NISARG UPADHYAYA 19CS30031

① We have $||A||_2 = \max_{x \neq 0} \frac{||Ax||_2}{||x||_2}$ , $A \in \mathbb{R}^{n \times n}$

We have $\frac{||A\hat{x}||_2}{||\hat{x}||_2} \leq \max_{x \neq 0} \frac{||Ax||_2}{||x||_2}$ for any $\hat{x} \neq 0$, $\hat{x} \in \mathbb{R}^n$

$\Rightarrow \frac{||A\hat{x}||_2}{||\hat{x}||_2} \leq ||A||_2$

$\Rightarrow ||A\hat{x}||_2 \leq ||A||_2 \, ||\hat{x}||_2$ for any $\hat{x} \neq 0$, $\hat{x} \in \mathbb{R}^n$

For $\hat{x} = 0$, this is trivially true. So we have the following result:

$\Rightarrow ||Ax||_2 \leq ||A||_2 \, ||x||_2 \quad \forall x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$ — ①

Now let B be a matrix, $B \in \mathbb{R}^{n \times n}$

Let $\hat{x} = Bx$. Clearly $\hat{x} \in \mathbb{R}^n$

From ① we have,

$||A\hat{x}||_2 \leq ||A||_2 \, ||\hat{x}||_2 = ||A||_2 \, ||Bx||_2 \leq ||A||_2 \, ||B||_2 \, ||x||_2$

Thus we have

$$\frac{\|A\hat{x}\|_2}{\|x\|_2} \leq \|A\|_2 \|B\|_2$$

$$\Rightarrow \frac{\|ABx\|_2}{\|x\|_2} \leq \|A\|_2 \|B\|_2 \quad - (11)$$

Hence, using (11)

$$\|AB\|_2 = \max_{x \neq 0} \frac{\|ABx\|_2}{\|x\|_2} \leq \|A\|_2 \|B\|_2$$

It is less than $\|A\|_2 \|B\|_2$ in the general case. Hence even for the maximum value it will be less.

Hence, proved.

This holds true for Frobenius norm as well and we can prove it using CST.

Let $AB = C$, $A, B, C \in \mathbb{R}^{n \times n}$

then $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Using CST

$$|c_{ij}|^2 = \left|\sum_{k=1}^{n} a_{ik} b_{kj}\right|^2 \leq \sum_{k=1}^{n} |a_{ik}|^2 \sum_{k=1}^{n} |b_{kj}|^2 \quad -(1)$$

Hence, $\|AB\|_F^2 = \|C\|_F = \sum_{i=1}^{n}\sum_{j=1}^{n} |c_{ij}|^2$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\sum_{k=1}^{n} |a_{ik}|^2 \sum_{k=1}^{n} |b_{kj}|^2\right)$$

[Using (1)]

$$= \left(\sum_{i=1}^{n}\sum_{k=1}^{n} |a_{ik}|^2\right)\left(\sum_{j=1}^{n}\sum_{k=1}^{n} |b_{kj}|^2\right)$$

$$= \|A\|_F^2 \, \|B\|_F^2 .$$

Hence $\|AB\|_F \leq \|A\|_F \|B\|_F$

Hence, proved