Chapter 7

Analytic Geometry

7.1 One-Dimensional Coordinate System

Point coordinates: x_0, x_1, x_2, y_0, y_1, y_2
Real number: λ
Distance between two points: d

607. Distance Between Two Points
$$d = AB = |x_2 - x_1| = |x_1 - x_2|$$

Figure 86.

608. Dividing a Line Segment in the Ratio λ
$$x_0 = \frac{x_1 + \lambda x_2}{1+\lambda},\ \lambda = \frac{AC}{CB},\ \lambda \neq -1.$$

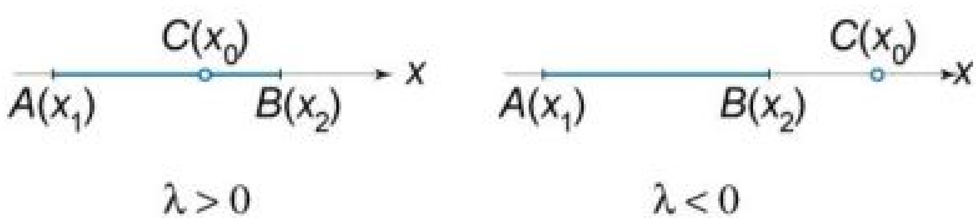


Figure 87.

609. Midpoint of a Line Segment

$$x_0 = \frac{x_1 + x_2}{2},\ \lambda = 1.$$

7.2 Two-Dimensional Coordinate System

Point coordinates: x_0, x_1, x_2, y_0, y_1, y_2
Polar coordinates: r, φ
Real number: λ
Positive real numbers: a, b, c,
Distance between two points: d
Area: S

610. Distance Between Two Points

$$d = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

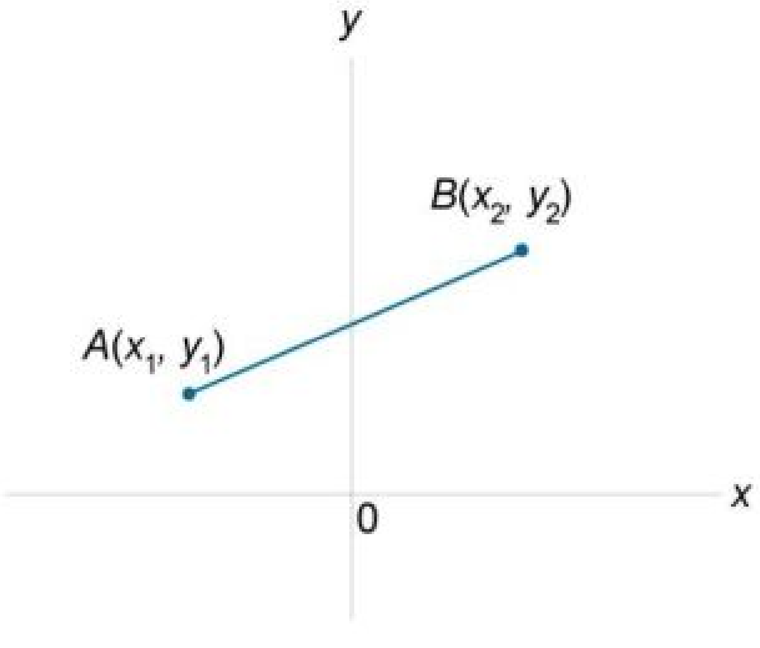


Figure 88.

611. Dividing a Line Segment in the Ratio λ

$$x_0 = \frac{x_1 + \lambda x_2}{1+\lambda},\ y_0 = \frac{y_1 + \lambda y_2}{1+\lambda},$$

$$\lambda = \frac{AC}{CB},\ \lambda \neq -1.$$

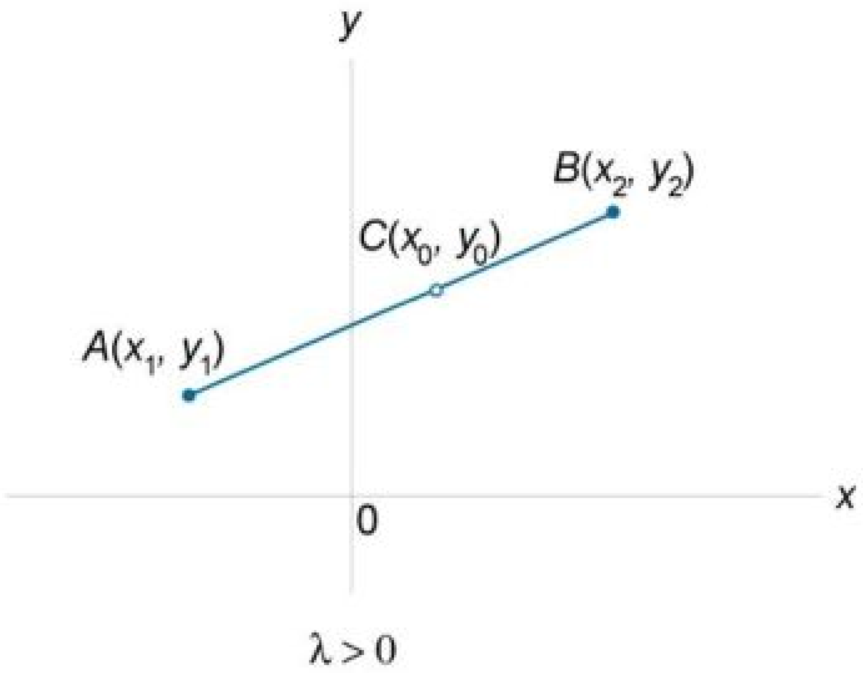


Figure 89.

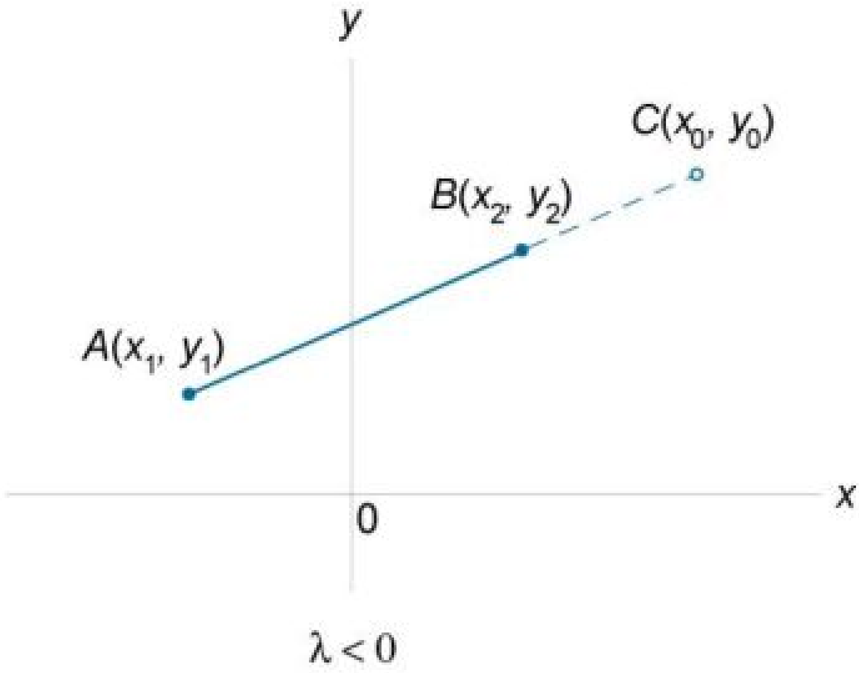


Figure 90.

612. Midpoint of a Line Segment

$$x_0 = \frac{x_1 + x_2}{2},\ y_0 = \frac{y_1 + y_2}{2},\ \lambda = 1.$$

613. Centroid (Intersection of Medians) of a Triangle

$$x_0 = \frac{x_1 + x_2 + x_3}{3},\ y_0 = \frac{y_1 + y_2 + y_3}{3},$$

where $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ are vertices of the triangle ABC.

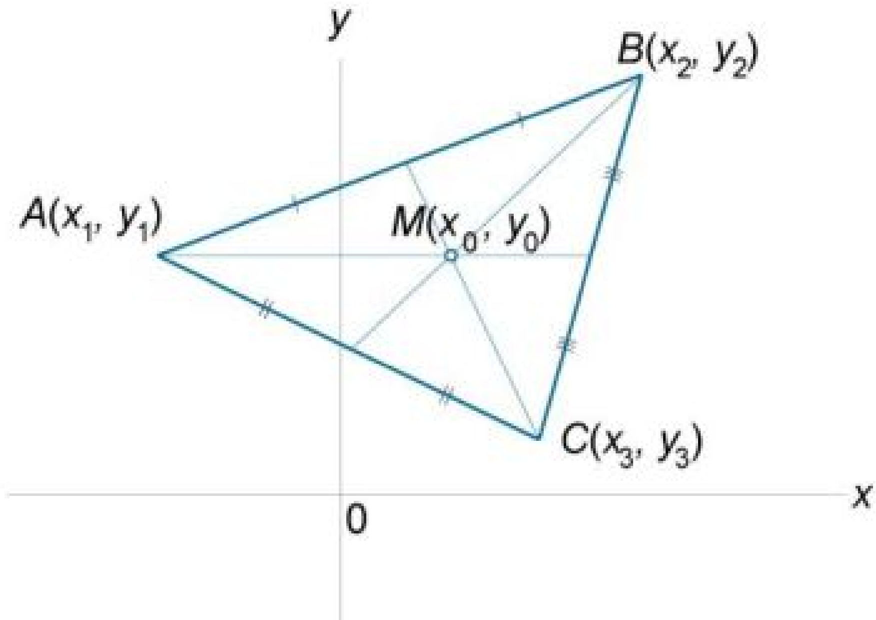


Figure 91.

614. Incenter (Intersection of Angle Bisectors) of a Triangle

$$x_0 = \frac{ax_1 + bx_2 + cx_3}{a + b + c}, \quad y_0 = \frac{ay_1 + by_2 + cy_3}{a + b + c},$$

where $a = BC$, $b = CA$, $c = AB$.

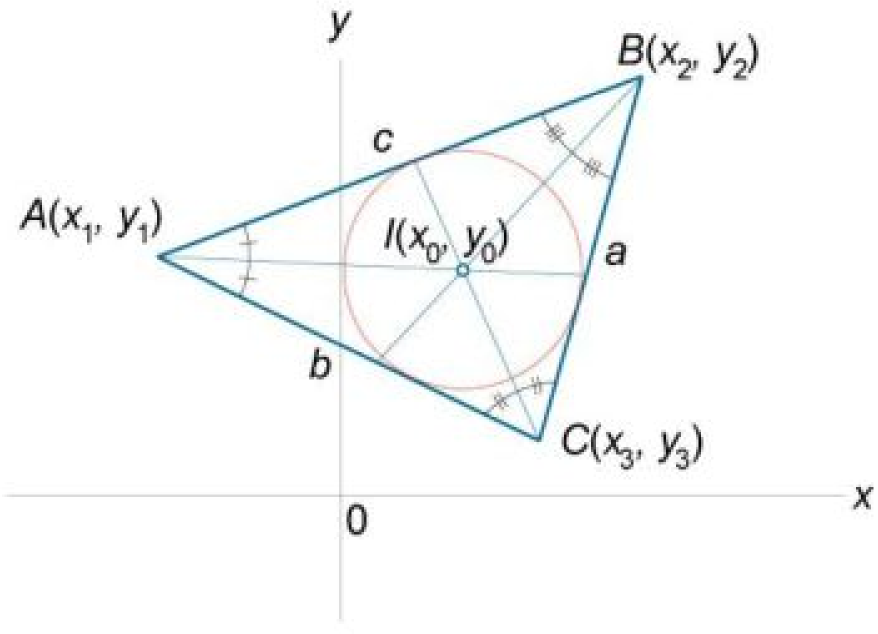


Figure 92.

615. Circumcenter (Intersection of the Side Perpendicular Bisectors) of a Triangle

$$x_0 = \frac{\begin{vmatrix} x_1^2+y_1^2 & y_1 & 1 \\ x_2^2+y_2^2 & y_2 & 1 \\ x_3^2+y_3^2 & y_3 & 1 \end{vmatrix}}{2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \quad y_0 = \frac{\begin{vmatrix} x_1 & x_1^2+y_1^2 & 1 \\ x_2 & x_2^2+y_2^2 & 1 \\ x_3 & x_3^2+y_3^2 & 1 \end{vmatrix}}{2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}$$

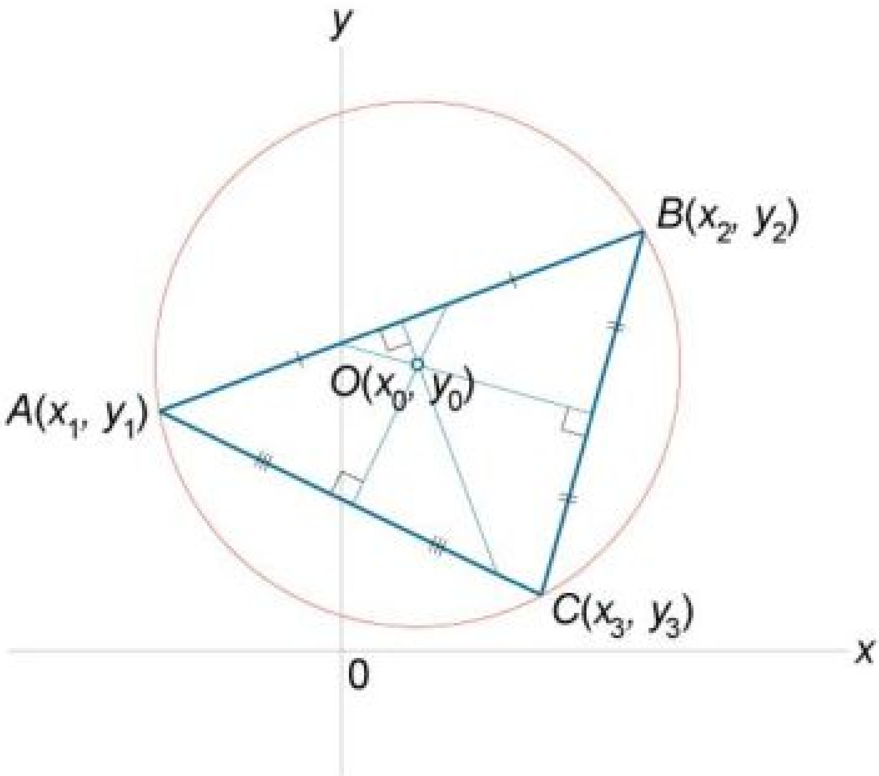


Figure 93.

616. Orthocenter (Intersection of Altitudes) of a Triangle

$$x_0 = \frac{\begin{vmatrix} y_1 & x_2x_3 + y_1^2 & 1 \\ y_2 & x_3x_1 + y_2^2 & 1 \\ y_3 & x_1x_2 + y_3^2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \quad y_0 = \frac{\begin{vmatrix} x_1^2 + y_2y_3 & x_1 & 1 \\ x_2^2 + y_3y_1 & x_2 & 1 \\ x_3^2 + y_1y_2 & x_3 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}$$

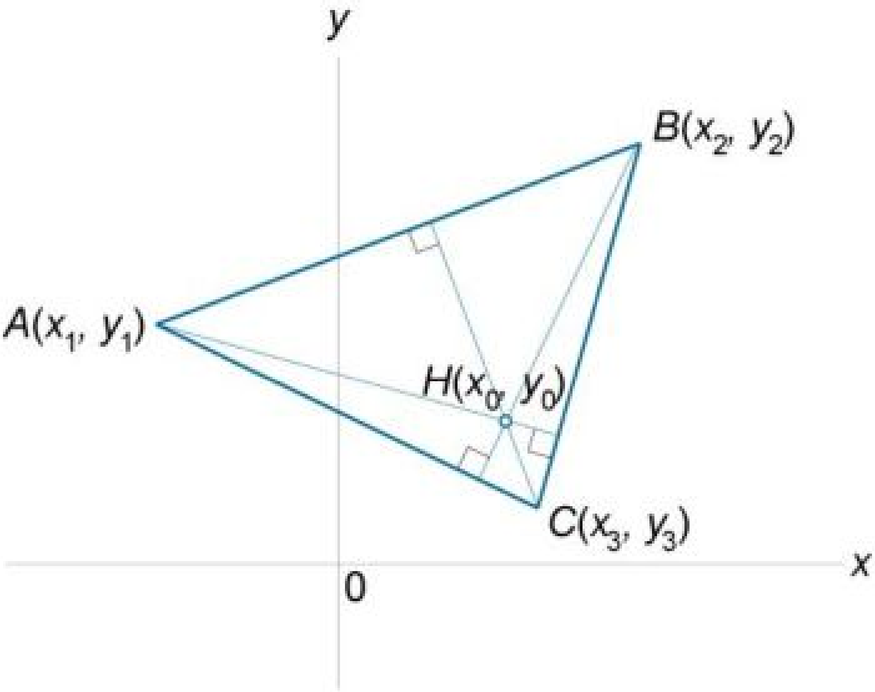


Figure 94.

617. Area of a Triangle

$$S = (\pm)\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = (\pm)\frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

618. Area of a Quadrilateral

$$S = (\pm)\frac{1}{2}[(x_1 - x_2)(y_1 + y_2) + (x_2 - x_3)(y_2 + y_3) +$$
$$+ (x_3 - x_4)(y_3 + y_4) + (x_4 - x_1)(y_4 + y_1)]$$

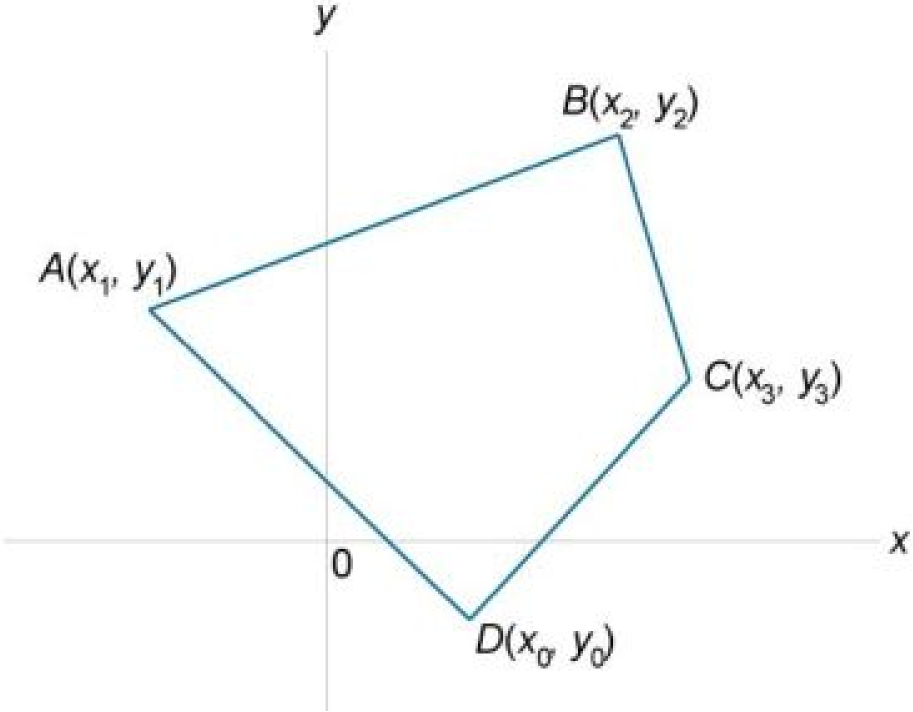


Figure 95.

Note: In formulas 617, 618 we choose the sign (+) or (–) so that to get a positive answer for area.

619. Distance Between Two Points in Polar Coordinates

$$d = AB = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\varphi_2 - \varphi_1)}$$

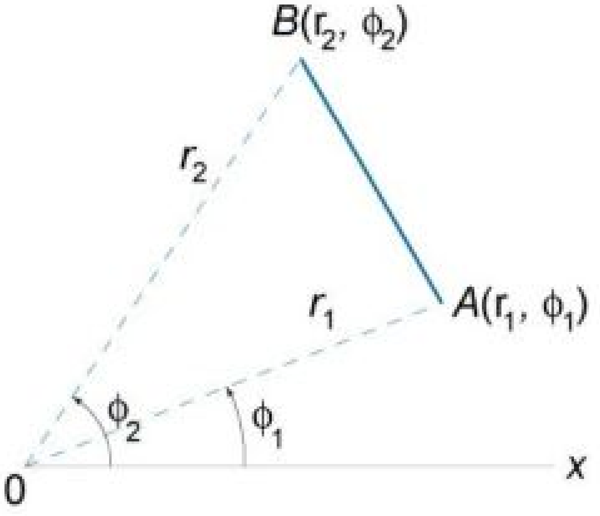


Figure 96.

620. Converting Rectangular Coordinates to Polar Coordinates
$x = r\cos\varphi$, $y = r\sin\varphi$.

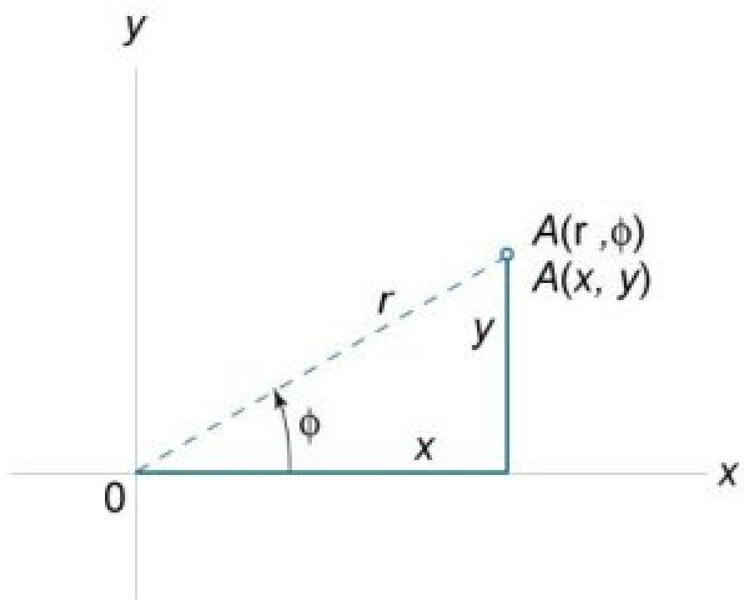


Figure 97.

621. Converting Polar Coordinates to Rectangular Coordinates
$r = \sqrt{x^2 + y^2}$, $\tan\varphi = \frac{y}{x}$.

7.3 Straight Line in Plane

Point coordinates: $X, Y, x, x_0, x_1, y_0, y_1, a_1, a_2, \ldots$
Real numbers: $k, a, b, p, t, A, B, C, A_1, A_2, \ldots$
Angles: α, β
Angle between two lines: φ
Normal vector: $\vec{n}$
Position vectors: $\vec{r}, \vec{a}, \vec{b}$

622. General Equation of a Straight Line
$Ax + By + C = 0$

623. Normal Vector to a Straight Line
The vector $\vec{n}(A, B)$ is normal to the line $Ax + By + C = 0$.

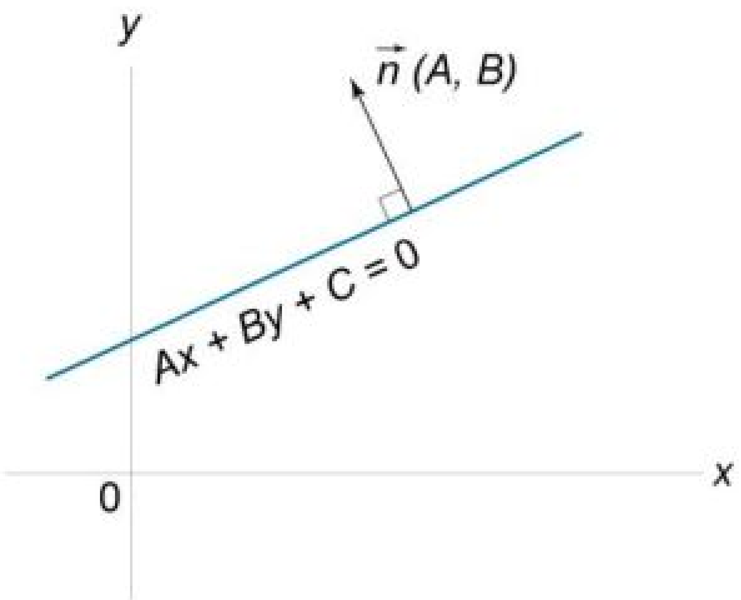


Figure 98.

624. Explicit Equation of a Straight Line (Slope-Intercept Form)
$y = kx + b$.

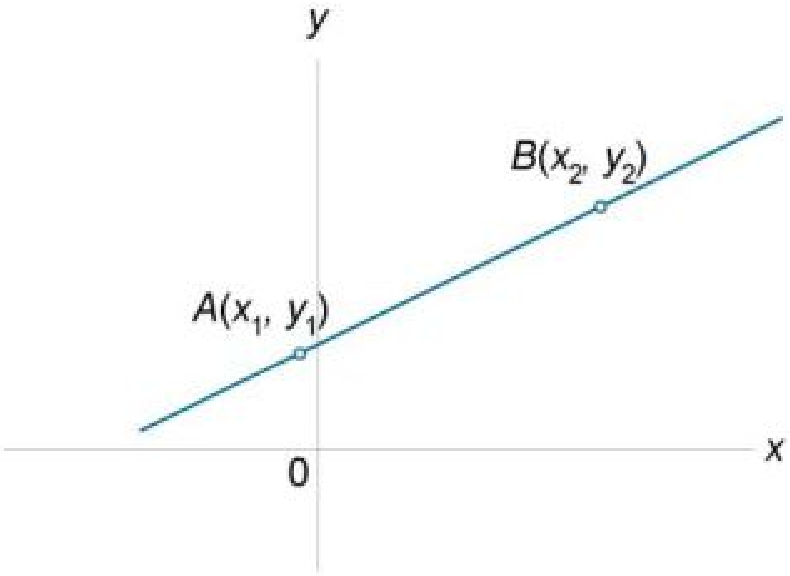


Figure 102.

628. Intercept Form

$$\frac{x}{a}+\frac{y}{b}=1$$

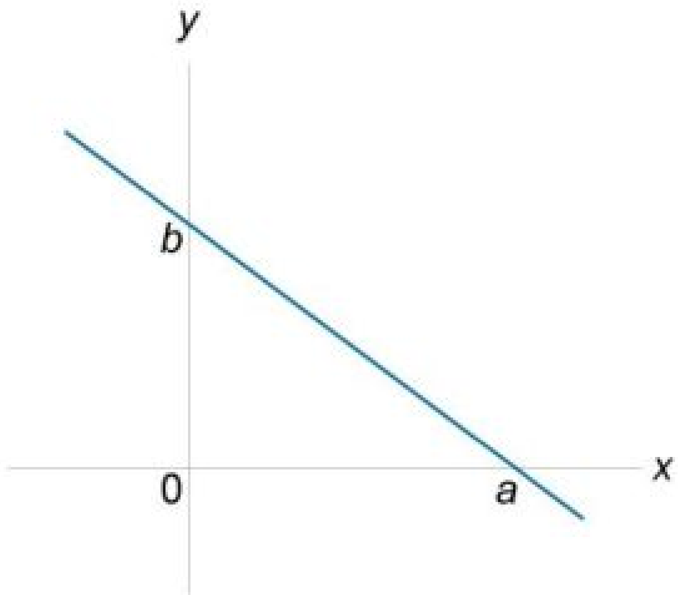


Figure 103.

The gradient of the line is $k = \tan\alpha$.

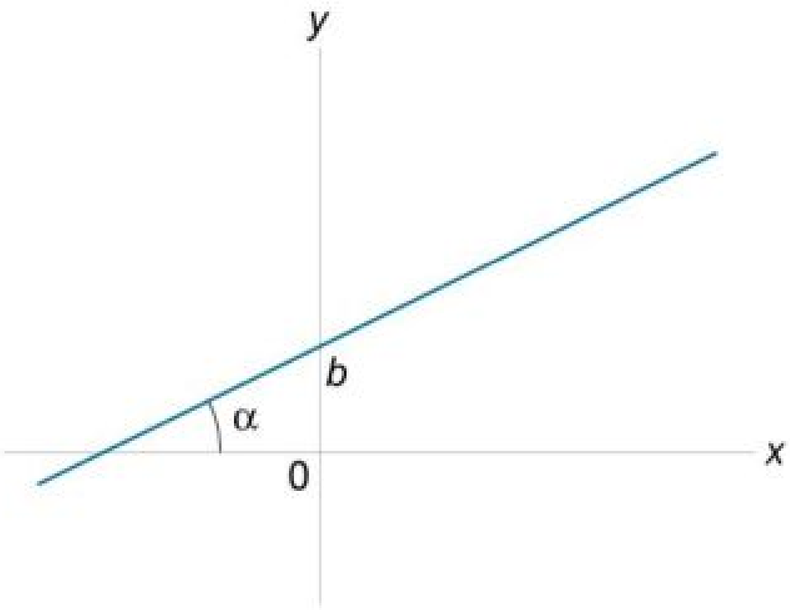


Figure 99.

625. Gradient of a Line

$$k = \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$$

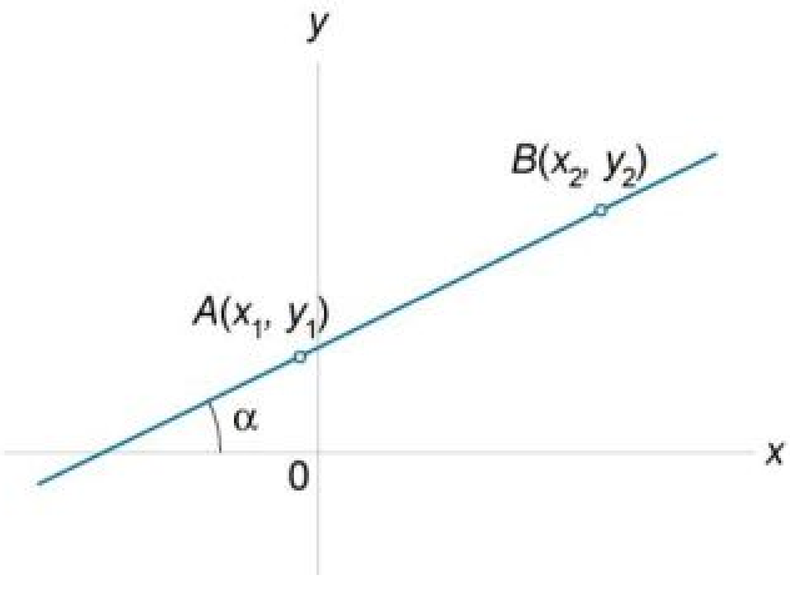


Figure 100.

626. Equation of a Line Given a Point and the Gradient

$y = y_0 + k(x - x_0)$,

where k is the gradient, $P(x_0, y_0)$ is a point on the line.

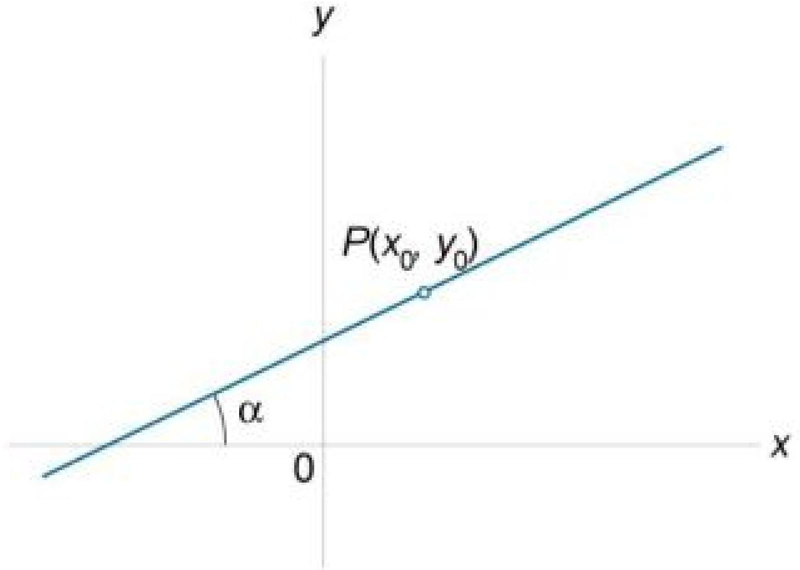


Figure 101.

627. Equation of a Line That Passes Through Two Points

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

or

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

629. Normal Form

$$x \cos\beta + y \sin\beta - p = 0$$

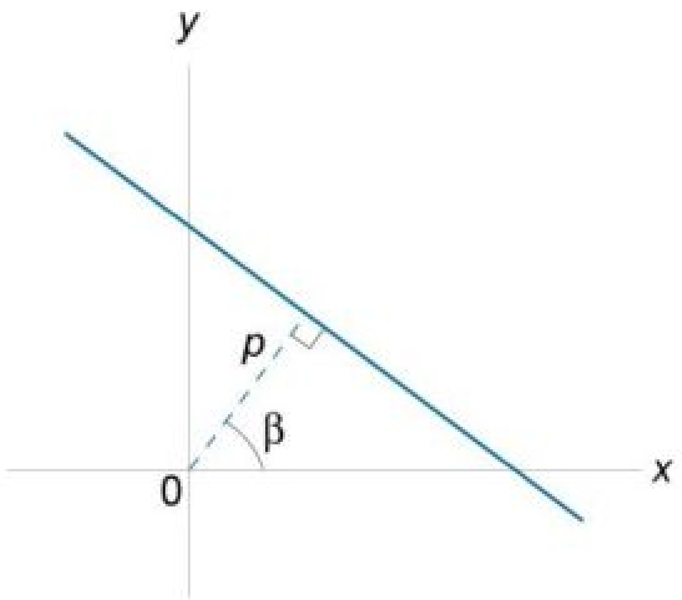


Figure 104.

630. Point Direction Form

$$\frac{x - x_1}{X} = \frac{y - y_1}{Y},$$

where (X, Y) is the direction of the line and $P_1(x_1, y_1)$ lies on the line.

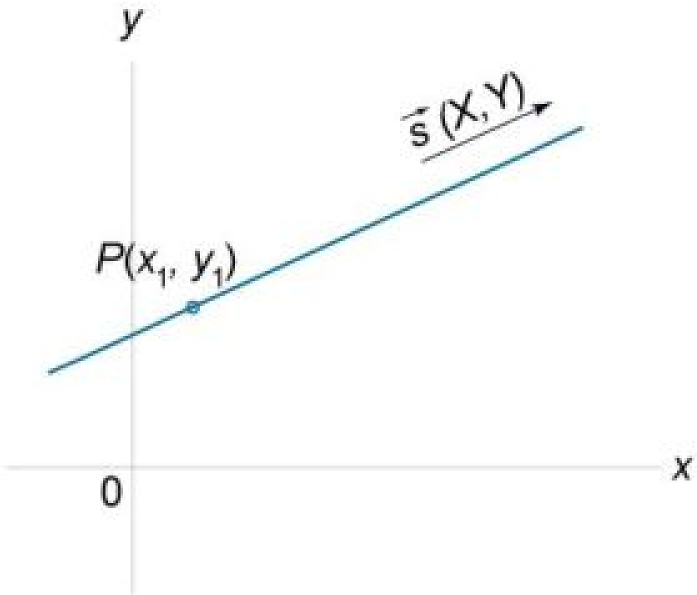


Figure 105.

631. Vertical Line

$x = a$

632. Horizontal Line

$y = b$

633. Vector Equation of a Straight Line

$\vec{r} = \vec{a} + t\vec{b}$,

where

O is the origin of the coordinates,

X is any variable point on the line,

$\vec{a}$ is the position vector of a known point A on the line ,

$\vec{b}$ is a known vector of direction, parallel to the line,

t is a parameter,

$\vec{r} = \overrightarrow{OX}$ is the position vector of any point X on the line.

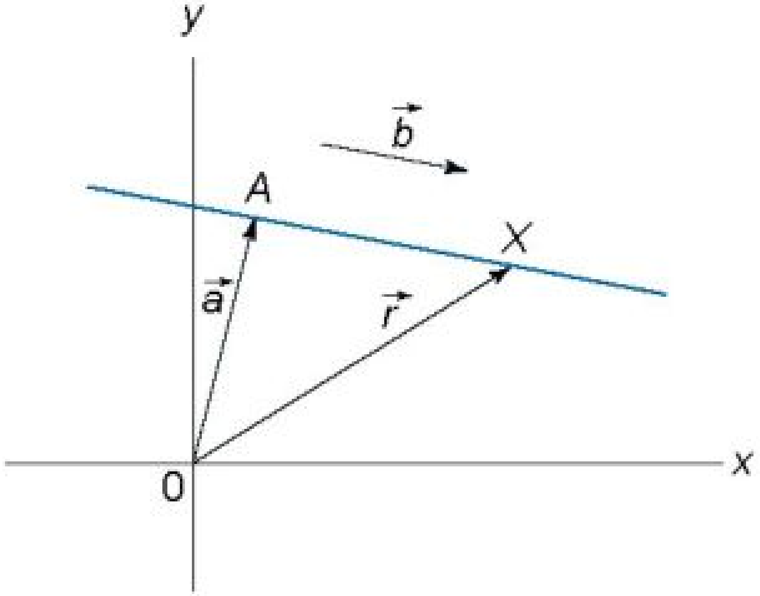


Figure 106.

634. Straight Line in Parametric Form

$$\begin{cases} x = a_1 + tb_1 \\ y = a_2 + tb_2 \end{cases},$$

where

(x, y) are the coordinates of any unknown point on the line,

(a_1, a_2) are the coordinates of a known point on the line,

(b_1, b_2) are the coordinates of a vector parallel to the line,

t is a parameter.

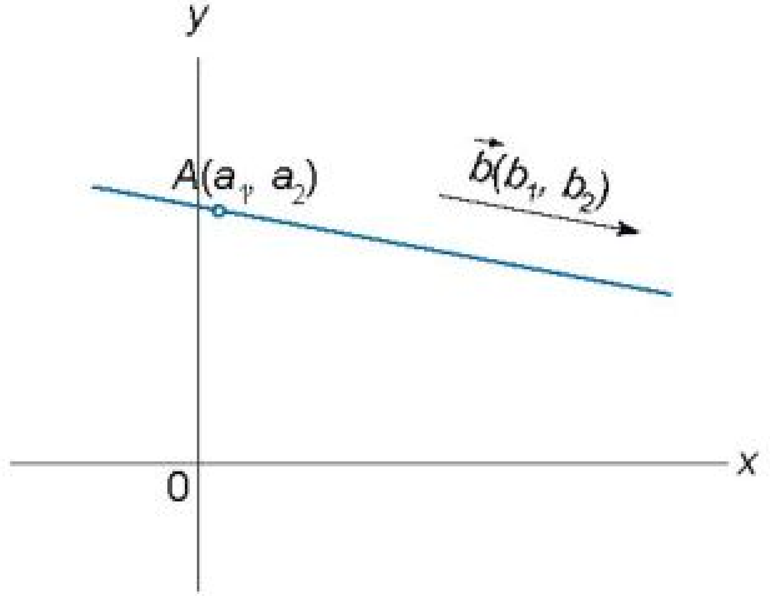


Figure 107.

635. Distance From a Point To a Line

The distance from the point $P(a, b)$ to the line $Ax + By + C = 0$ is

$$d = \frac{|Aa + Bb + C|}{\sqrt{A^2 + B^2}}.$$

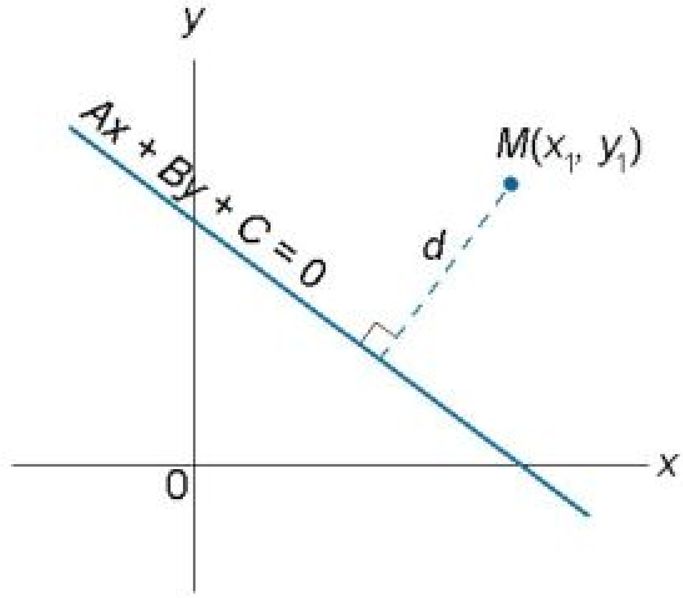


Figure 108.

636. Parallel Lines

Two lines $y = k_1 x + b_1$ and $y = k_2 x + b_2$ are parallel if $k_1 = k_2$.

Two lines $A_1 x + B_1 y + C_1 = 0$ and $A_2 x + B_2 y + C_2 = 0$ are parallel if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2}.$$

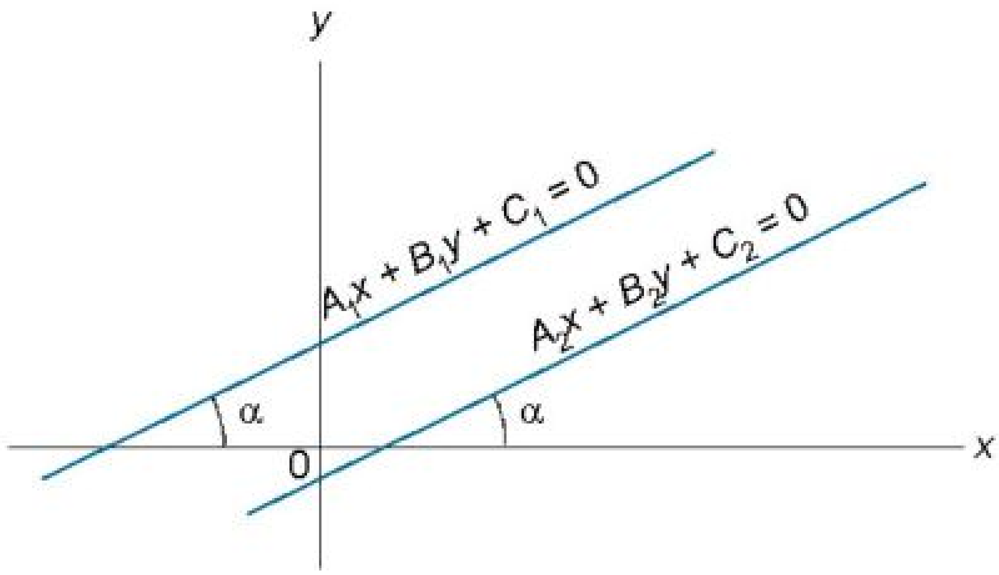


Figure 109.

637. Perpendicular Lines

Two lines $y = k_1 x + b_1$ and $y = k_2 x + b_2$ are perpendicular if

$k_2 = -\frac{1}{k_1}$ or, equivalently, $k_1 k_2 = -1$.

Two lines $A_1 x + B_1 y + C_1 = 0$ and $A_2 x + B_2 y + C_2 = 0$ are perpendicular if

$$A_1 A_2 + B_1 B_2 = 0.$$

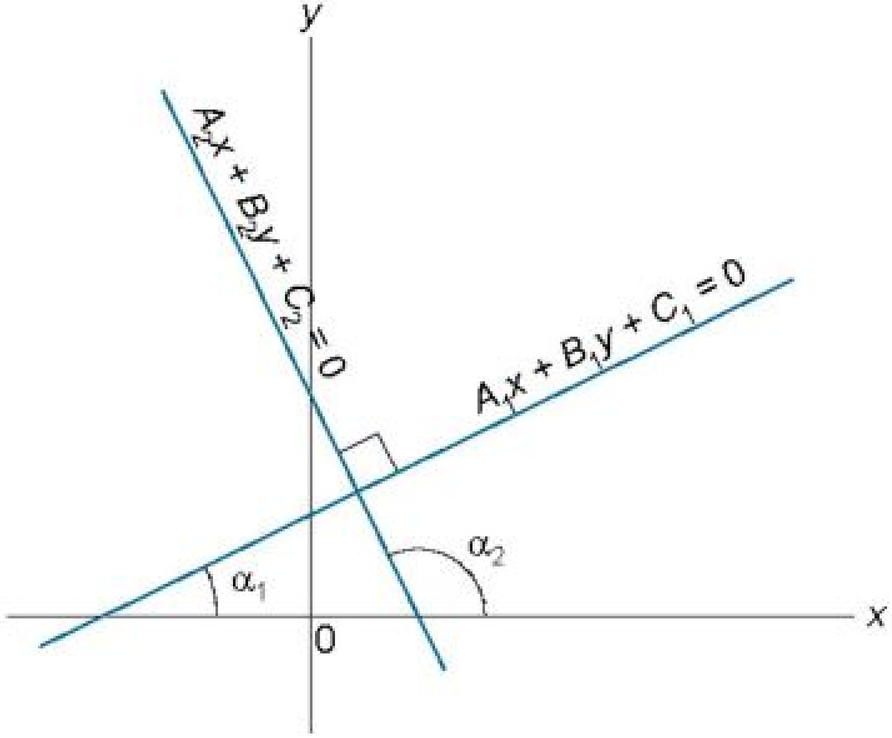


Figure 110.

638. Angle Between Two Lines

$$\tan\varphi = \frac{k_2 - k_1}{1 + k_1 k_2},$$

$$\cos\varphi = \frac{A_1 A_2 + B_1 B_2}{\sqrt{A_1^2 + B_1^2} \cdot \sqrt{A_2^2 + B_2^2}}.$$

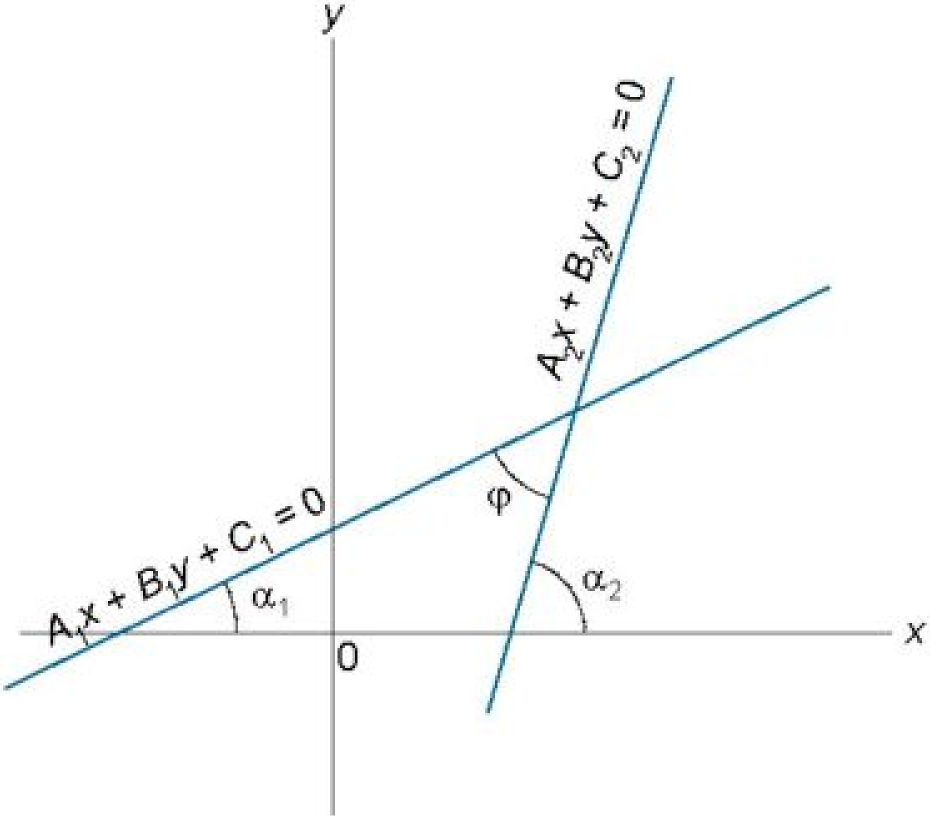


Figure 111.

639. Intersection of Two Lines

If two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ intersect, the intersection point has coordinates

$$x_0 = \frac{-C_1B_2 + C_2B_1}{A_1B_2 - A_2B_1}, \ y_0 = \frac{-A_1C_2 + A_2C_1}{A_1B_2 - A_2B_1}.$$

7.4 Circle

Radius: R

Center of circle: (a, b)

Point coordinates: $x, y, x_1, y_1, \ldots$

Real numbers: A, B, C, D, E, F, t

640. Equation of a Circle Centered at the Origin (Standard Form)

$$x^2 + y^2 = R^2$$

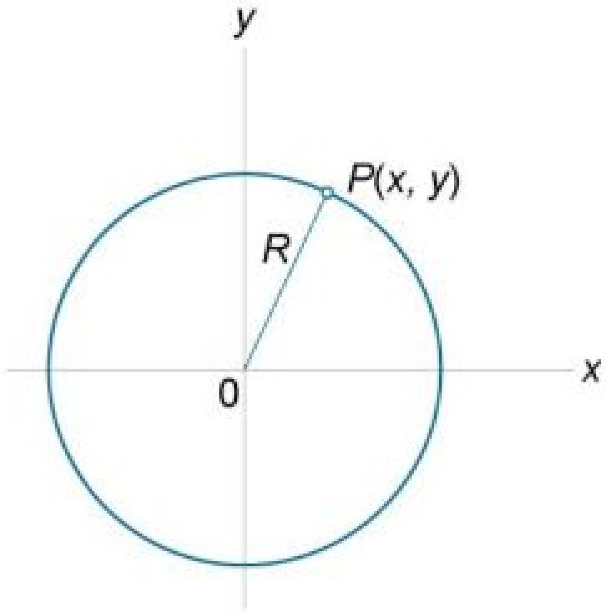


Figure 112.

641. Equation of a Circle Centered at Any Point (a, b)

$$(x - a)^2 + (y - b)^2 = R^2$$

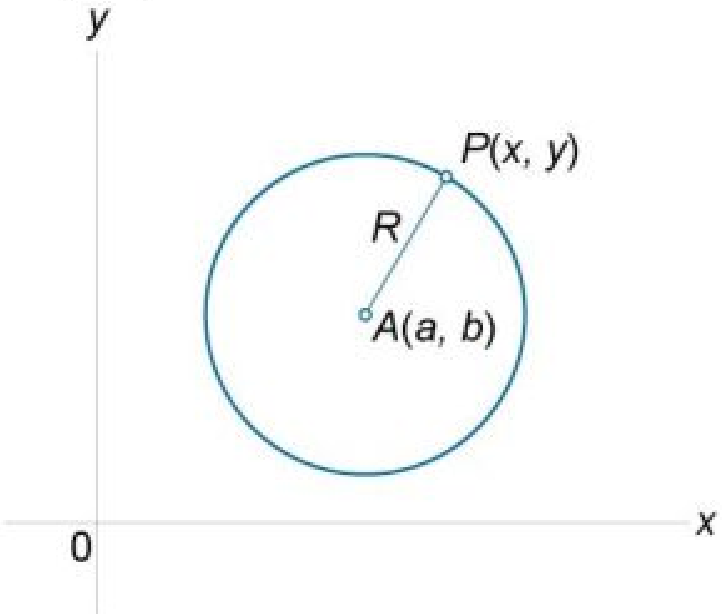


Figure 113.

642. Three Point Form

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

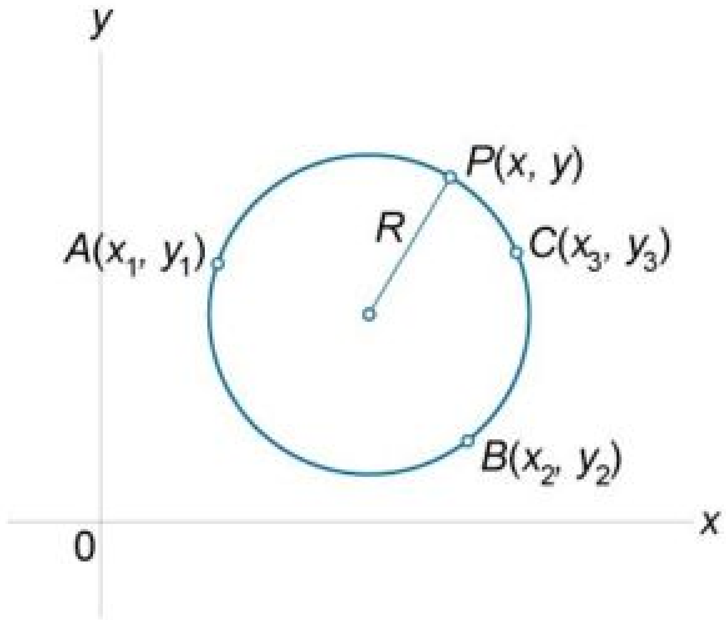


Figure 114.

643. Parametric Form

$$\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, \ 0 \le t \le 2\pi .$$

644. General Form

$Ax^2 + Ay^2 + Dx + Ey + F = 0$ (A nonzero, $D^2 + E^2 > 4AF$).

The center of the circle has coordinates (a, b), where

$$a = -\frac{D}{2A}, \ b = -\frac{E}{2A}.$$

The radius of the circle is

$$R = \sqrt{\frac{D^2 + E^2 - 4AF}{2|A|}}.$$

7.5 Ellipse

Semimajor axis: a
Semiminor axis: b
Foci: $F_1(-c, 0)$, $F_2(c, 0)$
Distance between the foci: 2c
Eccentricity: e
Real numbers: A, B, C, D, E, F, t
Perimeter: L
Area: S

645. Equation of an Ellipse (Standard Form)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

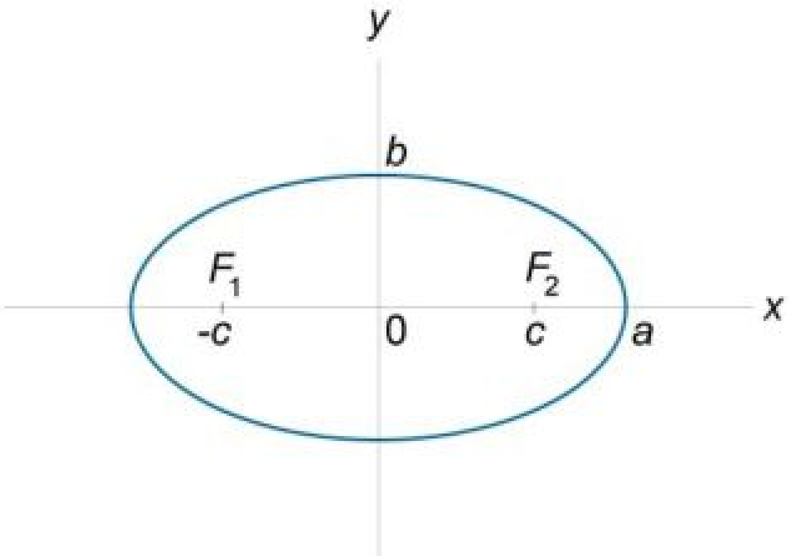


Figure 115.

646. $r_1 + r_2 = 2a$,
where r_1, r_2 are distances from any point $P(x, y)$ on the ellipse to the two foci.

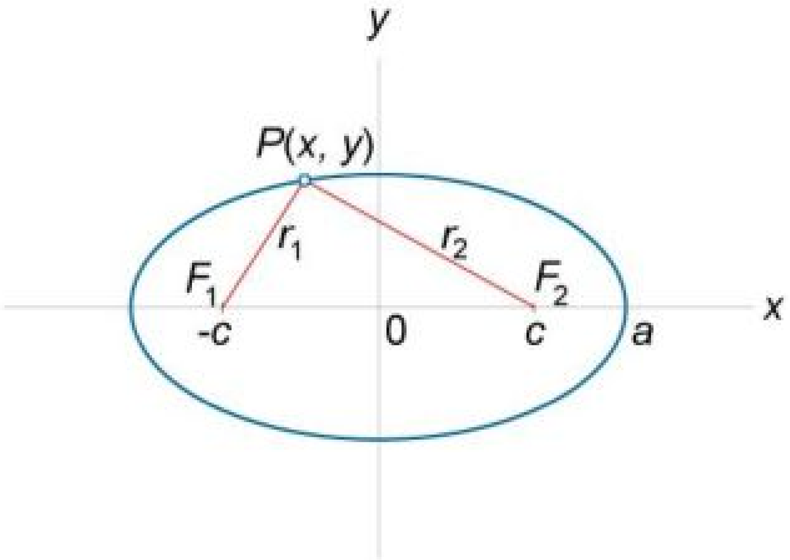


Figure 116.

647. $a^2 = b^2 + c^2$

648. Eccentricity

$$e = \frac{c}{a} < 1$$

649. Equations of Directrices

$$x = \pm\frac{a}{e} = \pm\frac{a^2}{c}$$

650. Parametric Form

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, \quad 0 \le t \le 2\pi.$$

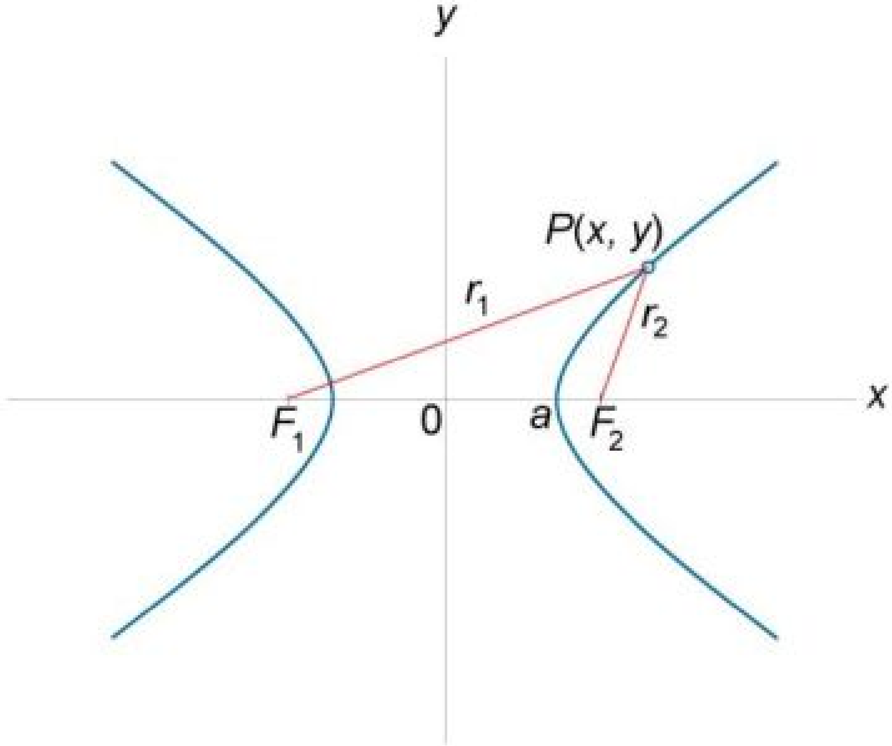


Figure 118.

658. Equations of Asymptotes

$$y = \pm \frac{b}{a} x$$

659. $c^2 = a^2 + b^2$

660. Eccentricity

$$e = \frac{c}{a} > 1$$

661. Equations of Directrices

$$x = \pm \frac{a}{e} = \pm \frac{a^2}{c}$$

651. General Form
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,
where $B^2 - 4AC < 0$.

652. General Form with Axes Parallel to the Coordinate Axes
$Ax^2 + Cy^2 + Dx + Ey + F = 0$,
where $AC > 0$.

653. Circumference
$L = 4aE(e)$,
where the function E is the complete elliptic integral of the second kind.

654. Approximate Formulas of the Circumference
$L = \pi\left(1.5(a+b) - \sqrt{ab}\right)$,
$L = \pi\sqrt{2\left(a^2 + b^2\right)}$.

655. $S = \pi ab$

7.6 Hyperbola

Transverse axis: a
Conjugate axis: b
Foci: $F_1(-c, 0)$, $F_2(c, 0)$
Distance between the foci: 2c
Eccentricity: e
Asymptotes: s, t
Real numbers: A, B, C, D, E, F, t, k

656. Equation of a Hyperbola (Standard Form)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

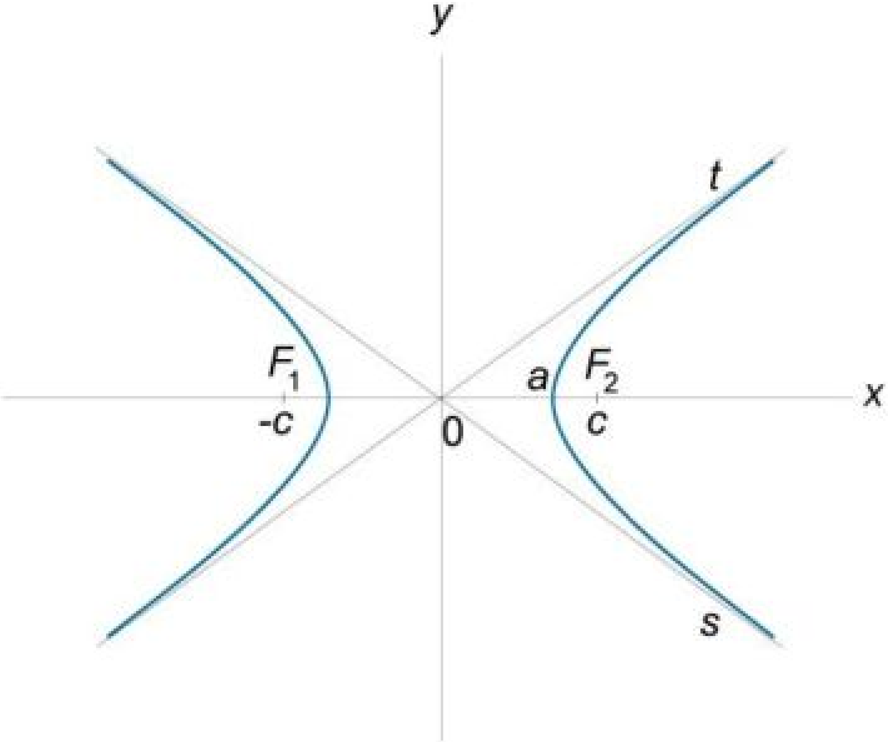


Figure 117.

657. $|r_1-r_2|=2a$,

where r_1, r_2 are distances from any point $P(x,y)$ on the hyperbola to the two foci.

662. Parametric Equations of the Right Branch of a Hyperbola

$$\begin{cases} x = a\cosh t \\ y = b\sinh t \end{cases}, \ 0 \le t \le 2\pi.$$

663. General Form

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,

where $B^2 - 4AC > 0$.

664. General Form with Axes Parallel to the Coordinate Axes

$Ax^2 + Cy^2 + Dx + Ey + F = 0$,

where $AC < 0$.

665. Asymptotic Form

$$xy = \frac{e^2}{4},$$

or

$$y = \frac{k}{x}, \text{ where } k = \frac{e^2}{4}.$$

In this case , the asymptotes have equations $x = 0$ and $y = 0$.

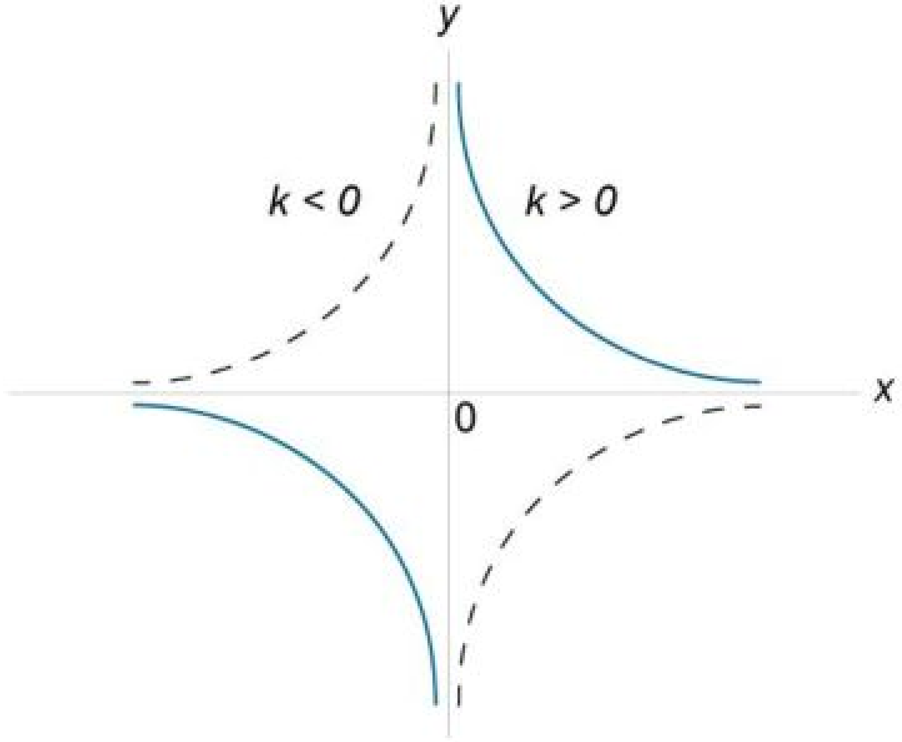


Figure 119.

7.7 Parabola

Focal parameter: p
Focus: F
Vertex: $M(x_0, y_0)$
Real numbers: $A, B, C, D, E, F, p, a, b, c$

666. Equation of a Parabola (Standard Form)

$$y^2 = 2px$$

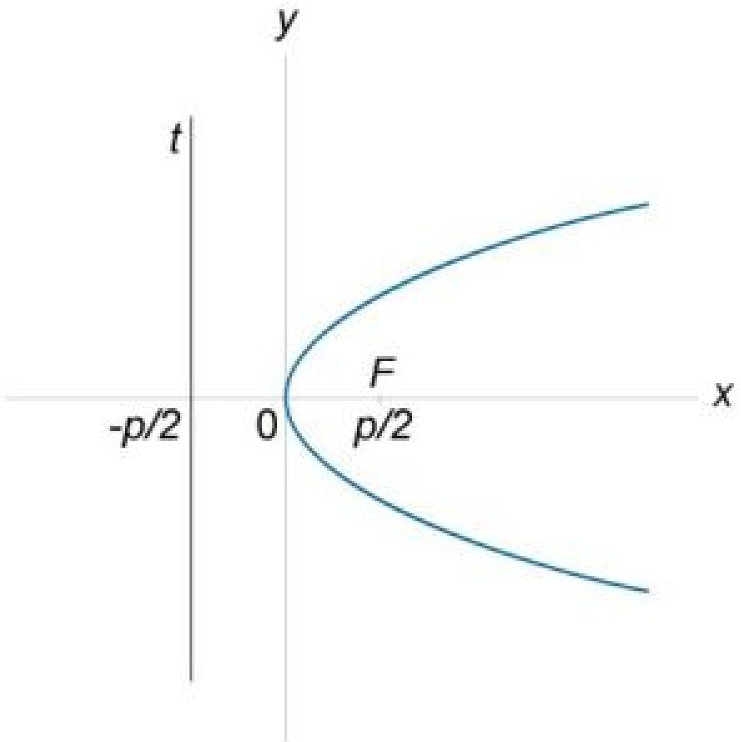


Figure 120.

Equation of the directrix

$$x = -\frac{p}{2},$$

Coordinates of the focus

$$F\left(\frac{p}{2}, 0\right),$$

Coordinates of the vertex

$$M(0,0).$$

667. General Form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where $B^2 - 4AC = 0$.

668. $y = ax^2$, $p = \frac{1}{2a}$.

Equation of the directrix

$y = -\frac{p}{2}$,

Coordinates of the focus

$F\left(0, \frac{p}{2}\right)$,

Coordinates of the vertex

$M(0, 0)$.

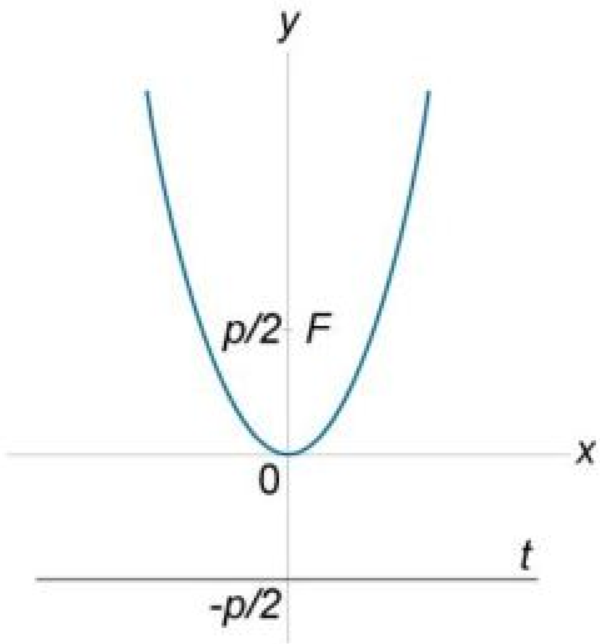


Figure 121.

669. General Form, Axis Parallel to the y-axis

$Ax^2 + Dx + Ey + F = 0$ (A, E nonzero),

$y = ax^2 + bx + c$, $p = \frac{1}{2a}$.

Equation of the directrix

$y = y_0 - \frac{p}{2}$,

Coordinates of the focus

$F\left(x_0, y_0 + \frac{p}{2}\right)$,

Coordinates of the vertex

$x_0 = -\frac{b}{2a}$, $y_0 = ax_0^2 + bx_0 + c = \frac{4ac - b^2}{4a}$.

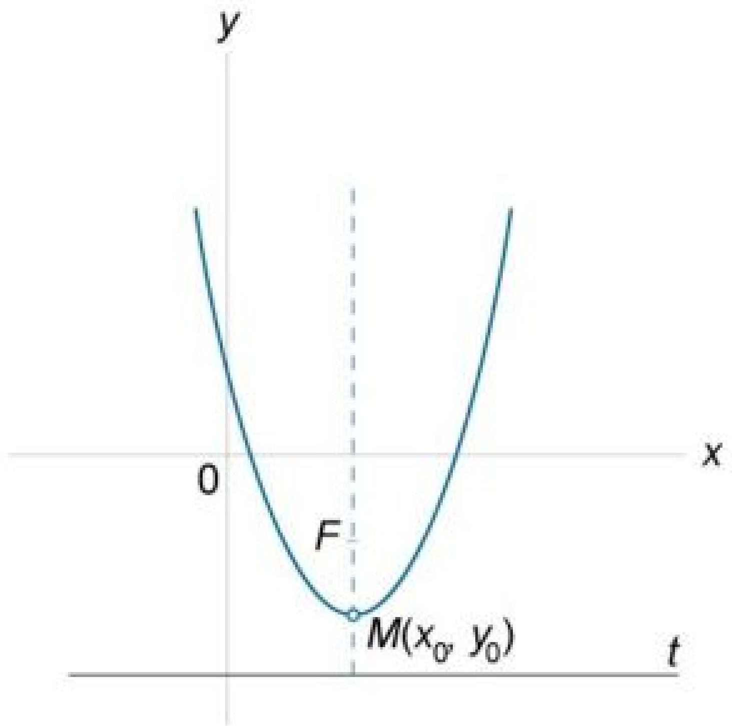


Figure 122.

7.8 Three-Dimensional Coordinate System

Point coordinates: x_0, y_0, z_0, x_1, y_1, z_1, ...

Real number: λ

Distance between two points: d

Area: S

Volume: V

670. Distance Between Two Points

$$d = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

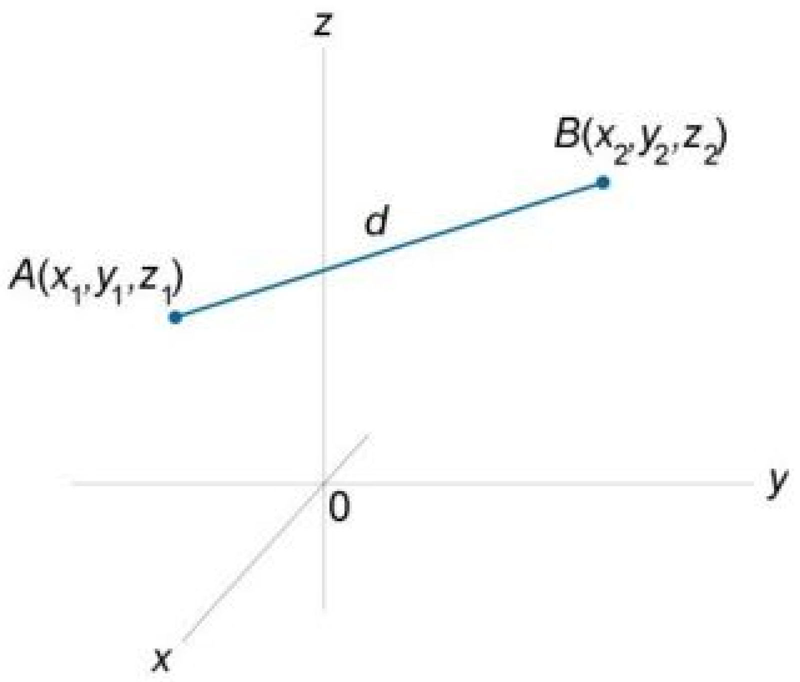


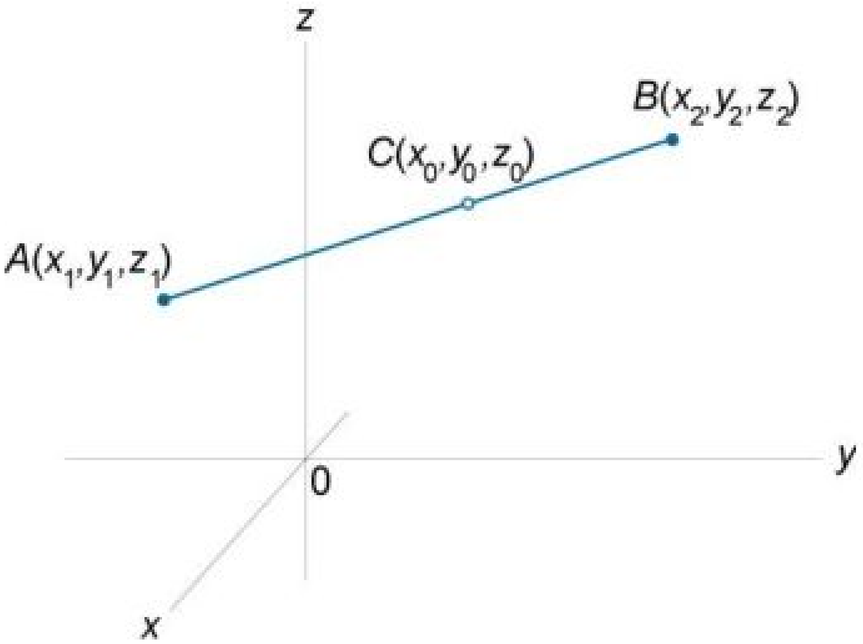
Figure 123.

671. Dividing a Line Segment in the Ratio λ

$$x_0 = \frac{x_1 + \lambda x_2}{1+\lambda},\ y_0 = \frac{y_1 + \lambda y_2}{1+\lambda},\ z_0 = \frac{z_1 + \lambda z_2}{1+\lambda},$$

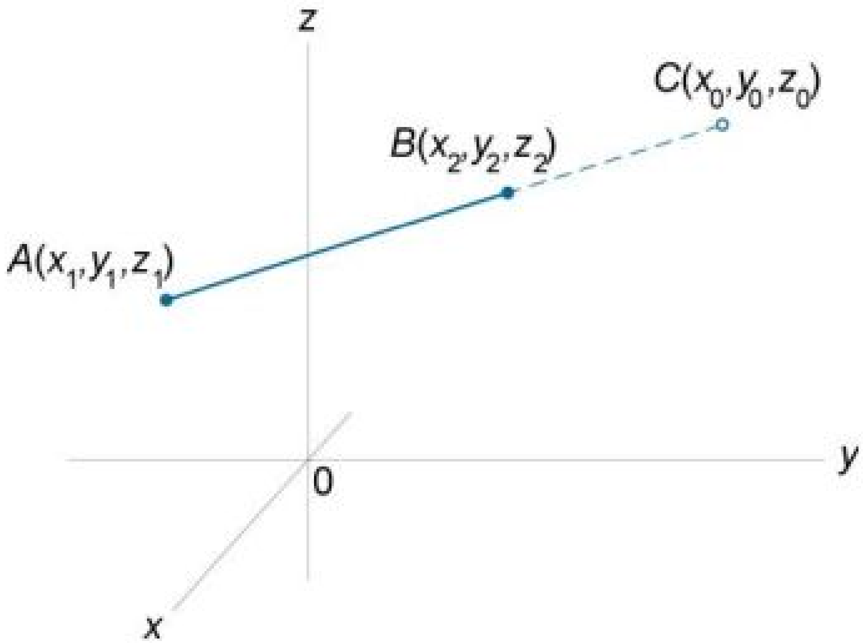
where

$$\lambda = \frac{AC}{CB},\ \lambda \neq -1.$$



$\lambda > 0$

Figure 124.



$\lambda < 0$

Figure 125.

672. Midpoint of a Line Segment

$$x_0 = \frac{x_1 + x_2}{2},\ y_0 = \frac{y_1 + y_2}{2},\ z_0 = \frac{z_1 + z_2}{2},\ \lambda = 1.$$

673. Area of a Triangle

The area of a triangle with vertices $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ is given by

$$S = \frac{1}{2}\sqrt{\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2}.$$

674. Volume of a Tetrahedron

The volume of a tetrahedron with vertices $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, and $P_4(x_4, y_4, z_4)$ is given by

$$V = \pm\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix},$$

or

$$V = \pm\frac{1}{6}\begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix}.$$

Note: We choose the sign (+) or (–) so that to get a positive answer for volume.

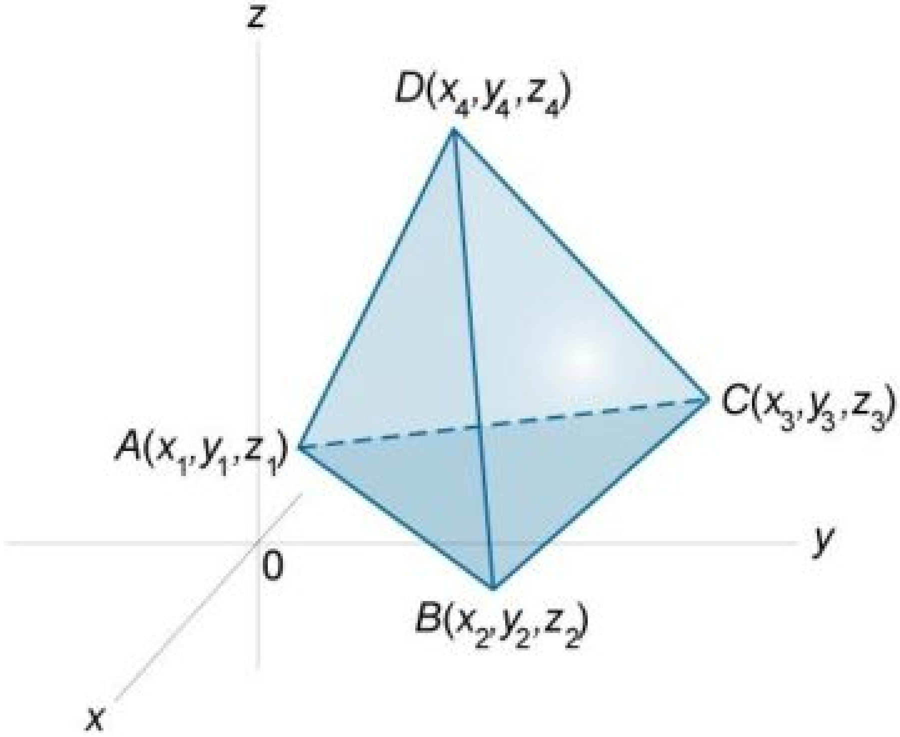


Figure 126.

7.9 Plane

Point coordinates: x, y, z, x_0, y_0, z_0, x_1, y_1, z_1, ...
Real numbers: A, B, C, D, A_1, A_2, a, b, c, a_1, a_2, λ, p, t, ...
Normal vectors: $\vec{n}$, $\vec{n}_1$, $\vec{n}_2$
Direction cosines: $\cos\alpha$, $\cos\beta$, $\cos\gamma$
Distance from point to plane: d

675. General Equation of a Plane
$Ax + By + Cz + D = 0$

676. Normal Vector to a Plane
The vector $\vec{n}(A, B, C)$ is normal to the plane
$Ax + By + Cz + D = 0$.

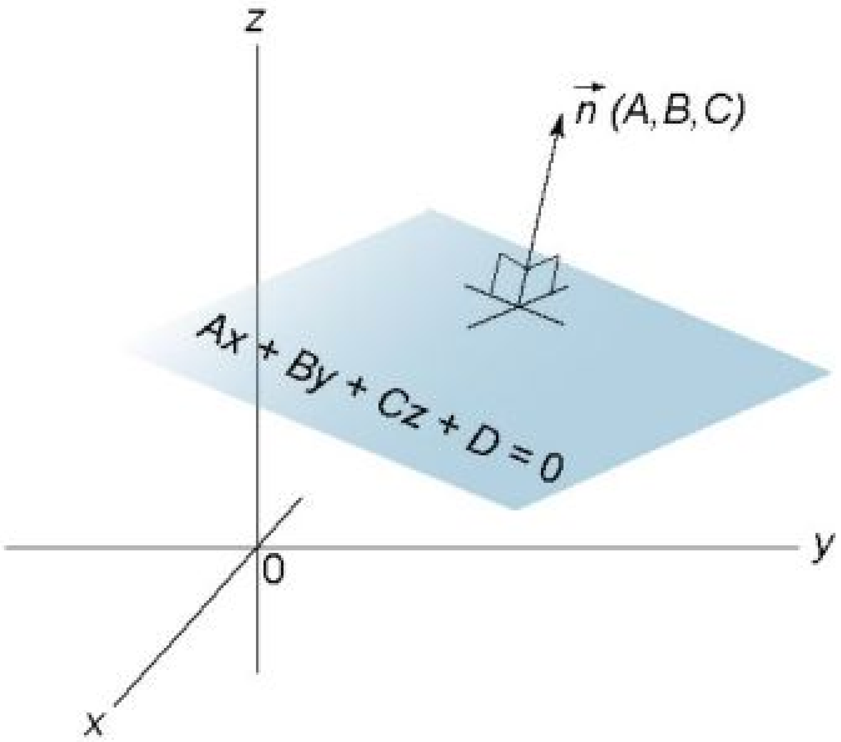


Figure 127.

677. Particular Cases of the Equation of a Plane
$Ax + By + Cz + D = 0$

If $A = 0$, the plane is parallel to the x-axis.
If $B = 0$, the plane is parallel to the y-axis.
If $C = 0$, the plane is parallel to the z-axis.
If $D = 0$, the plane lies on the origin.

If $A = B = 0$, the plane is parallel to the xy-plane.
If $B = C = 0$, the plane is parallel to the yz-plane.
If $A = C = 0$, the plane is parallel to the xz-plane.

678. Point Direction Form

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$,

where the point $P(x_0, y_0, z_0)$ lies in the plane, and the vector (A, B, C) is normal to the plane.

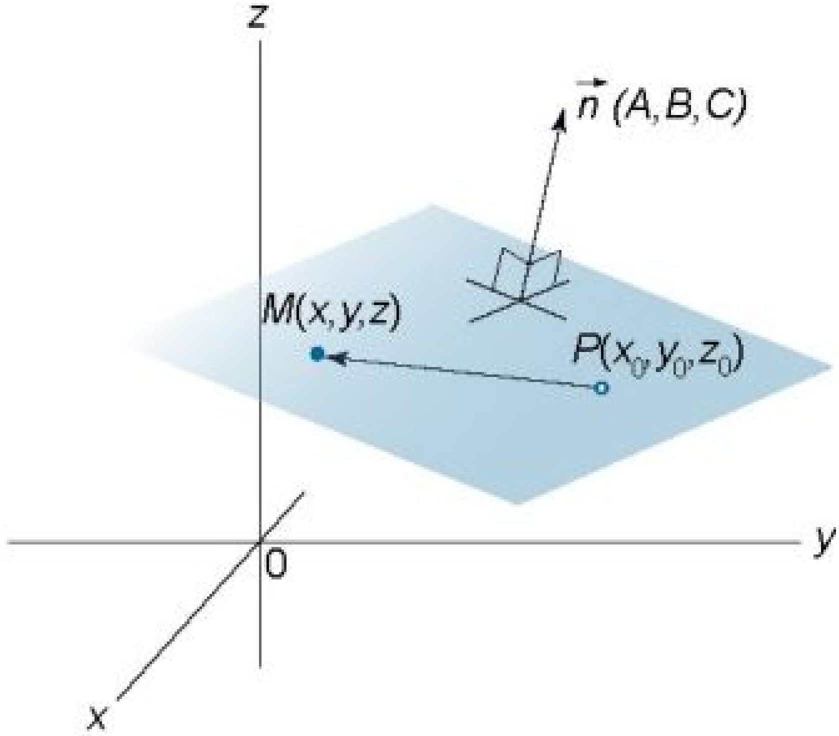


Figure 128.

679. Intercept Form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

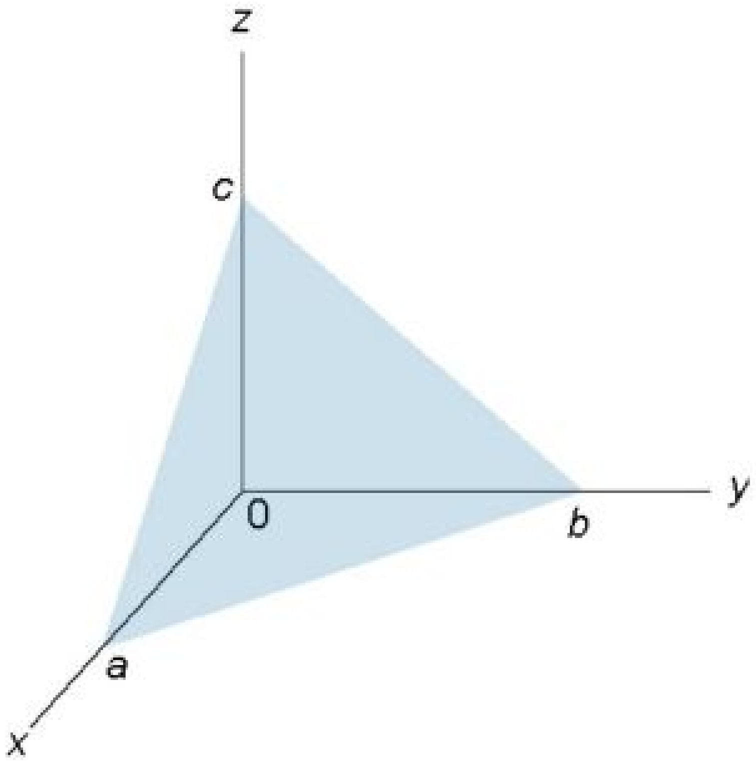


Figure 129.

680. Three Point Form

$$\begin{vmatrix} x-x_3 & y-y_3 & z-z_3 \\ x_1-x_3 & y_1-y_3 & z_1-z_3 \\ x_2-x_3 & y_2-y_3 & z_2-z_3 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

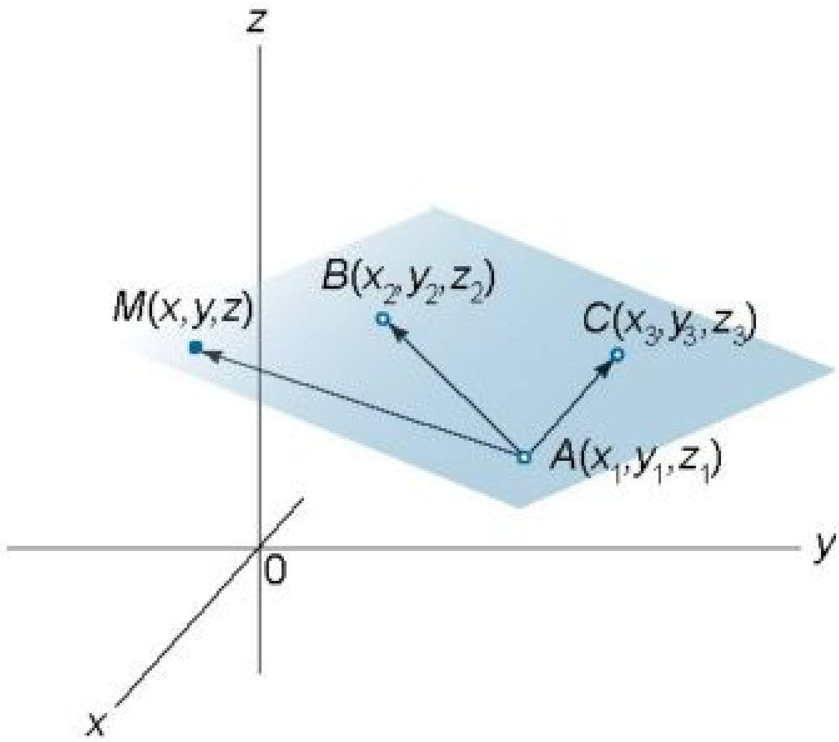


Figure 130.

681. Normal Form

$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0$,

where p is the perpendicular distance from the origin to the plane, and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of any line normal to the plane.

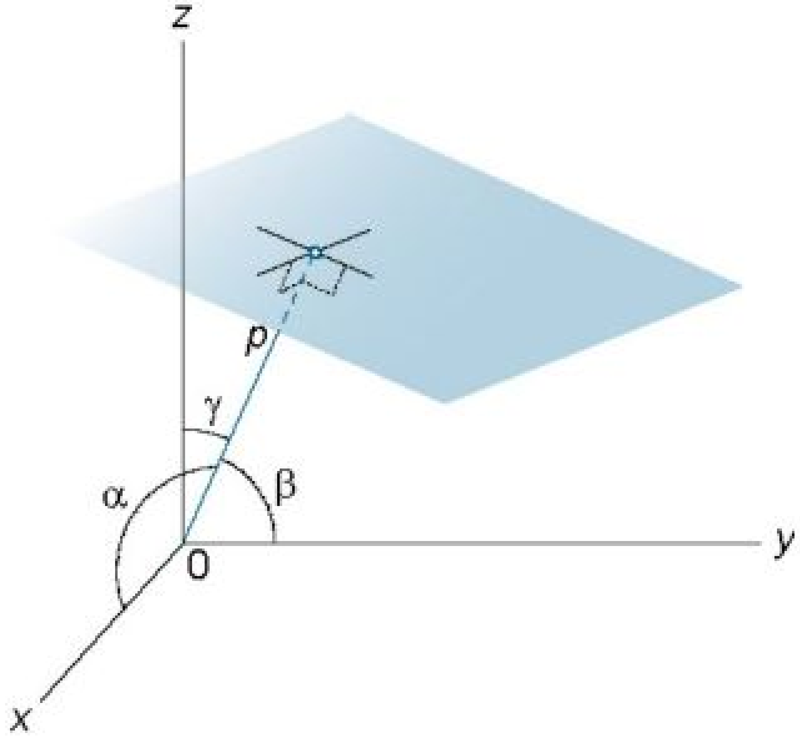


Figure 131.

682. Parametric Form

$$\begin{cases} x = x_1 + a_1 s + a_2 t \\ y = y_1 + b_1 s + b_2 t, \\ z = z_1 + c_1 s + c_2 t \end{cases}$$

where (x, y, z) are the coordinates of any unknown point on the line , the point $P(x_1, y_1, z_1)$ lies in the plane, the vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) are parallel to the plane.

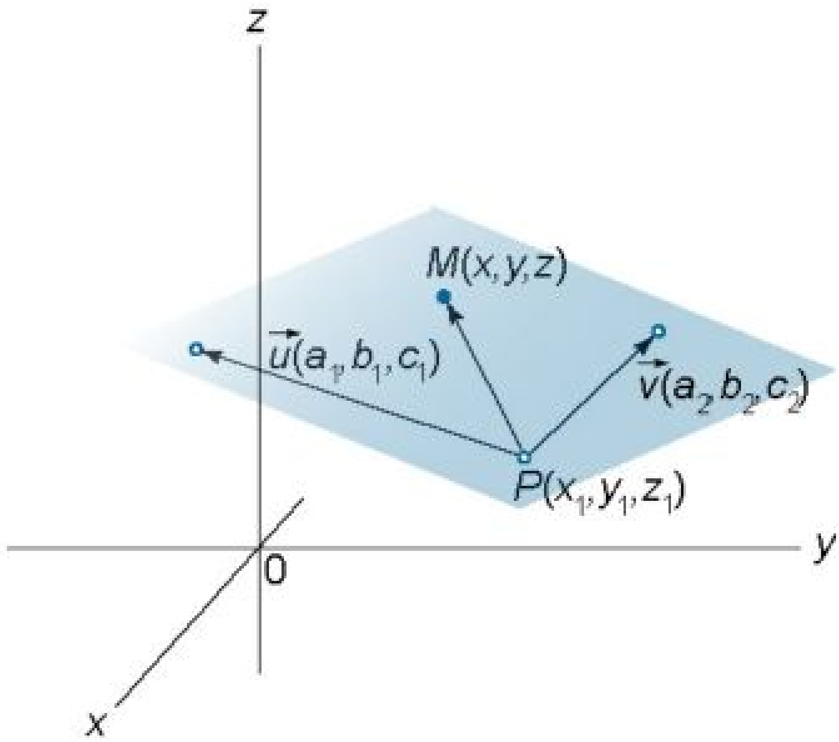


Figure 132.

683. Dihedral Angle Between Two Planes

If the planes are given by

$A_1x + B_1y + C_1z + D_1 = 0$,

$A_2x + B_2y + C_2z + D_2 = 0$,

then the dihedral angle between them is

$$\cos\varphi = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

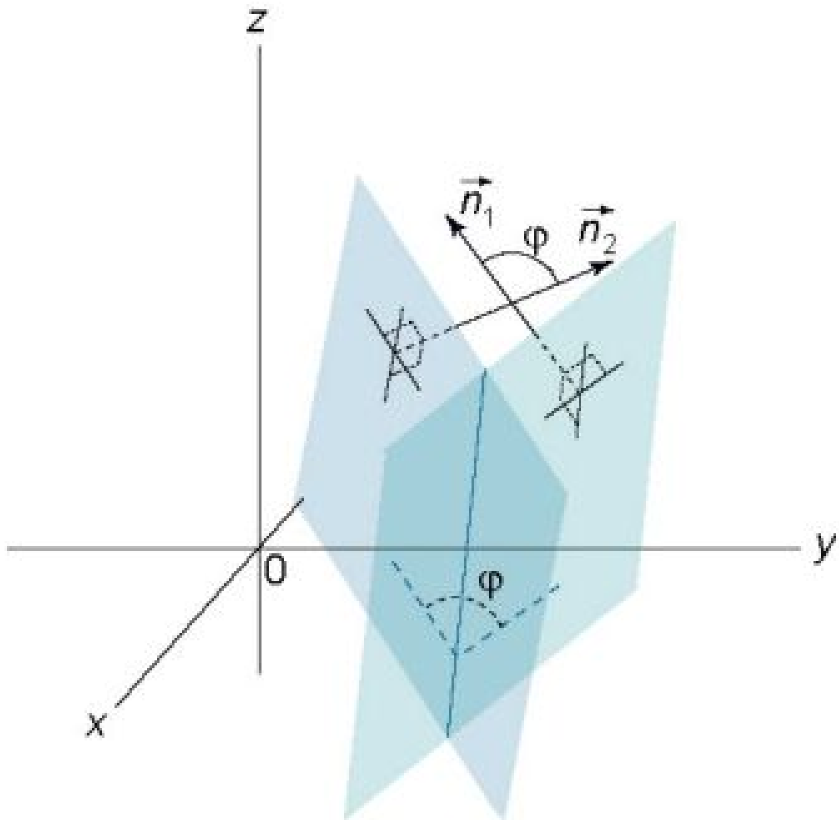


Figure 133.

684. Parallel Planes

Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are parallel if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

685. Perpendicular Planes

Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are perpendicular if

$$A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

686. Equation of a Plane Through $P(x_1, y_1, z_1)$ and Parallel To the Vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) (Fig.132)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

687. Equation of a Plane Through $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$, and Parallel To the Vector (a,b,c)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix} = 0$$

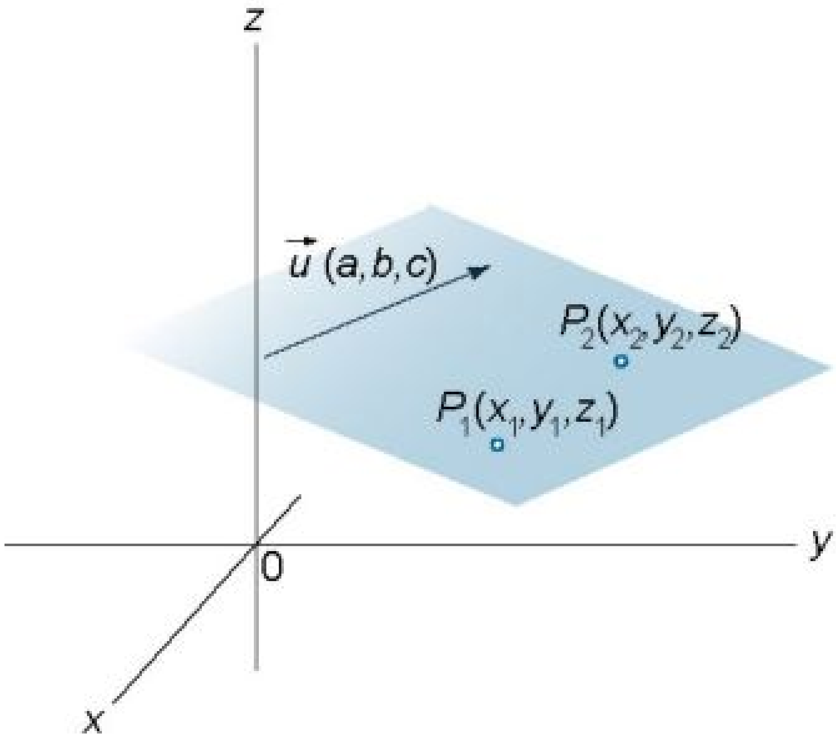


Figure 134.

688. Distance From a Point To a Plane
The distance from the point $P_1(x_1,y_1,z_1)$ to the plane $Ax+By+Cz+D=0$ is

$$d = \left| \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}} \right|.$$

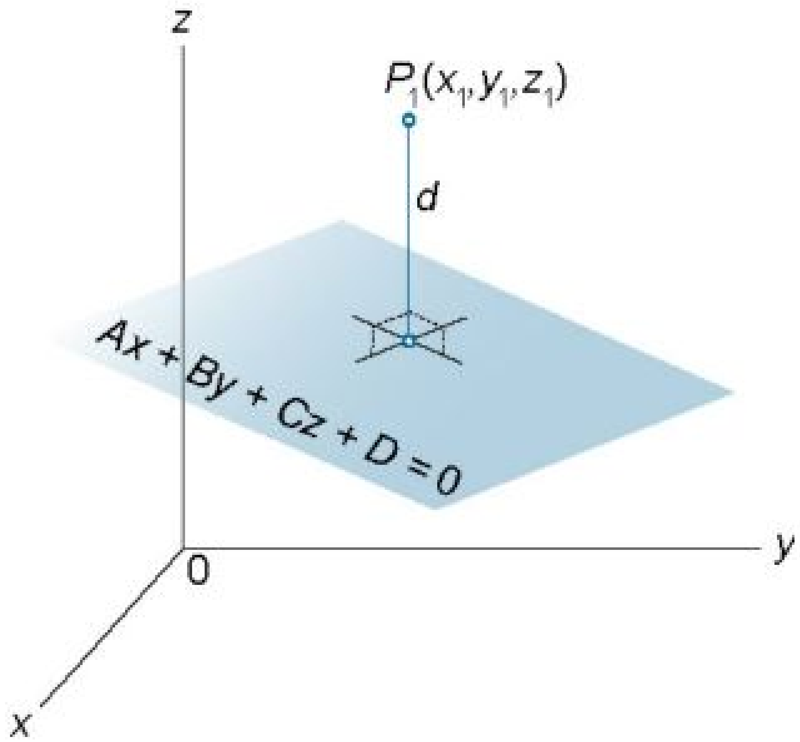


Figure 135.

689. Intersection of Two Planes

If two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ intersect, the intersection straight line is given by

$$\begin{cases} x = x_1 + at \\ y = y_1 + bt, \\ z = z_1 + ct \end{cases}$$

or

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

where

$$a = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix},\ b = \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix},\ c = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix},$$

$$x_1 = \frac{b\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix} - c\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix}}{a^2 + b^2 + c^2},$$

$$y_1 = \frac{c\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix} - a\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix}}{a^2 + b^2 + c^2},$$

$$z_1 = \frac{a\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix} - b\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix}}{a^2 + b^2 + c^2}.$$

7.10 Straight Line in Space

Point coordinates: x, y, z, x_1, y_1, z_1, ...
Direction cosines: $\cos\alpha$, $\cos\beta$, $\cos\gamma$
Real numbers: A, B, C, D, a, b, c, a_1, a_2, t, ...
Direction vectors of a line: $\vec{s}$, $\vec{s}_1$, $\vec{s}_2$
Normal vector to a plane: $\vec{n}$
Angle between two lines: φ

690. Point Direction Form of the Equation of a Line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

where the point $P_1(x_1, y_1, z_1)$ lies on the line, and (a, b, c) is the direction vector of the line.

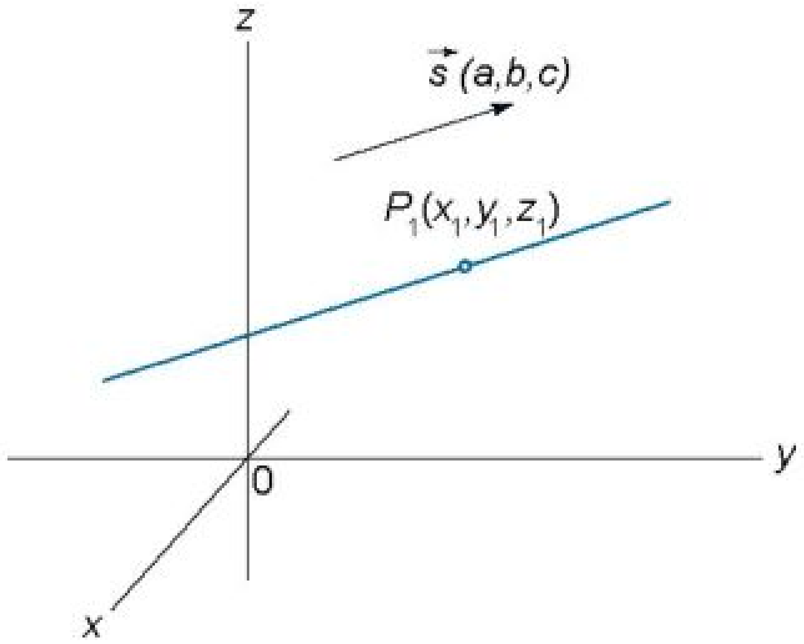


Figure 136.

691. Two Point Form

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

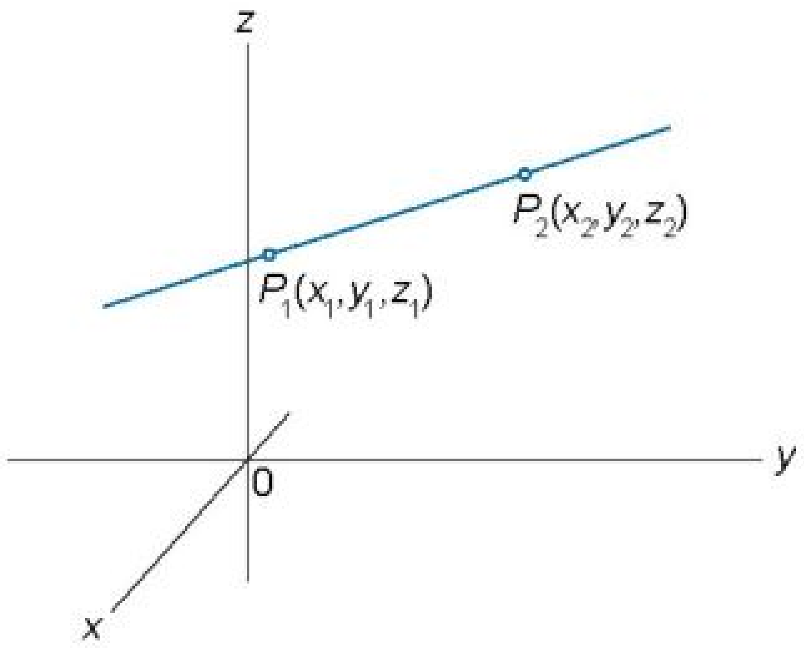


Figure 137.

692. Parametric Form

$$\begin{cases} x = x_1 + t\cos\alpha \\ y = y_1 + t\cos\beta \\ z = z_1 + t\cos\gamma \end{cases},$$

where the point $P_1(x_1, y_1, z_1)$ lies on the straight line, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the direction vector of the line, the parameter t is any real number.

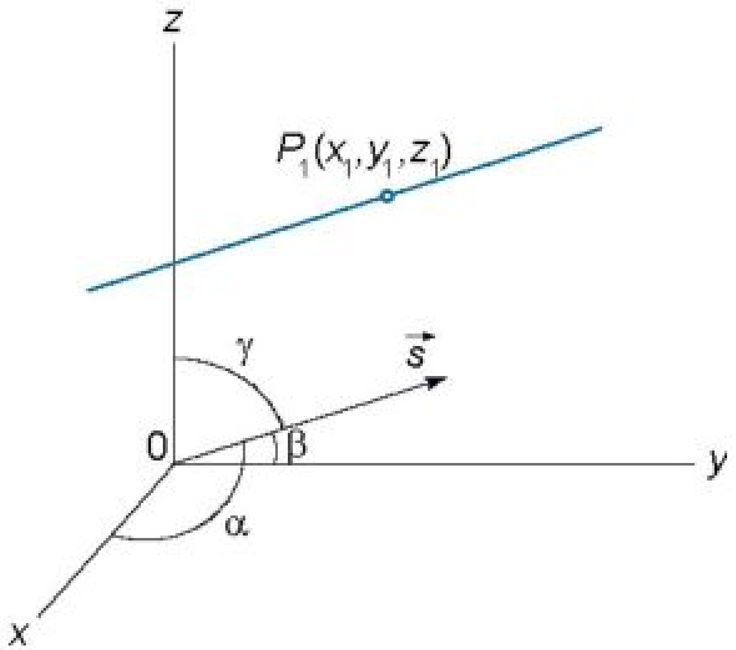


Figure 138.

693. Angle Between Two Straight Lines

$$\cos\varphi = \frac{\vec{s}_1 \cdot \vec{s}_2}{|\vec{s}_1| \cdot |\vec{s}_2|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

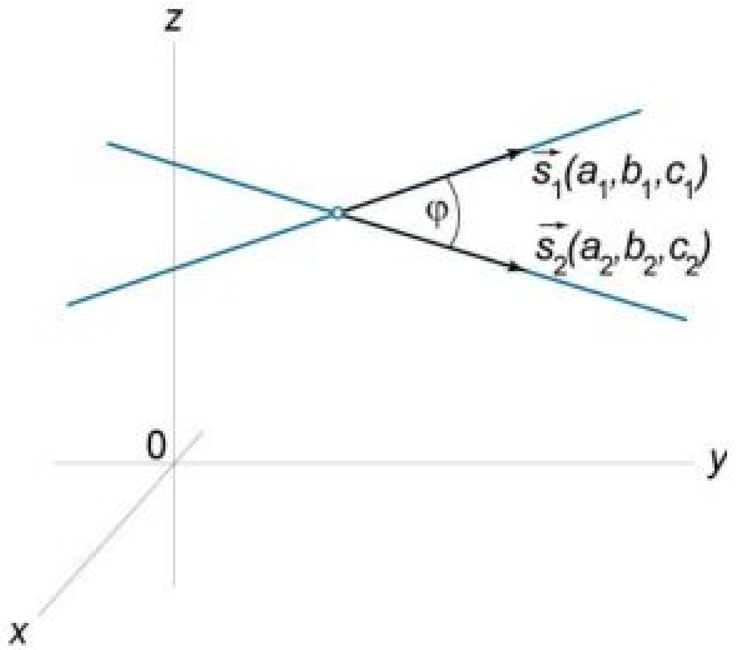


Figure 139.

694. Parallel Lines

Two lines are parallel if

$\vec{s}_1 \parallel \vec{s}_2$,

or

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

695. Perpendicular Lines

Two lines are parallel if

$\vec{s}_1 \cdot \vec{s}_2 = 0$,

or

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

696. Intersection of Two Lines

Two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and

$$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2} \text{ intersect if}$$

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0.$$

697. Parallel Line and Plane

The straight line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and the plane $Ax+By+Cz+D=0$ are parallel if

$\vec{n}\cdot\vec{s}=0$,

or

$Aa+Bb+Cc=0$.

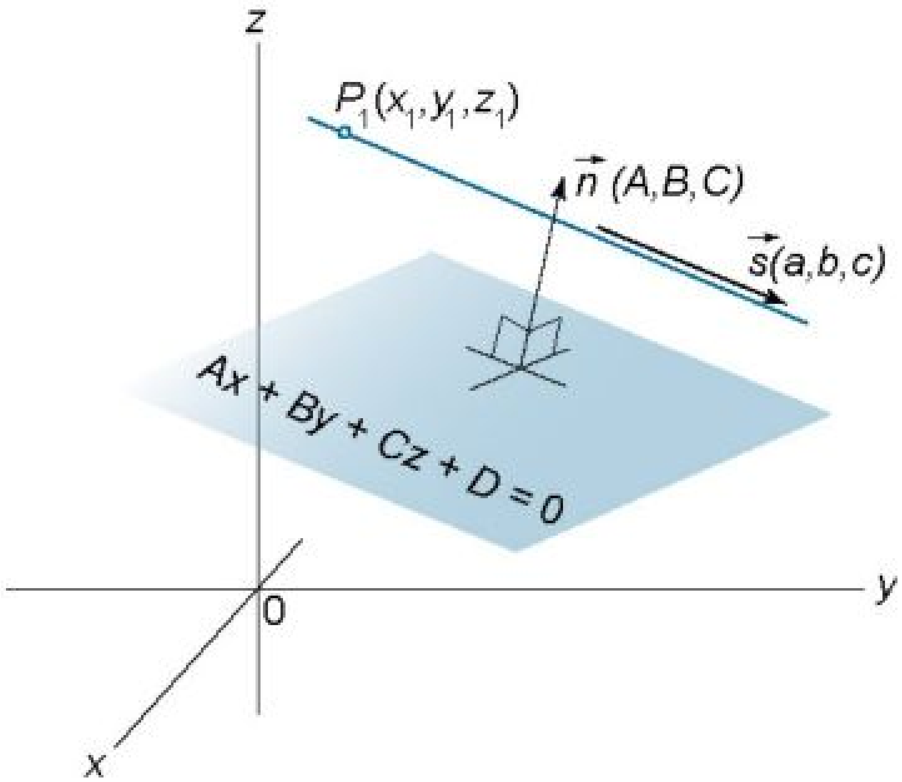


Figure 140.

698. Perpendicular Line and Plane

The straight line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and the plane $Ax+By+Cz+D=0$ are perpendicular if

$\vec{n}\,||\,\vec{s}$,

or

$\frac{A}{a}=\frac{B}{b}=\frac{C}{c}$.

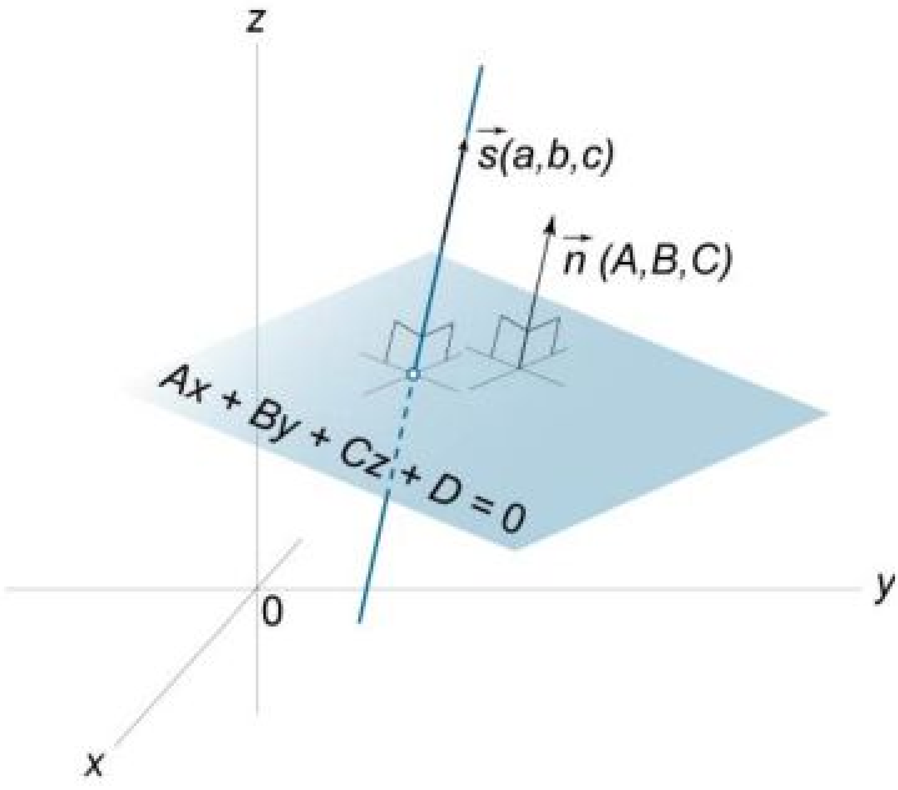


Figure 141.

7.11 Quadric Surfaces

Point coordinates of the quadric surfaces: x, y, z

Real numbers: A, B, C, a, b, c, $k_1, k_2, k_3, \ldots$

699. General Quadratic Equation

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy + 2Px + 2Qy + 2Rz + D = 0$$

700. Classification of Quadric Surfaces

Case	Rank(e)	Rank(E)	Δ	k signs	Type of Surface
1	3	4	<0	Same	Real Ellipsoid
2	3	4	>0	Same	Imaginary Ellipsoid
3	3	4	>0	Different	Hyperboloid of 1 Sheet
4	3	4	<0	Different	Hyperboloid of 2 Sheets
5	3	3		Different	Real Quadric Cone
6	3	3		Same	Imaginary Quadric Cone
7	2	4	<0	Same	Elliptic Paraboloid
8	2	4	>0	Different	Hyperbolic Paraboloid
9	2	3		Same	Real Elliptic Cylinder
10	2	3		Same	Imaginary Elliptic Cylinder
11	2	3		Different	Hyperbolic Cylinder
12	2	2		Different	Real Intersecting Planes
13	2	2		Same	Imaginary Intersecting Planes
14	1	3			Parabolic Cylinder
15	1	2			Real Parallel Planes
16	1	2			Imaginary Parallel Planes
17	1	1			Coincident Planes

Here

$$e = \begin{pmatrix} A & H & G \\ H & B & F \\ G & F & C \end{pmatrix}, \; E = \begin{pmatrix} A & H & Q & P \\ H & B & F & Q \\ G & F & C & R \\ P & Q & R & D \end{pmatrix}, \; \Delta = \det(E),$$

k_1, k_2, k_3 are the roots of the equation,

$$\begin{vmatrix} A-x & H & G \\ H & B-x & F \\ G & F & C-x \end{vmatrix} = 0.$$

701. Real Ellipsoid (Case 1)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

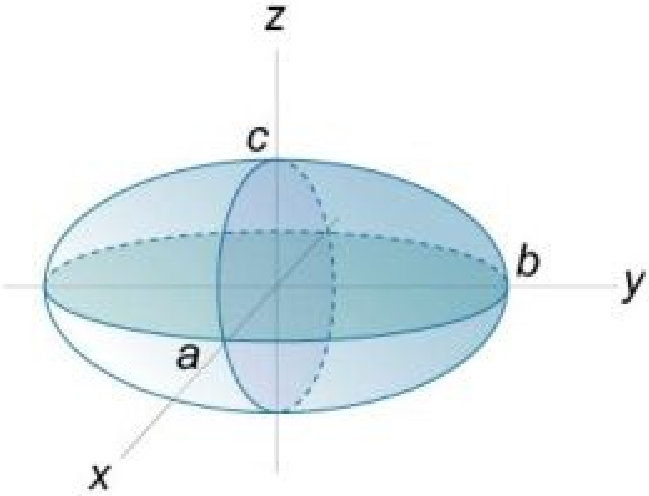


Figure 142.

702. Imaginary Ellipsoid (Case 2)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$$

703. Hyperboloid of 1 Sheet (Case 3)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

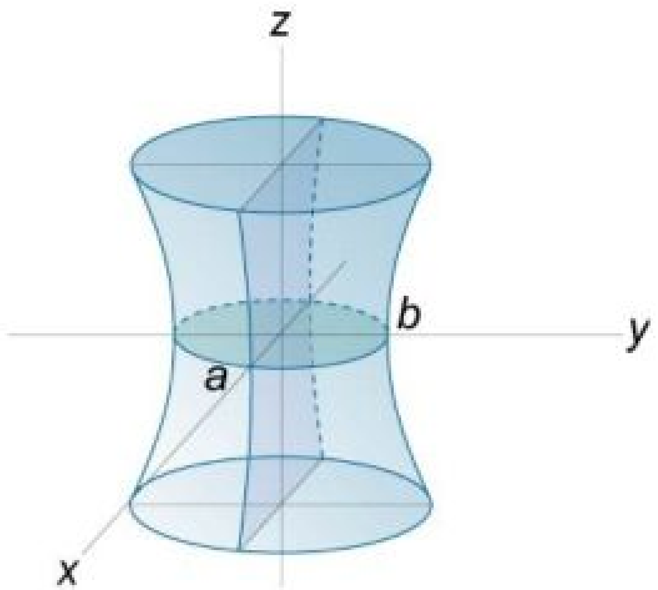


Figure 143.

704. Hyperboloid of 2 Sheets (Case 4)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

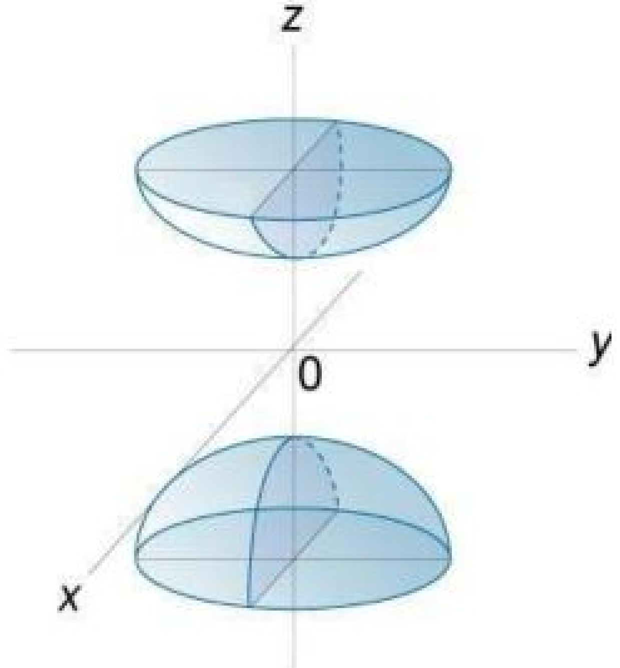


Figure 144.

705. Real Quadric Cone (Case 5)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

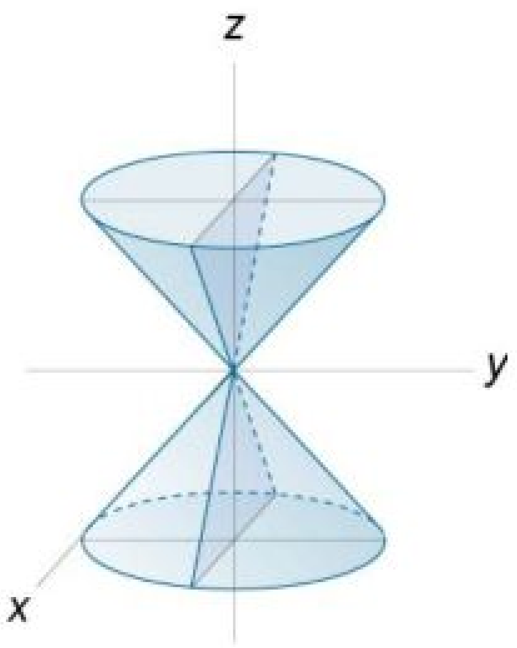


Figure 145.

706. Imaginary Quadric Cone (Case 6)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

707. Elliptic Paraboloid (Case 7)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$$

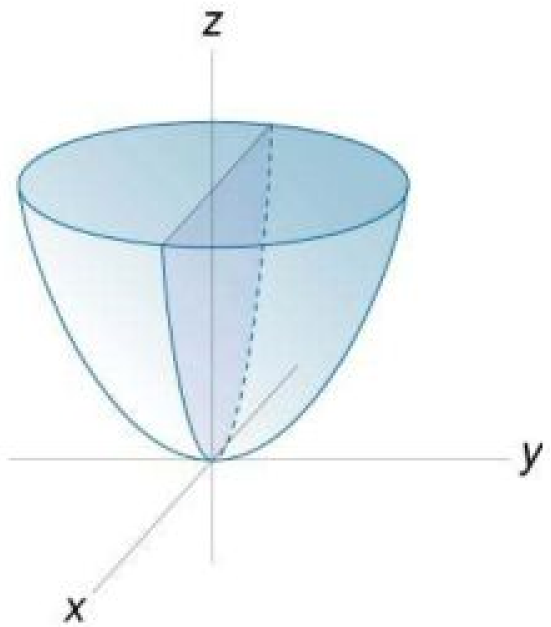


Figure 146.

708. Hyperbolic Paraboloid (Case 8)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0$$

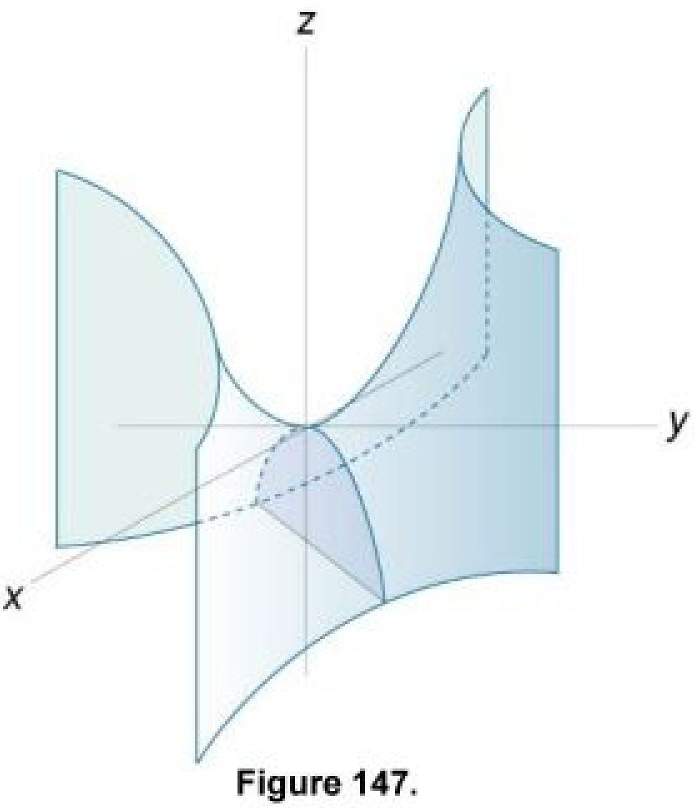


Figure 147.

709. Real Elliptic Cylinder (Case 9)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

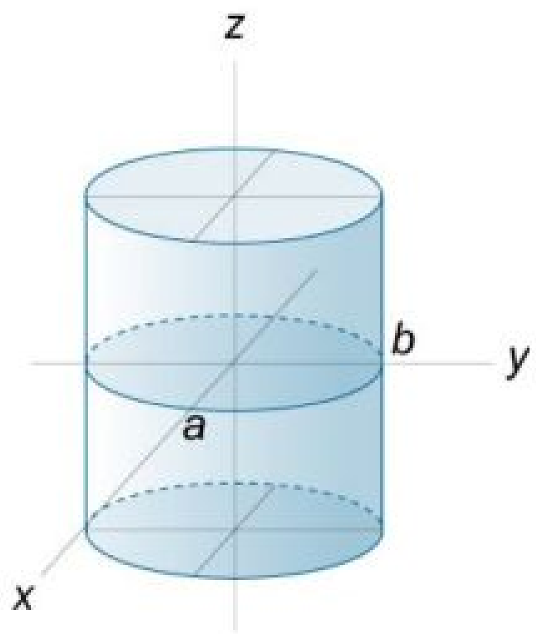


Figure 148.

710. Imaginary Elliptic Cylinder (Case 10)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$$

711. Hyperbolic Cylinder (Case 11)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

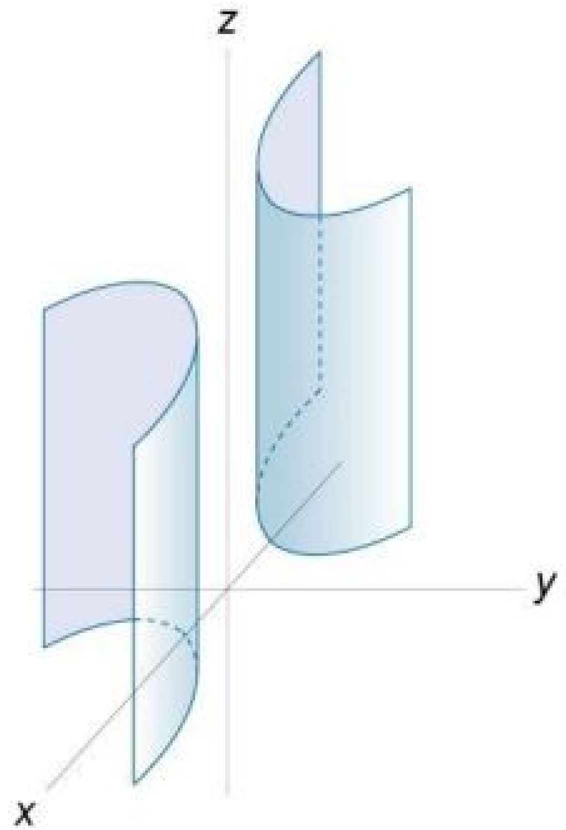


Figure 149.

712. Real Intersecting Planes (Case 12)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

713. Imaginary Intersecting Planes (Case 13)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$

714. Parabolic Cylinder (Case 14)

$$\frac{x^2}{a^2} - y = 0$$

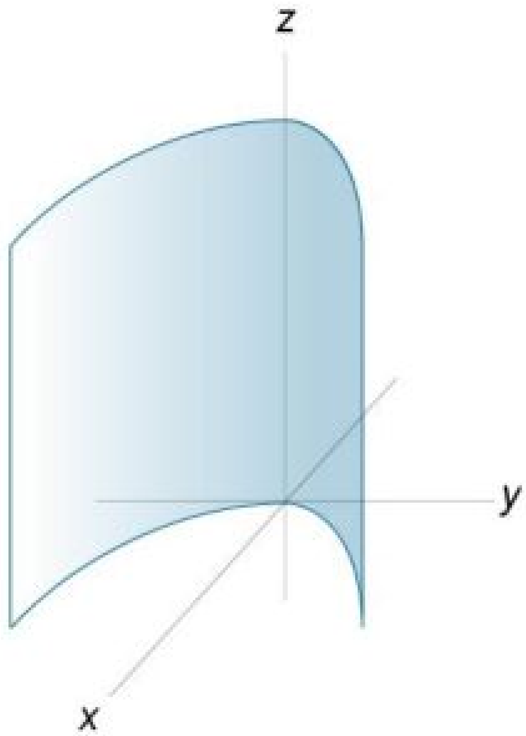


Figure 150.

715. Real Parallel Planes (Case 15)

$$\frac{x^2}{a^2} = 1$$

716. Imaginary Parallel Planes (Case 16)

$$\frac{x^2}{a^2} = -1$$

717. Coincident Planes (Case 17)

$$x^2 = 0$$

7.12 Sphere

Radius of a sphere: R
Point coordinates: x, y, z, x_1, y_1, z_1, ...
Center of a sphere: (a, b, c)
Real numbers: A, D, E, F, M

718. Equation of a Sphere Centered at the Origin (Standard Form)

$$x^2 + y^2 + z^2 = R^2$$

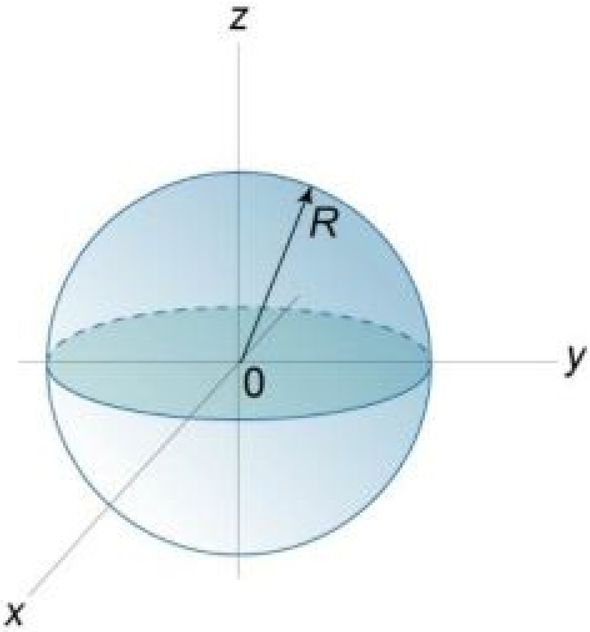


Figure 151.

719. Equation of a Circle Centered at Any Point (a, b, c)

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2$$

720. Diameter Form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0,$$

where
$P_1(x_1,y_1,z_1)$, $P_2(x_2,y_2,z_2)$ are the ends of a diameter.

721. Four Point Form

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+x_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+x_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+x_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+x_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

722. General Form

$Ax^2+Ay^2+Az^2+Dx+Ey+Fz+M=0$ (A is nonzero).
The center of the sphere has coordinates (a,b,c), where

$$a=-\frac{D}{2A},\ b=-\frac{E}{2A},\ c=-\frac{F}{2A}.$$

The radius of the sphere is

$$R=\frac{\sqrt{D^2+E^2+F^2-4A^2M}}{2A}.$$