Chapter 11
Series

11.1 Arithmetic Series

Initial term: a_1
Nth term: a_n
Difference between successive terms: d
Number of terms in the series: n
Sum of the first n terms: S_n

1184. $a_n = a_{n-1} + d = a_{n-2} + 2d = \ldots = a_1 + (n-1)d$

1185. $a_1 + a_n = a_2 + a_{n-1} = \ldots = a_i + a_{n+1-i}$

1186. $a_i = \dfrac{a_{i-1} + a_{i+1}}{2}$

1187. $S_n = \dfrac{a_1 + a_n}{2} \cdot n = \dfrac{2a_1 + (n-1)d}{2} \cdot n$

11.2 Geometric Series

Initial term: a_1
Nth term: a_n
Common ratio: q
Number of terms in the series: n
Sum of the first n terms: S_n
Sum to infinity: S

1188. $a_n = qa_{n-1} = a_1 q^{n-1}$

1189. $a_1 \cdot a_n = a_2 \cdot a_{n-1} = \ldots = a_i \cdot a_{n+1-i}$

1190. $a_i = \sqrt{a_{i-1} \cdot a_{i+1}}$

1191. $S_n = \dfrac{a_n q - a_1}{q-1} = \dfrac{a_1\left(q^n - 1\right)}{q-1}$

1192. $S = \lim\limits_{n\to\infty} S_n = \dfrac{a_1}{1-q}$

For $|q| < 1$, the sum S converges as $n \to \infty$.

11.3 Some Finite Series

Number of terms in the series: n

1193. $1+2+3+\ldots+n=\frac{n(n+1)}{2}$

1194. $2+4+6+\ldots+2n=n(n+1)$

1195. $1+3+5+\ldots+(2n-1)=n^2$

1196. $k+(k+1)+(k+2)+\ldots+(k+n-1)=\frac{n(2k+n-1)}{2}$

1197. $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

1198. $1^3+2^3+3^3+\ldots+n^3=\left[\frac{n(n+1)}{2}\right]^2$

1199. $1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{n(4n^2-1)}{3}$

1200. $1^3+3^3+5^3+\ldots+(2n-1)^3=n^2(2n^2-1)$

1201. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}+\ldots=2$

1202. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots=1$

1203. $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{(n-1)!}+\ldots=e$

11.4 Infinite Series

Sequence: $\{a_n\}$
First term: a_1
Nth term: a_n

1204. Infinite Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

1205. Nth Partial Sum

$$S_n = \sum_{n=1}^{n} a_n = a_1 + a_2 + \ldots + a_n$$

1206. Convergence of Infinite Series

$$\sum_{n=1}^{\infty} a_n = L, \text{ if } \lim_{n\to\infty} S_n = L$$

1207. Nth Term Test

- If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
- If $\lim_{n\to\infty} a_n \neq 0$, then the series is divergent.

11.5 Properties of Convergent Series

Convergent Series: $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$

Real number: c

1208. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B$

1209. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n = cA$.

11.6 Convergence Tests

1210. The Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that $0 < a_n \le b_n$ for all n.

- If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is also convergent.
- If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ is also divergent.

1211. The Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that a_n and b_n are positive for all n.

- If $0 < \lim\limits_{n\to\infty} \frac{a_n}{b_n} < \infty$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are either both convergent or both divergent.
- If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ then $\sum_{n=1}^{\infty} b_n$ convergent implies that $\sum_{n=1}^{\infty} a_n$ is also convergent.

- If $\lim_{n\to\infty}\frac{a_n}{b_n}=\infty$ then $\sum_{n=1}^{\infty} b_n$ divergent implies that $\sum_{n=1}^{\infty} a_n$ is also divergent.

1212. p-series

p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p>1$ and diverges for $0<p\le 1$.

1213. The Integral Test

Let $f(x)$ be a function which is continuous, positive, and decreasing for all $x\ge 1$. The series

$$\sum_{n=1}^{\infty} f(n)=f(1)+f(2)+f(3)+\ldots+f(n)+\ldots$$

converges if $\int_1^{\infty} f(x)dx$ converges, and diverges if

$\int_1^{n} f(x)dx \to \infty$ as $n\to\infty$.

1214. The Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

- If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}<1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}>1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1$ then $\sum_{n=1}^{\infty} a_n$ may converge or diverge and the ratio test is inconclusive; some other tests must be used.

1215. The Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

- If $\lim_{n \to \infty} \sqrt[n]{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\lim_{n \to \infty} \sqrt[n]{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n \to \infty} \sqrt[n]{a_n} = 1$ then $\sum_{n=1}^{\infty} a_n$ may converge or diverge, but no conclusion can be drawn from this test.

11.7 Alternating Series

1216. The Alternating Series Test (Leibniz's Theorem)

Let $\{a_n\}$ be a sequence of positive numbers such that
$a_{n+1} < a_n$ for all n.
$\lim_{n \to \infty} a_n = 0$.

Then the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ both converge.

1217. Absolute Convergence

- A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

- If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is convergent.

1218. Conditional Convergence

A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if the series is convergent but is not absolutely convergent.

11.8 Power Series

Real numbers: x, x_0

Power series: $\sum_{n=0}^{\infty} a_n x^n$, $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

Whole number: n

Radius of Convergence: R

1219. Power Series in x

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

1220. Power Series in $(x - x_0)$

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots + a_n (x - x_0)^n + \ldots$$

1221. Interval of Convergence

The set of those values of x for which the function $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is convergent is called the interval of convergence.

1222. Radius of Convergence

If the interval of convergence is $(x_0 - R, x_0 + R)$ for some $R \geq 0$, the R is called the radius of convergence. It is given as

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}} \text{ or } R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

11.9 Differentiation and Integration of Power Series

Continuous function: $f(x)$

Power series: $\sum_{n=0}^{\infty} a_n x^n$

Whole number: n

Radius of Convergence: R

1223. Differentiation of Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$ for $|x| < R$.

Then, for $|x| < R$, $f(x)$ is continuous, the derivative $f'(x)$ exists and

$$f'(x) = \frac{d}{dx} a_0 + \frac{d}{dx} a_1 x + \frac{d}{dx} a_2 x^2 + \ldots$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \ldots = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

1224. Integration of Power Series

Let $f(x)=\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$ for $|x|<R$.

Then, for $|x|<R$, the indefinite integral $\int f(x)dx$ exists and

$$\int f(x)dx = \int a_0 dx + \int a_1 x dx + \int a_2 x^2 dx + \ldots$$

$$= a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \ldots = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} + C.$$

11.10 Taylor and Maclaurin Series

Whole number: n
Differentiable function: $f(x)$
Remainder term: R_n

1225. Taylor Series

$$f(x)=\sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!} = f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\ldots$$

$$+\frac{f^{(n)}(a)(x-a)^n}{n!}+R_n.$$

1226. The Remainder After n+1 Terms is given by

$$R_n = \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}, \quad a<\xi<x.$$

1227. Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots + \frac{f^{(n)}(0)x^n}{n!} + R_n$$

11.11 Power Series Expansions for Some Functions

Whole number: n
Real number: x

1228. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$

1229. $a^x = 1 + \frac{x \ln a}{1!} + \frac{(x \ln a)^2}{2!} + \frac{(x \ln a)^3}{3!} + \ldots + \frac{(x \ln a)^n}{n!} + \ldots$

1230. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^n x^{n+1}}{n+1} \pm \ldots, \; -1 < x \le 1.$

1231. $\ln \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots \right), \; |x| < 1.$

1232. $\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 \ldots \right], \; x > 0.$

1233. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} \pm \ldots$

1234. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} \pm \ldots$

1235. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots,\ |x| < \frac{\pi}{2}.$

1236. $\cot x = \frac{1}{x} - \left(\frac{x}{3} + \frac{x^3}{45} + \frac{2x^5}{945} + \frac{2x^7}{4725} + \ldots\right),\ |x| < \pi.$

1237. $\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \ldots + \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)x^{2n+1}}{2 \cdot 4 \cdot 6 \ldots (2n)(2n+1)} + \ldots,$
$|x| < 1.$

1238. $\arccos x = \frac{\pi}{2} - \left(x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \ldots + \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)x^{2n+1}}{2 \cdot 4 \cdot 6 \ldots (2n)(2n+1)} + \ldots\right),$
$|x| < 1.$

1239. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^n x^{2n+1}}{2n+1} \pm \ldots,\ |x| \le 1.$

1240. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$

1241. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots$

11.12 Binomial Series

Whole numbers: n, m
Real number: x
Combinations: ${}^{n}C_{m}$

1242. $(1+x)^n = 1 + {}^{n}C_1 x + {}^{n}C_2 x^2 + \ldots + {}^{m}C_n x^m + \ldots + x^n$

1243. ${}^{n}C_{m} = \dfrac{n(n-1)\ldots[n-(m-1)]}{m!}, \ |x|<1.$

1244. $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots, \ |x|<1.$

1245. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots, \ |x|<1.$

1246. $\sqrt{1+x} = 1 + \dfrac{x}{2} - \dfrac{x^2}{2\cdot 4} + \dfrac{1\cdot 3x^3}{2\cdot 4\cdot 6} - \dfrac{1\cdot 3\cdot 5x^4}{2\cdot 4\cdot 6\cdot 8} + \ldots, \ |x|\le 1.$

1247. $\sqrt[3]{1+x} = 1 + \dfrac{x}{3} - \dfrac{1\cdot 2x^2}{3\cdot 6} + \dfrac{1\cdot 2\cdot 5x^3}{3\cdot 6\cdot 9} - \dfrac{1\cdot 2\cdot 5\cdot 8x^4}{3\cdot 6\cdot 9\cdot 12} + \ldots, \ |x|\le 1.$

11.13 Fourier Series

Integrable function: $f(x)$
Fourier coefficients: a_0, a_n, b_n
Whole number: n

1248. $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$

1249. $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$

1250. $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$