Chapter 10
Differential Equations

Functions of one variable: $y, p, q, u, g, h, G, H, r, z$
Arguments (independent variables): x, y
Functions of two variables: $f(x,y)$, $M(x,y)$, $N(x,y)$

First order derivative: y', u', $\dot{y}$, $\frac{dy}{dt}$, ...

Second order derivatives: y'', $\ddot{y}$, $\frac{d^2I}{dt^2}$, ...

Partial derivatives: $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$, ...

Natural number: n
Particular solutions: y_1, y_p
Real numbers: k, t, C, C_1, C_2, p, q, α, β
Roots of the characteristic equations: λ_1, λ_2
Time: t
Temperature: T, S
Population function: $P(t)$
Mass of an object: m
Stiffness of a spring: k
Displacement of the mass from equilibrium: y
Amplitude of the displacement: A
Frequency: ω
Damping coefficient: γ
Phase angle of the displacement: δ
Angular displacement: θ
Pendulum length: L

Acceleration of gravity: g
Current: I
Resistance: R
Inductance: L
Capacitance: C

10.1 First Order Ordinary Differential Equations

1164. Linear Equations

$$\frac{dy}{dx} + p(x)y = q(x).$$

The general solution is

$$y = \frac{\int u(x)q(x)dx + C}{u(x)},$$

where

$$u(x) = \exp\left(\int p(x)dx\right).$$

1165. Separable Equations

$$\frac{dy}{dx} = f(x,y) = g(x)h(y)$$

The general solution is given by

$$\int \frac{dy}{h(y)} = \int g(x)dx + C,$$

or

$$H(y) = G(x) + C.$$

1166. Homogeneous Equations

The differential equation $\frac{dy}{dx} = f(x, y)$ is homogeneous, if the function $f(x, y)$ is homogeneous, that is
$f(tx, ty) = f(x, y)$.

The substitution $z = \frac{y}{x}$ (then $y = zx$) leads to the separable equation

$$x\frac{dz}{dx} + z = f(1, z).$$

1167. Bernoulli Equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n.$$

The substitution $z = y^{1-n}$ leads to the linear equation

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x).$$

1168. Riccati Equation

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2$$

If a particular solution y_1 is known, then the general solution can be obtained with the help of substitution

$z = \frac{1}{y - y_1}$, which leads to the first order linear equation

$$\frac{dz}{dx} = -[q(x) + 2y_1 r(x)]z - r(x).$$

1169. Exact and Nonexact Equations

The equation

$M(x,y)dx + N(x,y)dy = 0$

is called exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

and nonexact otherwise.

The general solution is

$\int M(x,y)dx + \int N(x,y)dy = C$.

1170. Radioactive Decay

$$\frac{dy}{dt} = -ky,$$

where $y(t)$ is the amount of radioactive element at time t, k is the rate of decay.

The solution is

$y(t) = y_0 e^{-kt}$, where $y_0 = y(0)$ is the initial amount.

1171. Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - S),$$

where $T(t)$ is the temperature of an object at time t, S is the temperature of the surrounding environment, k is a positive constant.

The solution is

$T(t) = S + (T_0 - S)e^{-kt}$,

where $T_0 = T(0)$ is the initial temperature of the object at time $t = 0$.

1172. Population Dynamics (Logistic Model)

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where $P(t)$ is population at time t, k is a positive constant, M is a limiting size for the population.

The solution of the differential equation is

$P(t) = \dfrac{MP_0}{P_0 + (M - P_0)e^{-kt}}$, where $P_0 = P(0)$ is the initial population at time $t = 0$.

10.2 Second Order Ordinary Differential Equations

1173. Homogeneous Linear Equations with Constant Coefficients

$y'' + py' + qy = 0$.

The characteristic equation is

$\lambda^2 + p\lambda + q = 0$.

If λ_1 and λ_2 are distinct real roots of the characteristic equation, then the general solution is

$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$, where

C_1 and C_2 are integration constants.

If $\lambda_1 = \lambda_2 = -\dfrac{p}{2}$, then the general solution is

$$y = (C_1 + C_2 x)e^{-\frac{p}{2}x}.$$

If λ_1 and λ_2 are complex numbers:

$\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, where

$\alpha = -\frac{p}{2}$, $\beta = \frac{\sqrt{4q - p^2}}{2}$,

then the general solution is

$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$.

1174. Inhomogeneous Linear Equations with Constant Coefficients

$y'' + py' + qy = f(x)$.

The general solution is given by

$y = y_p + y_h$, where

y_p is a particular solution of the inhomogeneous equation and y_h is the general solution of the associated homogeneous equation (see the previous topic 1173).

If the right side has the form

$f(x) = e^{\alpha x}(P_1(x)\cos\beta x + P_1(x)\sin\beta x)$,

then the particular solution y_p is given by

$y_p = x^k e^{\alpha x}(R_1(x)\cos\beta x + R_2(x)\sin\beta x)$,

where the polynomials $R_1(x)$ and $R_2(x)$ have to be found by using the method of undetermined coefficients.

- If $\alpha + \beta i$ is not a root of the characteristic equation, then the power $k = 0$,
- If $\alpha + \beta i$ is a simple root, then $k = 1$,
- If $\alpha + \beta i$ is a double root, then $k = 2$.

1175. Differential Equations with y Missing

$y'' = f(x, y')$.

Set $u = y'$. Then the new equation satisfied by v is

$u' = f(x, u)$,

which is a first order differential equation.

1176. Differential Equations with x Missing

$y'' = f(y, y')$.

Set $u = y'$. Since

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy},$$

we have

$$u\frac{du}{dy} = f(y, u),$$

which is a first order differential equation.

1177. Free Undamped Vibrations

The motion of a Mass on a Spring is described by the equation

$$m\ddot{y} + ky = 0,$$

where

m is the mass of the object,

k is the stiffness of the spring,

y is displacement of the mass from equilibrium.

The general solution is

$$y = A\cos(\omega_0 t - \delta),$$

where

A is the amplitude of the displacement,

ω_0 is the fundamental frequency, the period is $T = \frac{2\pi}{\omega_0}$,

δ is phase angle of the displacement.

This is an example of simple harmonic motion.

1178. Free Damped Vibrations

$m\ddot{y} + \gamma\dot{y} + ky = 0$, where

γ is the damping coefficient.

There are 3 cases for the general solution:

Case 1. $\gamma^2 > 4km$ (overdamped)
$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$,
where

$$\lambda_1 = \frac{-\gamma - \sqrt{\gamma^2 - 4km}}{2m},\ \lambda_2 = \frac{-\gamma + \sqrt{\gamma^2 - 4km}}{2m}.$$

Case 2. $\gamma^2 = 4km$ (critically damped)
$y(t) = (A + Bt)e^{\lambda t}$,
where

$$\lambda = -\frac{\gamma}{2m}.$$

Case 3. $\gamma^2 < 4km$ (underdamped)
$y(t) = e^{-\frac{\gamma}{2m}t} A\cos(\omega t - \delta)$, where
$\omega = \sqrt{4km - \gamma^2}$.

1179. Simple Pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0,$$

where θ is the angular displacement, L is the pendulum length, g is the acceleration of gravity.

The general solution for small angles θ is
$\theta(t) = \theta_{max} \sin\sqrt{\frac{g}{L}}t$, the period is $T = 2\pi\sqrt{\frac{L}{g}}$.

1180. RLC Circuit

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = V'(t) = \omega E_0 \cos(\omega t),$$

where I is the current in an RLC circuit with an ac voltage source $V(t) = E_0 \sin(\omega t)$.

The general solution is
$I(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + A\sin(\omega t - \varphi)$,
where

$$r_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L},$$

$$A = \frac{\omega E_0}{\sqrt{\left(L\omega^2 - \frac{1}{C}\right)^2 + R^2\omega^2}},$$

$$\varphi = \arctan\left(\frac{L\omega}{R} - \frac{1}{RC\omega}\right),$$

C_1, C_2 are constants depending on initial conditions.

10.3. Some Partial Differential Equations

1181. The Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

applies to potential energy function $u(x, y)$ for a conservative force field in the xy-plane. Partial differential equations of this type are called elliptic.

1182. The Heat Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

applies to the temperature distribution $u(x,y)$ in the xy-plane when heat is allowed to flow from warm areas to cool ones. The equations of this type are called parabolic.

1183. The Wave Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial t^2}$$

applies to the displacement $u(x,y)$ of vibrating membranes and other wave functions. The equations of this type are called hyperbolic.