# Lecture 2

vectors

- vector addition
- scalar multiplication

Vector space.

Let $V$ be a set. $V$ is called a vector space over $\mathbb{R}$ if the following properties are satisfied.

Define an operation (addition) on $V \times \cancel{V} \longrightarrow V$ ✱

Define another operation (scalar multiplication) on $\mathbb{R} \times V \longrightarrow V$ ✱

**Addition**

- for every $v_1, v_2 \in V$
  $v_1 + v_2 = v_2 + v_2$
- for every $v_1, v_2, v_3 \in V$
  $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$
- $\exists\ 0 \in V$ such that $v + 0 = 0 + v = v$ $\forall v \in V$.
- $\exists\ w \in V$ for every $v \in V$ s.t.
  $w + v = v + w = 0$

**Scalar multiplication**

- $\forall \alpha, \beta \in \mathbb{R}$, $v \in V$
  $\alpha(\beta v) = (\alpha\beta) v$
- for $1 \in \mathbb{R}$, $1 . v = v$ $\forall v \in V$
- $(\alpha + \beta) v = \alpha v + \beta v$ $\forall v \in V$
- $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$
  $\forall v_1, v_2 \in V$

Examples: 1) $\mathbb{R}^2$, $\mathbb{R}^3$, in general $\mathbb{R}^n$ for any $n \in \mathbb{N}$ over $\mathbb{R}$.

2) $S = \{ (x_1, x_2, x_3, \ldots) \mid x_i \in \mathbb{R}$ for each $i \in \mathbb{N} \}$

$x \in S$, $x = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

$y = (1, 1, 1, \ldots)$

$0 = (0, 0, 0, 0, \ldots) \Leftarrow$ "Zero" vector

for $x, y \in S$, $x + y = (x_1 + y_1, x_2 + y_2, \ldots)$

for $\alpha \in \mathbb{R}$, $x \in S$, $\alpha x = (\alpha x_1, \alpha x_2, \ldots)$

Claim: $S$ is a vector space over $\mathbb{R}$.

3) $\mathcal{F} = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous on } \mathbb{R} \}$

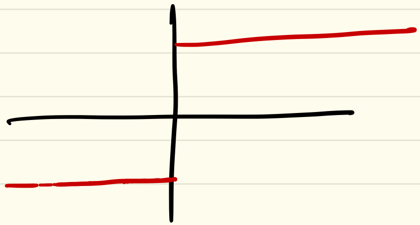
$f(x) =$

$$f(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

RELU

$f(x) \in \mathcal{F}$

$g(x) =$



$$f(x) = \begin{cases} -1 & \text{for } x \leq 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

$g(x) \notin \mathcal{F}$

Addition : $f(x) + g(x) =: (f+g)(x)$

scalar multiplication: $\alpha f(x) =: (\alpha f)(x)$

$\mathcal{F}$ is also a real vector space.

[ $\mathcal{F}$ is a vector space over $\mathbb{R}$.

4) $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; x_1 + x_2 = 0 \right\} \subseteq \mathbb{R}^2$

Is $W$ a vector space over $\mathbb{R}$??

Vector addition: componentwise

scalar multiplication: — " —

i) $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in W$

Ans: $W$ is a vector space over $\mathbb{R}$.

special!! Vector subspaces.

$\boxed{W \subseteq \mathbb{R}^2}$

$\mathbb{R}^2$, in itself, a vector space over $\mathbb{R}$.

5) $Z = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; x_1 + x_2 = 1 \right\} \subseteq \mathbb{R}^2$

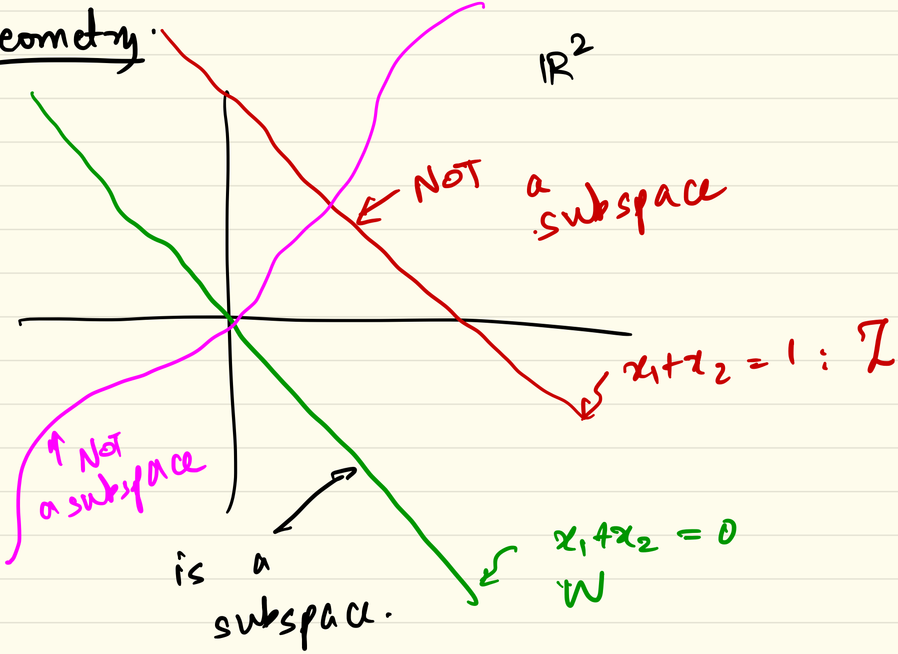
Is $Z$ a vector space over $\mathbb{R}$??

Ans: NO !!

$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$

**Vector subspace:** Let $V$ be a vector space over $\mathbb{R}$. Let $W \subseteq V$. Then $W$ is called a vector subspace if $W$ is a vector space over $\mathbb{R}$.

**Geometry:**




All the lines passing thru' origin will be subspaces of $\mathbb{R}^2$.

All the subspaces in $\mathbb{R}^2$.

- $\{0\}$
- $\mathbb{R}^2$
- any line passing thru' origin.

All the subspaces of $\mathbb{R}^3$

- $\{0\}$
- $\mathbb{R}^3$
- any line passing thru' origin.
- any plane passing thru' origin.

$\longrightarrow \mathbb{R}^n$

Back to the example of $\mathbb{R}^n$.

Inner product: For $x, y \in \mathbb{R}^n$, $x^T y = \sum_{i=1}^{n} x_i y_i$

$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

Properties:

1) $x^T y = y^T x$ for every $x, y \in \mathbb{R}^n$

2) $(\alpha a^T) b = \alpha (a^T b)$ for $a, b \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$

3) $(a+b)^T c = a^T c + b^T c$

$$(x+y)^T(a+b) = x^T a + x^T b + y^T a + y^T b$$

Examples:

1) Inner product with unit vectors.

Let $e_i \in \mathbb{R}^n$. $\forall x \in \mathbb{R}^n$

$$e_i^T x = x_i \quad \text{the } i^{th} \text{ co-ordinate of } x.$$

2) Sum: $1_n \in \mathbb{R}^n$

$$\forall x \in \mathbb{R}^n, \quad 1_n^T x = \sum_{i=1}^{n} x_i$$

3) Average:

$$\left(\frac{1_n}{n}\right)^T x = \sum_{i=1}^{n} \frac{x_i}{n} = \operatorname{avg}(x)$$

- Sum of squares: $\forall x \in \mathbb{R}^n$

$$x^T x = x_1^2 + x_2^2 + \cdots + x_n^2$$

Block - vector inner product.

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$x_1, y_1 \in \mathbb{R}^{n_1}, \quad x_2, y_2 \in \mathbb{R}^{n_2}, \ldots, \quad x_m, y_m \in \mathbb{R}^{n_m}$$

$$x^T y = x_1^T y_1 + x_2^T y_2 + \cdots + x_m^T y_m$$

Applications:

1) Co-occurrance: 10 : objects, 2 : sets.

$x$ and $y$ — $x, y \in \mathbb{R}^{10}$

set 1 — set 2

$x$ & $y$ are vectors of zeros & ones.

$x^T y$ = number of common objects in both the sets.

2) Weights, features:

Let $x$ be the feature vector.

$y$ be the weight vector / scores vector.

$y^T x$ = credit score

3) Probability & expected values.

$p$ : p.m.f.

$x$ : values of quantity at the p.m.f. support.

$x^T p$ : expectation of quantity.

$$p = \begin{bmatrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix} \in \mathbb{R}^6 \qquad x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

$$x^T p = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

4) Polynomial Evaluation:

$$p(x) = p_0 + p_1 x + p_2 x^2 + \dots + p_n x^n$$

$$p = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix}$$

Evaluate $p(x)$ at $x = x_0$

$$e = \begin{bmatrix} 1 \\ x_0 \\ x_0^2 \\ \vdots \\ x_0^n \end{bmatrix}$$

$$e^T p = p(x) \Big|_{x = x_0}$$

Complexity: number of operation (additions / multiplications) involved.

$x, y \in \mathbb{R}^n$ $\qquad x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$ : $n$ operations ($n$ additions)

$\alpha x = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$ : $n$ operations (multiplications)

$$x^T y = \underbrace{x_1 y_1} + \underbrace{x_2 y_2} + \cdots + \underbrace{x_n y_n}$$

$n$ : multiplications
$n-1$ : additions

$\Big]$ $2n - 1$ : Total operations.

Definition of vector spaces / subspaces examples.

inner product : examples

applications.

computational complexity.