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Report On

Modeling & Computation Lab
Lab 07

Author: Nisarg Jain - 23N0454
Course Instructor: Prof J. Venkateswaran, Prof N. Hemachandra

Abstract

Report on Lab 07 includes Question 1 and Question 2 of Lab 07 Homework. All images are taken from .ipynb snip. All the data is available in the ipynb file attached. Assumptions made during the solving of questions are written inline of the discussion for the question.

Contents

List of Figures

1 Question 1 : LP- Reformulation

Consider the following optimization problem:

$$min_{\bar{x};y} \quad 0.043x_1 + 0.027x_2 + 0.025x_3 + 0.022x_4 + 0.045x_5 - 0.0275y \tag{1}$$

subject to

$$x_1 + 3x_2 + x_3 + 7x_4 + x_5 - y \leq 100 \tag{2}$$

$$x_2 + 2x_3 + 4x_4 \geq 60 \tag{3}$$

$$\frac{2x_1 + 2x_2 + x_3 + x_4 + 5x_5}{x_1 - x_2 + x_3 - x_4 + x_5} \leq 2.8 \tag{4}$$

$$\frac{2x_1 + 15x_2 + 4x_3 + 3x_4 - 8x_5}{x_1 + x_2 + x_3 - x_5} \leq 3 \tag{5}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \tag{6}$$

$$y \in [0, 2.5] \tag{7}$$

1.1 Rewriting

Rewrite this optimization problem as a linear program and include it in your report. Did you make any assumptions when you constructed the linear program? If so, explain those assumptions.
To rewrite this as linear programming, we need the equation (4) and (5) in linear form. We do this by taking different cases of denominators of both, in total there are 4 cases.

- Case 1: $x_1 - x_2 + x_3 - x_4 + x_5 \geq 0$ and $x_1 + x_2 + x_3 - x_5 \geq 0$
  then,
  $$2x_1 + 2x_2 + x_3 + x_4 + 5x_5 \leq 2.8 * (x_1 - x_2 + x_3 - x_4 + x_5)$$
  $$-0.8x_1 + 4.8x_2 - 1.8x_3 + 3.8x_4 + 2.2x_5 \leq 0$$

  and,
  $$2x_1 + 15x_2 + 4x_3 + 3x_4 - 8x_5 \leq 3 * (x_1 + x_2 + x_3 - x_5)$$
  $$-x_1 + 12x_2 + x_3 + 3x_4 - 5x_5 \leq 0$$

- Case 2: $x_1 - x_2 + x_3 - x_4 + x_5 \leq 0$ and $x_1 + x_2 + x_3 - x_5 \leq 0$
  then,
  $$2x_1 + 2x_2 + x_3 + x_4 + 5x_5 \geq 2.8 * (x_1 - x_2 + x_3 - x_4 + x_5)$$
  $$-0.8x_1 + 4.8x_2 - 1.8x_3 + 3.8x_4 + 2.2x_5 \geq 0$$

  and,
  $$2x_1 + 15x_2 + 4x_3 + 3x_4 - 8x_5 \geq 3 * (x_1 + x_2 + x_3 - x_5)$$
  $$-x_1 + 12x_2 + x_3 + 3x_4 - 5x_5 \geq 0$$

- Case 3: $x_1 - x_2 + x_3 - x_4 + x_5 \leq 0$ and $x_1 + x_2 + x_3 - x_5 \geq 0$
  then,
$$2x_1 + 2x_2 + x_3 + x_4 + 5x_5 \geq 2.8 * (x_1 - x_2 + x_3 - x_4 + x_5)$$
$$-0.8x_1 + 4.8x_2 - 1.8x_3 + 3.8x_4 + 2.2x_5 \geq 0$$

  and,
$$2x_1 + 15x_2 + 4x_3 + 3x_4 - 8x_5 \leq 3 * (x_1 + x_2 + x_3 - x_5)$$
$$-x_1 + 12x_2 + x_3 + 3x_4 - 5x_5 \leq 0$$

- Case 4: $x_1 - x_2 + x_3 - x_4 + x_5 \geq 0$ and $x_1 + x_2 + x_3 - x_5 \leq 0$
  then,
$$2x_1 + 2x_2 + x_3 + x_4 + 5x_5 \leq 2.8 * (x_1 - x_2 + x_3 - x_4 + x_5)$$
$$-0.8x_1 + 4.8x_2 - 1.8x_3 + 3.8x_4 + 2.2x_5 \leq 0$$

  and,
$$2x_1 + 15x_2 + 4x_3 + 3x_4 - 8x_5 \geq 3 * (x_1 + x_2 + x_3 - x_5)$$
$$-x_1 + 12x_2 + x_3 + 3x_4 - 5x_5 \geq 0$$

Using these 4 cases, we can rewrite the LP as 4 LP's based on constraints, and then we can find the solution for each of these.
The 4 LP formulations are :

- Case 1:

$$min_{\bar{x};y} \;\; 0.043x_1 + 0.027x_2 + 0.025x_3 + 0.022x_4 + 0.045x_5 - 0.0275y$$

  subject to
$$x_1 + 3x_2 + x_3 + 7x_4 + x_5 - y \leq 100$$
$$x_2 + 2x_3 + 4x_4 \geq 60$$
$$x_1 - x_2 + x_3 - x_4 + x_5 \geq 0$$
$$x_1 + x_2 + x_3 - x_5 \geq 0$$
$$-0.8x_1 + 4.8x_2 - 1.8x_3 + 3.8x_4 + 2.2x_5 \leq 0$$
$$-x_1 + 12x_2 + x_3 + 3x_4 - 5x_5 \leq 0$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$
$$y \in [0, 2.5]$$

  Similarly,

- Case 2:

$$min_{\bar{x};y} \quad 0.043x_1 + 0.027x_2 + 0.025x_3 + 0.022x_4 + 0.045x_5 - 0.0275y$$

subject to

$$x_1 + 3x_2 + x_3 + 7x_4 + x_5 - y \leq 100$$
$$x_2 + 2x_3 + 4x_4 \geq 60$$
$$x_1 - x_2 + x_3 - x_4 + x_5 \leq 0$$
$$x_1 + x_2 + x_3 - x_5 \leq 0$$
$$-0.8x_1 + 4.8x_2 - 1.8x_3 + 3.8x_4 + 2.2x_5 \geq 0$$
$$-x_1 + 12x_2 + x_3 + 3x_4 - 5x_5 \geq 0$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$
$$y \in [0, 2.5]$$

- Case 3:

$$min_{\bar{x};y} \quad 0.043x_1 + 0.027x_2 + 0.025x_3 + 0.022x_4 + 0.045x_5 - 0.0275y$$

subject to

$$x_1 + 3x_2 + x_3 + 7x_4 + x_5 - y \leq 100$$
$$x_2 + 2x_3 + 4x_4 \geq 60$$
$$x_1 - x_2 + x_3 - x_4 + x_5 \leq 0$$
$$x_1 + x_2 + x_3 - x_5 \geq 0$$
$$-0.8x_1 + 4.8x_2 - 1.8x_3 + 3.8x_4 + 2.2x_5 \geq 0$$
$$-x_1 + 12x_2 + x_3 + 3x_4 - 5x_5 \leq 0$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$
$$y \in [0, 2.5]$$

- Case 4:

$$min_{\bar{x};y} \quad 0.043x_1 + 0.027x_2 + 0.025x_3 + 0.022x_4 + 0.045x_5 - 0.0275y$$

subject to

$$x_1 + 3x_2 + x_3 + 7x_4 + x_5 - y \leq 100$$
$$x_2 + 2x_3 + 4x_4 \geq 60$$
$$x_1 - x_2 + x_3 - x_4 + x_5 \geq 0$$
$$x_1 + x_2 + x_3 - x_5 \leq 0$$
$$-0.8x_1 + 4.8x_2 - 1.8x_3 + 3.8x_4 + 2.2x_5 \leq 0$$
$$-x_1 + 12x_2 + x_3 + 3x_4 - 5x_5 \geq 0$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$
$$y \in [0, 2.5]$$

In all these cases, note that denominator being 0 is not considered. Nor we can model that non equility in pyomo. Hence, it is an assumption and hope that denominator does not become 0.

1.2 Solving

The optimization problem has been in the ipynb file for all four cases.

1.3 Reporting

Report the optimal solution value, the values of variables at the optimal solution, and the activities of all constraints of the LP model

1.3.1 Optimal Solutions

- Case 1: The Optimal Solution value is 0.8523724489795922.
  The Solver Status is : ok
  The Solver Termination Condition is : optimal

- Case 2: The Optimal Solution value is 0.3595833333333333.
  The Solver Status is : ok
  The Solver Termination Condition is : optimal

- Case 1: The Optimal Solution value is None.
  The Solver Status is : ok
  The Solver Termination Condition is : Other

- Case 1: The Optimal Solution value is None.
  The Solver Status is : ok
  The Solver Termination Condition is : Other

1.3.2 Variable Values

- Decision Variables for Case 1:
  x1 = 0.0
  x2 = 0.0
  x3 = 19.5918367346939
  x4 = 5.20408163265306
  x5 = 7.04081632653061
  y = 2.5

- Decision Variables for Case 2:
  x1 = 0.0
  x2 = 0.0
  x3 = 1.66666666666667

x4 = 14.1666666666667
x5 = 1.66666666666666
y = 2.5

- Decision Variables for Case 3 are all None.
- Decision Variables for Case 4 are all None.

1.3.3 Constraint Values

- Constraint Values for Case 1:
  Constraint 1: 60.56122448979593
  Constraint 2: 60.00000000000004
  Constraint 3: 21.42857142857145
  Constraint 4: 12.55102040816329
  Constraint 5: -4.796163466380676e-14
  Constraint 6: 2.842170943040401e-14
- Constraint Values for Case 2:
  Constraint 1: 100.00000000000023
  Constraint 2: 60.00000000000014
  Constraint 3: -10.83333333333337
  Constraint 4: 9.992007221626409e-15
  Constraint 5: 54.5000000000001
  Constraint 6: 35.83333333333347
- Problem is not Optimal.
- Problem is not Optimal.

1.4 Assumptions

Explain how you would modify the model if no assumptions are made during the construction of the linear program. Note: You need not solve the modified model. If we don't make any assumptions than we need to remove the assumption that denominator of the constraint will not be zero. We will need to constrain using the strict inequality of 0. To handle this we can take a really small $\epsilon = 0.0001$ and make an inequality using this. We would than model the problem as: Example Case:

$$min_{\bar{x};y} \;\; 0.043x_1 + 0.027x_2 + 0.025x_3 + 0.022x_4 + 0.045x_5 - 0.0275y$$

subject to

$$x_1 + 3x_2 + x_3 + 7x_4 + x_5 - y \leq 100$$

$$x_2 + 2x_3 + 4x_4 \geq 60$$

$$x_1 - x_2 + x_3 - x_4 + x_5 \geq \epsilon$$

$$x_1 + x_2 + x_3 - x_5 \geq \epsilon$$
$$-0.8x_1 + 4.8x_2 - 1.8x_3 + 3.8x_4 + 2.2x_5 \leq 0$$
$$-x_1 + 12x_2 + x_3 + 3x_4 - 5x_5 \leq 0$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$
$$y \in [0, 2.5]$$

Similar thing can be done for other cases as well.

2 Question 2

ABC Industries Ltd. has two refineries situated in India (say, Domestic (D) and Export (E)). Each refinery makes two products: #1 and #2. Standard profit contributions towards revenue are Rs. 10 per unit and Rs. 15 per unit, respectively. Each factory uses a two-stage process: Distillation and Treatment. The following table summarises the processing capacity (in hours per week) of both D and E refineries:

	Refinery D	Refinery E
Distillation	80	60
Treatment	60	75

Table 1: Figure 1

ABC Industries Ltd. has 120 units of raw crude oil available each week, which is processed into two products (#1 and #2) as the finished product. Below is the time required (in hours) for each type of product (#1 and #2) in the Distillation and Treatment process.

	Refinery D	Refinery D	Refinery E	Refinery E
Product	#1	#2	#1	#2
Distillation	4	2	5	3
Treatment	2	5	5	6

Table 2: Figure 1

2.1 Decentralized

Formulate the optimization model of Refinery D and E at the decentralized level and describe your formulation along with the variables, objective function, and constraints. Solve your formulated model of product mix to maximize profit using Pyomo and report your solution. Assume that 50-50 % is the raw crude oil distribution among D and E refineries.

To model this first since we are given 50-50 distribution. Then each of the refineries will get 60 60 units of crude oil. Using this limitation, we create our LP. For Refinery D, notice that, it has about 80 hours of capacity of Distillation and 60 hours for Treatment. If we decide our decision variables, as amount of product #1 to produce and amount of product #2 to produce as dx_1 and dx_2 then,
Objective is to maximize the profit which will be $10 * dx_1 + 15 * dx_2$
subject to:

- Constraint 1: Total Time for Distillation for D for products must be less than 80.
  $4 * dx_1 + 2 * dx_2 <= 80$
- Constraint 2: Total Time for Treatment for D for products must be less than 60.
  $2 * dx_1 + 5 * dx_2 <= 60$
- Constraint 3: Total amount of product that can be produced must be less than 60.
  $dx_1 + dx_2 <= 60$
- Constraint 4: Also both must be greater than 0.
  $dx_1 >= 0$ and $dx_2 >= 0$

Similarly we can do for the Refinery E:

- $5 * ex_1 + 3 * ex_2 \leq 60$
- $5 * ex_1 + 6 * ex_2 \leq 75$
- $ex_1 + ex_2 \leq 60$
- $ex_1 \geq 0$ and $ex_2 \geq 0$

2.1.1 Solution

- For Refinery D our solution is :
  Profit = 250.0
  Decision Variables:
  Product 1 = 17.5
  Product 2 = 5.0
- For Refinery E our solution is :
  Profit = 187.5
  Decision Variables:
  Product 1 = 0
  Product 2 = 12.5

- Total Profit is 437.5.

2.2 Centralized

Formulate the optimization model of Refinery D and E at a centralized level. Solve your formulated product mix model to maximize profit for ABC Industries Ltd. Report your solution and describe your formulation.

For the centralized case most of our formulation regarding the refinery specific capacity will remain the same. Just our distribution of crude oil will now be flexible in both the refineries.

Our decision variable will be: dx_1, dx_2, ex_1, ex_2 for the quantities of Product 1 and Product 2 amount of D and Product 1 and Product 2 amount for E.

Our objective function will be to maximize: $10 * dx_1 + 15 * dx_2 + 10 * ex_1 + 15 * ex_2$

Constraints will be:

- $4 * dx_1 + 2 * dx_2 <= 80$
- $2 * dx_1 + 5 * dx_2 <= 60$
- $dx_1 >= 0$ and $dx_2 >= 0$
- $5 * ex_1 + 3 * ex_2 \leq 60$
- $5 * ex_1 + 6 * ex_2 \leq 75$
- $ex_1 \geq 0$ and $ex_2 \geq 0$
- The only constraint which will change is total amount of products must be less than 120.
  $dx_1 + dx_2 + ex_1 + ex_2 \leq 120$

2.2.1 Solutions

Profit = 437.5
Decision Variables:
Product 1 for D= 17.5
Product 2 for D = 5.0
Product 1 for E = 0.0
Product 2 for E = 12.5

Which is surprising because the answer is same as when we split it by 50-50 distribution.

2.3 Best Allocation

We see that even though we remove the constraint of distribution still the maximum profit we are able to achieve is still 437.5. So to plot the profit with respect to distribution, I created about 100 ratio values to distribute the crude oil on and found the following results.
We can argue that if we supply less than 20% of crude oil to the Refinery D than our profits are not optimal. Similar is the case for more than 85%. Between these two r's our profits are optimal. This is the case because our refinery's are capacity constrained and only become supply constrained when r is less than 20% and more than about 85%

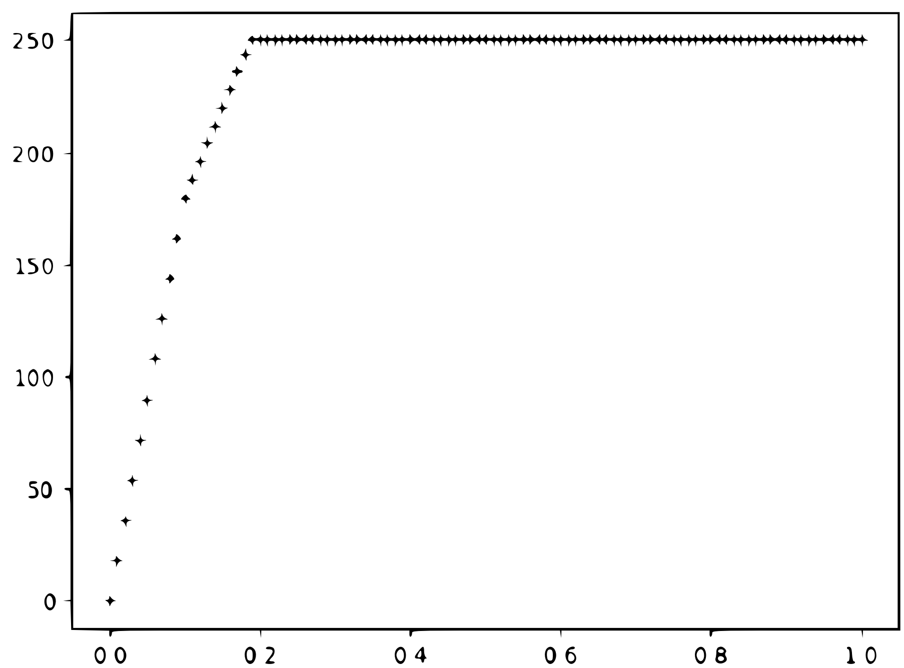

Figure 1: The Graph of Profit of Refinery D w.r.t r from [0,1]

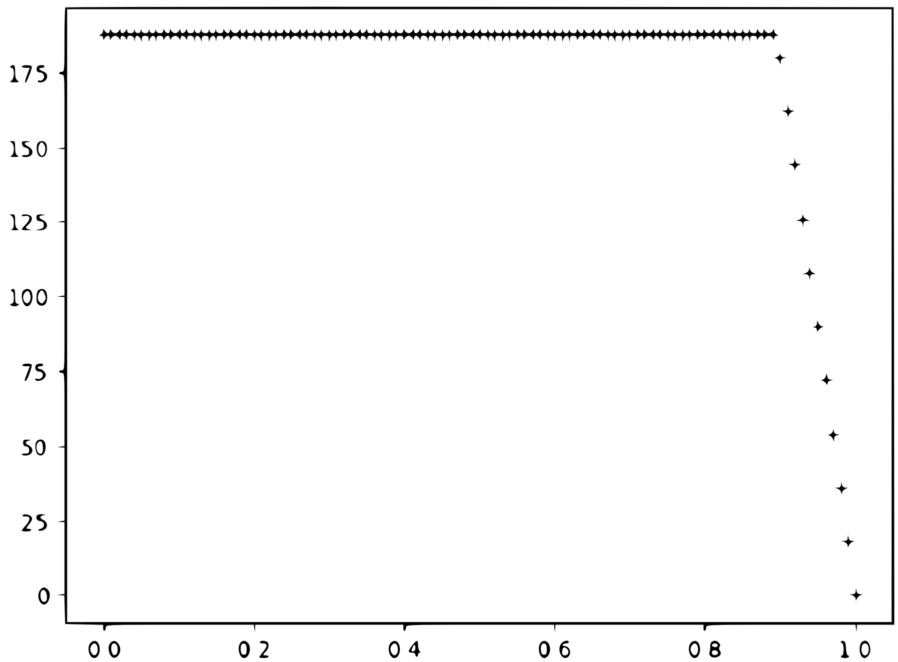

Figure 2: The Graph of Profit of Refinery E w.r.t r from [0,1]

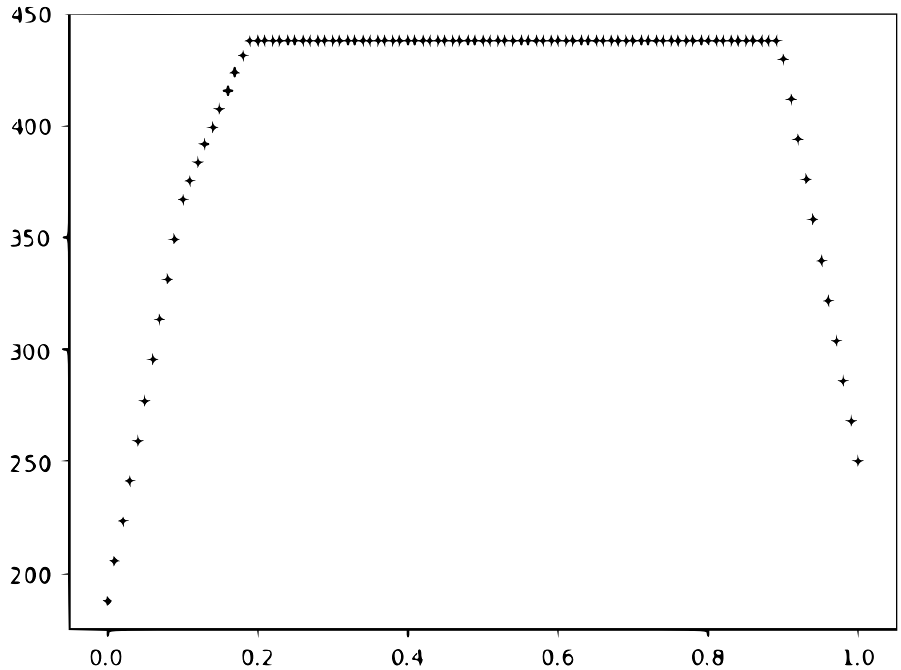

Figure 3: The Graph of Total Profit of D and E w.r.t r from [0,1]

2.4 Managerial Insights

To formulate the Dual LP for Refinery D in the first problem, we see that our LP in primal form is:

$$max(10 * dx_1 + 15 * dx_2)$$

subject to:

$$4 * dx_1 + 2 * dx_2 <= 80$$

$$2 * dx_1 + 5 * dx_2 <= 60$$

$$dx_1 + dx_2 <= 60$$

$$dx_1 >= 0$$

and

$$dx_2 >= 0$$

using this we calculate the Dual:

$$min(80 * x_1 + 60 * x_2 + 60 * x_3)$$

subject to:

$$4 * x_1 + 2 * x_2 + x_3 >= 10$$

$$2 * x_1 + 5 * x_2 + x_3 >= 15$$

$$x_1 >= 0$$

and

$$x_2 >= 0$$

and

$$x_3 >= 0$$

Solving it we get,
profit = 250.0
Decision Variables:
x1 = 1.25
x2 = 2.5
x3 = 0.0
Constraints:
Constraint 1 = 10.0
Constraint 2 = 15.0
which is the same objective value as primal problem.
As a manager this gives us a different way of looking at a problem. In Primal we were concerned about the max profit which was at industry level. In dual instead we are required to minimize the time of production with respect to process itself. We minimize

the time constrained on the fact that profit should be greater than 10 and 15. This way of looking is beneficial on process level, whereas for financial analysts of the company primal problem is beneficial.