A4) (i)

$\Rightarrow$ Existence of additive identity.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad 0+0+0=0$$

Hence $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{L}$

$\Rightarrow$ Closed under addition.

Let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ & $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathcal{L}$

Then $\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} \in \mathcal{L}$

because $(x_1+y_1) + (x_2+y_2) + (x_3+y_3)$

$= (x_1+x_2+x_3) + (y_1+y_2+y_3)$

$= 0 + 0 = 0.$

$\Rightarrow$ Closed under multiplication

Let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathcal{L}$ then for some $\alpha \in \mathbb{R}$

$\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix} \in \mathcal{L}$ because

$\alpha x_1 + \alpha x_2 + \alpha x_3 = \alpha(x_1+x_2+x_3) = \alpha \cdot 0 = 0.$

As all 3 conditions are satisfied it is subspace of $\mathbb{R}^3$.

(ii) By observation a certain orthonormal basis of L are

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

The vector of all 1's is perpendicular to both.

$1_n^T A = 1_n^T B = 0$.

(This holds for any vector in plane $\mathcal{L}$)

* This directly stems from the property $x_1 + x_2 + x_3 = 0$. *

So a unit vector pependicular to the given plane $\mathcal{L} \Rightarrow \frac{1_n}{\|1_n\|} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = u$

So a reflector can be $I - 2uu^T$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$