Linear algebra for AI and ML

(November 11)

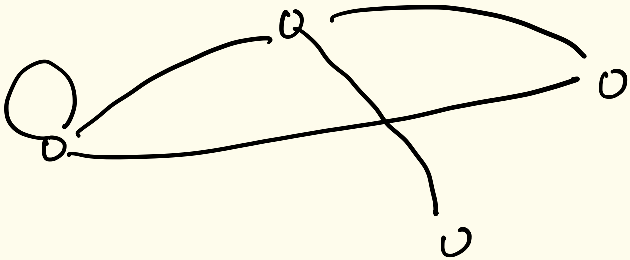
Given $A \in \mathbb{R}^{n\times n}$

eigenvalues and eigenvectors

scaling

invariant directions

Markov chain: with finite states



G

P = Probability transition matrix

Eigenvector corresponding to the eigenvalue 1 is called stationary distⁿ. .

$$\frac{d}{dt} x = \underbrace{A}\, x \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$e^{At}$$

$$x = \alpha_1 e^{\lambda_1 t} v_1 + \cdots$$

Given a matrix A, compute its eigenvalues and corresponding eigenvectors.

[QR - algorithm] ← iterative process

[Krylov subspace]

David Watkins / Golub [Computional linear algebra]

Q: $A \in \mathbb{R}^{n \times n}$ $\longrightarrow$ compute eigenvalues / eigenvectors

Q: Given a set of numbers (eigenvalues) and a set of vectors (corresponding eigenvectors), construct a matrix A such that this matrix has these numbers as its eigenvalues and these vectors as corresponding eigenvectors.

(Inverse eigenvalue problem)

<u>Observe:</u> If (λ, x) is an eigenpair of A

then $$Ax = \lambda x$$

unknown

i) unknown variables are n^2. (entries in the matrix A)

ii) if 'm' pairs of eigenvalues and eigenvectors are given $(1 \leq m \leq n)$, then we have mn number of relationships. (equations in unknowns)

iii) Relationships (eq^{ns}) are linear.

Ex: let $\left(\underbrace{2}_{\lambda}, \underbrace{\begin{pmatrix}-1\\1\end{pmatrix}}_{x}\right)$ be given.

Compute $A \in \mathbb{R}^{2\times 2}$ s.t A has $\left(2, \begin{pmatrix}-1\\1\end{pmatrix}\right)$ as its eigen pair.

$$\underbrace{\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}$$

$\rightarrow$ Eigenvalue - eigenvector relationship

$$\Rightarrow \quad Ax = \lambda x$$

$$\Rightarrow \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$$

$$\Rightarrow \begin{aligned} -a_{11} + a_{12} &= -2\\ -a_{21} + a_{22} &= 2\end{aligned}$$

$$\Rightarrow \begin{bmatrix}-1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{bmatrix}a_{11}\\a_{12}\\a_{21}\\a_{22}\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$$

$$\begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \end{matrix}} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \end{matrix}} \\ \boxed{\begin{matrix} a_{12} \\ a_{22} \end{matrix}} \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

vectorization of a matrix

<u>Observation:</u> In general, solving the inverse eigenvalue problem is equivalent to solving an underdetermined (partial) or square (complete) system of linear eqns.

$m < n$ n

Ex: P: probability transition matrix. (n states)

$x \in \mathbb{R}^n$ s.t.

$Px = x$ (x is a stationary distribution)

Bernoulli($p = 1/2$)

$x = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

Uniform(6)

$x = \begin{bmatrix} 1/6 \\ 1/6 \\ \vdots \\ 1/6 \end{bmatrix}$

$N(0,1)$, $\underbrace{x \in \mathbb{R}^n \quad \text{where} \quad n = 2^5}$

$P_{ij} \geq 0$

M.T. Chu

Boyd

(SIAM Review)