Chapter 2

Algebra

2.1 Factoring Formulas

Real numbers: a, b, c
Natural number: n

65. $a^2 - b^2 = (a+b)(a-b)$

66. $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

67. $a^3 + b^3 = (a+b)(a^2-ab+b^2)$

68. $a^4 - b^4 = (a^2-b^2)(a^2+b^2) = (a-b)(a+b)(a^2+b^2)$

69. $a^5 - b^5 = (a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$

70. $a^5 + b^5 = (a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$

71. If n is odd, then
$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots - ab^{n-2} + b^{n-1})$.

72. If n is even, then
$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$,

$$a^n + b^n = (a+b)\left(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots + ab^{n-2} - b^{n-1}\right).$$

2.2 Product Formulas

Real numbers: a, b, c
Whole numbers: n, k

73. $(a-b)^2 = a^2 - 2ab + b^2$

74. $(a+b)^2 = a^2 + 2ab + b^2$

75. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

76. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

77. $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

78. $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

79. Binomial Formula

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_{n-1} ab^{n-1} + {}^nC_n b^n,$$

where ${}^nC_k = \dfrac{n!}{k!(n-k)!}$ are the binomial coefficients.

80. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

81. $(a+b+c+\ldots+u+v)^2 = a^2 + b^2 + c^2 + \ldots + u^2 + v^2 +$
$+ 2(ab + ac + \ldots + au + av + bc + \ldots + bu + bv + \ldots + uv)$

2.3 Powers

Bases (positive real numbers): a, b
Powers (rational numbers): n, m

82. $a^m a^n = a^{m+n}$

83. $\frac{a^m}{a^n} = a^{m-n}$

84. $(ab)^m = a^m b^m$

85. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

86. $\left(a^m\right)^n = a^{mn}$

87. $a^0 = 1,\ a \neq 0$

88. $a^1 = 1$

89. $a^{-m} = \frac{1}{a^m}$

90. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

2.4 Roots

Bases: a, b
Powers (rational numbers): n, m
$a, b \geq 0$ for even roots ($n = 2k$, $k \in N$)

91. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

92. $\sqrt[n]{a}\,\sqrt[m]{b} = \sqrt[nm]{a^m b^n}$

93. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, $b \neq 0$

94. $\frac{\sqrt[n]{a}}{\sqrt[m]{b}} = \frac{\sqrt[nm]{a^m}}{\sqrt[nm]{b^n}} = \sqrt[nm]{\frac{a^m}{b^n}}$, $b \neq 0$.

95. $\left(\sqrt[n]{a^m}\right)^p = \sqrt[n]{a^{mp}}$

96. $\left(\sqrt[n]{a}\right)^n = a$

97. $\sqrt[n]{a^m} = \sqrt[np]{a^{mp}}$

98. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

99. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

100. $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

101. $\dfrac{1}{\sqrt[n]{a}} = \dfrac{\sqrt[n]{a^{n-1}}}{a}, \; a \neq 0.$

102. $\sqrt{a \pm \sqrt{b}} = \sqrt{\dfrac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\dfrac{a - \sqrt{a^2 - b}}{2}}$

103. $\dfrac{1}{\sqrt{a} \pm \sqrt{b}} = \dfrac{\sqrt{a} \mp \sqrt{b}}{a - b}$

2.5 Logarithms

Positive real numbers: x, y, a, c, k
Natural number: n

104. Definition of Logarithm
$y = \log_a x$ if and only if $x = a^y$, $a > 0$, $a \neq 1$.

105. $\log_a 1 = 0$

106. $\log_a a = 1$

107. $\log_a 0 = \begin{cases} -\infty & \text{if } a > 1 \\ +\infty & \text{if } a < 1 \end{cases}$

108. $\log_a (xy) = \log_a x + \log_a y$

109. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$

110. $\log_a\left(x^n\right) = n\log_a x$

111. $\log_a \sqrt[n]{x} = \frac{1}{n}\log_a x$

112. $\log_a x = \frac{\log_c x}{\log_c a} = \log_c x \cdot \log_a c,\ c > 0,\ c \neq 1.$

113. $\log_a c = \frac{1}{\log_c a}$

114. $x = a^{\log_a x}$

115. Logarithm to Base 10

$\log_{10} x = \log x$

116. Natural Logarithm

$\log_e x = \ln x,$

where $e = \lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^k = 2.718281828\ldots$

117. $\log x = \frac{1}{\ln 10}\ln x = 0.434294 \ln x$

118. $\ln x = \frac{1}{\log e}\log x = 2.302585 \log x$

2.6 Equations

Real numbers: a, b, c, p, q, u, v
Solutions: x_1, x_2, y_1, y_2, y_3

119. Linear Equation in One Variable

$$ax + b = 0,\ x = -\frac{b}{a}.$$

120. Quadratic Equation

$$ax^2 + bx + c = 0,\ x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

121. Discriminant

$$D = b^2 - 4ac$$

122. Viete's Formulas

If $x^2 + px + q = 0$, then

$$\begin{cases} x_1 + x_2 = -p \\ x_1 x_2 = q \end{cases}.$$

123. $ax^2 + bx = 0,\ x_1 = 0,\ x_2 = -\frac{b}{a}.$

124. $ax^2 + c = 0,\ x_{1,2} = \pm\sqrt{-\frac{c}{a}}.$

125. Cubic Equation. Cardano's Formula.

$$y^3 + py + q = 0,$$

$$y_1 = u + v,\ y_{2,3} = -\frac{1}{2}(u+v) \pm \frac{\sqrt{3}}{2}(u+v)i,$$

where

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2}},\ v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2}}.$$

2.7 Inequalities

Variables: x, y, z

Real numbers: $\begin{cases} a, b, c, d \\ a_1, a_2, a_3, \ldots, a_n \end{cases}$, m, n

Determinants: D, D_x, D_y, D_z

126. Inequalities, Interval Notations and Graphs

Inequality	Interval Notation	Graph
$a \le x \le b$	$[a, b]$	
$a < x \le b$	$(a, b]$	
$a \le x < b$	$[a, b)$	
$a < x < b$	(a, b)	
$-\infty < x \le b$, $x \le b$	$(-\infty, b]$	
$-\infty < x < b$, $x < b$	$(-\infty, b)$	
$a \le x < \infty$, $x \ge a$	$[a, \infty)$	
$a < x < \infty$, $x > a$	(a, ∞)	

127. If $a > b$, then $b < a$.

128. If $a > b$, then $a - b > 0$ or $b - a < 0$.

129. If $a > b$, then $a + c > b + c$.

130. If $a > b$, then $a - c > b - c$.

131. If $a > b$ and $c > d$, then $a + c > b + d$.

132. If $a > b$ and $c > d$, then $a - d > b - c$.

133. If $a > b$ and $m > 0$, then $ma > mb$.

134. If $a > b$ and $m > 0$, then $\frac{a}{m} > \frac{b}{m}$.

135. If $a > b$ and $m < 0$, then $ma < mb$.

136. If $a > b$ and $m < 0$, then $\frac{a}{m} < \frac{b}{m}$.

137. If $0 < a < b$ and $n > 0$, then $a^n < b^n$.

138. If $0 < a < b$ and $n < 0$, then $a^n > b^n$.

139. If $0 < a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$.

140. $\sqrt{ab} \le \frac{a+b}{2}$,
where $a > 0$, $b > 0$; an equality is valid only if $a = b$.

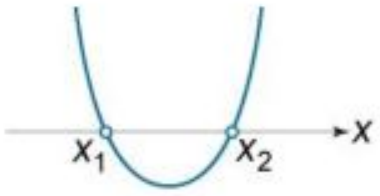
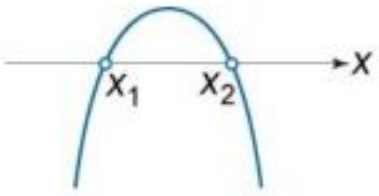
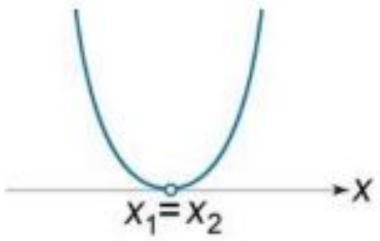
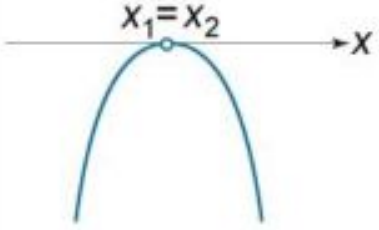
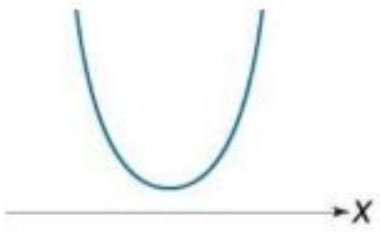
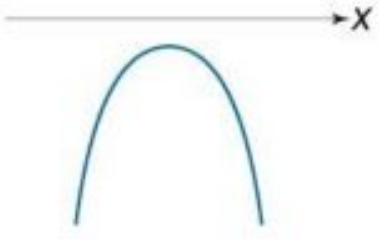
141. $a + \frac{1}{a} \ge 2$, where $a > 0$; an equality takes place only at $a = 1$.

142. $\sqrt[n]{a_1 a_2 \ldots a_n} \le \dfrac{a_1 + a_2 + \ldots + a_n}{n}$, where $a_1, a_2, \ldots, a_n > 0$.

143. If $ax + b > 0$ and $a > 0$, then $x > -\dfrac{b}{a}$.

144. If $ax + b > 0$ and $a < 0$, then $x < -\dfrac{b}{a}$.

145. $ax^2 + bx + c > 0$

	$a > 0$	$a < 0$
$D > 0$	 $x < x_1,\ x > x_2$	 $x_1 < x < x_2$
$D = 0$	 $x_1 < x,\ x > x_1$	 $x \in \varnothing$
$D < 0$	 $-\infty < x < \infty$	 $x \in \varnothing$

146. $|a+b| \le |a| + |b|$

147. If $|x| < a$, then $-a < x < a$, where $a > 0$.

148. If $|x| > a$, then $x < -a$ and $x > a$, where $a > 0$.

149. If $x^2 < a$, then $|x| < \sqrt{a}$, where $a > 0$.

150. If $x^2 > a$, then $|x| > \sqrt{a}$, where $a > 0$.

151. If $\dfrac{f(x)}{g(x)} > 0$, then $\begin{cases} f(x) \cdot g(x) > 0 \\ g(x) \neq 0 \end{cases}$.

152. $\dfrac{f(x)}{g(x)} < 0$, then $\begin{cases} f(x) \cdot g(x) < 0 \\ g(x) \neq 0 \end{cases}$.

2.8 Compound Interest Formulas

Future value: A
Initial deposit: C
Annual rate of interest: r
Number of years invested: t
Number of times compounded per year: n

153. General Compound Interest Formula

$$A = C\left(1 + \frac{r}{n}\right)^{nt}$$

154. Simplified Compound Interest Formula
If interest is compounded once per year, then the previous formula simplifies to:
$A = C(1+r)^t$.

155. Continuous Compound Interest
If interest is compounded continually ($n \to \infty$), then
$A = Ce^{rt}$.