Linear algebra for AI and ML

August 27

(Lecture #6)

$Ax = b$ $\qquad A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

- left invertibility
- right invertibility
- invertibility of matrix A

→ generalize the concept of inverse of a real number. $(a \neq 0)$

(A has lin. indep. columns,) $\Longleftrightarrow$ (A^TA is invertible)

$A \in \mathbb{R}^{m \times n}$

A has lin. cols. $\begin{bmatrix} A \end{bmatrix}$

$B = (A^TA)^{-1} A^T$ ← pseudo inverse of A

B

$\begin{bmatrix} \ \end{bmatrix}_{n \times n} \begin{bmatrix} \ \end{bmatrix}_{n \times m} = \begin{bmatrix} \ \end{bmatrix}_{n \times m}$

$\begin{bmatrix} A^T \end{bmatrix}_{n \times m} \begin{bmatrix} A \end{bmatrix}_{m \times n} = \begin{bmatrix} A^TA \end{bmatrix}_{n \times n}$

computations:

$$Ax = b \qquad (A \in \mathbb{R}^{m\times m}) \quad \text{square matrix}$$

$$b \in \mathbb{R}^m$$

Particular case:

$$\underbrace{\overset{U}{\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ & u_{22} & \cdots & u_{2m} \\ 0 & & \ddots & \vdots \\ & & & u_{mm} \end{bmatrix}}}_{\text{upper triangular matrix}} x = b$$

If $\underbrace{u_{ii} \neq 0,}$ for $i = 1, 2, \dots, m$

$$x_m = \frac{1}{u_{mm}} b_m$$

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ 0 & u_{22} & \cdots & u_{2m} \\ \vdots & & & \\ 0 & 0 & \cdots & u_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & & \\ & & \\ u_{mm} x_m & & \end{bmatrix} = \begin{bmatrix} \vdots \\ \\ b_m \end{bmatrix}$$

$$u_{m-1,m-1} \, x_{m-1} + u_{m-1,m} \, x_m = b_{m-1}$$

$$x_{m-1} = \frac{1}{u_{m-1,m-1}} \left[b_{m-1} - u_{m-1,m} \, x_m \right]$$

$$\vdots$$

(backward substitution)

$$x = U^{-1} b$$

Given $Ax = b$

can we transform this system of linear equations in cannonical form (upper triangular system or lower triangular system) provided that the solution of the transformed system does not change.

- Gaussian elimination
- LU decomposition
- QR decomposition

$A = QR$; $Ax = b \Rightarrow QRx = b$
$\Rightarrow Rx = Q^T b$

$A = LU$

$=$ 

L U

$Ax = L(\underbrace{Ux}_{y}) = Ly = b$

- Cholesky decomposition
- LU decomposition ←
- QR decomposition ←
- Eigenvalue, eigenvector decomposition
- SVD

$\Updownarrow$

Low rank approximations

LU decomposition:

$$A \in \mathbb{R}^{m \times m}$$

$$\underbrace{L_{m-1} \cdots L_2 L_1}_{\underbrace{L^{-1}}} A = U$$

$$L^{-1} A = U$$

$$\Rightarrow A = LU$$

Construct

$L_{m-1}, L_{m-2}, \ldots, L_2, L_1$

s.t.

U: upper triangular.

To prove: L is lower triangular.

setting: $L = \underbrace{L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1}}$

$$\underset{A}{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}} \xrightarrow{L_1} \underset{L_1 A}{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}} \xrightarrow{L_2} \underset{L_2 L_1 A}{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}} \xrightarrow{L_3}$$

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} = U$$

$$\underbrace{L_3 L_2 L_1}_{L^{-1}} A$$

$$A = LU = \underbrace{L_1^{-1} L_2^{-1} L_3^{-1}}_{L} U$$

Ex:

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

$$L_1 A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

$$\left\{\begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \\ R_4 - 3R_1 \end{matrix}\right\} \text{ on } I$$

$$\underset{\ell_{21}}{2} = , \quad \underset{\ell_{31}}{4} = , \quad \underset{\ell_{41}}{3} =$$

$$= \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}$$

$$L_2 L_1 A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -3 & 1 & \\ & -4 & & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & 3 & 5 & 5 \\ & 4 & 6 & 8 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}}$$

$$\left\{ \begin{array}{l} R_3 - 3R_2 \checkmark \\ R_4 - 4R_2 \checkmark \end{array} \right.$$

$$L_3 L_2 L_1 A = \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix}}_{L_3} \underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}}_{L_2 L_1 A} = \underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{U}$$

$$L_3 L_2 L_1 A = U$$

$$A = \underbrace{L_1^{-1} L_2^{-1} L_3^{-1}}_{= L} U$$

To check:

i) L_1, L_2, L_3 are invertible

ii) Compute $L_1^{-1}, L_2^{-1}, L_3^{-1}$

iii) show $L_1^{-1} L_2^{-1} L_3^{-1}$ is a lower triangular matrix.

H.W.

→

i) Since L_1, L_2, L_3 are lower triangular and L_1, L_2, L_3 diagonals are all 1's $(\neq 0)$; L_1, L_2, L_3 are invertible.

ii) Inverses of L_1, L_2, L_3

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ -4 & & 1 & \\ -3 & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & & 1 & \\ 3 & & & 1 \end{bmatrix}$$

L_1

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -3 & 1 & \\ & -4 & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 3 & 1 & \\ & 4 & & 1 \end{bmatrix}$$

L_2

Similarly for L_3 also.

iii) To show $L_1^{-1} L_2^{-1} L_3^{-1}$ is lower triangular

$$\underbrace{\begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & & 1 & \\ 3 & & & 1 \end{bmatrix}}_{L_1^{-1}} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & 3 & 1 & \\ & 4 & & 1 \end{bmatrix}}_{L_2^{-1}} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix}}_{L_3^{-1}}$$

$$= \underbrace{\begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 3 & 1 & \\ 3 & 4 & 1 & 1 \end{bmatrix}}_{L_1^{-1} L_2^{-1} L_3^{-1}}$$

$\therefore$ lower triangular

General case of $A \in \mathbb{R}^{m \times m}$:

Let y_k denote the k^{th} column of matrix $\underbrace{A \text{ at the beginning of step } k}_{L_{k-1} \cdots L_1 A}$.

Choose L_k s.t.

$$y_k = \begin{bmatrix} y_{1k} \\ \vdots \\ y_{kk} \\ \boxed{y_{k+1,k}} \\ \vdots \\ \boxed{y_{m,k}} \end{bmatrix} \xrightarrow{L_k} L_k y_k = \begin{bmatrix} y_{1k} \\ \vdots \\ \boxed{y_{kk}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$l_{jk} = \frac{y_{jk}}{y_{kk}} \qquad k < j \leq m$$

$$L_k = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -l_{k+1,k} & \ddots & \\ & & \vdots & & \\ & & -l_{mk} & & 1 \end{bmatrix}_{m \times m}$$

k^{th} column

<u>**Observation:**</u> L_k can be inverted by just changing the sign (negating the entries) below diagonal.

Define

$$l_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{k+1,k} \\ \vdots \\ l_{m,k} \end{bmatrix}_{m\times 1} \in \mathbb{R}^m \qquad k \text{ entries are zero.}$$

$$e_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow k^{th} \text{ position.}$$

Then **check:** $L_k = I - l_k e_k^T$

$$k^{th} \text{ position}$$

$$\begin{bmatrix} \cancel{0} \\ \cancel{0} \\ \vdots \\ \cancel{0} \\ \cancel{l_{k+1,k}} \\ \vdots \\ l_{m,k} \end{bmatrix}_{m\times 1} \begin{bmatrix} 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}_{1\times m} = \begin{array}{c} \qquad\qquad k^{th} \text{ column} \\ \begin{bmatrix} 0 & 0 \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots 0 & & 0 \\ 0 & 0 & -l_{k+1,k} & 0 \cdots & 0 \\ \vdots & & & & \\ 0 & 0 \cdots & l_{m,k} & 0 \cdots & 0 \end{bmatrix}_{m\times m} \end{array} \leftarrow k^{th} \text{ row}$$

Observe: $e_k^T l_k = 0$ ✓

$$[0 \; 0 \cdots \underset{\substack{\uparrow \\ k^{th} \text{ position}}}{1} \; 0 \cdots 0] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{k+1,k} \\ \vdots \\ l_{m,k} \end{bmatrix} \begin{matrix} \} \, k \text{ position} \\ \\ \} \, m-k \text{ position} \end{matrix} = 0$$

Check:

$$\underbrace{(I - l_k e_k^T)}_{L_k} \underbrace{(I + l_k e_k^T)}_{L_k^{-1}} = I + \cancel{l_k e_k^T} - \cancel{l_k e_k^T} - l_k \underbrace{(e_k^T l_k)}_{0} e_k^T$$

$$= I$$

Given: $A \in \mathbb{R}^{m \times m}$, $b \in \mathbb{R}^m$

A is invertible (A has full col. rank / cols of A are lin. indep.)

(previous)

$\downarrow$

$\exists\ L, U$ s.t. $A = LU$

$$Ax = b$$

$$L\boxed{Ux} = b$$

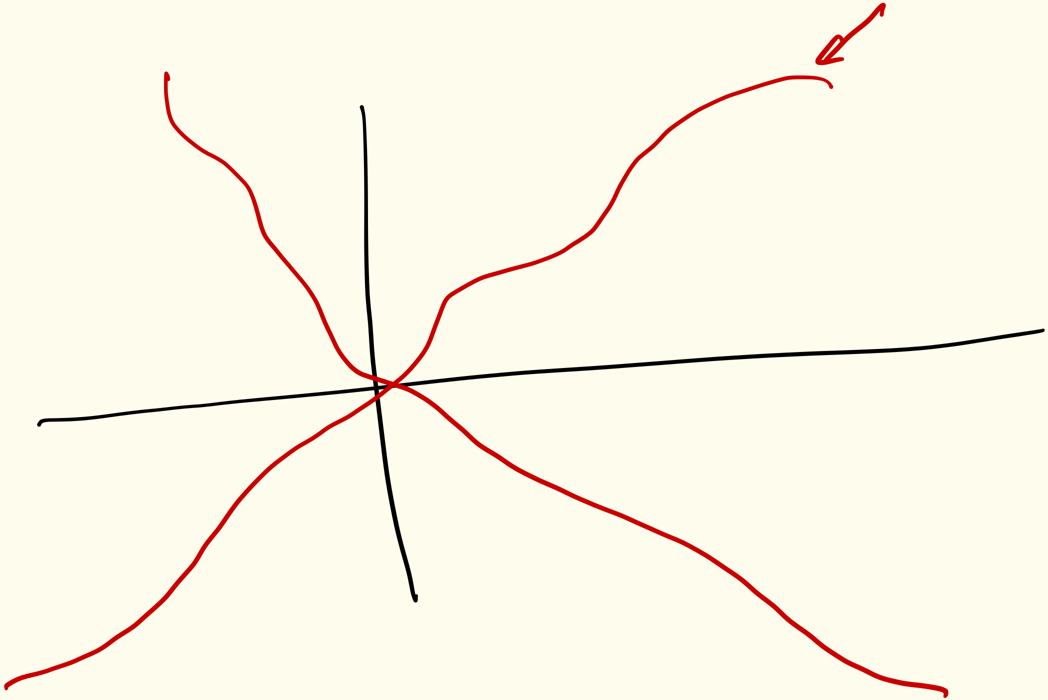
$$\| \quad y$$

First solve:

$Ly = b$ by forward substitution

$Ux = y$ by backward substitution

Observation: Gaussian elimination $\equiv LU$



$\mathbb{R}^{m^2}$

y_{11}

10^{2}

y_{j1}

y_{11}

$\approx 10^{6}$

10^{-4}

LU

$\boxed{\text{with partial pivoting}}$

Numerical linear algebra

Trefethen Bau

Backward stability of computations 10^{2}

A. Willikinson

D. Watkins : Matrix Computations

Computational Linear Algebra Spring