Chapter 8

Differential Calculus

Functions: f, g, y, u, v
Argument (independent variable): x
Real numbers: a, b, c, d
Natural number: n
Angle: α
Inverse function: f^{-1}

8.1 Functions and Their Graphs

723. Even Function
$f(-x) = f(x)$

724. Odd Function
$f(-x) = -f(x)$

725. Periodic Function
$f(x + nT) = f(x)$

726. Inverse Function
$y = f(x)$ is any function, $x = g(y)$ or $y = f^{-1}(x)$ is its inverse function.

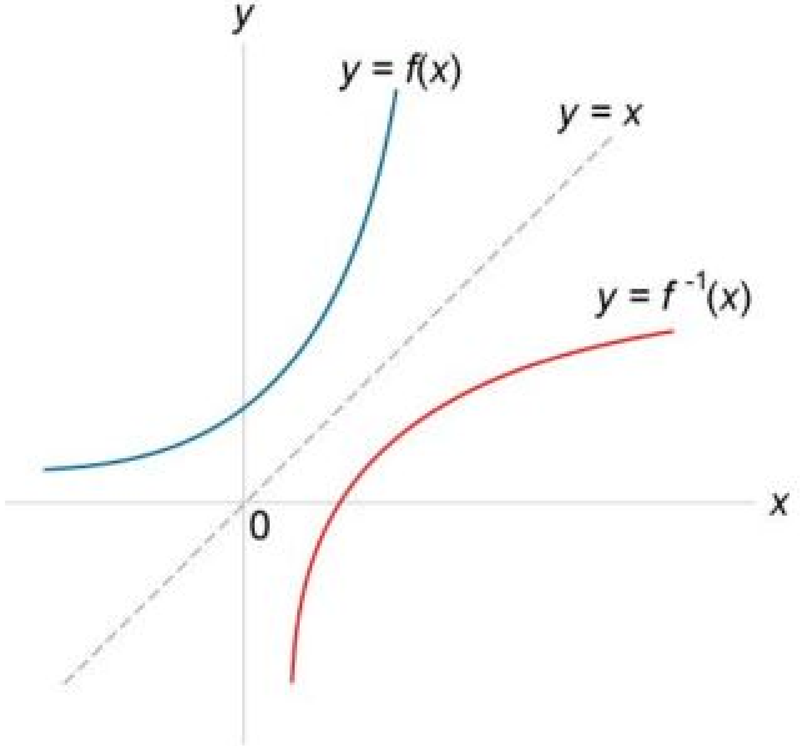

Figure 152.

727. Composite Function

$y = f(u),\ u = g(x),\ y = f(g(x))$ is a composite function.

728. Linear Function

$y = ax + b,\ \ x \in R,\ \ a = \tan\alpha$ is the slope of the line, b is the y-intercept.

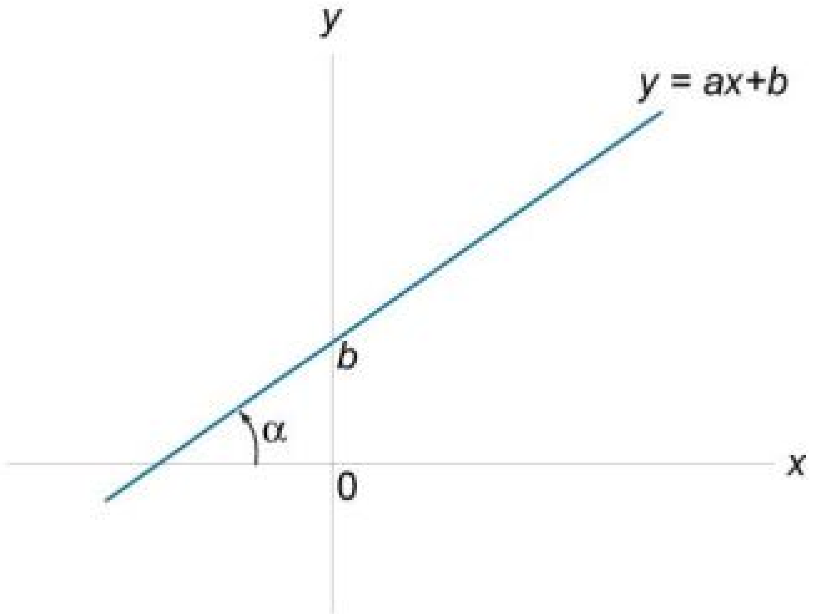

Figure 153.

729. Quadratic Function
$y = x^2,\ x \in R$.

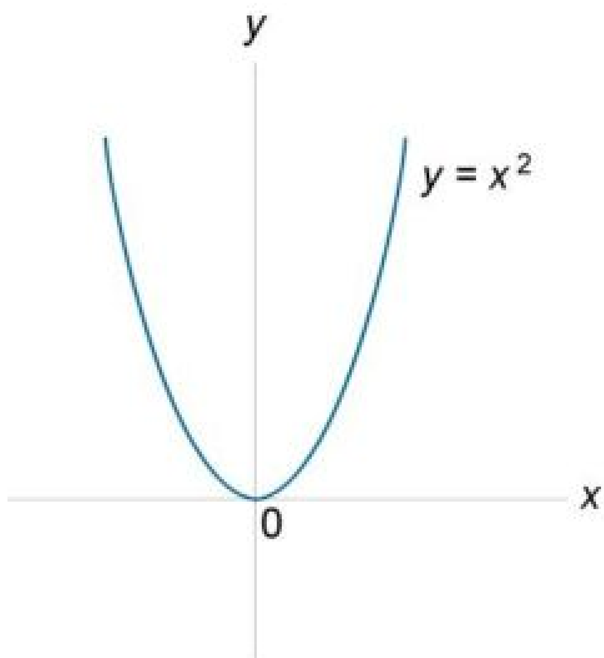

Figure 154.

730. $y = ax^2 + bx + c$, $x \in R$.

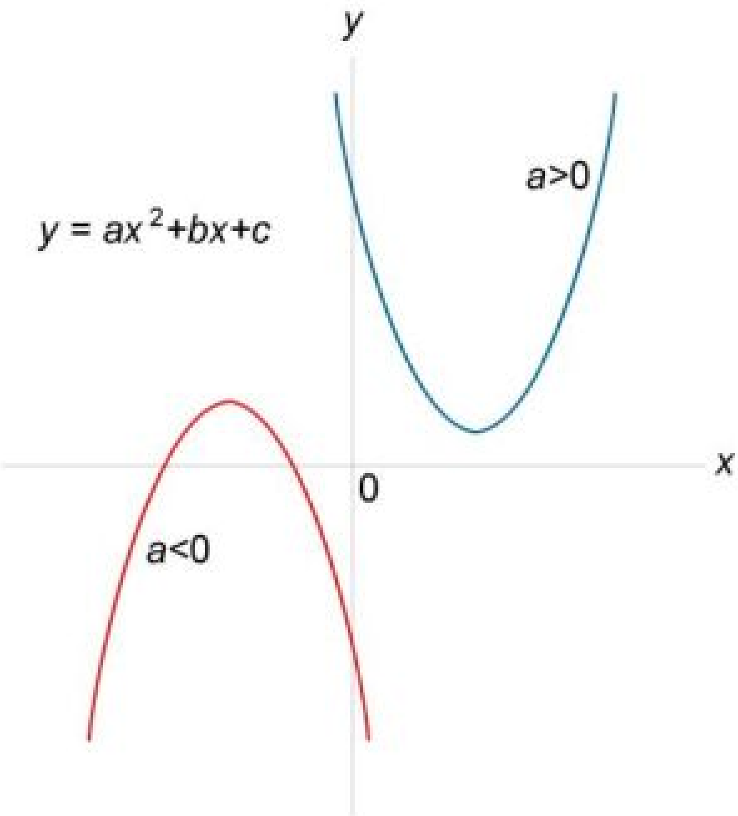

Figure 155.

731. Cubic Function
$y = x^3$, $x \in R$.

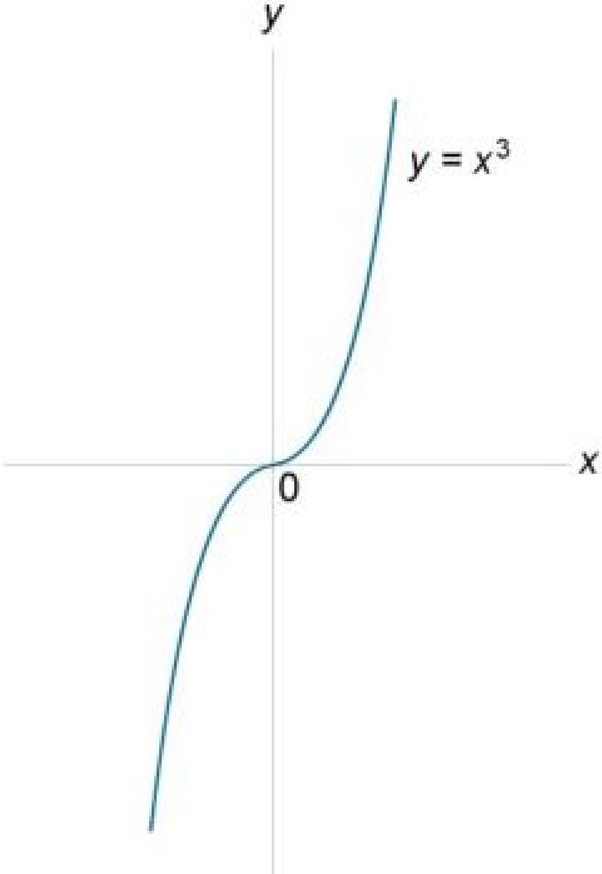

Figure 156.

732. $y = ax^3 + bx^2 + cx + d,\ x \in R.$

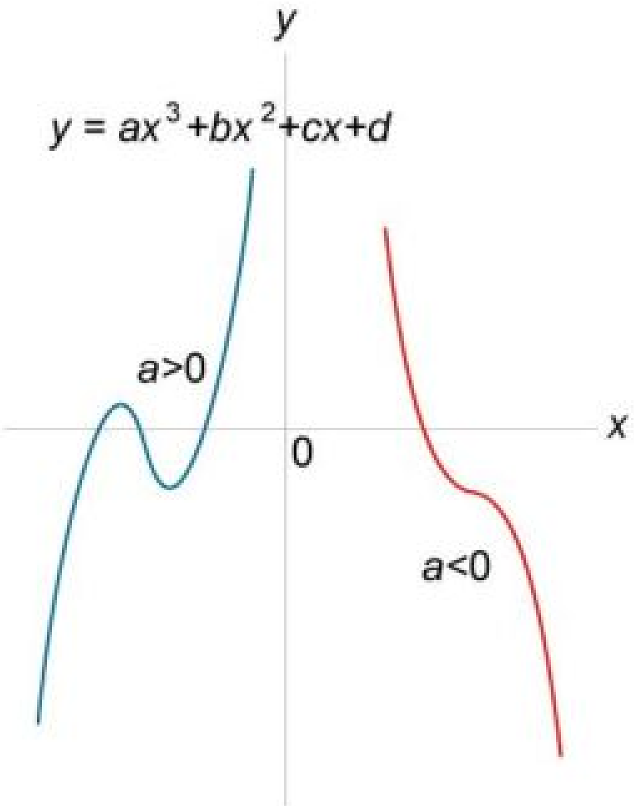

Figure 157.

733. Power Function

$y = x^n$, $n \in N$.

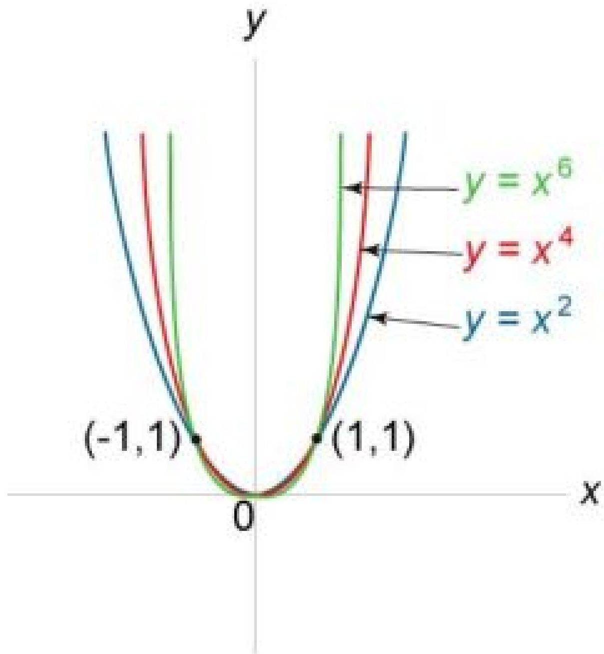

Figure 158.

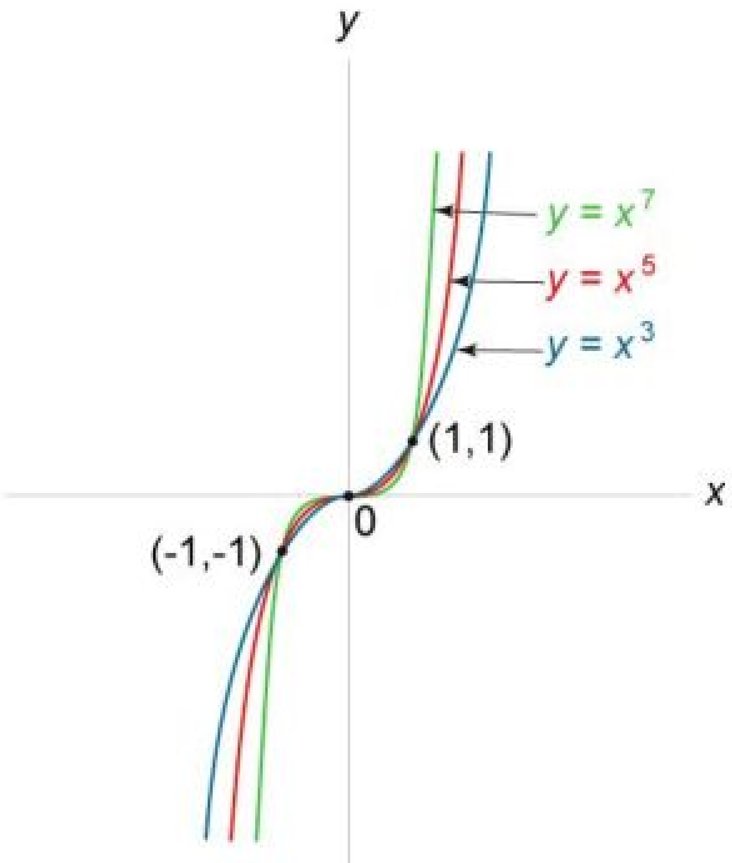

Figure 159.

734. Square Root Function

$y = \sqrt{x}$, $x \in [0, \infty)$.

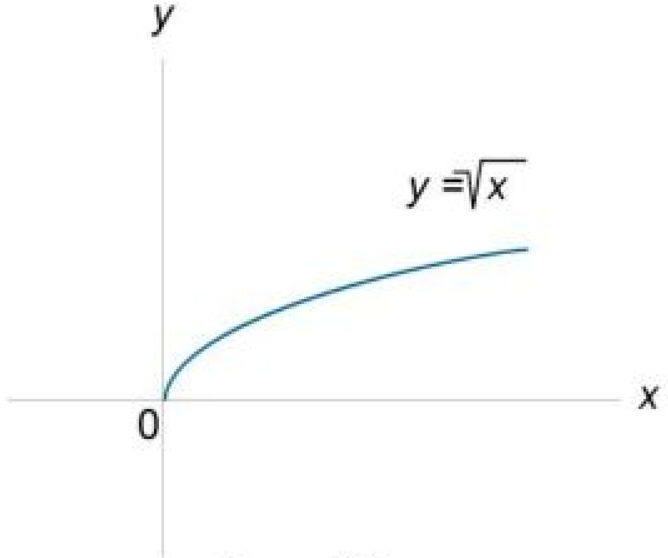

Figure 160.

735. Exponential Functions

$y = a^x$, $a > 0$, $a \neq 1$,

$y = e^x$ if $a = e$, $e = 2.71828182846\ldots$

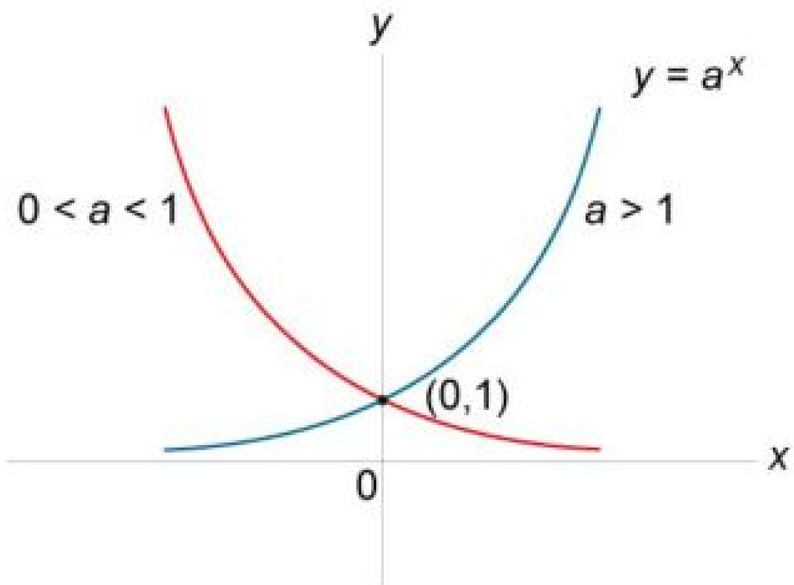

Figure 161.

736. Logarithmic Functions

$y = \log_a x$, $x \in (0, \infty)$, $a > 0$, $a \neq 1$,

$y = \ln x$ if $a = e$, $x > 0$.

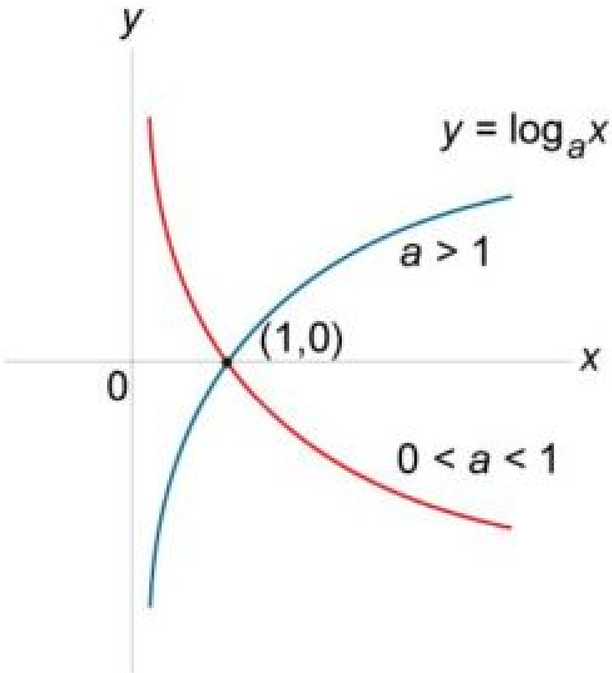

Figure 162.

737. Hyperbolic Sine Function

$$y = \sinh x, \ \sinh x = \frac{e^x - e^{-x}}{2}, \ x \in R.$$

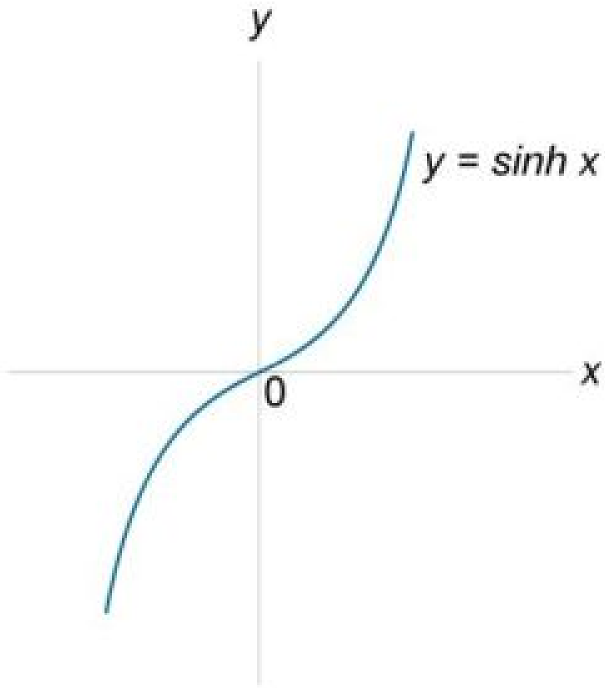

Figure 163.

738. Hyperbolic Cosine Function

$$y = \cosh x\,,\ \cosh x = \frac{e^{x} + e^{-x}}{2}\,,\ x \in R\,.$$

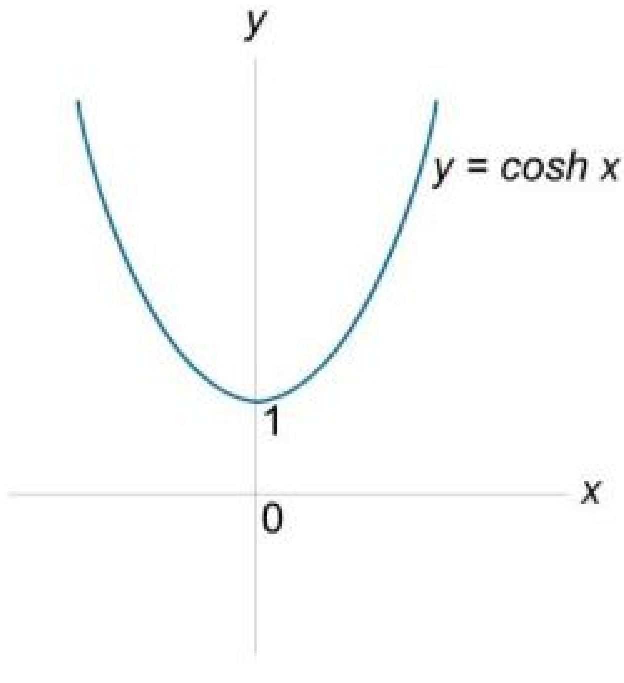

Figure 164.

739. Hyperbolic Tangent Function

$$y = \tanh x\,,\ y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\,,\ x \in R\,.$$

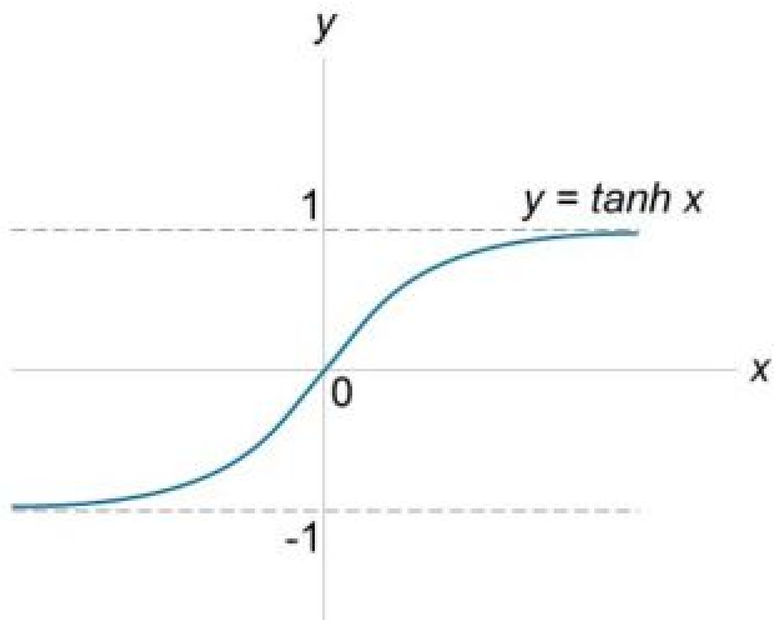

Figure 165.

740. Hyperbolic Cotangent Function

$$y = \coth x\,,\ y = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}\,,\ x \in R\,,\ x \neq 0\,.$$

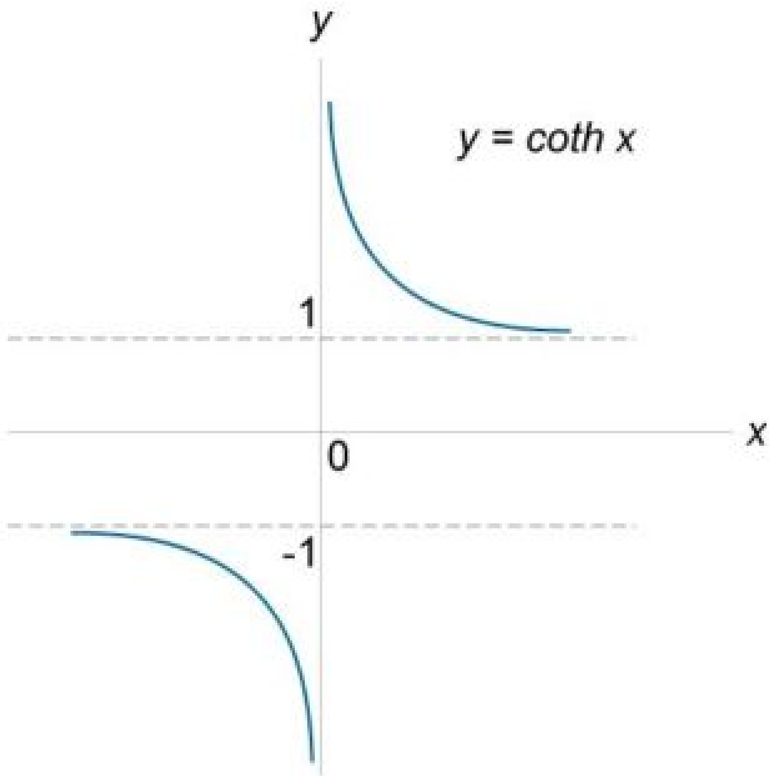

Figure 166.

741. Hyperbolic Secant Function

$$y = \operatorname{sech} x, \; y = \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \; x \in R.$$

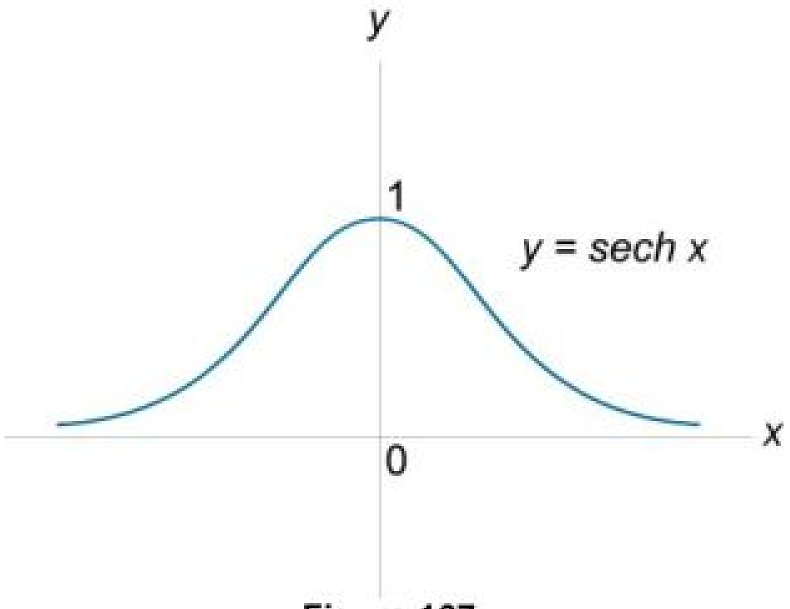

Figure 167.

742. Hyperbolic Cosecant Function

$$y = \operatorname{csch} x,\ y = \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}},\ x \in R,\ x \neq 0.$$

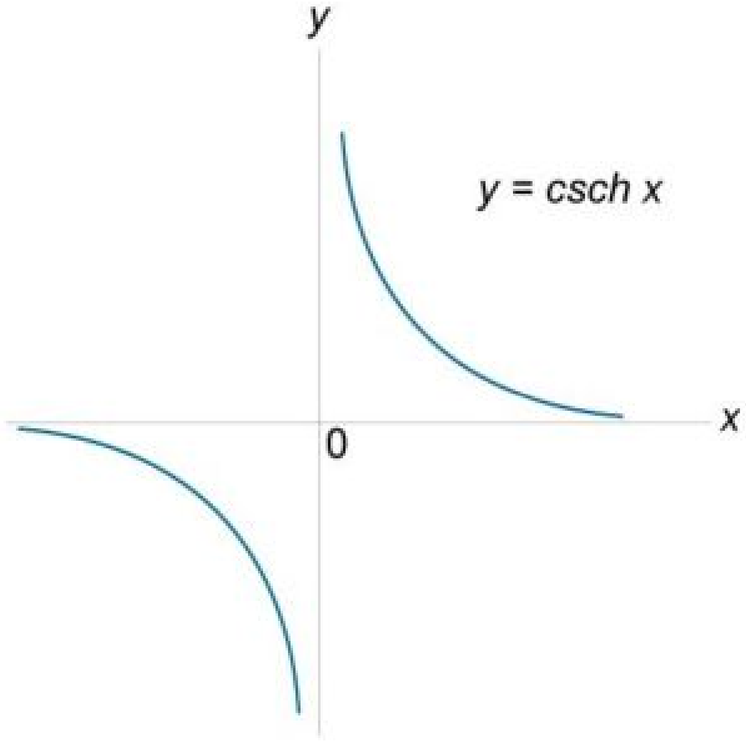

Figure 168.

743. Inverse Hyperbolic Sine Function

$$y = \operatorname{arcsinh} x,\ x \in R.$$

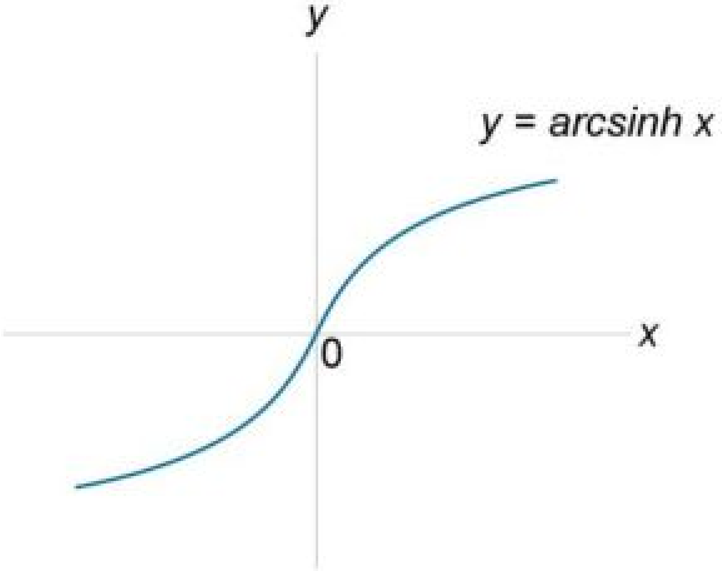

Figure 169.

744. Inverse Hyperbolic Cosine Function
$y = \operatorname{arccosh} x,\ x \in [1, \infty)$.

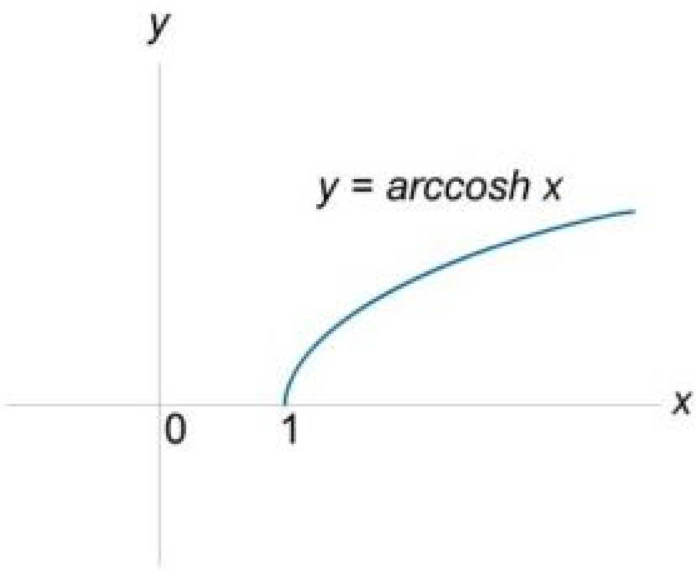

Figure 170.

745. Inverse Hyperbolic Tangent Function
$y = \operatorname{arctanh} x,\ x \in (-1, 1)$.

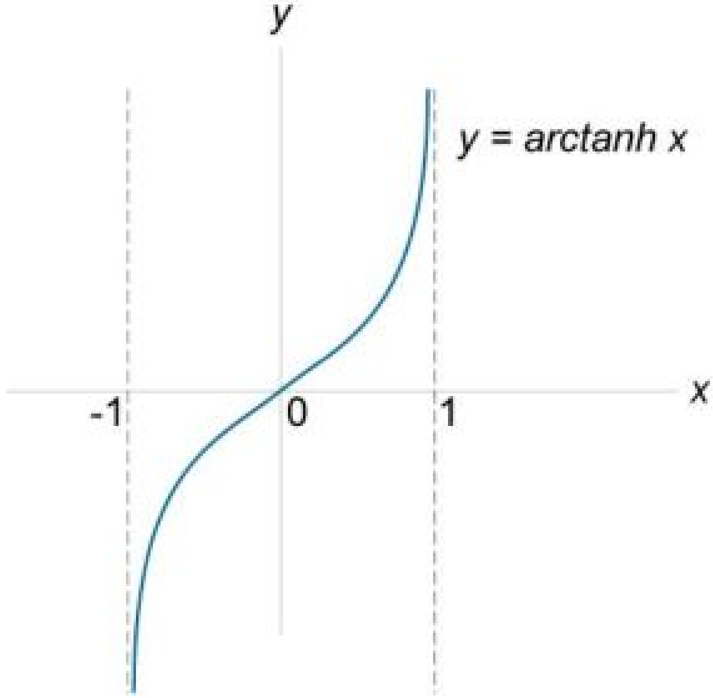

Figure 171.

746. Inverse Hyperbolic Cotangent Function
$y = \operatorname{arccoth} x$, $x \in (-\infty, -1) \cup (1, \infty)$.

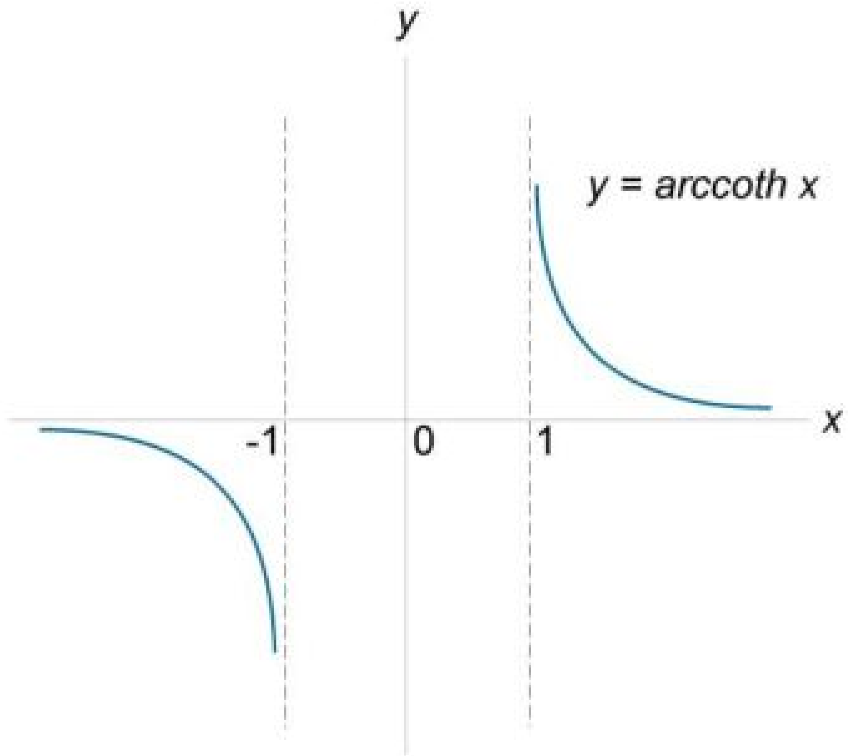

Figure 172.

747. Inverse Hyperbolic Secant Function
$y = \operatorname{arcsech} x, \ x \in (0, 1]$.

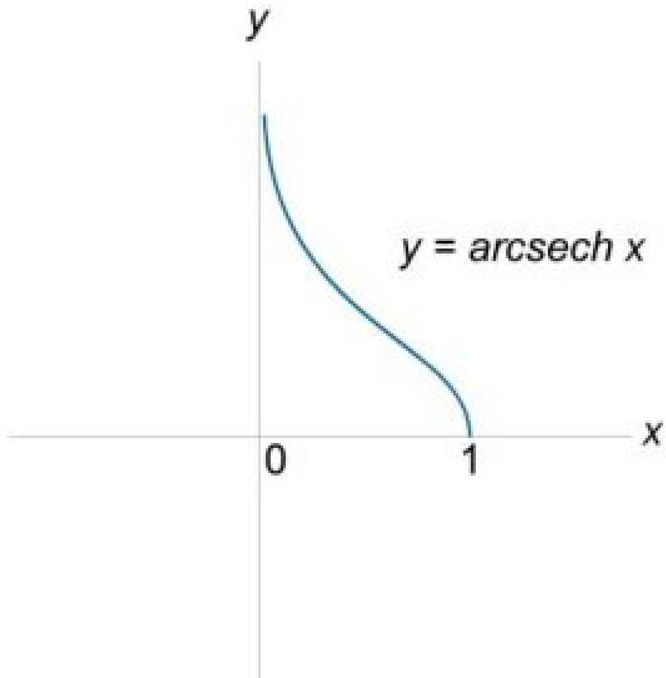

Figure 173.

748. Inverse Hyperbolic Cosecant Function
$y = \operatorname{arccsch} x,\ x \in R,\ x \neq 0.$

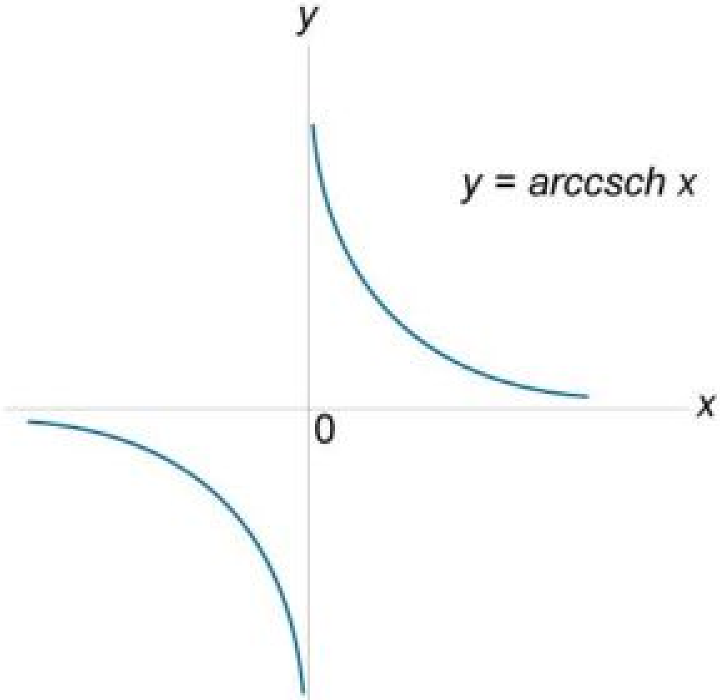

Figure 174.

8.2 Limits of Functions

Functions: $f(x)$, $g(x)$
Argument: x
Real constants: a, k

749. $\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$

750. $\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)$

751. $\lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)$

752. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$, if $\lim_{x\to a}g(x)\neq 0$.

753. $\lim_{x\to a}[kf(x)]=k\lim_{x\to a}f(x)$

754. $\lim_{x\to a}f(g(x))=f\left(\lim_{x\to a}g(x)\right)$

755. $\lim_{x\to a}f(x)=f(a)$, if the function $f(x)$ is continuous at $x=a$.

756. $\lim_{x\to 0}\frac{\sin x}{x}=1$

757. $\lim_{x\to 0}\frac{\tan x}{x}=1$

758. $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$

759. $\lim\limits_{x\to 0}\frac{\tan^{-1}x}{x}=1$

760. $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=1$

761. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$

762. $\lim\limits_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}=e^{k}$

763. $\lim\limits_{x\to 0}a^{x}=1$

8.3 Definition and Properties of the Derivative

Functions: f, g, y, u, v
Independent variable: x
Real constant: k
Angle: α

764. $y'(x)=\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}$

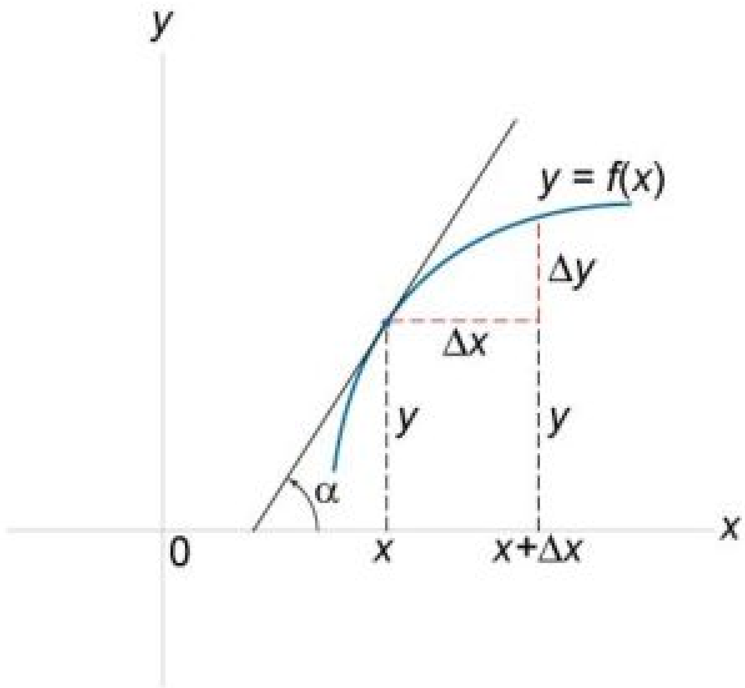

Figure 175.

765. $\frac{dy}{dx} = \tan\alpha$

766. $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

767. $\frac{d(u-v)}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

768. $\frac{d(ku)}{dx} = k\frac{du}{dx}$

769. Product Rule

$$\frac{d(u \cdot v)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

770. Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{\frac{du}{dx}\cdot v-u\cdot\frac{dv}{dx}}{v^2}$$

771. Chain Rule

$y=f(g(x)),\ u=g(x),$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

772. Derivative of Inverse Function

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}},$$

where $x(y)$ is the inverse function of $y(x)$.

773. Reciprocal Rule

$$\frac{d}{dx}\left(\frac{1}{y}\right)=-\frac{\frac{dy}{dx}}{y^2}$$

774. Logarithmic Differentiation

$y=f(x),\ \ln y=\ln f(x),$

$$\frac{dy}{dx}=f(x)\cdot\frac{d}{dx}[\ln f(x)].$$

8.4 Table of Derivatives

Independent variable: x
Real constants: C, a, b, c
Natural number: n

775. $\frac{d}{dx}(C) = 0$

776. $\frac{d}{dx}(x) = 1$

777. $\frac{d}{dx}(ax + b) = a$

778. $\frac{d}{dx}(ax^2 + bx + c) = ax + b$

779. $\frac{d}{dx}(x^n) = nx^{n-1}$

780. $\frac{d}{dx}(x^{-n}) = -\frac{n}{x^{n+1}}$

781. $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

782. $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

783. $\frac{d}{dx}\left(\sqrt[n]{x}\right) = \frac{1}{n\sqrt[n]{x^{n-1}}}$

784. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

785. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$, $a > 0$, $a \neq 1$.

786. $\frac{d}{dx}\left(a^x\right)=a^x \ln a$, $a>0$, $a\neq 1$.

787. $\frac{d}{dx}\left(e^x\right)=e^x$

788. $\frac{d}{dx}(\sin x)=\cos x$

789. $\frac{d}{dx}(\cos x)=-\sin x$

790. $\frac{d}{dx}(\tan x)=\frac{1}{\cos^2 x}=\sec^2 x$

791. $\frac{d}{dx}(\cot x)=-\frac{1}{\sin^2 x}=-\csc^2 x$

792. $\frac{d}{dx}(\sec x)=\tan x\cdot\sec x$

793. $\frac{d}{dx}(\csc x)=-\cot x\cdot\csc x$

794. $\frac{d}{dx}(\arcsin x)=\frac{1}{\sqrt{1-x^2}}$

795. $\frac{d}{dx}(\arccos x)=-\frac{1}{\sqrt{1-x^2}}$

796. $\frac{d}{dx}(\arctan x)=\frac{1}{1+x^2}$

797. $\frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$

798. $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$

799. $\frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

800. $\frac{d}{dx}(\sinh x) = \cosh x$

801. $\frac{d}{dx}(\cosh x) = \sinh x$

802. $\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

803. $\frac{d}{dx}(\coth x) = -\frac{1}{\sinh^2 x} = -\operatorname{csch}^2 x$

804. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \cdot \tanh x$

805. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \cdot \coth x$

806. $\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}}$

807. $\frac{d}{dx}(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}}$

$$(uv)''' = u'''v + 3u''v' + 3u'v'' + uv'''$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{1 \cdot 2}u^{(n-2)}v'' + \ldots + uv^{(n)}$$

816. $$\left(x^{m}\right)^{(n)} = \frac{m!}{(m-n)!}x^{m-n}$$

817. $$\left(x^{n}\right)^{(n)} = n!$$

818. $$(\log_a x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^{n}\ln a}$$

819. $$(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^{n}}$$

820. $$\left(a^{x}\right)^{(n)} = a^{x}\ln^{n} a$$

821. $$\left(e^{x}\right)^{(n)} = e^{x}$$

822. $$\left(a^{mx}\right)^{(n)} = m^{n}a^{mx}\ln^{n} a$$

823. $$(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$$

824. $$(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$$

8.6 Applications of Derivative

Functions: f, g, y
Position of an object: s
Velocity: v
Acceleration: w
Independent variable: x
Time: t
Natural number: n

825. Velocity and Acceleration
$s = f(t)$ is the position of an object relative to a fixed coordinate system at a time t,
$v = s' = f'(t)$ is the instantaneous velocity of the object,
$w = v' = s'' = f''(t)$ is the instantaneous acceleration of the object.

826. Tangent Line
$$y - y_0 = f'(x_0)(x - x_0)$$

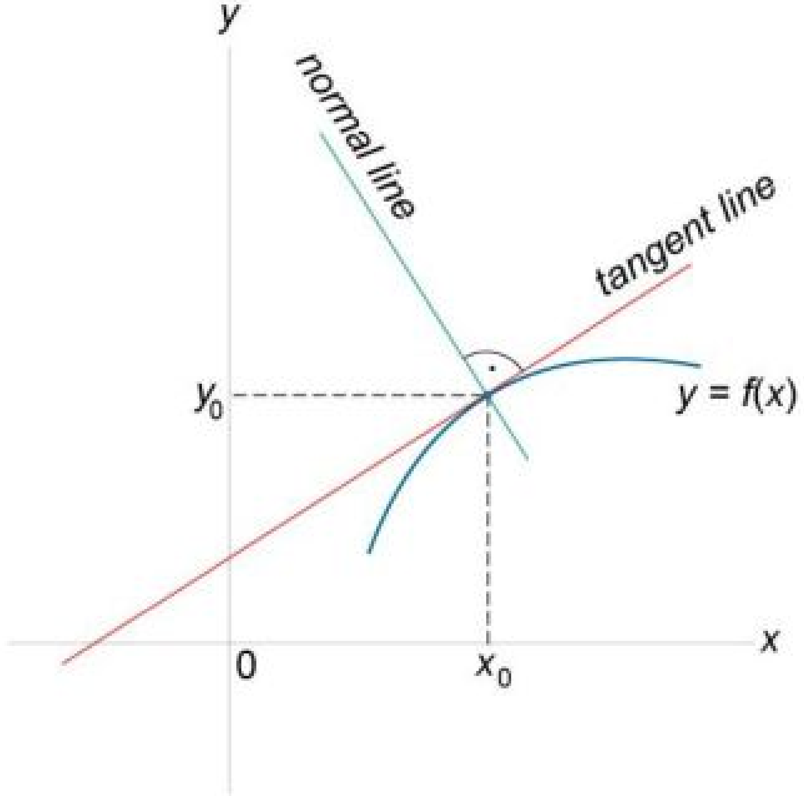

Figure 176.

827. Normal Line

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0) \text{ (Fig 176)}$$

828. Increasing and Decreasing Functions.

If $f'(x_0) > 0$, then f(x) is increasing at x_0. (Fig 177, $x < x_1$, $x_2 < x$),

If $f'(x_0) < 0$, then f(x) is decreasing at x_0. (Fig 177, $x_1 < x < x_2$),

If $f'(x_0)$ does not exist or is zero, then the test fails.

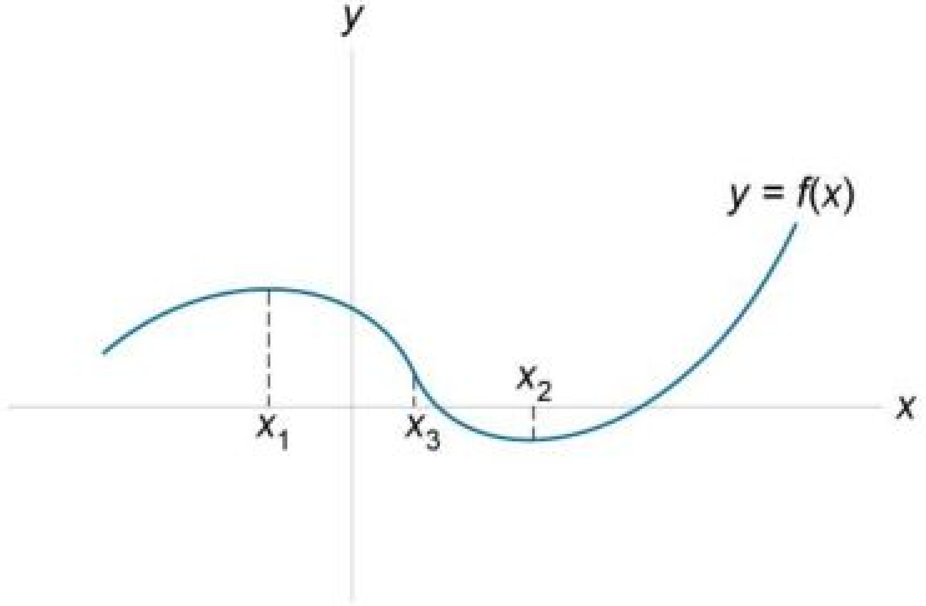

Figure 177.

829. Local extrema
A function f(x) has a local maximum at x_1 if and only if there exists some interval containing x_1 such that $f(x_1) \geq f(x)$ for all x in the interval (Fig.177).

A function f(x) has a local minimum at x_2 if and only if there exists some interval containing x_2 such that $f(x_2) \leq f(x)$ for all x in the interval (Fig.177).

830. Critical Points
A critical point on f(x) occurs at x_0 if and only if either $f'(x_0)$ is zero or the derivative doesn't exist.

831. First Derivative Test for Local Extrema.
If f(x) is increasing ($f'(x) > 0$) for all x in some interval $(a, x_1]$ and f(x) is decreasing ($f'(x) < 0$) for all x in some interval $[x_1, b)$, then f(x) has a local maximum at x_1 (Fig.177).

832. If f(x) is decreasing ($f'(x)<0$) for all x in some interval $(a,x_2]$ and f(x) is increasing ($f'(x)>0$) for all x in some interval $[x_2,b)$, then f(x) has a local minimum at x_2. (Fig.177).

833. Second Derivative Test for Local Extrema.
If $f'(x_1)=0$ and $f''(x_1)<0$, then f(x) has a local maximum at x_1.
If $f'(x_2)=0$ and $f''(x_2)>0$, then f(x) has a local minimum at x_2. (Fig.177)

834. Concavity.
f(x) is concave upward at x_0 if and only if $f'(x)$ is increasing at x_0 (Fig.177, $x_3<x$).
f(x) is concave downward at x_0 if and only if $f'(x)$ is decreasing at x_0. (Fig.177, $x<x_3$).

835. Second Derivative Test for Concavity.
If $f''(x_0)>0$, then f(x) is concave upward at x_0.
If $f''(x_0)<0$, then f(x) is concave downward at x_0.
If $f''(x)$ does not exist or is zero, then the test fails.

836. Inflection Points
If $f'(x_3)$ exists and $f''(x)$ changes sign at $x=x_3$, then the point $(x_3,f(x_3))$ is an inflection point of the graph of $f(x)$. If $f''(x_3)$ exists at the inflection point, then $f''(x_3)=0$ (Fig.177).

837. L'Hopital's Rule

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to c}f(x)=\lim_{x\to c}g(x)=\begin{cases}0\\ \infty\end{cases}.$$

8.7 Differential

Functions: f, u, v
Independent variable: x
Derivative of a function: $y'(x)$, $f'(x)$
Real constant: C
Differential of function $y = f(x)$: dy
Differential of x: dx
Small change in x: Δx
Small change in y: Δy

838. $dy = y'dx$

839. $f(x + \Delta x) = f(x) + f'(x)\Delta x$

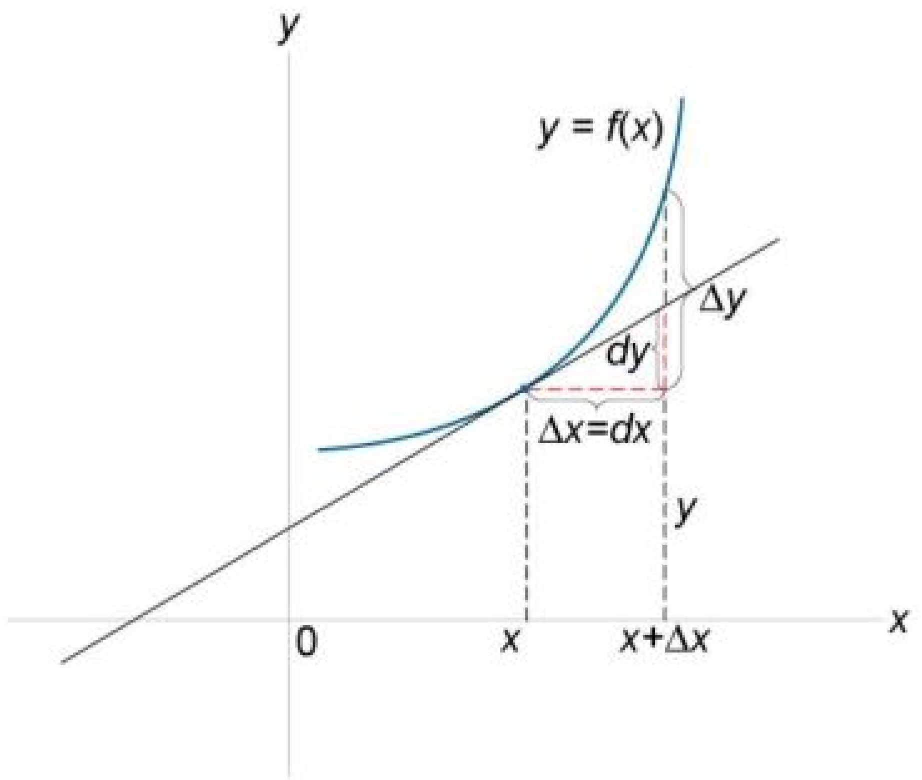

Figure 178.

840. Small Change in y
$\Delta y = f(x + \Delta x) - f(x)$

841. $d(u + v) = du + dv$

842. $d(u - v) = du - dv$

843. $d(Cu) = Cdu$

844. $d(uv) = vdu + udv$

845. $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$

8.8 Multivariable Functions

Functions of two variables: $z(x,y)$, $f(x,y)$, $g(x,y)$, $h(x,y)$
Arguments: x, y, t
Small changes in x, y, z, respectively: Δx, Δy, Δz.

846. First Order Partial Derivatives
The partial derivative with respect to x
$\frac{\partial f}{\partial x} = f_x$ (also $\frac{\partial z}{\partial x} = z_x$),
The partial derivative with respect to y
$\frac{\partial f}{\partial y} = f_y$ (also $\frac{\partial z}{\partial y} = z_y$).

847. Second Order Partial Derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y \partial x}=f_{xy},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x \partial y}=f_{yx}.$$

If the derivatives are continuous, then

$$\frac{\partial^2 f}{\partial y \partial x}=\frac{\partial^2 f}{\partial x \partial y}.$$

848. Chain Rules

If $f(x,y)=g(h(x,y))$ (g is a function of one variable h), then

$$\frac{\partial f}{\partial x}=g'(h(x,y))\frac{\partial h}{\partial x},\ \frac{\partial f}{\partial y}=g'(h(x,y))\frac{\partial h}{\partial y}.$$

If $h(t)=f(x(t),y(t))$, then $h'(t)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$.

If $z=f(x(u,v),y(u,v))$, then

$$\frac{\partial z}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u},\ \frac{\partial z}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

849. Small Changes

$$\Delta z \approx \frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y$$

850. Local Maxima and Minima

$f(x,y)$ has a local maximum at (x_0, y_0) if $f(x,y) \le f(x_0, y_0)$ for all (x,y) sufficiently close to (x_0, y_0).

$f(x,y)$ has a local minimum at (x_0, y_0) if $f(x,y) \ge f(x_0, y_0)$ for all (x,y) sufficiently close to (x_0, y_0).

851. Stationary Points

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 .$$

Local maxima and local minima occur at stationary points.

852. Saddle Point

A stationary point which is neither a local maximum nor a local minimum

853. Second Derivative Test for Stationary Points

Let (x_0, y_0) be a stationary point ($\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$).

$$D = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}.$$

If $D > 0$, $f_{xx}(x_0, y_0) > 0$, (x_0, y_0) is a point of local minima.

If $D > 0$, $f_{xx}(x_0, y_0) < 0$, (x_0, y_0) is a point of local maxima.

If $D < 0$, (x_0, y_0) is a saddle point.

If $D = 0$, the test fails.

854. Tangent Plane

The equation of the tangent plane to the surface $z = f(x,y)$ at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

855. Normal to Surface
The equation of the normal to the surface $z = f(x, y)$ at (x_0, y_0, z_0) is

$$\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1}.$$

8.9 Differential Operators

Unit vectors along the coordinate axes: $\vec{i}$, $\vec{j}$, $\vec{k}$
Scalar functions (scalar fields): $f(x, y, z)$, $u(x_1, x_2, \ldots, x_n)$
Gradient of a scalar field: $\text{grad}\, u$, ∇u
Directional derivative: $\frac{\partial f}{\partial l}$
Vector function (vector field): $\vec{F}(P, Q, R)$
Divergence of a vector field: $\text{div}\, \vec{F}$, $\nabla \cdot \vec{F}$
Curl of a vector field: $\text{curl}\, \vec{F}$, $\nabla \times \vec{F}$
Laplacian operator: ∇^2

856. Gradient of a Scalar Function

$$\text{grad}\, f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right),$$

$$\text{grad}\, u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right).$$

857. Directional Derivative

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos\alpha + \frac{\partial f}{\partial y} \cos\beta + \frac{\partial f}{\partial z} \cos\gamma,$$

where the direction is defined by the vector
$\vec{l}(\cos\alpha, \cos\beta, \cos\gamma)$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

858. Divergence of a Vector Field

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

859. Curl of a Vector Field

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

860. Laplacian Operator

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

861. $\operatorname{div}(\operatorname{curl}\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) \equiv 0$

862. $\operatorname{curl}(\operatorname{grad} f) = \nabla \times (\nabla f) \equiv 0$

863. $\operatorname{div}(\operatorname{grad} f) = \nabla \cdot (\nabla f) = \nabla^2 f$

864. $\operatorname{curl}(\operatorname{curl}\vec{F}) = \operatorname{grad}(\operatorname{div}\vec{F}) - \nabla^2\vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

808. $\frac{d}{dx}(\operatorname{arctanh} x) = \frac{1}{1 - x^2}$, $|x| < 1$.

809. $\frac{d}{dx}(\operatorname{arccoth} x) = -\frac{1}{x^2 - 1}$, $|x| > 1$.

810. $\frac{d}{dx}(u^v) = vu^{v-1} \cdot \frac{du}{dx} + u^v \ln u \cdot \frac{dv}{dx}$

8.5 Higher Order Derivatives

Functions: f, y, u, v
Independent variable: x
Natural number: n

811. Second derivative

$$f'' = (f')' = \left(\frac{dy}{dx}\right)' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

812. Higher-Order derivative

$$f^{(n)} = \frac{d^n y}{dx^n} = y^{(n)} = \left(f^{(n-1)}\right)'$$

813. $(u + v)^{(n)} = u^{(n)} + v^{(n)}$

814. $(u - v)^{(n)} = u^{(n)} - v^{(n)}$

815. Leibnitz's Formulas

$$(uv)'' = u''v + 2u'v' + uv''$$