cdots \alpha(a_n+b_n)x^n$$
$$= (\alpha a_0 + \alpha a_1 x + \cdots \alpha a_n x^n) + (\alpha b_0 + \alpha b_1 x + \cdots \alpha b_n x^n)$$
$$= \alpha a + \alpha b$$

4. Multiplicative identity

$$1 \in \mathbb{R}$$

$$1 \cdot a = (1 \cdot a_0) + (1 \cdot a_1)x + \cdots (1 \cdot a_n)x^n$$
$$= a_0 + a_1 x + \cdots a_n x^n$$
$$= a$$

As all the properties are satisfied $P_n(\mathbb{R})$ is a vector space.

(b) For linear functional homogenity + additivity, i.e., superposition should hold.

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

$$f(p(x)) = \frac{d}{dx} p(x) \Big|_{x=0}$$

$$\Rightarrow f(\alpha p(x) + \beta q(x)) = \frac{d}{dx} [\alpha p(x) + \beta q(x)] \Big|_{x=0}$$

$$= \alpha \frac{d}{dx} p(x) \Big|_{x=0} + \beta \frac{d}{dx} q(x) \Big|_{x=0}$$

$$= \alpha f(p(x)) + \beta f(q(x)).$$

Hence, proved.

(c) Let $p(x) = p_0 + p_1 x + \cdots p_n x^n$.

$$f(p(x)) = \frac{d}{dx} p(x) \Big|_{x=0} = p_1 + 2p_2 x + \cdots n p_n x^{n-1} \Big|_{x=0}$$

$$= p_1$$

So, $p = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix}$ and $e_1^T = [0 \ 1 \ 0 \ \cdots \ 0]$

Note $e_1 \to$ 0-based indexing
$\to$ length is $n+1$

Then $f(p(x)) = e_1^T p$.