NISARG UPADHYAYA 19CS30031

(6) (a) A $\longrightarrow$ MN×4 Matrix whose entries are as follows.

$$A = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_1 & y_N & x_1 y_N \\ 1 & x_2 & y_1 & x_2 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_M & y_N & x_M y_N \end{bmatrix}$$

$\Theta \longrightarrow 4 \times 1$ Matrix

$$\Theta = \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \end{bmatrix}$$

$G \longrightarrow MN \times 1$ Matrix

$$G = \begin{bmatrix} F_{11} \\ F_{12} \\ \vdots \\ F_{1N} \\ F_{21} \\ F_{22} \\ \vdots \\ F_{MN} \end{bmatrix}$$

(b) We can see that A has 4 colums.

For a unique solution we need linearly independent columns. From independence-dimension inequality this means we require at least 4 rows.

$$\Rightarrow MN = 4$$

Possible candidates are $M=4, N=1$

$M=2, N=2$

$M=1, N=4$

For $M=4, N=1$ and $M=1, N=4$ following matrices are generated

$$\underbrace{\begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_1 & x_2y_1 \\ 1 & x_3 & y_1 & x_3y_1 \\ 1 & x_4 & y_1 & x_4y_1 \end{bmatrix}}_{\text{Dependent}} \qquad \underbrace{\begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_1 & y_2 & x_1y_2 \\ 1 & x_1 & y_3 & x_1y_3 \\ 1 & x_1 & y_4 & x_1y_4 \end{bmatrix}}_{\text{Dependent}}$$

Hence, these dont generate unique solutions.

The remaining option is $\boxed{M=2, N=2}$ which is the final answer.

$$\begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_1 & y_2 & x_1y_2 \\ 1 & x_2 & y_1 & x_2y_1 \\ 1 & x_2 & y_2 & x_2y_2 \end{bmatrix}$$

Proof on following page.

On reducing this matrix to row-echelon form.

$R_2 \to R_2 - R_1$
$$\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 0 & 0 & y_2 - y_1 & x_1(y_2 - y_1) \\ 1 & x_2 & y_1 & x_2 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \end{bmatrix}$$

$R_4 \to R_4 - R_3$
$$\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 0 & 0 & y_2 - y_1 & x_1(y_2 - y_1) \\ 1 & x_2 & y_1 & x_2 y_1 \\ 0 & 0 & y_2 - y_1 & x_2(y_2 - y_1) \end{bmatrix}$$

$R_3 \to R_3 - R_1$
$$\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 0 & 0 & y_2 - y_1 & x_1(y_2 - y_1) \\ 0 & x_2 - x_1 & 0 & (x_2 - x_1) y_1 \\ 0 & 0 & y_2 - y_1 & x_2(y_2 - y_1) \end{bmatrix}$$

$R_4 \to R_4 - R_2$
$$\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 0 & 0 & y_2 - y_1 & x_1(y_2 - y_1) \\ 0 & x_2 - x_1 & 0 & (x_2 - x_1) y_1 \\ 0 & 0 & 0 & (x_2 - x_1)(y_2 - y_1) \end{bmatrix}$$

$R_2 \leftrightarrow R_3$
$$\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 0 & x_2 - x_1 & 0 & (x_2 - x_1) y_1 \\ 0 & 0 & y_2 - y_1 & x_1(y_2 - y_1) \\ 0 & 0 & 0 & (x_2 - x_1)(y_2 - y_1) \end{bmatrix}$$

Now, because $x_1 < x_2$ & $y_1 < y_2$ all the diagonal entries are non-zero and hence this has a full column rank. The colums are linearly independent and we have a unique sol^n.