Chapter 5

Matrices and Determinants

Matrices: A, B, C
Elements of a matrix: a_i, b_i, a_{ij}, b_{ij}, c_{ij}
Determinant of a matrix: $\det A$
Minor of an element a_{ij}: M_{ij}
Cofactor of an element a_{ij}: C_{ij}
Transpose of a matrix: A^T, $\widetilde{A}$
Adjoint of a matrix: $\operatorname{adj} A$
Trace of a matrix: $\operatorname{tr} A$
Inverse of a matrix: A^{-1}
Real number: k
Real variables: x_i
Natural numbers: m, n

5.1 Determinants

513. Second Order Determinant

$$\det A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

514. Third Order Determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

515. Sarrus Rule (Arrow Rule)

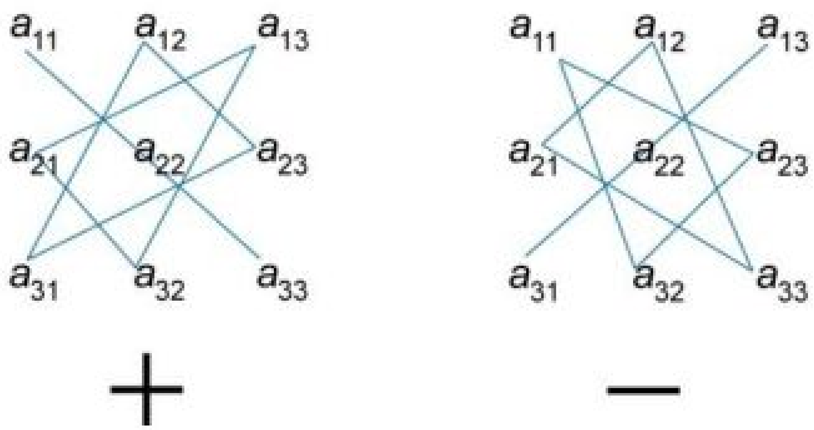

Figure 72.

516. N-th Order Determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$

517. Minor

The minor M_{ij} associated with the element a_{ij} of n-th order matrix A is the $(n-1)$-th order determinant derived from the matrix A by deletion of its i-th row and j-th column.

518. Cofactor

$$C_{ij} = (-1)^{i+j} M_{ij}$$

519. Laplace Expansion of n-th Order Determinant

Laplace expansion by elements of the i-th row

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}\,,\ i = 1, 2, \ldots, n\,.$$

Laplace expansion by elements of the j-th column

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij}\,,\ j = 1, 2, \ldots, n\,.$$

5.2 Properties of Determinants

520. The value of a determinant remains unchanged if rows are changed to columns and columns to rows.

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

521. If two rows (or two columns) are interchanged, the sign of the determinant is changed.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$$

522. If two rows (or two columns) are identical, the value of the determinant is zero.

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = 0$$

523. If the elements of any row (or column) are multiplied by a common factor, the determinant is multiplied by that factor.

$$\begin{vmatrix} ka_1 & kb_1 \\ a_2 & b_2 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

524. If the elements of any row (or column) are increased (or decreased)by equal multiples of the corresponding elements of any other row (or column), the value of the determinant is unchanged.

$$\begin{vmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

5.3 Matrices

525. Definition

An $m \times n$ matrix A is a rectangular array of elements (numbers or functions) with m rows and n columns.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

526. Square matrix is a matrix of order $n \times n$.

527. A square matrix $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$, i.e. it is symmetric about the leading diagonal.

528. A square matrix $[a_{ij}]$ is skew-symmetric if $a_{ij} = -a_{ji}$.

529. Diagonal matrix is a square matrix with all elements zero except those on the leading diagonal.

530. Unit matrix is a diagonal matrix in which the elements on the leading diagonal are all unity. The unit matrix is denoted by I.

531. A null matrix is one whose elements are all zero.

5.4 Operations with Matrices

532. Two matrices A and B are equal if, and only if, they are both of the same shape $m \times n$ and corresponding elements are equal.

533. Two matrices A and B can be added (or subtracted) of, and only if, they have the same shape $m \times n$. If

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$B = [b_{ij}] = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix},$$

then

$$A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}.$$

534. If k is a scalar, and $A=[a_{ij}]$ is a matrix, then

$$kA=[ka_{ij}]=\begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}.$$

535. Multiplication of Two Matrices
Two matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second.

If

$$A=[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

$$B=[b_{ij}]=\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{bmatrix},$$

then

$$AB = C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & \vdots & & \vdots \\ b_{m1} & c_{m2} & \dots & c_{mk} \end{bmatrix},$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{\lambda=1}^{n} a_{i\lambda}b_{\lambda j}$$

$(i = 1, 2, \dots, m;\ j = 1, 2, \dots, k)$.

Thus if

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix},\ B = [b_i] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 & a_{12}b_2 & a_{13}b_3 \\ a_{21}b_1 & a_{22}b_2 & a_{23}b_3 \end{bmatrix}.$$

536. Transpose of a Matrix
If the rows and columns of a matrix are interchanged, then the new matrix is called the transpose of the original matrix. If A is the original matrix, its transpose is denoted A^T or $\tilde{A}$.

537. The matrix A is orthogonal if $AA^T = I$.

538. If the matrix product AB is defined, then
$(AB)^T = B^T A^T$.

539. Adjoint of Matrix
If A is a square $n \times n$ matrix, its adjoint, denoted by $\operatorname{adj} A$, is the transpose of the matrix of cofactors C_{ij} of A:
$\operatorname{adj} A = [C_{ij}]^T$.

540. Trace of a Matrix
If A is a square $n \times n$ matrix, its trace, denoted by $\operatorname{tr} A$, is defined to be the sum of the terms on the leading diagonal:
$\operatorname{tr} A = a_{11} + a_{22} + \ldots + a_{nn}$.

541. Inverse of a Matrix
If A is a square $n \times n$ matrix with a nonsingular determinant $\det A$, then its inverse A^{-1} is given by
$$A^{-1} = \frac{\operatorname{adj} A}{\det A}.$$

542. If the matrix product AB is defined, then
$(AB)^{-1} = B^{-1}A^{-1}$.

543. If A is a square $n \times n$ matrix, the eigenvectors X satisfy the equation
$AX = \lambda X$,
while the eigenvalues λ satisfy the characteristic equation
$|A - \lambda I| = 0$.

5.5 Systems of Linear Equations

Variables: $x, y, z, x_1, x_2, \ldots$
Real numbers: $a_1, a_2, a_3, b_1, a_{11}, a_{12}, \ldots$

Determinants: D, D_x , D_y , D_z

Matrices: A, B, X

544. $\begin{cases} a_1x + b_1y = d_1 \\ a_2x + b_2y = d_2 \end{cases}$,

$x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$ (Cramer's rule),

where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

$$D_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix} = d_1b_2 - d_2b_1,$$

$$D_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} = a_1d_2 - a_2d_1.$$

545. If $D \neq 0$, then the system has a single solution:

$x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$.

If $D = 0$ and $D_x \neq 0$ (or $D_y \neq 0$), then the system has no solution.

If $D = D_x = D_y = 0$, then the system has infinitely many solutions.

546. $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$,

$x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$ (Cramer's rule), where

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ D_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\ D_z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

547. If $D \neq 0$, then the system has a single solution:

$$x=\frac{D_x}{D},\ y=\frac{D_y}{D},\ z=\frac{D_z}{D}.$$

If $D=0$ and $D_x \neq 0$ (or $D_y \neq 0$ or $D_z \neq 0$), then the system has no solution.

If $D=D_x=D_y=D_z=0$, then the system has infinitely many solutions.

548. Matrix Form of a System of n Linear Equations in n Unknowns

The set of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

can be written in matrix form

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

i.e.

$$A \cdot X = B,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

549. Solution of a Set of Linear Equations $n \times n$

$X = A^{-1} \cdot B$,

where A^{-1} is the inverse of A.

Chapter 6

Vectors

Vectors: $\vec{u}$, $\vec{v}$, $\vec{w}$, $\vec{r}$, $\overrightarrow{AB}$, ...
Vector length: $|\vec{u}|$, $|\vec{v}|$, ...
Unit vectors: $\vec{i}$, $\vec{j}$, $\vec{k}$
Null vector: $\vec{0}$
Coordinates of vector $\vec{u}$: X_1, Y_1, Z_1
Coordinates of vector $\vec{v}$: X_2, Y_2, Z_2
Scalars: λ, μ
Direction cosines: $\cos\alpha$, $\cos\beta$, $\cos\gamma$
Angle between two vectors: θ

6.1 Vector Coordinates

550. Unit Vectors

$$\vec{i} = (1, 0, 0),$$
$$\vec{j} = (0, 1, 0),$$
$$\vec{k} = (0, 0, 1),$$
$$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1.$$

551. $\vec{r} = \overrightarrow{AB} = (x_1 - x_0)\vec{i} + (y_1 - y_0)\vec{j} + (z_1 - z_0)\vec{k}$

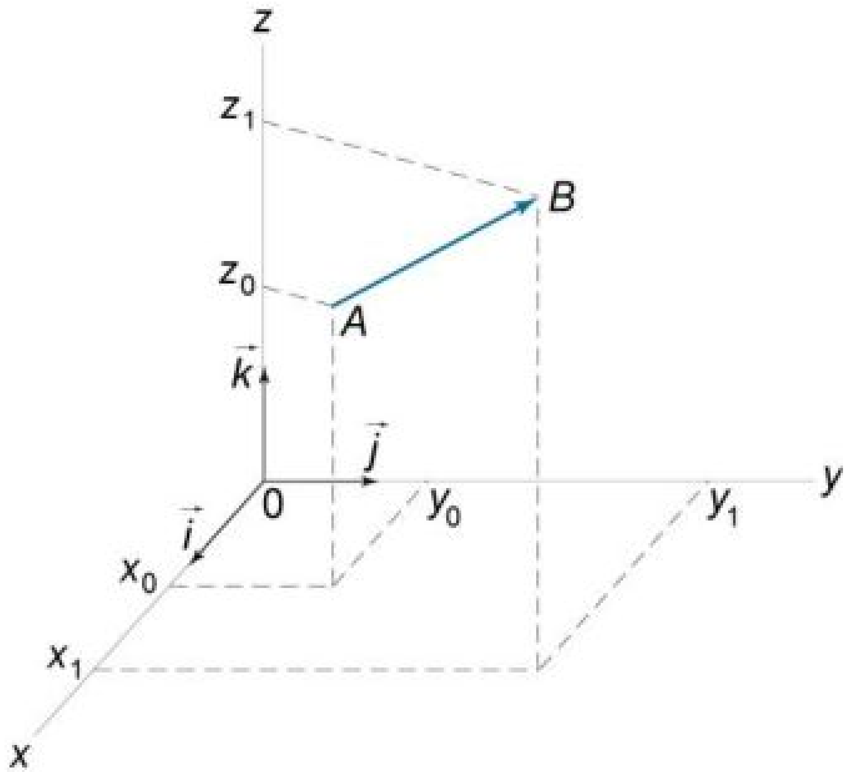

Figure 73.

552. $|\vec{r}| = \left|\vec{AB}\right| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$

553. If $\vec{AB} = \vec{r}$, then $\vec{BA} = -\vec{r}$.

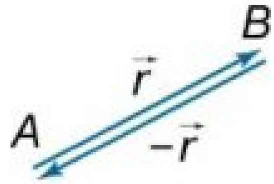

Figure 74.

554. $X = |\vec{r}|\cos\alpha$,

$Y = |\vec{r}|\cos\beta$,

$Z = |\vec{r}|\cos\gamma$.

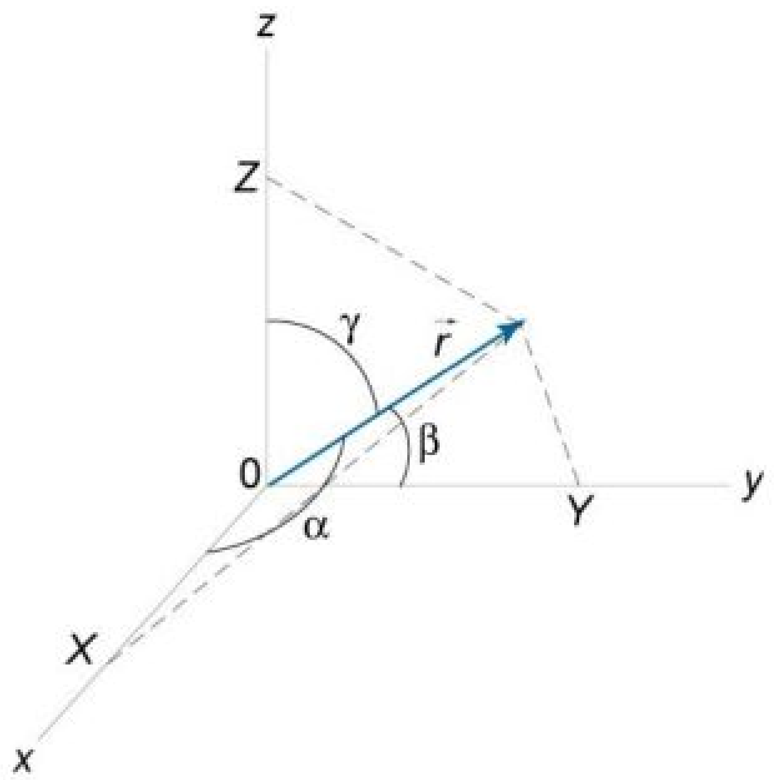

Figure 75.

555. If $\vec{r}(X, Y, Z) = \vec{r}_1(X_1, Y_1, Z_1)$, then $X = X_1$, $Y = Y_1$, $Z = Z_1$.

6.2 Vector Addition

556. $\vec{w} = \vec{u} + \vec{v}$

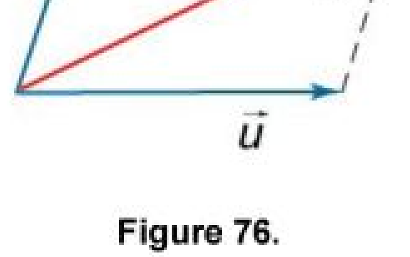

Figure 76.

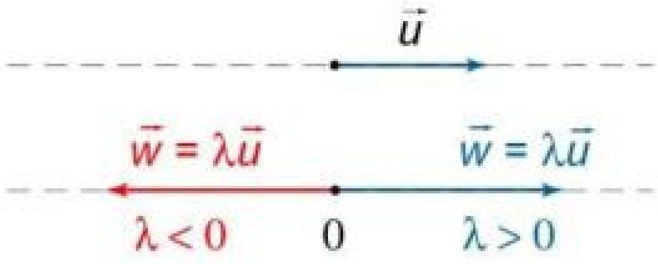

Figure 81.

567. $|\vec{w}| = |\lambda| \cdot |\vec{u}|$

568. $\lambda\vec{u} = (\lambda X, \lambda Y, \lambda Z)$

569. $\lambda\vec{u} = \vec{u}\lambda$

570. $(\lambda + \mu)\vec{u} = \lambda\vec{u} + \mu\vec{u}$

571. $\lambda(\mu\vec{u}) = \mu(\lambda\vec{u}) = (\lambda\mu)\vec{u}$

572. $\lambda(\vec{u} + \vec{v}) = \lambda\vec{u} + \lambda\vec{v}$

6.5 Scalar Product

573. Scalar Product of Vectors $\vec{u}$ and $\vec{v}$
$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$,
where θ is the angle between vectors $\vec{u}$ and $\vec{v}$.

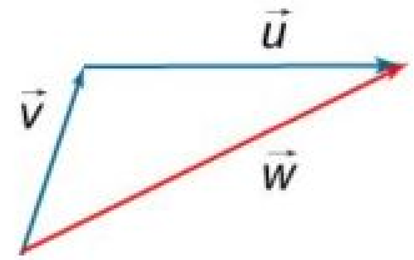

Figure 77.

557. $\vec{w} = \vec{u}_1 + \vec{u}_2 + \vec{u}_3 + \ldots + \vec{u}_n$

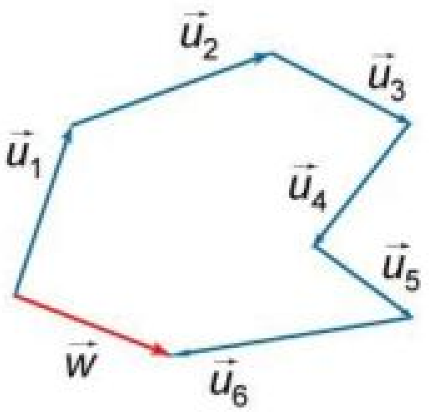

Figure 78.

558. Commutative Law

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

559. Associative Law

$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

560. $\vec{u} + \vec{v} = (X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2)$

6.3 Vector Subtraction

561. $\vec{w} = \vec{u} - \vec{v}$ if $\vec{v} + \vec{w} = \vec{u}$.

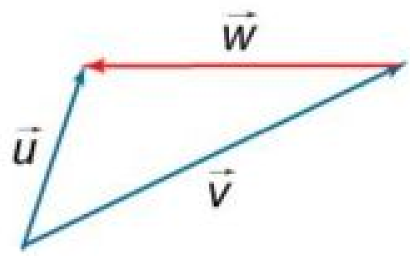

Figure 79.

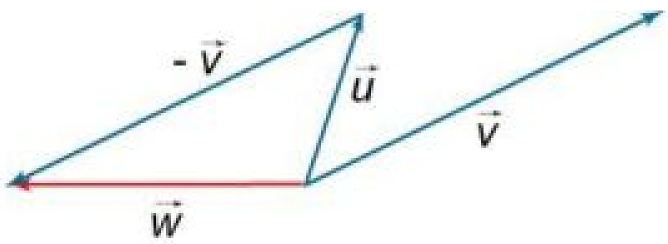

Figure 80.

562. $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$

563. $\vec{u} - \vec{u} = \vec{0} = (0, 0, 0)$

564. $|\vec{0}| = 0$

565. $\vec{u} - \vec{v} = (X_1 - X_2, Y_1 - Y_2, Z_1 - Z_2)$,

6.4 Scaling Vectors

566. $\vec{w} = \lambda \vec{u}$

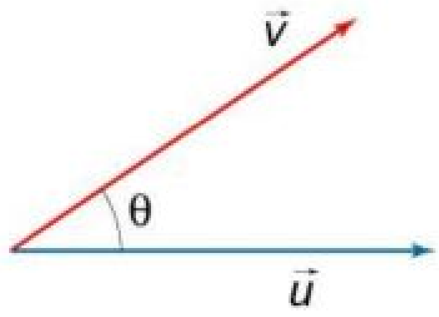

Figure 82.

574. Scalar Product in Coordinate Form
If $\vec{u} = (X_1, Y_1, Z_1)$, $\vec{v} = (X_2, Y_2, Z_2)$, then
$\vec{u} \cdot \vec{v} = X_1X_2 + Y_1Y_2 + Z_1Z_2$.

575. Angle Between Two Vectors
If $\vec{u} = (X_1, Y_1, Z_1)$, $\vec{v} = (X_2, Y_2, Z_2)$, then

$$\cos\theta = \frac{X_1X_2 + Y_1Y_2 + Z_1Z_2}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}\sqrt{X_2^2 + Y_2^2 + Z_2^2}}.$$

576. Commutative Property
$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

577. Associative Property
$(\lambda\vec{u}) \cdot (\mu\vec{v}) = \lambda\mu\vec{u} \cdot \vec{v}$

578. Distributive Property
$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

579. $\vec{u} \cdot \vec{v} = 0$ if $\vec{u}, \vec{v}$ are orthogonal ($\theta = \frac{\pi}{2}$).

580. $\vec{u} \cdot \vec{v} > 0$ if $0 < \theta < \frac{\pi}{2}$.

581. $\vec{u}\cdot\vec{v}<0$ if $\frac{\pi}{2}<\theta<\pi$.

582. $\vec{u}\cdot\vec{v}\le|\vec{u}|\cdot|\vec{v}|$

583. $\vec{u}\cdot\vec{v}=|\vec{u}|\cdot|\vec{v}|$ if $\vec{u}$, $\vec{v}$ are parallel ($\theta=0$).

584. If $\vec{u}=(X_1, Y_1, Z_1)$, then
$\vec{u}\cdot\vec{u}=\vec{u}^2=|\vec{u}|^2=X_1^{\ 2}+Y_1^{\ 2}+Z_1^{\ 2}$.

585. $\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1$

586. $\vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=0$

6.6 Vector Product

587. Vector Product of Vectors $\vec{u}$ and $\vec{v}$
$\vec{u}\times\vec{v}=\vec{w}$, where

- $|\vec{w}|=|\vec{u}|\cdot|\vec{v}|\cdot\sin\theta$, where $0\le\theta\le\frac{\pi}{2}$;
- $\vec{w}\perp\vec{u}$ and $\vec{w}\perp\vec{v}$;
- Vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ form a right-handed screw.

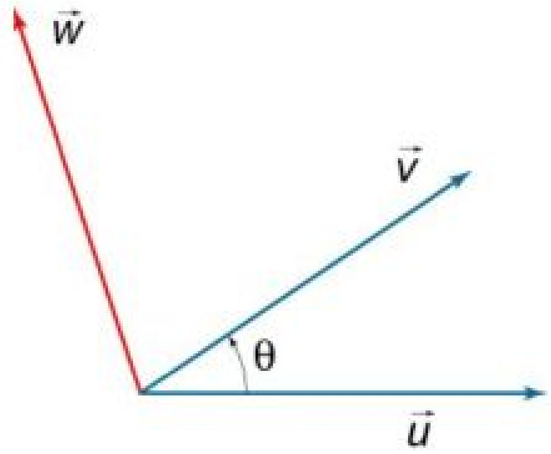

Figure 83.

588. $\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$

589. $\vec{w} = \vec{u} \times \vec{v} = \left(\begin{vmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{vmatrix}, - \begin{vmatrix} X_1 & Z_1 \\ X_2 & Z_2 \end{vmatrix}, \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix} \right)$

590. $S = |\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\theta$ (Fig.83)

591. Angle Between Two Vectors (Fig.83)
$\sin\theta = \dfrac{\vec{u} \times \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$

592. Noncommutative Property
$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

593. Associative Property
$(\lambda\vec{u}) \times (\mu\vec{v}) = \lambda\mu\vec{u} \times \vec{v}$

594. Distributive Property
$\vec{u}\times(\vec{v}+\vec{w})=\vec{u}\times\vec{v}+\vec{u}\times\vec{w}$

595. $\vec{u}\times\vec{v}=\vec{0}$ if $\vec{u}$ and $\vec{v}$ are parallel ($\theta=0$).

596. $\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=\vec{0}$

597. $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, $\vec{k}\times\vec{i}=\vec{j}$

6.7 Triple Product

598. Scalar Triple Product
$[\vec{u}\vec{v}\vec{w}]=\vec{u}\cdot(\vec{v}\times\vec{w})=\vec{v}\cdot(\vec{w}\times\vec{u})=\vec{w}\cdot(\vec{u}\times\vec{v})$

599. $[\vec{u}\vec{v}\vec{w}]=[\vec{w}\vec{u}\vec{v}]=[\vec{v}\vec{w}\vec{u}]=-[\vec{v}\vec{u}\vec{w}]=-[\vec{w}\vec{v}\vec{u}]=-[\vec{u}\vec{w}\vec{v}]$

600. $k\vec{u}\cdot(\vec{v}\times\vec{w})=k[\vec{u}\vec{v}\vec{w}]$

601. Scalar Triple Product in Coordinate Form

$$\vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix},$$

where
$\vec{u}=(X_1,Y_1,Z_1)$, $\vec{v}=(X_2,Y_2,Z_2)$, $\vec{w}=(X_3,Y_3,Z_3)$.

602. Volume of Parallelepiped
$V=|\vec{u}\cdot(\vec{v}\times\vec{w})|$

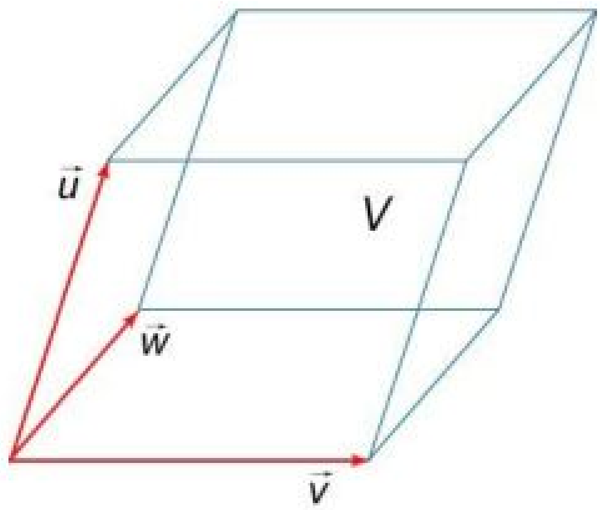

Figure 84.

603. Volume of Pyramid

$$V = \frac{1}{6}\left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$$

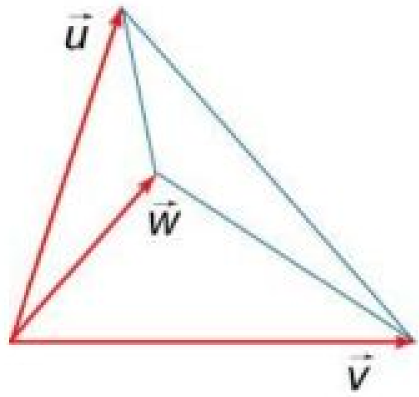

Figure 85.

604. If $\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)=0$, then the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent, so $\vec{w}=\lambda\vec{u}+\mu\vec{v}$ for some scalars λ and μ.

605. If $\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\neq 0$, then the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent.

606. Vector Triple Product

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$