Lab 11: Computational Quadratic Optimization

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Objective: In this lab, you will get familiarized with computational quadratic optimization by modeling real-life standard problems. To solve quadratic problems, use the existing Pyomo framework with **cbc** solver or qpsolvers or gurobipy instead of GLPK solver, as the nature of the objective function is quadratic (or can be non-convex) in nature.

Instructions: Below are some instructions. Please go through them carefully:

- Along with the .ipynb file, you must submit a report (**.pdf**, **not a photo-captured image** as .pdf file) file that answers all the questions from lab 11.
- Also, explicitly mention the assumptions used throughout your modeling technique.
- Use Pyomo, Numpy, qpsolvers, gurobipy and other library documentation if you need help.
- An introductory ready-to-use Pyomo framework along with instructions is available as an Introduction to Data Handling and Quadratic Optimization to solve optimization problems for a given data set. Also, use gurobi framework to solve standard optimization problems. Use this .ipynb file(s), save it to your iitb Gdrive, and understand how to handle and utilize the given dataset for optimization.
- Your task is to model the scenarios given below as optimization formulation and report your answers to questions.
- Use the traditional approach to name your files for submission:
  - **ROLLNUMBER_IE507_Lab11_Submission.ipynb** (can be multiple files)
  - **ROLLNUMBER_IE507_Lab11_Report.pdf**

Quadratic Programming

An optimization formulation of the type (1) below is called a quadratic program due to a quadratic objective function and linear constraints.

$$
\begin{aligned}
\min_{x} :& \frac{1}{2}x^T P x + q^T x + s \\
\textit{subject to} :& \\
& Gx \leq h
\end{aligned}
$$

where, $x \in \mathcal{R}^n$ is the decision variables. Further, $P \in \mathcal{R}^{n\times n}$, $s \in \mathcal{R}$, $G \in \mathcal{R}^{m\times n}$ where n is the number of decision variables and m is the number of constraints in the system.

We stick to the case when matrix P is positive definite; then the objective function $\frac{1}{2}x^T Px + q^T x + s$ is convex in x and hence has a unique global minimum. Also, recall that the positive definiteness of P is equivalent to all its eigenvalues being positive.

Convexity is a geometric aspect and convex functions have 'curvature', whose orientation is in a certain sense 'ordered'; plot a nonconvex function, say $\sin x$ and $\cdots$. We skip zeroth and first order characterisations of convexity. Also since, s independent to decision variable x, so we minimizing problem (1) above is equivalent to minimizing objective function $\frac{1}{2}x^T Px + q^T x$.

A (computational) consequence of this convexity is that any local minima is a global minima; so, any descent algorithm will eventual yield global minima.

Some further background: Recall, that if a Linear Program has an optimal value, then its dual is a LP and has an optimal value and both values are the same. Now, one can ask a question, about the class of optimization problems with such results; turns out that for finite dimensional optimization problems, convex programs have this property (under mild regularity conditions). Convex quadratic programs, being convex programs, have this property; we skip the details.

Converting non-linear program to linear program

Consider a standard example of absolute value given below:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise .} \end{cases} \tag{1}$$

It is known that the absolute function is non-linear and can be converted into a linear program with some manipulations. We will now consider the absolute value function appearing in the objective function in a standard optimization problem **(OP)**:

$$\begin{aligned} \min\ & |x_1| + |x_2| + |x_3| + |x_4| + |x_5| \\ \text{s.t. } & 85x_1 + 92x_2 + 45x_3 + 27x_4 + 31x_5 \geq 1 \\ & 92x_1 + 54x_2 + 22x_3 + 20x_4 + 7x_5 \geq 1 \\ & 96x_1 + 67x_2 + 29x_3 + 20x_4 + 11x_5 \geq 1 \\ & -91x_1 - 57x_2 - 33x_3 - 23x_4 - 12x_5 \geq 1 \\ & -99x_1 - 75x_2 - 26x_3 - 24x_4 - 41x_5 \geq 1 \\ & -98x_1 - 99x_2 - 57x_3 - 45x_4 - 65x_5 \geq 1 \end{aligned}$$

In this optimization problem **(OP)**, we wish to minimize the sum of absolute values of the decision variables. Such problems are helpful when the decision variables might take both positive and negative values, and we want to optimize the magnitudes of the decision variables. The following two approaches traditionally convert the above **(OP)** into a standard linear program.

Approach 1: OP1

We can use a substitution of the absolute function values of the variables to new variables as $u_i = |x_i|,\ \forall i \in \{1, \ldots, 5\}$. With this substitution, note that we have $u_i \geq 0,\ \forall i \in \{1, \ldots, 5\}$. Also, the following inequalities are satisfied: $x_i \leq u_i, -x_i \leq u_i,\ \forall i \in \{1, \ldots, 5\}$. Thus we can transform the problem **(OP)** as the following optimization problem **(OP1)**, where the absolute values of x_i are replaced with u_i and constraints related to the new variables u_i are introduced. Thus we have the problem **(OP1)**:

$$\begin{aligned} \min\ & u_1 + u_2 + u_3 + u_4 + u_5 \\ \text{s.t. } & 85x_1 + 92x_2 + 45x_3 + 27x_4 + 31x_5 \geq 1 \\ & 92x_1 + 54x_2 + 22x_3 + 20x_4 + 7x_5 \geq 1 \\ & 96x_1 + 67x_2 + 29x_3 + 20x_4 + 11x_5 \geq 1 \\ & -91x_1 - 57x_2 - 33x_3 - 23x_4 - 12x_5 \geq 1 \\ & -99x_1 - 75x_2 - 26x_3 - 24x_4 - 41x_5 \geq 1 \\ & -98x_1 - 99x_2 - 57x_3 - 45x_4 - 65x_5 \geq 1 \\ & u_i \geq x_i,\ \forall i \in \{1, \ldots, 5\} \\ & u_i \geq -x_i,\ \forall i \in \{1, \ldots, 5\} \\ & u_i \geq 0,\ \forall i \in \{1, \ldots, 5\}. \end{aligned} \tag{2}$$

Note that the number of decision variables in **(OP1)** is twice the number of decision variables in **(OP)**. However the objective function and the constraints in **(OP1)** are now linear, and thus **(OP1)** is a linear program.

Approach 2

In this approach, we will use some fundamental properties of real numbers. Indeed, for any real number x, we can write $x = a - b$ where $a \geq 0, b \geq 0$. Hence, we can substitute $|x| = |a - b|,\ a \geq 0, b \geq 0$.

Now we will consider another interesting property of real numbers: $|u - v| = u + v \iff u \geq 0, v \geq 0, uv = 0$.

Using this property we can write $|x| = a + b,\ a \geq 0,\ b \geq 0$. Note that this replacement will imply that $ab = 0$.

Thus we can transform the optimization problem **(OP)** into a new optimization problem **(OP2)** where $x_i = a_i - b_i$, and $|x_i| = a_i + b_i$, $\forall i \in \{1, \ldots, 5\}$ where $a_i \geq 0, b_i \geq 0, \forall i \in \{1, \ldots, 5\}$. Thus we have the new optimization problem **(OP2)**:

$$
\begin{aligned}
\min\ & (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + (a_5 + b_5) \\
\text{s.t. } & 85(a_1 - b_1) + 92(a_2 - b_2) + 45(a_3 - b_3) + 27(a_4 - b_4) + 31(a_5 - b_5) \geq 1 \\
& 92(a_1 - b_1) + 54(a_2 - b_2) + 22(a_3 - b_3) + 20(a_4 - b_4) + 7(a_5 - b_5) \geq 1 \\
& 96(a_1 - b_1) + 67(a_2 - b_2) + 29(a_3 - b_3) + 20(a_4 - b_4) + 11(a_5 - b_5) \geq 1 \\
& -91(a_1 - b_1) - 57(a_2 - b_2) - 33(a_3 - b_3) - 23(a_4 - b_4) - 12(a_5 - b_5) \geq 1 \\
& -99(a_1 - b_1) - 75(a_2 - b_2) - 26(a_3 - b_3) - 24(a_4 - b_4) - 41(a_5 - b_5) \geq 1 \\
& -98(a_1 - b_1) - 99(a_2 - b_2) - 57(a_3 - b_3) - 45(a_4 - b_4) - 65(a_5 - b_5) \geq 1 \\
& a_i \geq 0,\ b_i \geq 0,\ \forall i \in \{1, \ldots, 5\}.
\end{aligned}
$$

Question 0. Objective function with absolute value function

Consider the linear program formulation for **(OP)** non-linear program into **(OP1)** and **(OP2)**.

1. Create python DataFrame coeffOP1 and coeffOP2 for both **(OP1)** and **(OP2)** respectively, containing the coefficients of the objecive function and constraints.Further, create a Pyomo Framework using these created DataFrame(s) for both **(OP1)** and **(OP2)**.
2. Report the optimal objective function value, values of the decision variables x_i. Create a table representing u_i from **(OP1)** and a_i, b_i from **(OP2)**. Compare the x_i values obtained from solving problem **(OP1)** and problem**(OP1)**. Comment on your observations for the decision value of optimal decision variables obtained in both **(OP1)** and **(OP2)**.
3. Both **(OP1)** and **(OP2)** have same number of constraints. Comment on the activity of constraints of one formulation compared to other formulation.

Question 1. Best Fit: Minimizing Least Square Estimates

You are an asset manager for a large cooperation bank; over the year, the market has observed volatility in interest rates of Sovereign Bond issued by the central bank. A data set for interest rate has been provided to you for the last few years. Your team is interested in forecasting the central bank's future interest rate for these bonds based on historical information. Two factors, $\tilde{R}$ and R, significantly contribute to determining interest rates. $\tilde{R}$ represents the export-import ratio, and R is the rating from the international rating agency. So, as a rule of thumb, the bank feels that a linear model of the form $r = b_0 + b_1\tilde{R} + b_2 R$ would reasonably perform well for estimation.

You would like to select the "best" linear model in some sense. If you knew the three parameters b_0, b_1, and b_2, the observations $i \in \{1, 2, \cdots 20\}$ in the data would each provide a forecast of the interest rate as follows:

$$\hat{r}_i = b_0 + b_1\tilde{R} + b_2 R$$

However, since b_0, b_1, and b_2 cannot, in general, be chosen so that the actual interest rate r_i is precisely equal to the forecast interest rate $\hat{r}_i$ for all observations, you would like to minimize the residuals $\epsilon_i = r_i - \hat{r}_i$.

1. Write a general optimization problem using the actual objective provided in the description in matrix notations. You may assume that $b_0 \geq 0$ and b_1, b_2 do not have bounds on them. Is the optimization problem linear? Explain.

2. Solve the above formulation using the Pyomo/qpsolver framework and report your optimal solution.

3. Assume you are interested in minimizing the absolute deviation between actual and predicted interest rates. How will you modify the above formulation? It is known that such an approach will result in a non-linear program. Use **(OP1)** or **(OP2)** approach to determine decision variables.

4. Plot your given data along with the predicted data using your decision variables from mean square deviation and absolute deviation.

Question 2. Policy Design: Determining Pricing [1]

Note: Make sure the dimensional consistency and unit consistency are maintained for objective and constraint formulations.

Suppose you are employed as an auditor by the government to a leading Oil and Gas Company. As an auditor, you are bound by the government to design policies that the company needs to follow. The product portfolio for the company includes three finished products available for consumption by consumers as P_1, P_2, and P_3. The company acquires some raw material and converts it into by-products B_1 and B_2, which further are processed to produce P_1, P_2 , P_3^A and P_3^B . It is estimated that 600000 and 750000 (in tonnes) of B_1 and B_2 will be available for finished production in the coming year. Table 1 below provides the content of B_1 and B_2 (in %) in finished products.

	B_1	B_2
P_1	4 %	9 %
P_2	80 %	2 %
P_3^A	35 %	30 %
P_3^B	25 %	40 %

Table 1: Percentage of content of B_1 and B_2 in finished product

As an auditor, you are also responsible for cost accounting and price determination based on market situation. From experience, it is known that price and demand are relatively sensitive, and Table 2 below provides a relationship between price and demand as the coefficient of elasticity.

P_1	P_2	P_3^A	P_3^B	P_3^A to P_3^B	P_3^B to P_3^A
0.4	2.7	1.1	0.4	0.1	0.4

Table 2: Price (cross) elasticity to demand for finished products

Herein, the market study suggests that prices and demand are sensitive to each other. Suppose e represents the elasticity of demand for a product, i.e., how sensitive your demand is against the price of that particular product (Δ change in demand $w.r.t$ Δ change in price). Further, e_{ij} denotes the cross elasticity of product i to product j wherein the demand of product i impacts the price of product j (Δ change in demand for i $w.r.t$ Δ change in price of j).

From the previous year's data, consumption and price for each product are given in Table 3. Capacity constraints for the production of finished products from by-products restrict production of finished products. Also, to ensure a competitive advantage in the market, it is highly undesirable to allow certain price index

[1] Louwes, Stephanus Louwe, John CG Boot, and S. Wage. " A Quadratic-programming Approach to the Problem of the Optimal Use of Milk in the Netherlands." Journal of Farm Economics 45.2 (1963): 309-317.

	Consumption (in 1000 tons)	Price (in $/ton)
P_1	4820	297
P_2	320	720
P_3^A	210	1050
P_3^B	70	815

Table 3: Price and Demand of last year for finished products

to rise. Therefore, this limitation simply demands that the new prices must be such that the total cost of last year's revenue from consumption would not be increased. Your objective is to determine the optimal price of the product portfolio along with the production of finished products for the company under market conditions, given that capacity constraints are satisfied for maximum revenue.

Hint:

Use the following relationship(s):

$$\frac{dx_i}{x_i} = -e_i \frac{dp_i}{p_i} + e_{ij} \frac{dp_{ij}}{p_{ij}}$$

is a differential equation representing the relationship between price and elasticity. **Note**: To use the above expressions in constraints and objective functions will result in non-linearities. You can convert the above differential equation into difference equation as:

$$\frac{x_i - \bar{x}_i}{\bar{x}_i} = -e_i \frac{p_i - \bar{p}_i}{\bar{p}_i} + e_{ij} \frac{p_{ij} - \bar{p}_{ij}}{\bar{p}_{ij}}$$

1. Write your optimization formulation from the above description to determine the optimal price for the product portfolio. Describe your formulation (constraints, objective functions, and support for decision variables) and assumption along with the necessary assumption used.
2. Your formulation by nature involved a quadratic objective resulting quadratic optimization problem (due to price and quantity both being decision variables). Suggest how you will modify the above formulation to reduce your complexity. (**Hint:** Try to difference equation!!).
3. Suggest how you can check the nature of P for your formulation and comment on observation about the above formulation based on nature of P.
4. Model your formulation as a Quadratic Optimization Problem and solve it using *qpsolvers* or *gurobi* solver or any other appropriate approach. Report optimal pricing for product portfolio.
5. Imagine that change in disposable income can vary -10 % to 5 % from the previous year to the new year, thus leading to a rise/fall in inflation as consumers are willing to pay more/less. Plot a curve to show how sensitive your revenue from consumption is against changes in disposable income. Comment on the observations made from the plot.

Question 3. Markowitz Portfolio Optimization: An Introduction to Stochastic Programming

Build-up to Optimal Portfolio

Assume you are an asset manager, and your task is to make money from initial investment. To do so, you create a portfolio from available stocks to maximize your return at the time of dis-investment. A financial portfolio is usually a collection of investments in stocks, bonds, commodities, cash, and cash equivalents, including closed-end funds and exchange-traded funds (ETFs).

Let us say that there are n risky assets available for investment, which provides a random return at dis-investment given by vector $R = \begin{bmatrix} R_1 & R_2 & R_3 & . & . & . & R_n \end{bmatrix}$ with known mean and covariance of return as

$$\mathcal{M} = \mathbb{E}\left[R\right]$$

and

$$\begin{aligned} \Sigma &= cov(R, R) \\ &= \mathbb{E}\left[[R - \mathcal{M}][R - \mathcal{M}]^T\right] \end{aligned}$$

As a portfolio manager, your task is to identify $\Omega = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 & . & . & . & \omega_n \end{bmatrix}$, $\omega_i \geq 0 \; \forall i \in \{1, 2, ..., n\}$ such that

$$\sum_{i=1}^{n} \omega_i = 1$$

Therefore, portfolio's return will be given as $R_\Omega = \Omega^T R$. Further, the mean (expected) return for the portfolio is given as $\mathcal{M}_\Omega$ and variability in return as Σ_Ω, which are defined as:

$$\begin{aligned} \mathcal{M}_\Omega &= \mathbb{E}[R_\Omega] \\ &= \mathbb{E}[\Omega^T R] \\ &= \Omega^T \mathcal{M} \end{aligned}$$

$$\begin{aligned} \Sigma_\Omega &= Var(R_\Omega) \\ &= Cov(\Omega^T R, \Omega^T R) \\ &= \mathbb{E}\left[[R_\Omega - \mathcal{M}_\Omega][R_\Omega - \mathcal{M}_\Omega]^T\right] \\ &= \Omega^T \Sigma \Omega \end{aligned}$$

Strategy

As a rationale, your strategy is supposed to identify Ω, which provides higher expected returns, $\mathcal{M}_\Omega$, and Σ_Ω. This can be achieved using *risk minimization*, where you try to minimize your variance subject to conditions that your target (mean) returns as Ω_0. Unfortunately, there is always a trade-off between expected return and risk associated with your assets. An ideal strategy is to balance this trade-off by maximizing your return and minimizing your associated risk while deciding Ω.

Let Σ_0 be the minimum risk associated possible from our Ω, you can maximize your (mean) return. This approach is known as *Expected Return Maximization*. Below are two optimal formulations used to identify Ω using Risk Minimization and Expected Return Maximization:

Risk Minimization

$$\begin{aligned} &\min_{\Omega} : \frac{1}{2} \Omega^T \Sigma \Omega \\ &\textit{subject to}: \\ &\qquad \Omega^T \mathcal{M} \geq \mathcal{M}_0 \\ &\qquad \Omega^T \mathbf{1}_n = 1 \\ &\qquad \Omega \geq \underset{\sim}{0} \end{aligned}$$

Expected Return Maximization

$$\begin{aligned}\max_{\Omega} :& \Omega^T \mathcal{M} \\ \textit{subject to} :& \\ & \Omega^T \Sigma \Omega \leq \Sigma_0 \\ & \Omega^T \mathbf{1}_n = 1 \\ & \Omega \geq \underset{\sim}{0}\end{aligned}$$

Risk Aversion Optimization

Above, formulation of *Risk Minimization* assumes a certain threshold return, $\mathcal{M}_0$. Identifying this threshold is challenging, making it difficult to determine the optimal policy. A similar case is with *Expected Risk Maximization* where threshold risk Σ_0 is to be assumed. Thus, a *utility function* can be formulated using risk minimization and expected return maximization, such that it maximizes our utility from the portfolio R_Ω given below:

$$\begin{aligned}\max_{\Omega,\lambda} :& \left[\mathbb{E}[R_\Omega] - \frac{1}{2}\lambda Var(R_\Omega)\right] = \Omega^T \mathcal{M} - \frac{1}{2}\lambda \Omega^T \Sigma \Omega \\ \textit{subject to} :& \\ & \Omega^T \mathbf{1}_n = 1 \\ & \Omega \geq \underset{\sim}{0}\end{aligned}$$

The above formulation is a *Risk Aversion Optimization*. It is to be noted cautiously that all the portfolio selection problems above are quadratic formulations. A rich theory has already been developed to study these problems in classical literature. The above formulations are stochastic optimization problems where random variables (in the form of returns) are given with certain probabilities. We transformed these stochastic optimization problems into a more deterministic setup by considering the mean (expected return) and variance (variability in stocks) nature of portfolio selection. For brevity, we will refrain from more details about this portfolio selection problem; you can refer to *Investment Science by David G. Luenberger* for more details.

Question 3.A. Three- Portfolio Analysis: Visualizing Trade-off

Given below is a sample return and volatility of three stocks. You are required to perform the following tasks and report your observations.

$$\mathcal{M} = \begin{bmatrix} 4.27\% & 0.15\% & 2.85\% \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 10\% & 0.18\% & 0.11\% \\ 0.18\% & 10.44\% & 0.26\% \\ 0.11\% & 0.26\% & 14.11\% \end{bmatrix}$$

1. Recall your simulation-based best search in lab 06, where you performed a search for an optimal solution using a random search. Using the above data, use such an approach to determine a feasible region in a risk-return plot from *Risk Minimization* problem. Further, determine the Pareto-Frontier boundary for the trade-off. Use appropriate $\mathcal{M}_0$ based on your choice.

2. Based on your understanding, how can you solve the *Expected Return Maximization* formulation without using Lagrange's approach? Explain.

3. Formulate your optimization problem using the above data to perform quadratic optimization for *Risk Aversion*. Explain the significance of λ in the context of portfolio optimization. Also, plot your trajectory for Pareto-Frontier by assuming $\lambda = 10^{\frac{5\nu}{N}-1}$ where $\nu \in \{1, 2, \cdots N\}$. Set $N = 100$

4. Assume that the Ω is unrestricted. What will be some managerial insights that can be developed from optimal solutions with this relaxation?

Question 3.B. Taxation Planning: Trade-off for growth and stability [2] [3]

It is required to determine optimum levels for various state government taxes that minimize instability while meeting constraints on growth rates over time. Seven different taxes are considered: sales, motor fuel, alcoholic beverages, tobacco, motor vehicle, personal income, and corporate taxes. State government finance is based on the assumption of predictable and steady growth of each tax over time. Instability in tax revenue is measured by the degree to which the actual revenue differs from predicted revenue.

Using past data, a regression equation can be determined to measure the growth in tax revenue over time. Let s be the tax rate for a particular tax and S_t the expected tax revenue from this tax in year t. Then, the regression equation used is

$$log_e S_t = a + bt + cs$$

where a, b, c are parameters to be determined using past data to give the closest fit. Data for the past 10 years from a state is used for this parameter estimation. The parameter c can only be estimated if the rate s for that tax has changed during this period; this has happened only for the motor fuel and the tobacco taxes. The best-fit parameter values for the various taxes are given below in Table 4. The annual growth rate is the regression coefficient b multiplied by 100 to convert it to percent.

For 2024, the tax revenue from each tax as a function of the tax rate can be determined by estimating

Tax	a	b	c
Sales	12.61	0.108	
Motor Fuel	10.16	0.020	0.276
Alcoholic Beverages	10.97	0.044	
Tobacco	9.79	0.027	0.102
Motor Vehicle	10.37	0.036	
Personal Income	11.89	0.160	
Corporate	211.09	0.112	

Table 4: Regression Coefficients

the tax base. This data, available with the state, is given below in Table 5.

If s_j is the tax rate for tax j in 2024 as a fraction, x_j = tax revenue collected in 2024 in crore rupees for the

Tax	**Tax Base**
Sales	34329
Motor Fuel	3269
Alcoholic Beverages	811
Tobacco	702
Motor Vehicle	2935
Personal Income	30809
Corporate	4200

Table 5: Tax revenue generated from various taxes

j^{th} tax is expected to be: (tax base for tax j)s_j. Assume that the decision variable for the state government

[2] Murty, Katta G. Optimization for Decision Making Linear and Quadratic Models. Springer Science+ Business Media, 2010.
[3] White, Fred C. "Trade-off in growth and stability in state taxes." National Tax Journal 36.1 (1983): 103-114.

is to determine x_j for $j \in \{1, 2, \cdots 7\}$, let $x = (x_1, x_2, \cdots x_7)^T$ such that total tax revenue is $\sum_{j=1}^{7} x_j$. Then, the variability or instability in this revenue is measured by the quadratic function $Q(x) = x^T V x$ where V, the variance-covariance matrix estimated from past data,

$$V = \begin{bmatrix} 0.00070 & -0.00007 & 0.00108 & -0.00002 & 0.00050 & 0.00114 & 0.00105 \\ -0.00007 & 0.00115 & 0.00054 & -0.00002 & 0.00058 & -0.00055 & 0.00139 \\ 0.00108 & 0.00054 & 0.00279 & 0.00016 & 0.00142 & 0.00112 & 0.00183 \\ -0.00002, & -0.00002 & 0.00016 & 0.00010 & 0.00009 & -0.00007 & -0.00003 \\ 0.00050 & 0.00058 & 0.00142 & 0.00009 & 0.00156 & 0.00047 & 0.00177 \\ 0.00114 & -0.00055 & 0.00112 & -0.00007 & 0.00047 & 0.00274 & 0.00177 \\ 0.00105 & 0.00139 & 0.00183 & -0.00003 & 0.00177 & 0.00177 & 0.00652 \end{bmatrix}$$

The problem is to determine the vector x that minimizes $Q(x)$, subject to several constraints. One of the constraints is that the total expected tax revenue should be $T = 3300$ crore rupees. The second constraint is that a specified growth g in the total revenue should be maintained. It can be assumed that this overall growth rate is the function $\sum_{i=1}^{7} \frac{x_j b_j}{T}$, which is a weighted average of the growth rates of the various taxes. We would like to solve the problem by treating g as a non-negative of the taxes.

The other constraints are lower and upper bounds on tax revenue from tax x_j; these are of the form $0 \leq x_j \leq u_j; \forall j \in j \in \{1, 2, \cdots 7\}$ is twice the 2023 revenue from tax j where,

$$u = \left[2216, 490, 195, 168, 95, 2074, 504\right]$$

.

1. Formulate and write the above model description as a Quadratic Program. Using the tax base information above, determine the optimal tax rates for 2024, assuming $g = 9\%$ and $g = 13\%$, and report your solution.

2. Perform *Risk Aversion* optimization for the above formulation assuming $\lambda = 10^{\frac{5\nu}{N}-1}$ where $\nu \in \{1, 2, \cdots N\}$ and setting $N = 100$. You can use any approach of quadratic optimization or simulation-based best search.