A3)

19CS30031 NISARG UPADHYAYA

$\eta \in \mathbb{R}^{3\times 1} \rightarrow$ non zero.

A is full column rank. Hence, columns are linearly independent. ~~col space~~ —①

Let $A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$

~~They are~~ We are given $\eta^T A = 0$

$$\Rightarrow \eta^T \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} = [\eta^T a_1, \eta^T a_2]$$
$$= [0, 0]$$
$$\Rightarrow \eta^T a_1 = \eta^T a_2 = 0$$

So η is $\perp$ to both a_1 & a_2 —(ii)

As a_1 and a_2 are linearly independent (①) they span a certain subspace of $\mathbb{R}^3$ [It is a plane as $\dim(\text{colspace}(A)) = \text{rank}(A) = 2$].

η is a vector perpendicular to this plane. —(iii)

Now we are given $\eta^T b = 0$. This means η & b are perpendicular.

In a 3 dimensional space, a vector perpendicular to the normal of a plane lies in the plane.

From (iii) this plane is the subspace spanned by a_1 and a_2.

As b lies in this plane

$$b \in \text{span}\{a_1, a_2\}.$$

Also a_1 & a_2 are linearly independent.

$\Rightarrow$ Hence ~~[illegible]~~ for every such b there exist unique $\alpha_1, \alpha_2 \in \mathbb{R}$ s.t. $b = \alpha_1 a_1 + \alpha_2 a_2$.

This is equivalent to

$$\Rightarrow \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = b$$

$\Rightarrow$ $Ax = b$ has a unique soln.

Hence, proved.