NISARG UPADHYAYA 19CS30031

A7)

$A \in \mathbb{R}^{n \times n}$

Aim: To express A as LU where L is lower triangle and U is upper triangle.

$\Rightarrow$ Now we multiply A with a series of lower triangular matrices s.t. A will finally become an upper triangular matrix.

Each such L_{ij} will be responsible for converting the i,j^{th} element of A to 0.

We do this sequentially columnwise as follows:

$$\underset{A}{\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}} \longrightarrow \underset{L_{21}A}{\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ \times & \times & \times \end{bmatrix}} \longrightarrow \underset{L_{31}L_{21}A}{\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}}$$

$$\longrightarrow \underset{L_{32}L_{31}L_{21}A}{\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}}$$

Now we have

$L_{32}L_{31}L_{21}A = U$

we have $L = (L_{32}L_{31}L_{21})^{-1}$

We will show that L is indeed a lower triangular matrix.

Let X_k be the k^{th} column of A after first $k-1$ colums have been processed

$$\begin{bmatrix} (x_{1k}) \\ (x_{2k}) \\ (x_{3k}) \\ \vdots \\ (x_{nk}) \end{bmatrix}$$

Let $l_{yk} = \dfrac{x_{yk}}{x_{kk}}$ $\quad k+1 \le y \le n$

Then the matrix L_{ik} $(k+1 \le i \le n)$ is as follows

$$\begin{array}{r c}
 & \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & 0 & \ddots & \\ & & & \vdots & & \\ & & & \vdots & & \\ i^{th} \text{ row} \rightarrow & \cdots & -l_{ik} & & 1 & \\ & & & 0 & & \\ & & & 0 & & 1 \end{bmatrix} \\
 & \uparrow \\
 & k^{th} \text{ column}
\end{array}$$

Rest all entries 0.

Then we have $\hat{L}_k = L_{n,k} \cdot L_{n-1,k} \cdots L_{k+1,k}$

$$= \begin{array}{c}
\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & -l_{k+1,k} & \ddots & \\ & & & -l_{k+2,k} & & \ddots \\ & & & \vdots & & & 1 \\ & & & -l_{n-1,k} & & & & 1 \\ & & & -l_{n,k} & & & & \end{bmatrix} \\
\uparrow \\
k^{th} \text{ column.}
\end{array}$$

Multiplying all such $\hat{L}$ gives

$$\hat{L}_{n-1} \cdot \hat{L}_{n-2} \cdots \hat{L}_1 = \begin{bmatrix} 1 & & & & \\ -l_{2,1} & 1 & & & \\ -l_{3,1} & -l_{3,2} & \ddots & & \\ -l_{4,1} & -l_{4,2} & 1 & & \\ \vdots & \vdots & (-l_{k+1,k}) & \ddots & \\ \vdots & \vdots & \vdots & & 1 \\ -l_{n,1} & -l_{n,2} & (-l_{n,k}) & & (-l_{n,n-1}) & 1 \end{bmatrix}$$

And the inverse of this can be obtained by replacing all $-l_{ik}$ with l_{ik}

$(\hat{L}_{n-1} \hat{L}_{n-2} \cdots \hat{L}_1)^{-1} = 2I - (\hat{L}_{n-1} \hat{L}_{n-2} \cdots \hat{L}_1)$ which is lower triangular matrix L.