Chapter 9

Integral Calculus

Functions: f, g, u, v
Independent variables: x, t, ξ
Indefinite integral of a function: $\int f(x)dx$, $\int g(x)dx$, ...
Derivative of a function: $y'(x)$, $f'(x)$, $F'(x)$, ...
Real constants: C, a, b, c, d, k
Natural numbers: m, n, i, j

9.1 Indefinite Integral

865. $\int f(x)dx = F(x) + C$ if $F'(x) = f(x)$.

866. $\left(\int f(x)dx\right)' = f(x)$

867. $\int kf(x)dx = k\int f(x)dx$

868. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

869. $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$

870. $\int f(ax)dx = \frac{1}{a}F(ax) + C$

871. $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$

872. $\int f(x)f'(x)dx = \frac{1}{2}f^2(x) + C$

873. $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

874. Method of Substitution
$\int f(x)dx = \int f(u(t))u'(t)dt$ if $x = u(t)$.

875. Integration by Parts
$\int udv = uv - \int vdu$,
where $u(x)$, $v(x)$ are differentiable functions.

9.2 Integrals of Rational Functions

876. $\int adx = ax + C$

877. $\int xdx = \frac{x^2}{2} + C$

878. $\int x^2dx = \frac{x^3}{3} + C$

879. $\int x^p dx = \frac{x^{p+1}}{p+1} + C$, $p \neq -1$.

880. $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C,\ n \neq -1.$$

881. $$\int \frac{dx}{x} = \ln|x| + C$$

882. $$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

883. $$\int \frac{ax+b}{cx+d}\,dx = \frac{a}{c}x + \frac{bc-ad}{c^2}\ln|cx+d| + C$$

884. $$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{a-b}\ln\left|\frac{x+b}{x+a}\right| + C,\ a \neq b.$$

885. $$\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}\left(a+bx-a\ln|a+bx|\right) + C$$

886. $$\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\ln|a+bx|\right] + C$$

887. $$\int \frac{dx}{x(a+bx)} = \frac{1}{a}\ln\left|\frac{a+bx}{x}\right| + C$$

888. $$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$$

889. $$\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left(\ln|a+bx| + \frac{a}{a+bx}\right) + C$$

890. $$\int \frac{x^2 dx}{(a+bx)^2} = \frac{1}{b^3}\left(a + bx - 2a\ln|a+bx| - \frac{a^2}{a+bx}\right) + C$$

891. $$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} + \frac{1}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$$

892. $$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

893. $$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$

894. $$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

895. $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

896. $$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

897. $$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

898. $$\int \frac{x dx}{x^2+a^2} = \frac{1}{2}\ln\left(x^2+a^2\right) + C$$

899. $$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\arctan\left(x\sqrt{\frac{b}{a}}\right) + C,\ ab > 0.$$

900. $\int \frac{xdx}{a+bx^2} = \frac{1}{2b}\ln\left|x^2+\frac{a}{b}\right| + C$

901. $\int \frac{dx}{x\left(a+bx^2\right)} = \frac{1}{2a}\ln\left|\frac{x^2}{a+bx^2}\right| + C$

902. $\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab}\ln\left|\frac{a+bx}{a-bx}\right| + C$

903. $\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C$,
$b^2-4ac>0$.

904. $\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C$,
$b^2-4ac<0$.

9.3 Integrals of Irrational Functions

905. $\int \frac{dx}{\sqrt{ax+b}} = \frac{2}{a}\sqrt{ax+b} + C$

906. $\int \sqrt{ax+b}\,dx = \frac{2}{3a}(ax+b)^{3/2} + C$

907. $\int \frac{xdx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b} + C$

908. $\int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}(ax+b)^{3/2} + C$

909. $\int \frac{dx}{(x+c)\sqrt{ax+b}} = \frac{1}{\sqrt{b-ac}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b-ac}}{\sqrt{ax+b}+\sqrt{b-ac}}\right| + C,$
$b-ac>0.$

910. $\int \frac{dx}{(x+c)\sqrt{ax+b}} = \frac{1}{\sqrt{ac-b}}\arctan\sqrt{\frac{ax+b}{ac-b}} + C,$
$b-ac<0.$

911. $\int \sqrt{\frac{ax+b}{cx+d}}\,dx = \frac{1}{c}\sqrt{(ax+b)(cx+d)} -$
$- \frac{ad-bc}{c\sqrt{ac}}\ln\left|\sqrt{a(cx+d)}+\sqrt{c(ax+b)}\right| + C,\ a>0.$

912. $\int \sqrt{\frac{ax+b}{cx+d}}\,dx = \frac{1}{c}\sqrt{(ax+b)(cx+d)} -$
$- \frac{ad-bc}{c\sqrt{ac}}\arctan\sqrt{\frac{a(cx+d)}{c(ax+b)}} + C,\ (a<0,\ c>0).$

913. $\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)}{105b^3}\sqrt{(a+bx)^3} + C$

914. $\int \frac{x^2dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx} + C$

915. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right| + C,\ a>0.$

916. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}} \arctan \left| \frac{a+bx}{-a} \right| + C$, $a<0$.

917. $\int \sqrt{\frac{a-x}{b+x}}\, dx = \sqrt{(a-x)(b+x)} + (a+b)\arcsin\sqrt{\frac{x+b}{a+b}} + C$

918. $\int \sqrt{\frac{a+x}{b-x}}\, dx = -\sqrt{(a+x)(b-x)} - (a+b)\arcsin\sqrt{\frac{b-x}{a+b}} + C$

919. $\int \sqrt{\frac{1+x}{1-x}}\, dx = -\sqrt{1-x^2} + \arcsin x + C$

920. $\int \frac{dx}{\sqrt{(x-a)(b-a)}} = 2\arcsin\sqrt{\frac{x-a}{b-a}} + C$

921. $\int \sqrt{a+bx-cx^2}\, dx = \frac{2cx-b}{4c}\sqrt{a+bx-cx^2} +$

$$+\frac{b^2-4ac}{8\sqrt{c^3}} \arcsin \frac{2cx-b}{\sqrt{b^2+4ac}} + C$$

922. $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| + C$,
$a>0$.

923. $\int \frac{dx}{\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{a}} \arcsin \frac{2ax+b}{4a}\sqrt{b^2-4ac} + C$, $a<0$.

924. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$

925. $\int x\sqrt{x^2+a^2}\,dx = \frac{1}{3}\left(x^2+a^2\right)^{3/2} + C$

926. $\int x^2\sqrt{x^2+a^2}\,dx = \frac{x}{8}\left(2x^2+a^2\right)\sqrt{x^2+a^2} -$

$-\frac{a^4}{8}\ln\left|x+\sqrt{x^2+a^2}\right| + C$

927. $\int \frac{\sqrt{x^2+a^2}}{x^2}\,dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln\left|x+\sqrt{x^2+a^2}\right| + C$

928. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + C$

929. $\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} + a\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

930. $\int \frac{x\,dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2} + C$

931. $\int \frac{x^2\,dx}{\sqrt{x^2+a^2}} = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$

932. $\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

933. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

934. $\int x\sqrt{x^2-a^2}\,dx = \frac{1}{3}\left(x^2-a^2\right)^{3/2} + C$

935. $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} + a\arcsin\frac{a}{x} + C$

936. $\int \frac{\sqrt{x^2-a^2}}{x^2}dx = -\frac{\sqrt{x^2-a^2}}{x} + \ln\left|x+\sqrt{x^2-a^2}\right| + C$

937. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

938. $\int \frac{xdx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2} + C$

939. $\int \frac{x^2dx}{\sqrt{x^2-a^2}} = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

940. $\int \frac{dx}{x\sqrt{x^2-a^2}} = -\frac{1}{a}\arcsin\frac{a}{x} + C$

941. $\int \frac{dx}{(x+a)\sqrt{x^2-a^2}} = \frac{1}{a}\sqrt{\frac{x-a}{x+a}} + C$

942. $\int \frac{dx}{(x-a)\sqrt{x^2-a^2}} = -\frac{1}{a}\sqrt{\frac{x+a}{x-a}} + C$

943. $\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$

944. $\int \frac{dx}{\left(x^2-a^2\right)^{3/2}} = -\frac{x}{a^2\sqrt{x^2-a^2}} + C$

945. $$\int\left(x^2-a^2\right)^{3/2}dx=-\frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2}+$$
$$+\frac{3a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right|+C$$

946. $$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$$

947. $$\int x\sqrt{a^2-x^2}\,dx=-\frac{1}{3}\left(a^2-x^2\right)^{3/2}+C$$

948. $$\int x^2\sqrt{a^2-x^2}\,dx=\frac{x}{8}\left(2x^2-a^2\right)\sqrt{a^2-x^2}+\frac{a^4}{8}\arcsin\frac{x}{a}+C$$

949. $$\int\frac{\sqrt{a^2-x^2}}{x}dx=\sqrt{a^2-x^2}+a\ln\left|\frac{x}{a+\sqrt{a^2-x^2}}\right|+C$$

950. $$\int\frac{\sqrt{a^2-x^2}}{x^2}dx=-\frac{\sqrt{a^2-x^2}}{x}-\arcsin\frac{x}{a}+C$$

951. $$\int\frac{dx}{\sqrt{1-x^2}}=\arcsin x+C$$

952. $$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin\frac{x}{a}+C$$

953. $$\int\frac{x\,dx}{\sqrt{a^2-x^2}}=-\sqrt{a^2-x^2}+C$$

954. $$\int\frac{x^2dx}{\sqrt{a^2-x^2}}=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$$

955. $\int \frac{dx}{(x+a)\sqrt{a^2-x^2}} = -\frac{1}{2}\sqrt{\frac{a-x}{a+x}} + C$

956. $\int \frac{dx}{(x-a)\sqrt{a^2-x^2}} = -\frac{1}{2}\sqrt{\frac{a+x}{a-x}} + C$

957. $\int \frac{dx}{(x+b)\sqrt{a^2-x^2}} = \frac{1}{\sqrt{b^2-a^2}} \arcsin \frac{bx+a^2}{a(x+b)} + C$, $b > a$.

958. $\int \frac{dx}{(x+b)\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-b^2}} \ln\left|\frac{x+b}{\sqrt{a^2-b^2}\sqrt{a^2-x^2}+a^2+bx}\right| + C$,
$b < a$.

959. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

960. $\int \left(a^2-x^2\right)^{3/2} dx = \frac{x}{8}\left(5a^2-2x^2\right)\sqrt{a^2-x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a} + C$

961. $\int \frac{dx}{\left(a^2-x^2\right)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C$

9.4 Integrals of Trigonometric Functions

962. $\int \sin x dx = -\cos x + C$

963. $\int \cos x dx = \sin x + C$

964. $\int \sin^2 x\, dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

965. $\int \cos^2 x\, dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

966. $\int \sin^3 x\, dx = \frac{1}{3}\cos^3 x - \cos x + C = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x + C$

967. $\int \cos^3 x\, dx = \sin x - \frac{1}{3}\sin^3 x + C = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + C$

968. $\int \frac{dx}{\sin x} = \int \csc x\, dx = \ln\left|\tan\frac{x}{2}\right| + C$

969. $\int \frac{dx}{\cos x} = \int \sec x\, dx = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$

970. $\int \frac{dx}{\sin^2 x} = \int \csc^2 x\, dx = -\cot x + C$

971. $\int \frac{dx}{\cos^2 x} = \int \sec^2 x\, dx = \tan x + C$

972. $\int \frac{dx}{\sin^3 x} = \int \csc^3 x\, dx = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C$

973. $\int \frac{dx}{\cos^3 x} = \int \sec^3 x\, dx = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$

974. $\int \sin x \cos x\, dx = -\frac{1}{4}\cos 2x + C$

975. $\int \sin^2 x \cos x \, dx = \frac{1}{3}\sin^3 x + C$

976. $\int \sin x \cos^2 x \, dx = -\frac{1}{3}\cos^3 x + C$

977. $\int \sin^2 x \cos^2 x \, dx = \frac{x}{8} - \frac{1}{32}\sin 4x + C$

978. $\int \tan x dx = -\ln|\cos x| + C$

979. $\int \frac{\sin x}{\cos^2 x} dx = \frac{1}{\cos x} + C = \sec x + C$

980. $\int \frac{\sin^2 x}{\cos x} dx = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| - \sin x + C$

981. $\int \tan^2 x \, dx = \tan x - x + C$

982. $\int \cot x dx = \ln|\sin x| + C$

983. $\int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x} + C = -\csc x + C$

984. $\int \frac{\cos^2 x}{\sin x} dx = \ln\left|\tan\frac{x}{2}\right| + \cos x + C$

985. $\int \cot^2 x \, dx = -\cot x - x + C$

986. $\int \frac{dx}{\cos x \sin x} = \ln|\tan x| + C$

987. $\int \frac{dx}{\sin^2 x \cos x} = -\frac{1}{\sin x} + \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$

988. $\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln\left|\tan\frac{x}{2}\right| + C$

989. $\int \frac{dx}{\sin^2 x \cos^2 x} = \tan x - \cot x + C$

990. $\int \sin mx \sin nx\, dx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C$, $m^2 \neq n^2$.

991. $\int \sin mx \cos nx\, dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)} + C$, $m^2 \neq n^2$.

992. $\int \cos mx \cos nx\, dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C$, $m^2 \neq n^2$.

993. $\int \sec x \tan x dx = \sec x + C$

994. $\int \csc x \cot x dx = -\csc x + C$

995. $\int \sin x \cos^n x\, dx = -\frac{\cos^{n+1} x}{n+1} + C$

996. $\int \sin^n x \cos x\, dx = \frac{\sin^{n+1} x}{n+1} + C$

997. $\int \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2} + C$

998. $\int \arccos x \, dx = x \arccos x - \sqrt{1-x^2} + C$

999. $\int \arctan x \, dx = x \arctan x - \frac{1}{2}\ln\left(x^2+1\right) + C$

1000. $\int \operatorname{arccot} x \, dx = x \operatorname{arccot} x + \frac{1}{2}\ln\left(x^2+1\right) + C$

9.5 Integrals of Hyperbolic Functions

1001. $\int \sinh x dx = \cosh x + C$

1002. $\int \cosh x dx = \sinh x + C$

1003. $\int \tanh x \, dx = \ln \cosh x + C$

1004. $\int \coth x \, dx = \ln\left|\sinh x\right| + C$

1005. $\int \operatorname{sech}^2 x dx = \tanh x + C$

1006. $\int \operatorname{csch}^2 x dx = -\coth x + C$

1007. $\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$

1008. $\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$

9.6 Integrals of Exponential and Logarithmic Functions

1009. $\int e^{x} dx = e^{x} + C$

1010. $\int a^{x} dx = \frac{a^{x}}{\ln a} + C$

1011. $\int e^{ax}\, dx = \frac{e^{ax}}{a} + C$

1012. $\int x e^{ax}\, dx = \frac{e^{ax}}{a^{2}}(ax-1) + C$

1013. $\int \ln x\, dx = x \ln x - x + C$

1014. $\int \frac{dx}{x \ln x} = \ln\left|\ln x\right| + C$

1015. $\int x^{n} \ln x\, dx = x^{n+1}\left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^{2}}\right] + C$

1016. $\int e^{ax} \sin bx\, dx = \frac{a \sin bx - b \cos bx}{a^{2}+b^{2}} e^{ax} + C$

1017. $\int e^{ax}\cos bx\,dx = \frac{a\cos bx + b\sin bx}{a^2+b^2}e^{ax} + C$

9.7 Reduction Formulas

1018. $\int x^n e^{mx}dx = \frac{1}{m}x^n e^{mx} - \frac{n}{m}\int x^{n-1}e^{mx}dx$

1019. $\int \frac{e^{mx}}{x^n}dx = -\frac{e^{mx}}{(n-1)x^{n-1}} + \frac{m}{n-1}\int \frac{e^{mx}}{x^{n-1}}dx,\ n \neq 1.$

1020. $\int \sinh^n x dx = \frac{1}{n}\sinh^{n-1}x\cosh x - \frac{n-1}{n}\int \sinh^{n-2}x dx$

1021. $\int \frac{dx}{\sinh^n x} = -\frac{\cosh x}{(n-1)\sinh^{n-1}x} - \frac{n-2}{n-1}\int \frac{dx}{\sinh^{n-2}x},\ n \neq 1.$

1022. $\int \cosh^n x dx = \frac{1}{n}\sinh x\cosh^{n-1}x\cosh x + \frac{n-1}{n}\int \cosh^{n-2}x dx$

1023. $\int \frac{dx}{\cosh^n x} = -\frac{\sinh x}{(n-1)\cosh^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\cosh^{n-2}x},\ n \neq 1.$

1024. $\int \sinh^n x\cosh^m x dx = \frac{\sinh^{n+1}x\cosh^{m-1}x}{n+m}$

$$+\frac{m-1}{n+m}\int \sinh^n x\cosh^{m-2}x dx$$

1025. $\int \sinh^n x\cosh^m x dx = \frac{\sinh^{n-1}x\cosh^{m+1}x}{n+m}$

$$-\frac{n-1}{n+m}\int \sinh^{n-2} x \cosh^{m} x dx$$

1026. $\int \tanh^{n} x dx = -\frac{1}{n-1}\tanh^{n-1} x + \int \tanh^{n-2} x dx$, $n \neq 1$.

1027. $\int \coth^{n} x dx = -\frac{1}{n-1}\coth^{n-1} x + \int \coth^{n-2} x dx$, $n \neq 1$.

1028. $\int \operatorname{sech}^{n} x dx = \frac{\operatorname{sech}^{n-2} x \tanh x}{n-1} + \frac{n-2}{n-1}\int \operatorname{sech}^{n-2} x dx$, $n \neq 1$.

1029. $\int \sin^{n} x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$

1030. $\int \frac{dx}{\sin^{n} x} = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}$, $n \neq 1$.

1031. $\int \cos^{n} x dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x dx$

1032. $\int \frac{dx}{\cos^{n} x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}$, $n \neq 1$.

1033. $\int \sin^{n} x \cos^{m} x dx = \frac{\sin^{n+1} x \cos^{m-1} x}{n+m}$

$$+\frac{m-1}{n+m}\int \sin^{n} x \cos^{m-2} x dx$$

1034. $\int \sin^{n} x \cos^{m} x dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{n+m}$

$$+\frac{n-1}{n+m}\int \sin^{n-2} x \cos^m x dx$$

1035. $\int \tan^n x dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x dx$, $n \neq 1$.

1036. $\int \cot^n x dx = -\frac{1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x dx$, $n \neq 1$.

1037. $\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x dx$, $n \neq 1$.

1038. $\int \csc^n x dx = -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x dx$, $n \neq 1$.

1039. $\int x^n \ln^m x dx = \frac{x^{n+1}\ln^m x}{n+1} - \frac{m}{n+1}\int x^n \ln^{m-1} x dx$

1040. $\int \frac{\ln^m x}{x^n} dx = -\frac{\ln^m x}{(n-1)x^{n-1}} + \frac{m}{n-1}\int \frac{\ln^{m-1} x}{x^n} dx$, $n \neq 1$.

1041. $\int \ln^n x dx = x \ln^n x - n\int \ln^{n-1} x dx$

1042. $\int x^n \sinh x dx = x^n \cosh x - n\int x^{n-1} \cosh x dx$

1043. $\int x^n \cosh x dx = x^n \sinh x - n\int x^{n-1} \sinh x dx$

1044. $\int x^n \sin x dx = -x^n \cos x + n\int x^{n-1} \cos x dx$

1045. $\int x^n \cos x dx = x^n \sin x - n\int x^{n-1} \sin x dx$

1046. $\int x^n \sin^{-1} x dx = \frac{x^{n+1}}{n+1} \sin^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx$

1047. $\int x^n \cos^{-1} x dx = \frac{x^{n+1}}{n+1} \cos^{-1} x + \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx$

1048. $\int x^n \tan^{-1} x dx = \frac{x^{n+1}}{n+1} \tan^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{1+x^2} dx$

1049. $\int \frac{x^n dx}{ax^n + b} = \frac{x}{a} - \frac{b}{a} \int \frac{dx}{ax^n + b}$

1050. $\int \frac{dx}{(ax^2 + bx + c)^n} = \frac{-2ax - b}{(n-1)(b^2 - 4ac)(ax^2 + bx + c)^{n-1}}$
$- \frac{2(2n-3)a}{(n-1)(b^2 - 4ac)} \int \frac{dx}{(ax^2 + bx + c)^{n-1}}$, $n \neq 1$.

1051. $\int \frac{dx}{(x^2 + a^2)^n} = \frac{x}{2(n-1)a^2 (x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{(x^2 + a^2)^{n-1}}$,
$n \neq 1$.

1052. $\int \frac{dx}{(x^2 - a^2)^n} = -\frac{x}{2(n-1)a^2 (x^2 - a^2)^{n-1}}$
$- \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{(x^2 - a^2)^{n-1}}$, $n \neq 1$.

9.8 Definite Integral

Definite integral of a function: $\int_a^b f(x)dx$, $\int_a^b g(x)dx$, ...

Riemann sum: $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$

Small changes: Δx_i

Antiderivatives: $F(x)$, $G(x)$

Limits of integrations: a, b, c, d

1053. $\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \max \Delta x_i \to 0}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$,

where $\Delta x_i = x_i - x_{i-1}$, $x_{i-1} \le \xi_i \le x_i$.

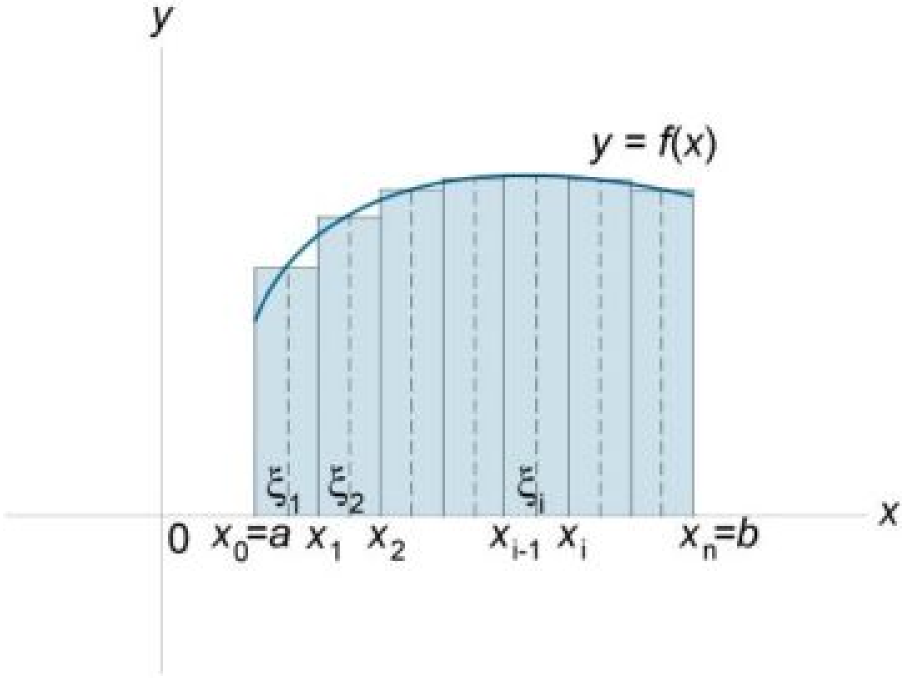

Figure 179.

1054. $\int_a^b 1dx = b - a$

1055. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

1056. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

1057. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

1058. $\int_a^a f(x)dx = 0$

1059. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

1060. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $a < c < b$.

1061. $\int_a^b f(x)dx \geq 0$ if $f(x) \geq 0$ on $[a, b]$.

1062. $\int_a^b f(x)dx \leq 0$ if $f(x) \leq 0$ on $[a, b]$.

1063. Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \text{ if } F'(x) = f(x).$$

1064. Method of Substitution

If $x = g(t)$, then

$$\int_a^b f(x)dx = \int_c^d f(g(t))g'(t)dt,$$

where

$$c = g^{-1}(a),\ d = g^{-1}(b).$$

1065. Integration by Parts

$$\int_a^b u\,dv = (uv)\Big|_a^b - \int_a^b v\,du$$

1066. Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right]$$

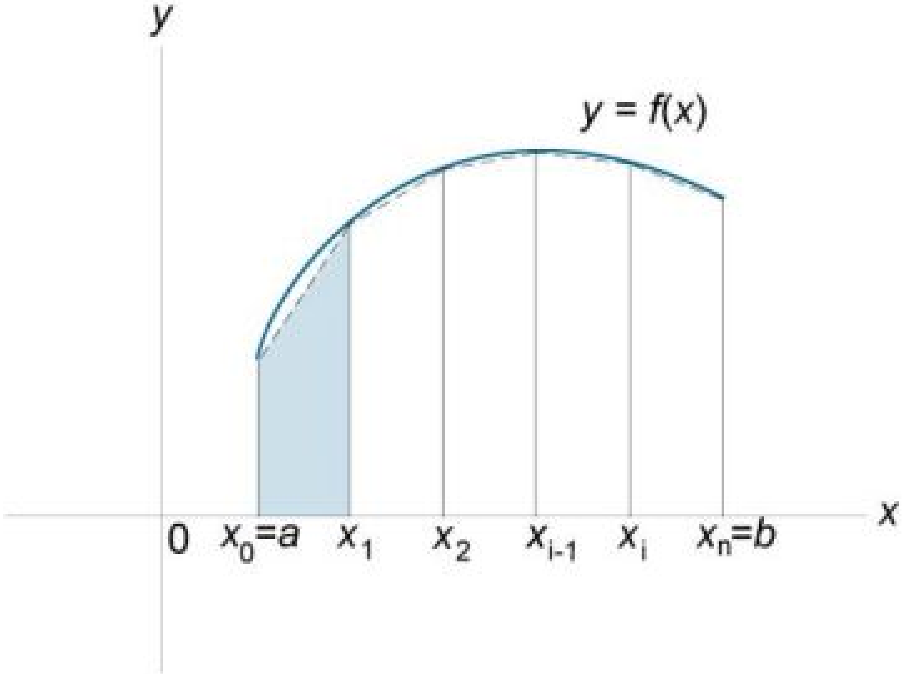

Figure 180.

1067. Simpson's Rule

$$\int_a^b f(x)dx = \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \right.$$
$$\left. + 2f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n)\right],$$

where

$$x_i = a + \frac{b-a}{n}i,\ i = 0, 1, 2, \ldots, n.$$

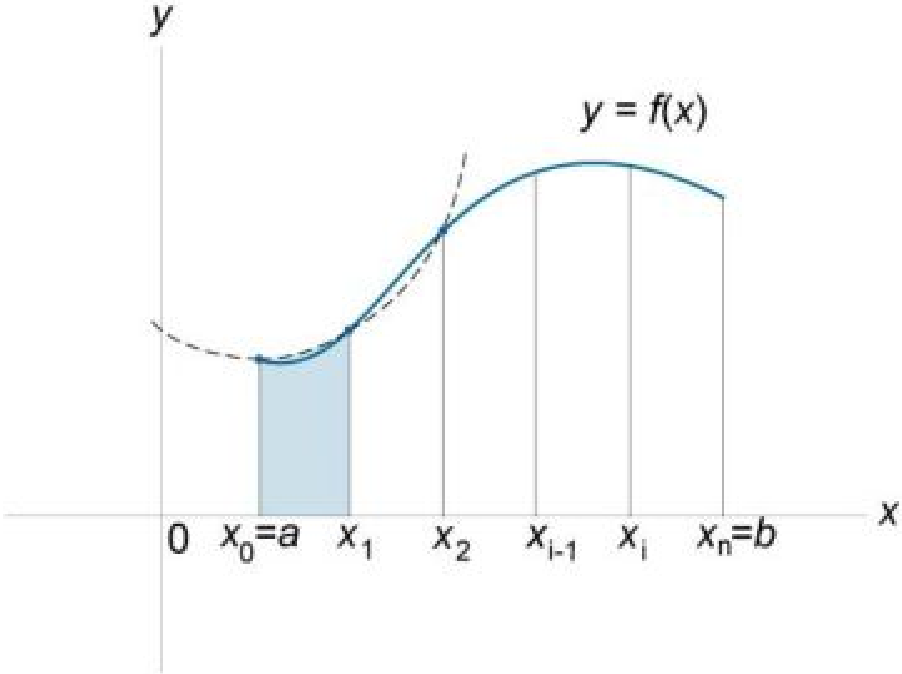

Figure 181.

1068. Area Under a Curve

$$S = \int_a^b f(x)dx = F(b) - F(a),$$

where $F'(x) = f(x)$.

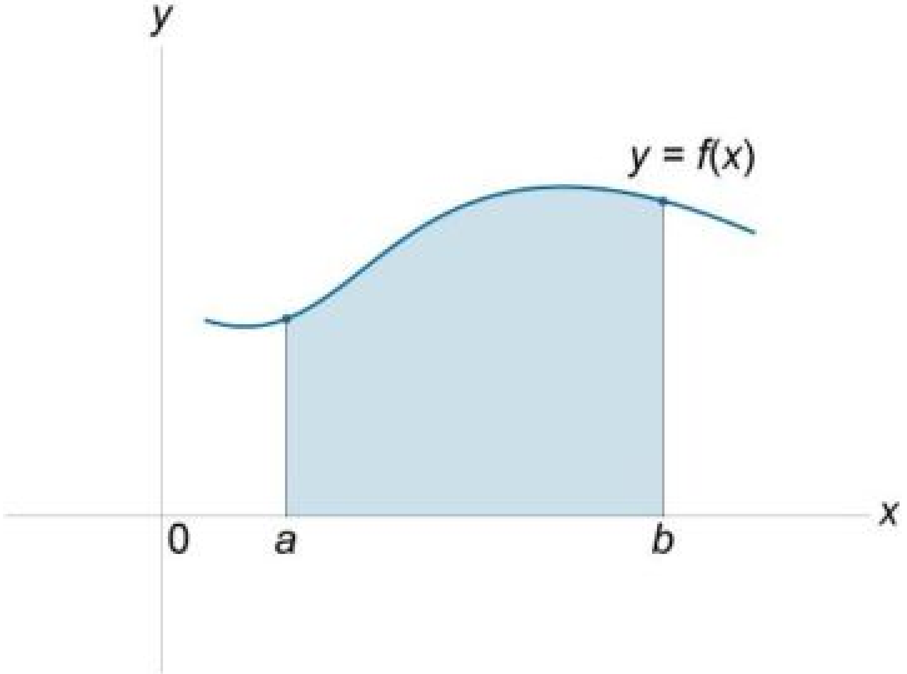

Figure 182.

1069. Area Between Two Curves

$$S = \int_a^b [f(x) - g(x)]dx = F(b) - G(b) - F(a) + G(a),$$

where $F'(x) = f(x),\ G'(x) = g(x)$.

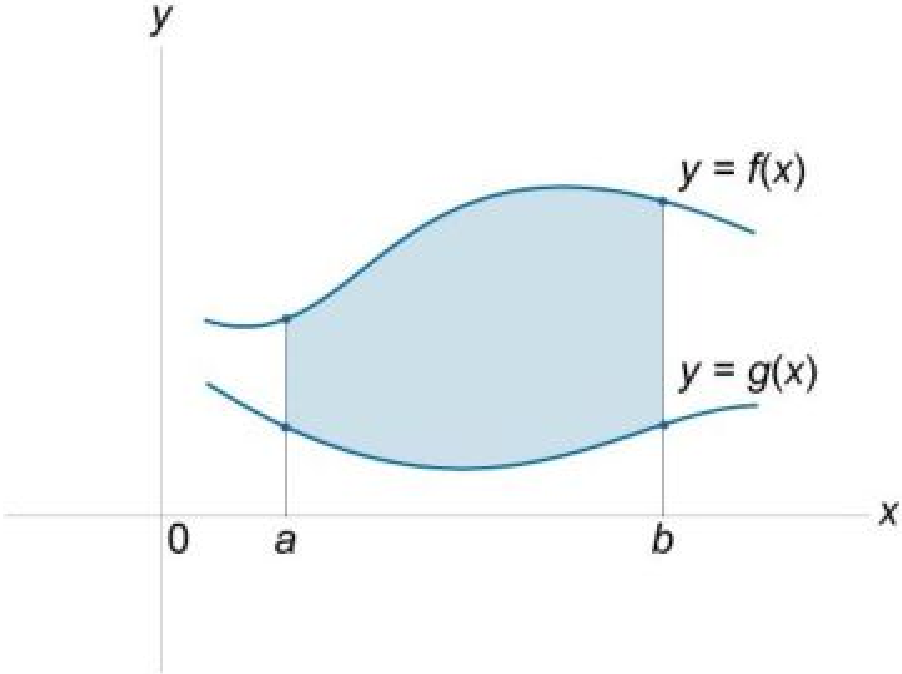

Figure 183.

9.9 Improper Integral

1070. The definite integral $\int_a^b f(x)dx$ is called an improper integral if

- a or b is infinite,
- $f(x)$ has one or more points of discontinuity in the interval $[a,b]$.

1071. If $f(x)$ is a continuous function on $[a,\infty)$, then

$$\int_a^\infty f(x)dx = \lim_{n\to\infty}\int_a^n f(x)dx.$$

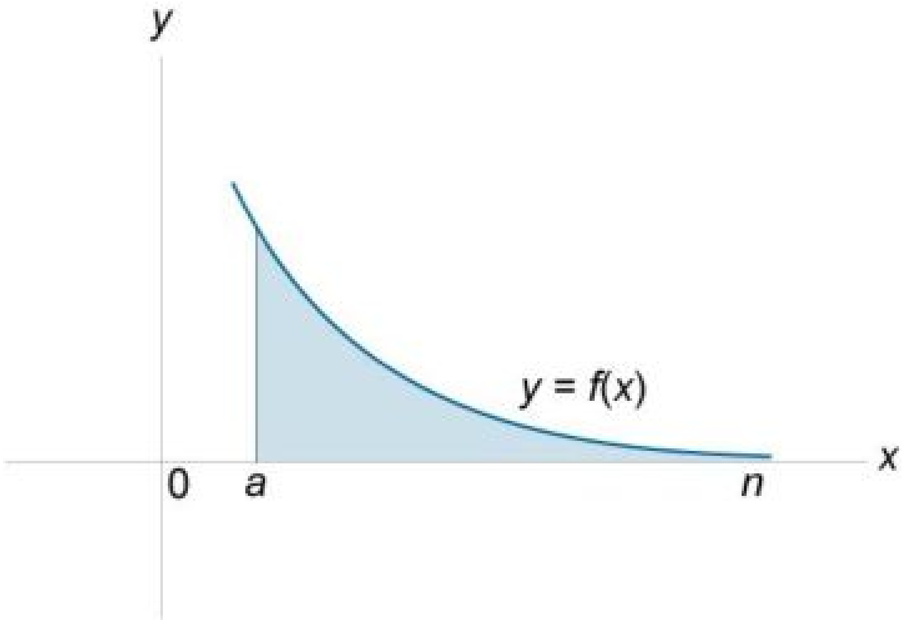

Figure 184.

1072. If $f(x)$ is a continuous function on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{n \to -\infty} \int_{n}^{b} f(x)dx.$$

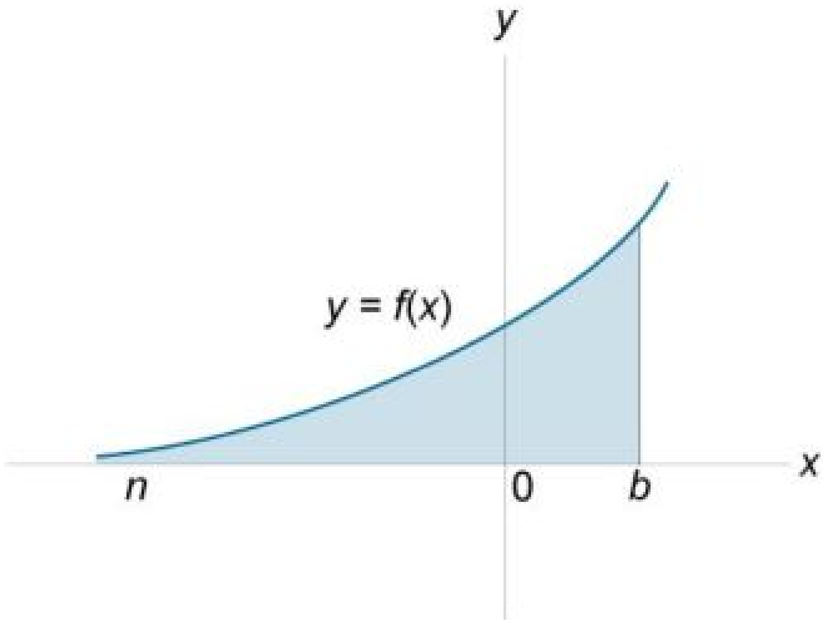

Figure 185.

Note : The improper integrals in 1071, 1072 are convergent if the limits exist and are finite; otherwise the integrals are divergent.

1073. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$

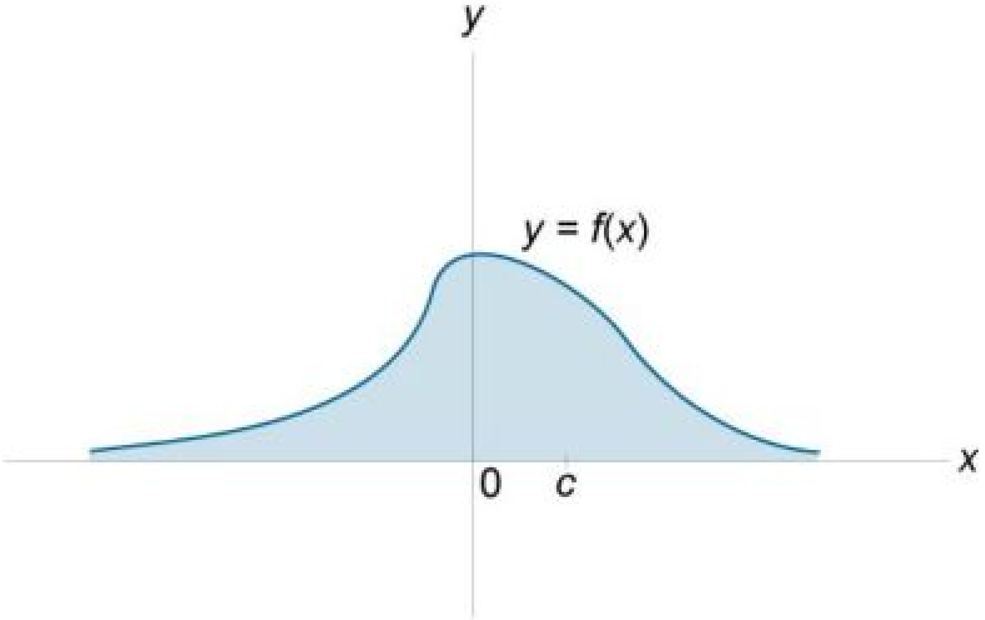

Figure 186.

If for some real number c, both of the integrals in the right side are convergent, then the integral $\int_{-\infty}^{\infty} f(x)dx$ is also convergent; otherwise it is divergent.

1074. Comparison Theorems
Let $f(x)$ and $g(x)$ be continuous functions on the closed interval $[a,\infty)$. Suppose that $0 \le g(x) \le f(x)$ for all x in $[a,\infty)$.

- If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is also convergent,
- If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is also divergent.

1075. Absolute Convergence

If $\int_a^\infty |f(x)|dx$ is convergent, then the integral $\int_a^\infty f(x)dx$ is absolutely convergent.

1076. Discontinuous Integrand

Let $f(x)$ be a function which is continuous on the interval $[a,b)$ but is discontinuous at $x = b$. Then

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)dx$$

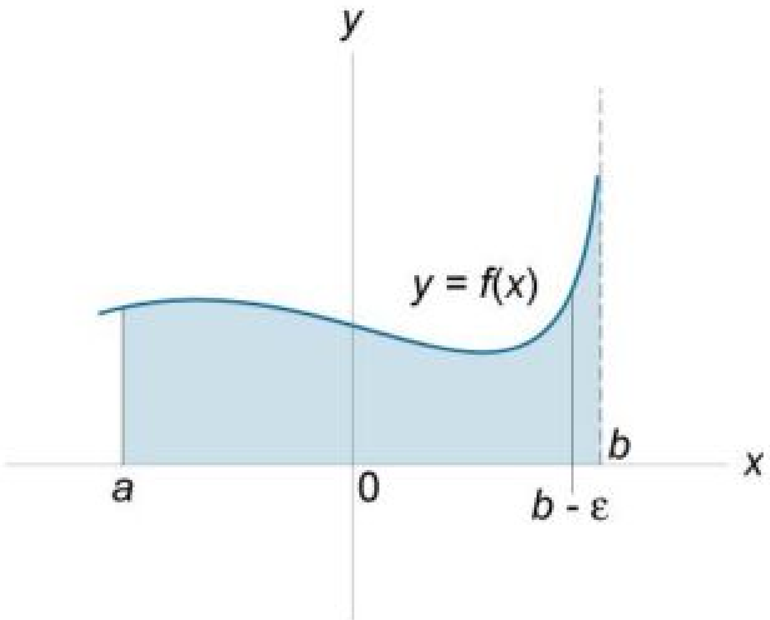

Figure 187.

1077. Let $f(x)$ be a continuous function for all real numbers x in the interval $[a,b]$ except for some point c in (a,b). Then
$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)dx + \lim_{\delta \to 0+} \int_{c+\delta}^b f(x)dx.$$

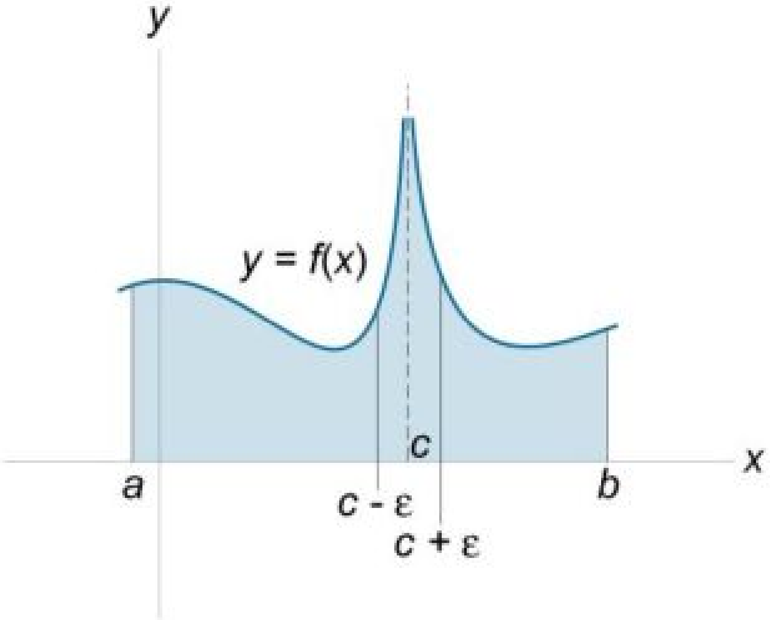

Figure 188.

9.10 Double Integral

Functions of two variables: $f(x,y)$, $f(u,v)$, ...

Double integrals: $\iint\limits_R f(x,y)dxdy$, $\iint\limits_R g(x,y)dxdy$, ...

Riemann sum: $\sum_{i=1}^{m}\sum_{j=1}^{n} f(u_i,v_j)\Delta x_i \Delta y_j$

Small changes: Δx_i, Δy_j

Regions of integration: R, S

Polar coordinates: r, θ

Area: A
Surface area: S
Volume of a solid: V
Mass of a lamina: m
Density: $\rho(x,y)$
First moments: M_x , M_y
Moments of inertia: I_x , I_y , I_0
Charge of a plate: Q
Charge density: $\sigma(x,y)$
Coordinates of center of mass: $\bar{x}$, $\bar{y}$
Average of a function: μ

1078. Definition of Double Integral
The double integral over a rectangle $[a, b]\times[c, d]$ is defined to be

$$\iint\limits_{[a,\,b]\times[c,\,d]} f(x,y)dA = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(u_i, v_j)\Delta x_i \Delta y_j \,,$$

where (u_i, v_j) is some point in the rectangle $(x_{i-1}, x_i)\times(y_{j-1}, y_j)$, and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$.

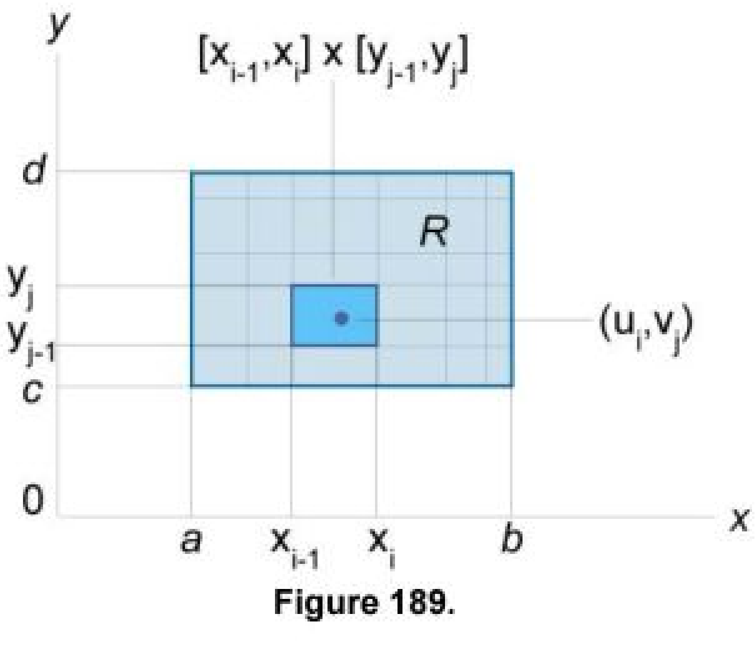

Figure 189.

The double integral over a general region R is

$$\iint\limits_R f(x,y)dA = \iint\limits_{[a,\, b]\times[c,\, d]} g(x,y)dA,$$

where rectangle $[a,\ b]\times[c,\ d]$ contains R,

$g(x,y) = f(x,y)$ if $f(x,y)$ is in R and $g(x,y) = 0$ otherwise.

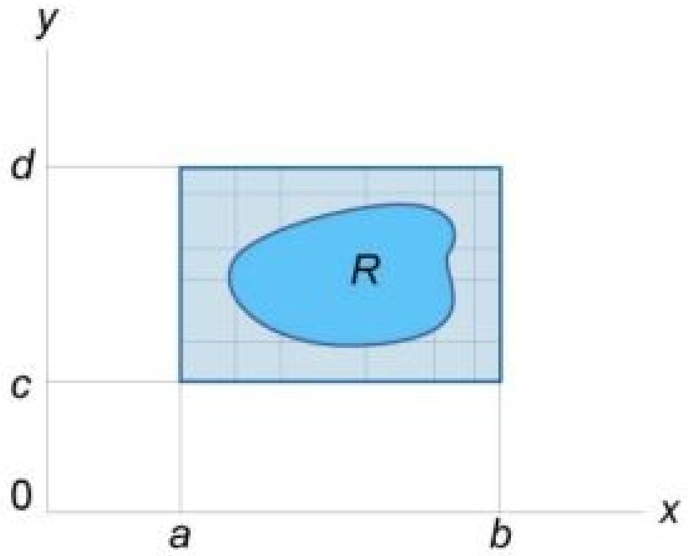

Figure 190.

1079. $\iint\limits_R [f(x,y) + g(x,y)]dA = \iint\limits_R f(x,y)dA + \iint\limits_R g(x,y)dA$

1080. $\iint\limits_R [f(x,y) - g(x,y)]dA = \iint\limits_R f(x,y)dA - \iint\limits_R g(x,y)dA$

1081. $\iint\limits_R kf(x,y)dA = k\iint\limits_R f(x,y)dA$,

where k is a constant.

1082. If $f(x,y) \le g(x,y)$ on R, then $\iint\limits_R f(x,y)dA \le \iint\limits_R g(x,y)dA$.

1083. If $f(x,y) \ge 0$ on R and $S \subset R$, then

$$\iint\limits_S f(x,y)\,dA \le \iint\limits_R f(x,y)\,dA\,.$$

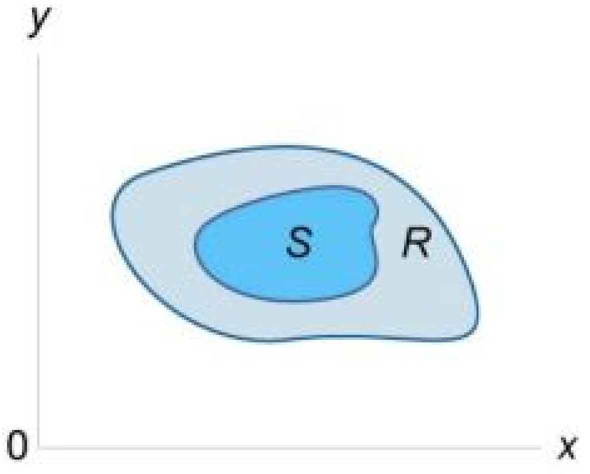

Figure 191.

1084. If $f(x,y) \ge 0$ on R and R and S are non-overlapping regions, then $\iint\limits_{R\cup S} f(x,y)\,dA = \iint\limits_R f(x,y)\,dA + \iint\limits_S f(x,y)\,dA$.
Here $R \cup S$ is the union of the regions R and S.

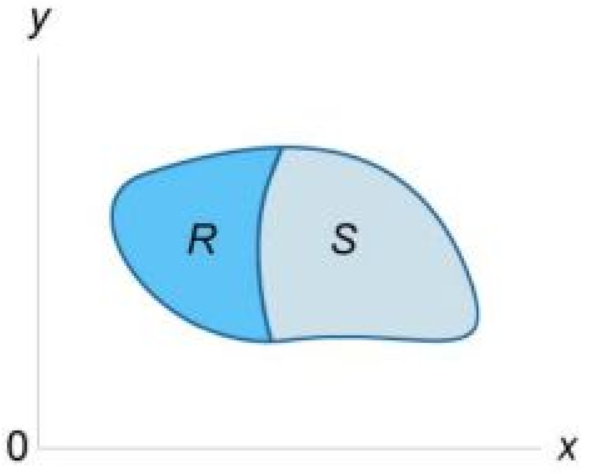

Figure 192.

1085. Iterated Integrals and Fubini's Theorem

$$\iint\limits_R f(x,y)\,dA = \int\limits_a^b \int\limits_{p(x)}^{q(x)} f(x,y)\,dy\,dx$$

for a region of type I,

$R = \{(x,y) \mid a \le x \le b,\ p(x) \le y \le q(x)\}$.

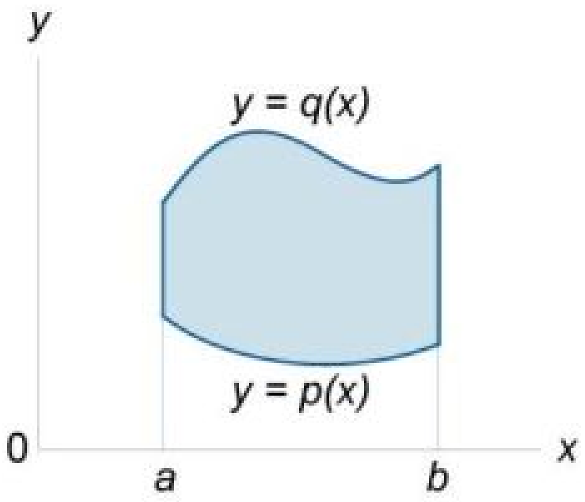

Figure 193.

$$\iint\limits_R f(x,y)\,dA = \int\limits_c^d \int\limits_{u(y)}^{v(y)} f(x,y)\,dx\,dy$$

for a region of type II,

$R = \{(x,y) \mid u(y) \le x \le v(y),\ c \le y \le d\}$.

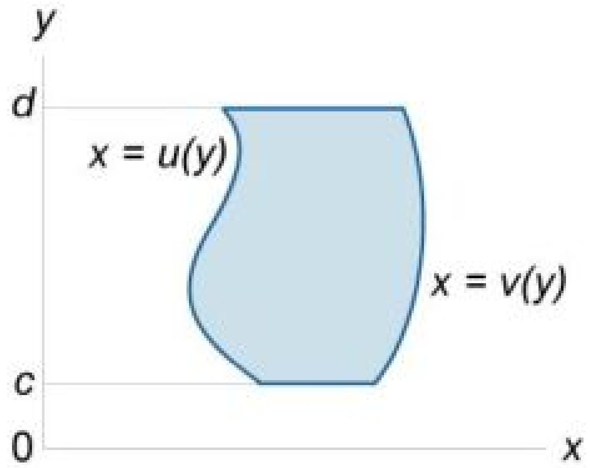

Figure 194.

1086. Double Integrals over Rectangular Regions

If R is the rectangular region $[a,b]\times[c,d]$, then

$$\iint\limits_R f(x,y)dxdy=\int\limits_a^b\left(\int\limits_c^d f(x,y)dy\right)dx=\int\limits_c^d\left(\int\limits_a^b f(x,y)dx\right)dy\,.$$

In the special case where the integrand $f(x,y)$ can be written as $g(x)h(y)$ we have

$$\iint\limits_R f(x,y)dxdy=\iint\limits_R g(x)h(y)dxdy=\left(\int\limits_a^b g(x)dx\right)\left(\int\limits_c^d h(y)dy\right).$$

1087. Change of Variables

$$\iint\limits_R f(x,y)dxdy=\iint\limits_S f[x(u,v),y(u,v)]\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv\,,$$

where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\neq 0$ is the jacobian of the transformations $(x,y)\rightarrow(u,v)$, and S is the pullback of R which can be computed by $x = x(u,v)$, $y = y(u,v)$ into the definition of R.

1088. Polar Coordinates
$x = r\cos\theta$, $y = r\sin\theta$.

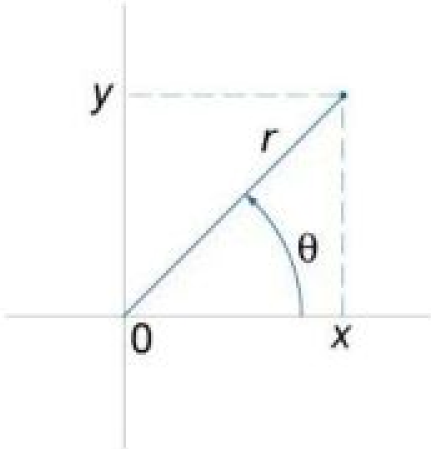

Figure 195.

1089. Double Integrals in Polar Coordinates

The Differential dxdy for Polar Coordinates is
$$dxdy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta = rdrd\theta .$$

Let the region R is determined as follows:
$0 \le g(\theta) \le r \le h(\theta)$, $\alpha \le \theta \le \beta$, where $\beta - \alpha \le 2\pi$.
Then
$$\iint\limits_R f(x,y)dxdy = \int\limits_{\alpha}^{\beta}\int\limits_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) rdrd\theta .$$

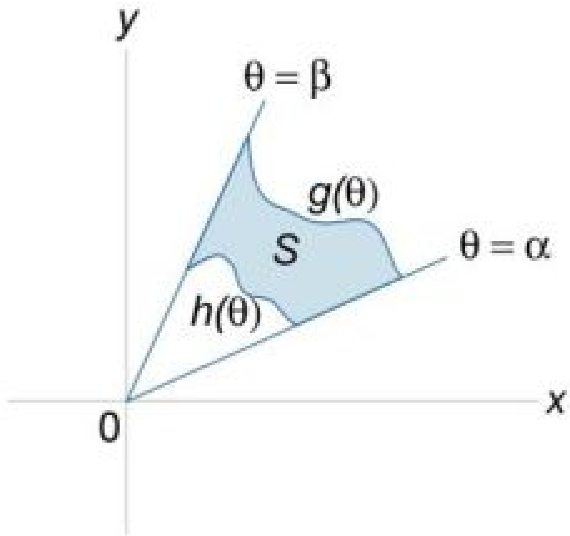

Figure 196.

If the region R is the polar rectangle given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $\beta - \alpha \le 2\pi$, then

$$\iint_R f(x, y)dxdy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta .$$

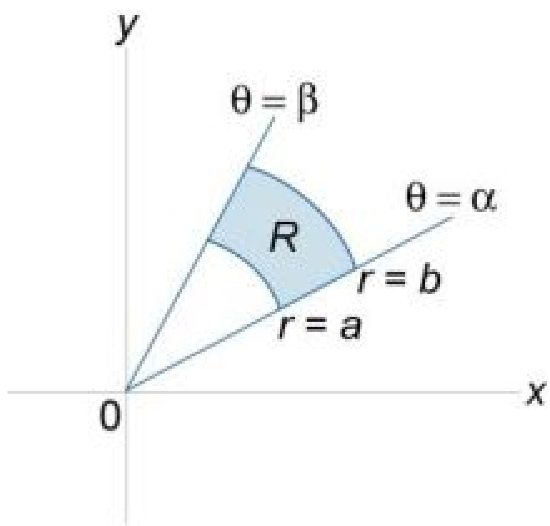

Figure 197.

1090. Area of a Region

$$A = \int_a^b \int_{g(x)}^{f(x)} dydx \quad \text{(for a type I region).}$$

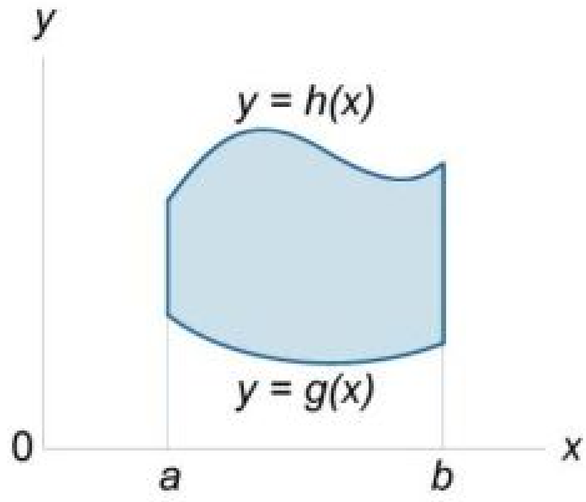

Figure 198.

$$A = \int_c^d \int_{p(y)}^{q(y)} dxdy \quad \text{(for a type II region).}$$

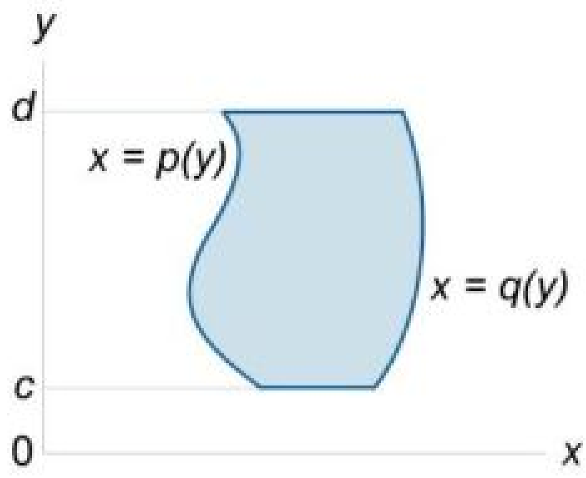

Figure 199.

1091. Volume of a Solid

$$V = \iint_R f(x,y)\,dA.$$

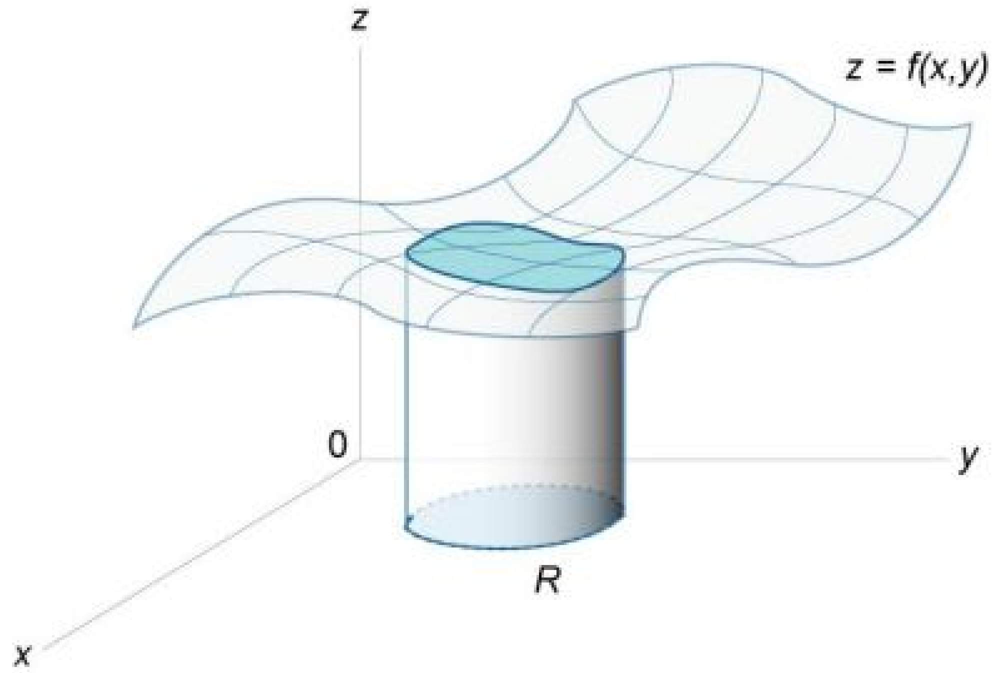

Figure 200.

If R is a type I region bounded by $x = a$, $x = b$, $y = h(x)$, $y = g(x)$, then

$$V = \iint_R f(x,y)\,dA = \int_a^b \int_{h(x)}^{g(x)} f(x,y)\,dy\,dx.$$

If R is a type II region bounded by $y = c$, $y = d$, $x = q(y)$, $x = p(y)$, then

$$V = \iint_R f(x,y)\,dA = \int_c^d \int_{p(y)}^{q(y)} f(x,y)\,dx\,dy.$$

If $f(x,y) \ge g(x,y)$ over a region R, then the volume of the solid between $z_1 = f(x,y)$ and $z_2 = g(x,y)$ over R is given by

$$V = \iint\limits_R [f(x,y) - g(x,y)]dA.$$

1092. Area and Volume in Polar Coordinates

If S is a region in the xy-plane bounded by $\theta = \alpha$, $\theta = \beta$, $r = h(\theta)$, $r = g(\theta)$,

then

$$A = \iint\limits_S dA = \int\limits_{\alpha}^{\beta}\int\limits_{h(\theta)}^{g(\theta)} r dr d\theta,$$

$$V = \iint\limits_S f(r,\theta) r dr d\theta.$$

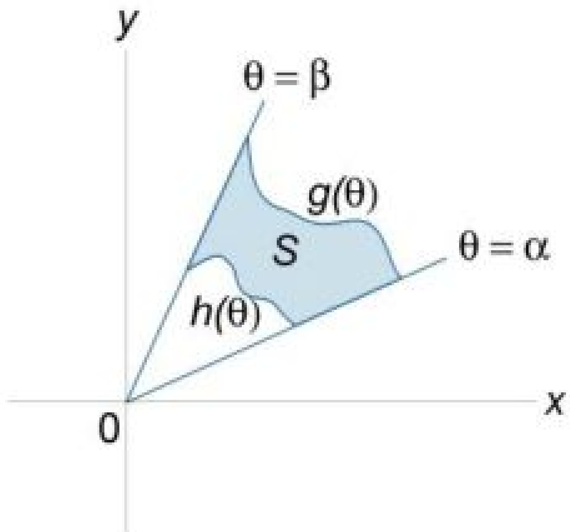

Figure 201.

1093. Surface Area

$$S = \iint\limits_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy$$

1094. Mass of a Lamina

$$m = \iint_R \rho(x,y)\,dA,$$

where the lamina occupies a region R and its density at a point (x,y) is $\rho(x,y)$.

1095. Moments

The moment of the lamina about the x-axis is given by formula

$$M_x = \iint_R y\rho(x,y)\,dA.$$

The moment of the lamina about the y-axis is

$$M_y = \iint_R x\rho(x,y)\,dA.$$

The moment of inertia about the x-axis is

$$I_x = \iint_R y^2\rho(x,y)\,dA.$$

The moment of inertia about the y-axis is

$$I_y = \iint_R x^2\rho(x,y)\,dA.$$

The polar moment of inertia is

$$I_0 = \iint_R \left(x^2 + y^2\right)\rho(x,y)\,dA.$$

1096. Center of Mass

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x,y)\,dA = \frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA},$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x,y)dA = \frac{\iint_R y\rho(x,y)dA}{\iint_R \rho(x,y)dA}.$$

1097. Charge of a Plate

$$Q = \iint_R \sigma(x,y)dA,$$

where electrical charge is distributed over a region R and its charge density at a point (x,y) is $\sigma(x,y)$.

1098. Average of a Function

$$\mu = \frac{1}{S}\iint_R f(x,y)dA,$$

where $S = \iint_R dA$.

9.11 Triple Integral

Functions of three variables: $f(x,y,z)$, $g(x,y,z)$, ...

Triple integrals: $\iiint_G f(x,y,z)dV$, $\iiint_G g(x,y,z)dV$, ...

Riemann sum: $\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(u_i, v_j, w_k)\Delta x_i \Delta y_j \Delta z_k$

Small changes: Δx_i, Δy_j, Δz_k

Limits of integration: a, b, c, d, r, s

Regions of integration: G, T, S

Cylindrical coordinates: r, θ, z

Spherical coordinates: r, θ, φ

Volume of a solid: V

Mass of a solid: m
Density: $\mu(x,y,z)$
Coordinates of center of mass: $\bar{x}$, $\bar{y}$, $\bar{z}$
First moments: M_{xy}, M_{yz}, M_{xz}
Moments of inertia: I_{xy}, I_{yz}, I_{xz}, I_x, I_y, I_z, I_0

1099. Definition of Triple Integral
The triple integral over a parallelepiped $[a, b]\times[c, d]\times[r, s]$ is defined to be

$$\iiint\limits_{[a,\, b]\times[c,\, d]\times[r,\, s]} f(x,y,z)dV = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0 \\ \max \Delta z_k \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(u_i, v_j, w_k)\Delta x_i \Delta y_j \Delta z_k,$$

where (u_i, v_j, w_k) is some point in the parallelepiped $(x_{i-1}, x_i)\times(y_{j-1}, y_j)\times(z_{k-1}, z_k)$, and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, $\Delta z_k = z_k - z_{k-1}$.

1100. $$\iiint\limits_G [f(x,y,z)+g(x,y,z)]dV = \iiint\limits_G f(x,y,z)dV + \iiint\limits_G g(x,y,z)dV$$

1101. $$\iiint\limits_G [f(x,y,z)-g(x,y,z)]dV = \iiint\limits_G f(x,y,z)dV - \iiint\limits_G g(x,y,z)dV$$

1102. $$\iiint\limits_G kf(x,y,z)dV = k\iiint\limits_G f(x,y,z)dV,$$

where k is a constant.

1103. If $f(x,y,z)\ge 0$ and G and T are nonoverlapping basic regions, then

$$\iiint\limits_{G\cup T} f(x,y,z)dV = \iiint\limits_G f(x,y,z)dV + \iiint\limits_T f(x,y,z)dV.$$

Here $G\cup T$ is the union of the regions G and T.

1104. Evaluation of Triple Integrals by Repeated Integrals

If the solid G is the set of points (x,y,z) such that $(x,y)\in R,\ \chi_1(x,y)\le z\le \chi_2(x,y)$, then

$$\iiint\limits_G f(x,y,z)dxdydz=\iint\limits_R\left[\int\limits_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)dz\right]dxdy,$$

where R is projection of G onto the xy-plane.

If the solid G is the set of points (x,y,z) such that $a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x), \chi_1(x,y)\le z\le \chi_2(x,y)$, then

$$\iiint\limits_G f(x,y,z)dxdydz=\int\limits_a^b\left[\int\limits_{\varphi_1(x)}^{\varphi_2(x)}\left(\int\limits_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)dz\right)dy\right]dx$$

1105. Triple Integrals over Parallelepiped

If G is a parallelepiped $[a,\ b]\times[c,\ d]\times[r,\ s]$, then

$$\iiint\limits_G f(x,y,z)dxdydz=\int\limits_a^b\left[\int\limits_c^d\left(\int\limits_r^s f(x,y,z)dz\right)dy\right]dx.$$

In the special case where the integrand $f(x,y,z)$ can be written as $g(x)h(y)k(z)$ we have

$$\iiint\limits_G f(x,y,z)dxdydz=\left(\int\limits_a^b g(x)dx\right)\left(\int\limits_c^d h(y)dy\right)\left(\int\limits_r^s k(z)dz\right).$$

1106. Change of Variables

$$\iiint\limits_G f(x,y,z)dxdydz=$$

$$=\iiint\limits_S f[x(u,v,w),y(u,v,w),z(u,v,w)]\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dxdydz,$$

where $\left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} \neq 0$ is the jacobian of the transformations $(x,y,z) \to (u,v,w)$, and S is the pull-back of G which can be computed by $x = x(u,v,w)$, $y = y(u,v,w)$
$z = z(u,v,w)$ into the definition of G.

1107. Triple Integrals in Cylindrical Coordinates
The differential dxdydz for cylindrical coordinates is

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| drd\theta dz = rdrd\theta dz.$$

Let the solid G is determined as follows:
$(x,y) \in R,\ \chi_1(x,y) \le z \le \chi_2(x,y)$,
where R is projection of G onto the xy-plane. Then

$$\iiint\limits_G f(x,y,z)dxdydz = \iiint\limits_S f(r\cos\theta, r\sin\theta, z)rdrd\theta dz$$

$$= \iint\limits_{R(r,\theta)} \left[\int\limits_{\chi_1(r\cos\theta, r\sin\theta)}^{\chi_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)dz \right] rdrd\theta.$$

Here S is the pullback of G in cylindrical coordinates.

1108. Triple Integrals in Spherical Coordinates
The Differential dxdydz for Spherical Coordinates is

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| drd\theta d\varphi = r^2 \sin\theta drd\theta d\varphi$$

$$\iiint\limits_G f(x,y,z)dxdydz =$$

$$= \iiint_S f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\varphi,$$

where the solid S is the pullback of G in spherical coordinates. The angle θ ranges from 0 to 2π, the angle φ ranges from 0 to π.

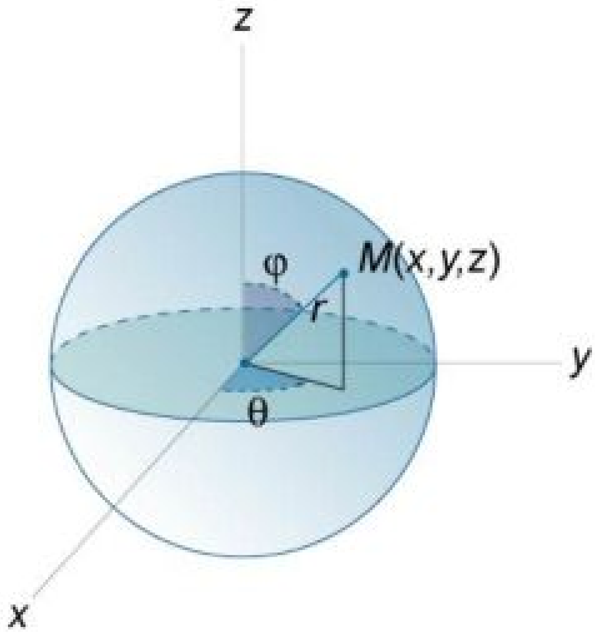

Figure 202.

1109. Volume of a Solid

$$V = \iiint_G dx\,dy\,dz$$

1110. Volume in Cylindrical Coordinates

$$V = \iiint_{S(r,\theta,z)} r\,dr\,d\theta\,dz$$

1111. Volume in Spherical Coordinates

$$V = \iiint_{S(r,\theta,\varphi)} r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

1112. Mass of a Solid

$$m = \iiint\limits_G \mu(x,y,z)dV,$$

where the solid occupies a region G and its density at a point (x,y,z) is $\mu(x,y,z)$.

1113. Center of Mass of a Solid

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \iiint\limits_G x\mu(x,y,z)\,dV,$$

$$M_{xz} = \iiint\limits_G y\mu(x,y,z)\,dV,$$

$$M_{xy} = \iiint\limits_G z\mu(x,y,z)\,dV$$

are the first moments about the coordinate planes $x=0$, $y=0$, $z=0$, respectively, $\mu(x,y,z)$ is the density function.

1114. Moments of Inertia about the xy-plane (or $z=0$), yz-plane ($x=0$), and xz-plane ($y=0$)

$$I_{xy} = \iiint\limits_G z^2\mu(x,y,z)\,dV,$$

$$I_{yz} = \iiint\limits_G x^2\mu(x,y,z)\,dV,$$

$$I_{xz} = \iiint\limits_G y^2\mu(x,y,z)\,dV.$$

1115. Moments of Inertia about the x-axis, y-axis, and z-axis

$$I_x = I_{xy} + I_{xz} = \iiint\limits_G \left(z^2 + y^2\right)\mu(x,y,z)\,dV,$$

$$I_y = I_{xy} + I_{yz} = \iiint\limits_G \left(z^2 + x^2\right)\mu(x,y,z)\,dV,$$

$$I_z = I_{xz} + I_{yz} = \iiint_G (y^2 + x^2)\mu(x,y,z)\,dV.$$

1116. Polar Moment of Inertia

$$I_0 = I_{xy} + I_{yz} + I_{xz} = \iiint_G (x^2 + y^2 + z^2)\mu(x,y,z)\,dV$$

9.12 Line Integral

Scalar functions: $F(x,y,z)$, $F(x,y)$, $f(x)$
Scalar potential: $u(x,y,z)$
Curves: C, C_1, C_2
Limits of integrations: a, b, α, β
Parameters: t, s
Polar coordinates: r, θ
Vector field: $\vec{F}(P,Q,R)$
Position vector: $\vec{r}(s)$
Unit vectors: $\vec{i}$, $\vec{j}$, $\vec{k}$, $\vec{\tau}$
Area of region: S
Length of a curve: L
Mass of a wire: m
Density: $\rho(x,y,z)$, $\rho(x,y)$
Coordinates of center of mass: $\bar{x}$, $\bar{y}$, $\bar{z}$
First moments: M_{xy}, M_{yz}, M_{xz}
Moments of inertia: I_x, I_y, I_z
Volume of a solid: V
Work: W
Magnetic field: $\vec{B}$
Current: I
Electromotive force: ε
Magnetic flux: ψ

1117. Line Integral of a Scalar Function
Let a curve C be given by the vector function $\vec{r} = \vec{r}(s)$, $0 \le s \le S$, and a scalar function F is defined over the curve C. Then

$$\int_0^S F(\vec{r}(s))ds = \int_C F(x,y,z)ds = \int_C F ds,$$

where ds is the arc length differential.

1118. $$\int_{C_1 \cup C_2} F\, ds = \int_{C_1} F\, ds + \int_{C_2} F\, ds$$

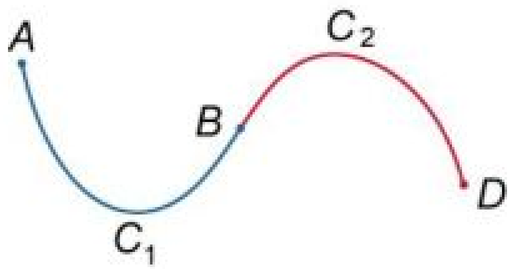

Figure 203.

1119. If the smooth curve C is parametrized by $\vec{r} = \vec{r}(t)$, $\alpha \le t \le \beta$, then

$$\int_C F(x,y,z)ds = \int_\alpha^\beta F(x(t),y(t),z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt.$$

1120. If C is a smooth curve in the xy-plane given by the equation $y = f(x)$, $a \le x \le b$, then

$$\int_C F(x,y)ds = \int_a^b F(x,f(x))\sqrt{1 + (f'(x))^2}\,dx.$$

1121. Line Integral of Scalar Function in Polar Coordinates

$$\int_C F(x,y)ds = \int_\alpha^\beta F(r\cos\theta, r\sin\theta)\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta\,,$$

where the curve C is defined by the polar function $r(\theta)$.

1122. Line Integral of Vector Field

Let a curve C be defined by the vector function $\vec{r} = \vec{r}(s)$, $0 \le s \le S$. Then

$$\frac{d\vec{r}}{ds} = \vec{\tau} = (\cos\alpha, \cos\beta, \cos\gamma)$$

is the unit vector of the tangent line to this curve.

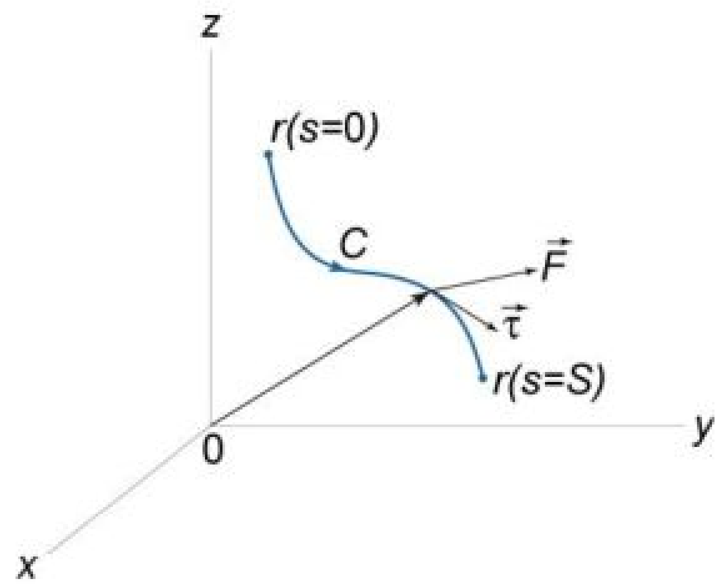

Figure 204.

Let a vector field $\vec{F}(P,Q,R)$ is defined over the curve C. Then the line integral of the vector field $\vec{F}$ along the curve C is

$$\int_C Pdx + Qdy + Rdz = \int_0^S (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds\,.$$

1123. Properties of Line Integrals of Vector Fields

$$\int_{-C} \left(\vec{F} \cdot d\vec{r}\right) = -\int_{C} \left(\vec{F} \cdot d\vec{r}\right),$$

where -C denote the curve with the opposite orientation.

$$\int_{C} \left(\vec{F} \cdot d\vec{r}\right) = \int_{C_1 \cup C_2} \left(\vec{F} \cdot d\vec{r}\right) = \int_{C_1} \left(\vec{F} \cdot d\vec{r}\right) + \int_{C_2} \left(\vec{F} \cdot d\vec{r}\right),$$

where C is the union of the curves C_1 and C_2.

1124. If the curve C is parameterized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $\alpha \le t \le \beta$, then

$$\int_{C} Pdx + Qdy + Rdz =$$

$$= \int_{\alpha}^{\beta} \left(P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt} \right) dt$$

1125. If C lies in the xy-plane and given by the equation $y = f(x)$, then

$$\int_{C} Pdx + Qdy = \int_{a}^{b} \left(P(x, f(x)) + Q(x, f(x)) \frac{df}{dx} \right) dx.$$

1126. Green's Theorem

$$\iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_{C} Pdx + Qdy,$$

where $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a continuous vector function with continuous first partial derivatives $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ in a some domain R, which is bounded by a closed, piecewise smooth curve C.

1127. Area of a Region R Bounded by the Curve C

$$S = \iint_R dxdy = \frac{1}{2}\oint_C xdy - ydx$$

1128. Path Independence of Line Integrals

The line integral of a vector function $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is said to be path independent, if and only if P, Q, and R are continuous in a domain D, and if there exists some scalar function $u = u(x, y, z)$ (a scalar potential) in D such that

$\vec{F} = \operatorname{grad} u$, or $\dfrac{\partial u}{\partial x} = P$, $\dfrac{\partial u}{\partial y} = Q$, $\dfrac{\partial u}{\partial z} = R$.

Then

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C Pdx + Qdy + Rdz = u(B) - u(A).$$

1129. Test for a Conservative Field

A vector field of the form $\vec{F} = \operatorname{grad} u$ is called a conservative field. The line integral of a vector function $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is path independent if and only if

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{0}.$$

If the line integral is taken in xy-plane so that

$$\int_C Pdx + Qdy = u(B) - u(A),$$

then the test for determining if a vector field is conservative can be written in the form

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

1130. Length of a Curve

$$L = \int_C ds = \int_\alpha^\beta \left| \frac{d\vec{r}}{dt}(t) \right| dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt,$$

where C ia a piecewise smooth curve described by the position vector $\vec{r}(t)$, $\alpha \le t \le \beta$.

If the curve C is two-dimensional, then

$$L = \int_C ds = \int_\alpha^\beta \left| \frac{d\vec{r}}{dt}(t) \right| dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

If the curve C is the graph of a function $y = f(x)$ in the xy-plane $(a \le x \le b)$, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

1131. Length of a Curve in Polar Coordinates

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta,$$

where the curve C is given by the equation $r = r(\theta)$, $\alpha \le \theta \le \beta$ in polar coordinates.

1132. Mass of a Wire

$$m = \int_C \rho(x, y, z) ds,$$

where $\rho(x, y, z)$ is the mass per unit length of the wire.

If C is a curve parametrized by the vector function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then the mass can be computed by the formula

$$m = \int_{\alpha}^{\beta} \rho(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt .$$

If C is a curve in xy-plane, then the mass of the wire is given by

$$m = \int_C \rho(x, y) ds ,$$

or

$$m = \int_{\alpha}^{\beta} \rho(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \text{ (in parametric form).}$$

1133. Center of Mass of a Wire

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \int_C x\rho(x, y, z) ds ,$$

$$M_{xz} = \int_C y\rho(x, y, z) ds ,$$

$$M_{xy} = \int_C z\rho(x, y, z) ds .$$

1134. Moments of Inertia

The moments of inertia about the x-axis, y-axis, and z-axis are given by the formulas

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z) ds ,$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z) ds ,$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z) ds .$$

1135. Area of a Region Bounded by a Closed Curve

$$S = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx .$$

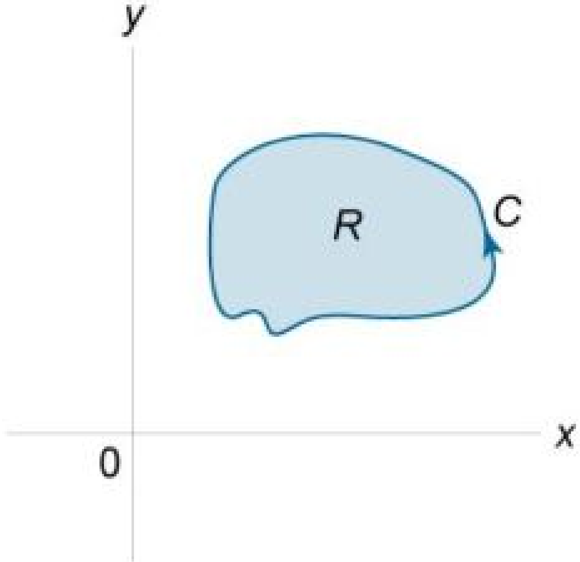

Figure 205.

If the closed curve C is given in parametric form $\vec{r}(t) = \langle x(t), y(t) \rangle$, then the area can be calculated by the formula

$$S = \int_\alpha^\beta x(t)\frac{dy}{dt}dt = -\int_\alpha^\beta y(t)\frac{dx}{dt}dt = \frac{1}{2}\int_\alpha^\beta \left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt} \right) dt .$$

1136. Volume of a Solid Formed by Rotating a Closed Curve about the x-axis

$$V = -\pi\oint_C y^2 dx = -2\pi\oint_C xy\,dy = -\frac{\pi}{2}\oint_C 2xy\,dy + y^2 dx$$

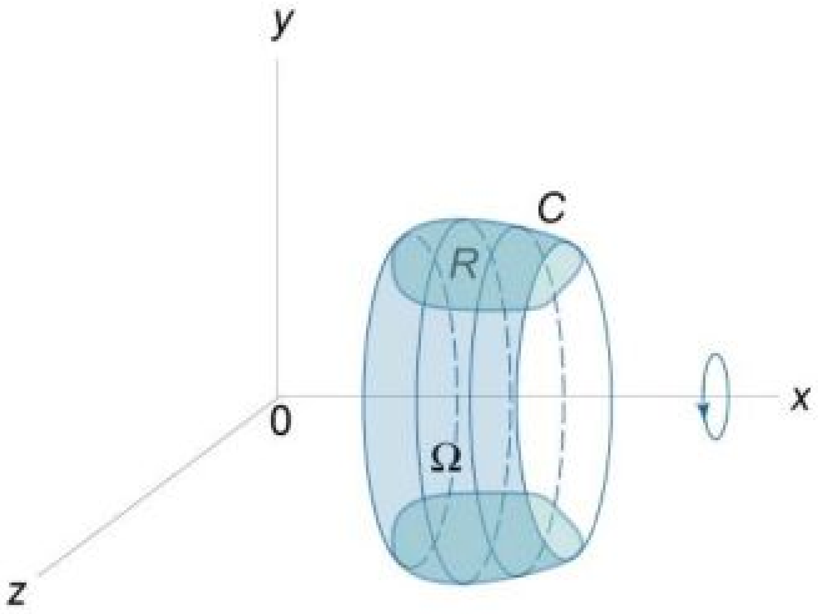

Figure 206.

1137. Work

Work done by a force $\vec{F}$ on an object moving along a curve C is given by the line integral

$$W = \int_C \vec{F} \cdot d\vec{r},$$

where $\vec{F}$ is the vector force field acting on the object, $d\vec{r}$ is the unit tangent vector.

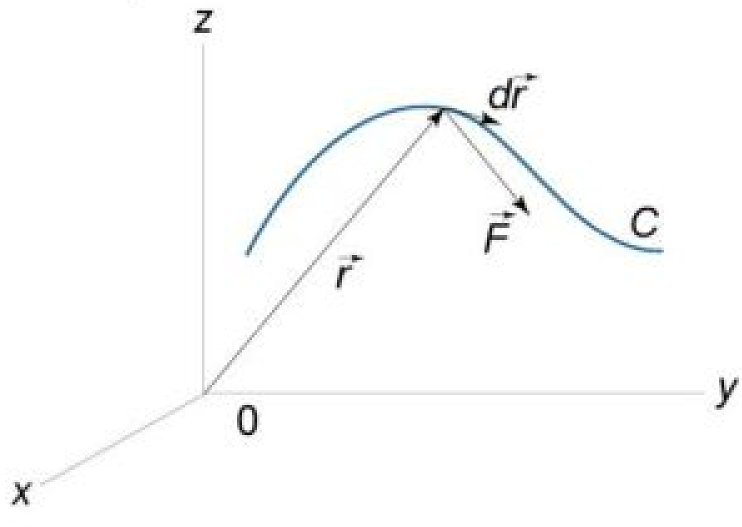

Figure 207.

If the object is moved along a curve C in the xy-plane, then
$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy\,,$$

If a path C is specified by a parameter t (t often means time), the formula for calculating work becomes
$$W = \int_\alpha^\beta \left[P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt} \right] dt,$$
where t goes from α to β.

If a vector field $\vec{F}$ is conservative and $u(x, y, z)$ is a scalar potential of the field, then the work on an object moving from A to B can be found by the formula
$$W = u(B) - u(A).$$

1138. Ampere's Law
$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I\,.$$
The line integral of a magnetic field $\vec{B}$ around a closed path C is equal to the total current I flowing through the area bounded by the path.

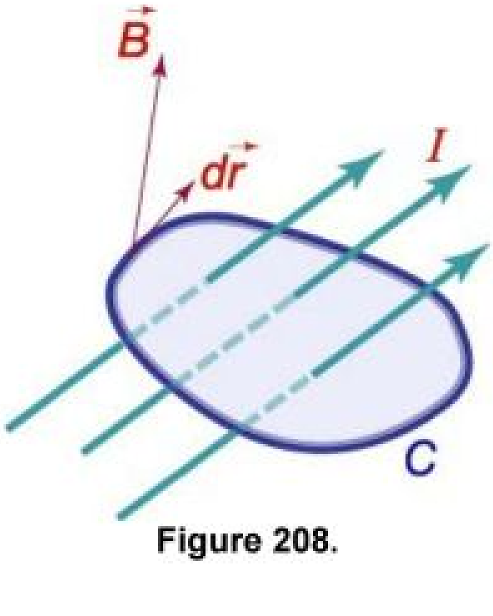

Figure 208.

1139. Faraday's Law

$$\varepsilon = \oint_C \vec{E} \cdot d\vec{r} = -\frac{d\psi}{dt}$$

The electromotive force (emf) ε induced around a closed loop C is equal to the rate of the change of magnetic flux ψ passing through the loop.

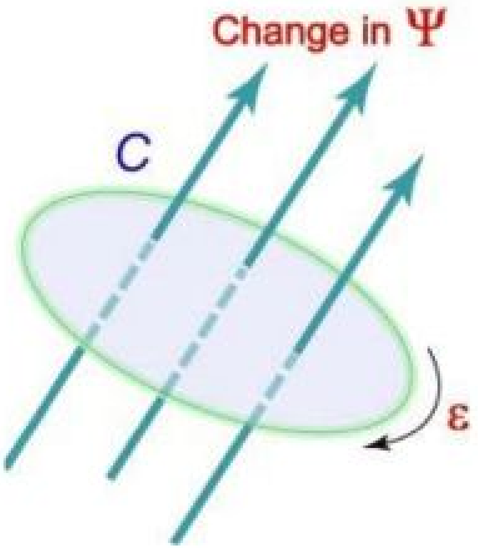

Figure 209.

9.13 Surface Integral

Scalar functions: $f(x,y,z)$, $z(x,y)$
Position vectors: $\vec{r}(u,v)$, $\vec{r}(x,y,z)$
Unit vectors: $\vec{i}$, $\vec{j}$, $\vec{k}$
Surface: S
Vector field: $\vec{F}(P,Q,R)$
Divergence of a vector field: $\operatorname{div}\vec{F} = \nabla \cdot \vec{F}$

Curl of a vector field: $\text{curl}\,\vec{F} = \nabla \times \vec{F}$

Vector element of a surface: $d\vec{S}$

Normal to surface: $\vec{n}$

Surface area: A

Mass of a surface: m

Density: $\mu(x, y, z)$

Coordinates of center of mass: $\bar{x}$, $\bar{y}$, $\bar{z}$

First moments: M_{xy}, M_{yz}, M_{xz}

Moments of inertia: I_{xy}, I_{yz}, I_{xz}, I_x, I_y, I_z

Volume of a solid: V

Force: $\vec{F}$

Gravitational constant: G

Fluid velocity: $\vec{v}(\vec{r})$

Fluid density: ρ

Pressure: $p(\vec{r})$

Mass flux, electric flux: Φ

Surface charge: Q

Charge density: $\sigma(x, y)$

Magnitude of the electric field: $\vec{E}$

1140. Surface Integral of a Scalar Function

Let a surface S be given by the position vector

$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$,

where (u, v) ranges over some domain $D(u, v)$ of the uv-plane.

The surface integral of a scalar function $f(x, y, z)$ over the surface S is defined as

$$\iint\limits_S f(x, y, z)\,dS = \iint\limits_{D(u,v)} f(x(u, v), y(u, v), z(u, v)) \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\,dv,$$

where the partial derivatives $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ are given by

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}(u,v)\cdot \vec{i} + \frac{\partial y}{\partial u}(u,v)\cdot \vec{j} + \frac{\partial z}{\partial u}(u,v)\cdot \vec{k},$$
$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v}(u,v)\cdot \vec{i} + \frac{\partial y}{\partial v}(u,v)\cdot \vec{j} + \frac{\partial z}{\partial v}(u,v)\cdot \vec{k}.$$

1143. If the surface S is given by the equation $z = z(x,y)$, where $z(x,y)$ is a differentiable function in the domain $D(x,y)$, then

- If S is oriented upward, i.e. the k-th component of the normal vector is positive, then
$$\iint_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint_S \vec{F}(x,y,z)\cdot \vec{n}dS$$
$$= \iint_{D(x,y)} \vec{F}(x,y,z)\cdot\left(-\frac{\partial z}{\partial x}\vec{i} - \frac{\partial z}{\partial y}\vec{j} + \vec{k}\right)dxdy,$$

- If S is oriented downward, i.e. the k-th component of the normal vector is negative, then
$$\iint_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint_S \vec{F}(x,y,z)\cdot \vec{n}dS$$
$$= \iint_{D(x,y)} \vec{F}(x,y,z)\cdot\left(\frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j} - \vec{k}\right)dxdy.$$

1144. $$\iint_S (\vec{F}\cdot\vec{n})dS = \iint_S Pdydz + Qdzdx + Rdxdy$$
$$= \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS,$$

where $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ are the components of the vector field $\vec{F}$.
$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the angles between the outer unit normal vector $\vec{n}$ and the x-axis, y-axis, and z-axis, respectively.

1145. If the surface S is given in parametric form by the vector $\vec{r}(x(u,v), y(u,v), z(u,v))$, then the latter formula can be written as

$$\iint_S (\vec{F}\cdot\vec{n})dS = \iint_S Pdydz + Qdzdx + Rdxdy = \iint_{D(u,v)} \begin{vmatrix} P & Q & R \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} dudv,$$

where (u,v) ranges over some domain $D(u,v)$ of the uv-plane.

1146. Divergence Theorem

$$\oint\!\!\!\oint_S \vec{F}\cdot d\vec{S} = \iiint_G (\nabla\cdot\vec{F})dV,$$

where

$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$

is a vector field whose components P, Q, and R have continuous partial derivatives,

$$\nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is the divergence of $\vec{F}$, also denoted $\mathrm{div}\vec{F}$. The symbol $\oint\!\!\!\oint$ indicates that the surface integral is taken over a closed surface.

1147. Divergence Theorem in Coordinate Form

$$\oint\!\!\!\oint_S Pdydz + Qdxdz + Rdxdy = \iiint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dxdydz.$$

1148. Stoke's Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S},$$

where

$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

is a vector field whose components P, Q, and R have continuous partial derivatives,

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

is the curl of $\vec{F}$, also denoted $\operatorname{curl} \vec{F}$.

The symbol $\oint$ indicates that the line integral is taken over a closed curve.

1149. Stoke's Theorem in Coordinate Form

$$\oint_C P\,dx + Q\,dy + R\,dz$$

$$= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

1150. Surface Area

$$A = \iint_S dS$$

1151. If the surface S is parameterized by the vector

$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$,

then the surface area is

$$A = \iint_{D(u,v)} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dudv,$$

where $D(u,v)$ is the domain where the surface $\vec{r}(u,v)$ is defined.

1152. If S is given explicitly by the function $z(x,y)$, then the surface area is

$$A = \iint\limits_{D(x,y)} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy,$$

where $D(x,y)$ is the projection of the surface S onto the xy-plane.

1153. Mass of a Surface

$$m = \iint\limits_{S} \mu(x,y,z)dS,$$

where $\mu(x,y,z)$ is the mass per unit area (density function).

1154. Center of Mass of a Shell

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \iint\limits_{S} x\mu(x,y,z)dS,$$

$$M_{xz} = \iint\limits_{S} y\mu(x,y,z)dS,$$

$$M_{xy} = \iint\limits_{S} z\mu(x,y,z)dS$$

are the first moments about the coordinate planes $x=0$, $y=0$, $z=0$, respectively. $\mu(x,y,z)$ is the density function.

1155. Moments of Inertia about the xy-plane (or $z=0$), yz-plane ($x=0$), and xz-plane ($y=0$)

$$I_{xy} = \iint\limits_{S} z^2\mu(x,y,z)dS,$$

$$I_{yz} = \iint\limits_{S} x^2\mu(x,y,z)dS,$$

$$I_{xz} = \iint_S y^2 \mu(x,y,z)\,dS.$$

1156. Moments of Inertia about the x-axis, y-axis, and z-axis

$$I_x = \iint_S \left(y^2 + z^2\right)\mu(x,y,z)\,dS,$$

$$I_y = \iint_S \left(x^2 + z^2\right)\mu(x,y,z)\,dS,$$

$$I_z = \iint_S \left(x^2 + y^2\right)\mu(x,y,z)\,dS.$$

1157. Volume of a Solid Bounded by a Closed Surface

$$V = \frac{1}{3}\left|\oiint_S x\,dydz + y\,dxdz + z\,dxdy\right|$$

1158. Gravitational Force

$$\vec{F} = Gm\iint_S \mu(x,y,z)\frac{\vec{r}}{r^3}\,dS,$$

where m is a mass at a point $\langle x_0, y_0, z_0\rangle$ outside the surface,

$\vec{r} = \langle x - x_0, y - y_0, z - z_0\rangle$,

$\mu(x,y,z)$ is the density function,

and G is gravitational constant.

1159. Pressure Force

$$\vec{F} = \iint_S p(\vec{r})\,d\vec{S},$$

where the pressure $p(\vec{r})$ acts on the surface S given by the position vector $\vec{r}$.

1160. Fluid Flux (across the surface S)

$$\Phi = \oiint_S \vec{v}(\vec{r})\cdot d\vec{S},$$

where $\vec{v}(\vec{r})$ is the fluid velocity.

1161. Mass Flux (across the surface S)

$$\Phi = \oiint_S \rho \vec{v}(\vec{r}) \cdot d\vec{S},$$

where $\vec{F} = \rho\vec{v}$ is the vector field, ρ is the fluid density.

1162. Surface Charge

$$Q = \iint_S \sigma(x, y) dS,$$

where $\sigma(x, y)$ is the surface charge density.

1163. Gauss' Law

The electric flux through any closed surface is proportional to the charge Q enclosed by the surface

$$\Phi = \oiint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0},$$

where

Φ is the electric flux,

$\vec{E}$ is the magnitude of the electric field strength,

$\varepsilon_0 = 8{,}85 \times 10^{-12} \frac{F}{m}$ is permittivity of free space.

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}(u,v)\vec{i} + \frac{\partial y}{\partial u}(u,v)\vec{j} + \frac{\partial z}{\partial u}(u,v)\vec{k},$$

$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v}(u,v)\vec{i} + \frac{\partial y}{\partial v}(u,v)\vec{j} + \frac{\partial z}{\partial v}(u,v)\vec{k}$$

and $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ is the cross product.

1141. If the surface S is given by the equation $z = z(x,y)$ where $z(x,y)$ is a differentiable function in the domain $D(x,y)$, then

$$\iint_S f(x,y,z)dS = \iint_{D(x,y)} f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy.$$

1142. Surface Integral of the Vector Field $\vec{F}$ over the Surface S

- If S is oriented outward, then

$$\iint_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint_S \vec{F}(x,y,z)\cdot \vec{n}dS$$

$$= \iint_{D(u,v)} \vec{F}(x(u,v),y(u,v),z(u,v))\cdot\left[\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right]dudv.$$

- If S is oriented inward, then

$$\iint_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint_S \vec{F}(x,y,z)\cdot \vec{n}dS$$

$$= \iint_{D(u,v)} \vec{F}(x(u,v),y(u,v),z(u,v))\cdot\left[\frac{\partial \vec{r}}{\partial v}\times\frac{\partial \vec{r}}{\partial u}\right]dudv.$$

$d\vec{S} = \vec{n}dS$ is called the vector element of the surface. Dot means the scalar product of the appropriate vectors. The partial derivatives $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ are given by