ECON675 – Assignment 3

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October 24, 2018

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1 Non-linear least squares

1.1 Identifiability

This is a standard M-estimation problem. The parameter vector $\boldsymbol{\beta}_0$ is assumed to solve the population problem

$$\boldsymbol{\beta}_0 = \arg\min_{\beta \in \mathbb{R}^d} \mathbb{E}[(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))^2].$$

For $\boldsymbol{\beta}_0$ to be identified, it must be the *unique* solution to the above population problem (i.e. the unique minimizer). In math, this means for all $\epsilon > 0$ and for some $\delta > 0$:

$$\sup_{||\beta - \beta_0|| > \epsilon} M(\boldsymbol{\beta}) \geq M(\boldsymbol{\beta}_0) + \delta$$

where $M(\boldsymbol{\beta}) = \mathbb{E}[(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))^2]$. Of course $\boldsymbol{\beta}_0$ can be written in closed form if $\mu(\cdot)$ is linear. In this case, we know that

$$\boldsymbol{\beta}_0 = \mathbb{E}[\boldsymbol{x}_i\boldsymbol{x}_i']^{-1}\mathbb{E}[\boldsymbol{x}_iy_i].$$

1.2 Asymptotic normality

The M-estimator is asymptotically normal if:

1. $\hat{\boldsymbol{\beta}} \to_p \boldsymbol{\beta}_0$

2. $\boldsymbol{\beta}_0 \in int(B)$ and $m(\boldsymbol{x}_i, \boldsymbol{\beta}) \equiv (y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))^2$ is 3 times continuously differentiable.

3. $\Sigma_0 = \mathbb{V}[\frac{\partial}{\partial\beta} m(\boldsymbol{x}_i; \beta_0)] < \infty$ and $H_0 = \mathbb{E}[\frac{\partial^2}{\partial\beta\partial\beta'} m(\boldsymbol{x}_i; \beta_0)]$ is full rank (and therefore invertible).

Now, the FOC for the M-estimation problem is

$$0 = \frac{1}{n} \sum_{i=1}^n (y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta})) \dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta})) \boldsymbol{x}_i \tag{1}$$

where $\dot{\mu} = \frac{\partial}{\partial\beta}\mu(\boldsymbol{x}_i'\boldsymbol{\beta})$. So, we've converted the M-estimation problem into a Z-estimation problem. Then we can use the standard asymptotic normality result to arrive at a precise form of the asymptotic variance:

$$\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \to_d \mathcal{N}(0, H_0^{-1}\Sigma_0 H_0^{-1}).$$

Now, taking the second derivative gives the Hessian

$$\begin{aligned} H_0 &= \mathbb{E}[\frac{\partial^2}{\partial\beta\partial\beta'} m(\boldsymbol{x}_i; \beta_0)] \\ &= \mathbb{E}\left[-\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))\boldsymbol{x}_i\boldsymbol{x}_i' + (y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))\ddot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))\boldsymbol{x}_i\boldsymbol{x}_i'\right] \\ &= -\mathbb{E}[\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i'] \end{aligned}$$

by LIE. And, the variance of the score is

$$\begin{aligned} \Sigma_0 &= \mathbb{V}[\frac{\partial}{\partial\beta} m(\boldsymbol{x}_i; \beta_0)] \\ &= \mathbb{E}\left[(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i'\right] \\ &= \mathbb{E}[\sigma^2(\boldsymbol{x}_i)\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i'] \end{aligned}$$

again by LIE. Then we have the asymptotic variance

$$\boldsymbol{V}_0 = H_0^{-1}\Sigma_0 H_0^{-1}.$$

1.3 Variance estimator under heteroskedasticity

Under heteroskedasticity we can use the sandwich variance estimator

$$\widehat{\boldsymbol{V}}_{HC} = \hat{H}^{-1}\hat{\Sigma}\hat{H}^{-1},$$

where

$$\hat{H} = \frac{1}{n} \sum_{i=1}^n \dot{\mu}(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}))^2\boldsymbol{x}_i\boldsymbol{x}_i'$$

$$\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n \hat{e}_i^2\dot{\mu}(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}))^2\boldsymbol{x}_i\boldsymbol{x}_i'$$

Now, to get an asymptotically valid CI for $||\boldsymbol{\beta}_0||^2$ we need to use the Delta Method. First, note that:

$$||\boldsymbol{\beta}_0||^2 = \boldsymbol{\beta}_0'\boldsymbol{\beta}_0$$

$$\implies \frac{\partial}{\partial\beta}||\boldsymbol{\beta}_0||^2 = 2\boldsymbol{\beta}_0$$

Then, using the Delta Method

$$\sqrt{n}(||\hat{\boldsymbol{\beta}}||^2 - ||\boldsymbol{\beta}_0||^2) \to_d 2\boldsymbol{\beta}_0\mathcal{N}(0, \boldsymbol{V}_0)$$
$$= \mathcal{N}(0, 4\boldsymbol{\beta}_0'\boldsymbol{V}_0\boldsymbol{\beta}_0)$$

Thus, an asymptotically valid 95% CI for $||\boldsymbol{\beta}_0||^2$ is

$$CI_{95} = \left[\hat{\boldsymbol{\beta}} - 1.96\sqrt{\frac{4\hat{\boldsymbol{\beta}}'\widehat{\boldsymbol{V}}_{HC}\hat{\boldsymbol{\beta}}}{n}}, \hat{\boldsymbol{\beta}} + 1.96\sqrt{\frac{4\hat{\boldsymbol{\beta}}'\widehat{\boldsymbol{V}}_{HC}\hat{\boldsymbol{\beta}}}{n}}\right]$$

1.4   Variance estimator under homoskedasticity

Using the above results, under homoskedasticity, the asymptotic variance collapses to

$$\boldsymbol{V}_0 = \mathbb{E}[\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i']^{-1}\sigma^2\mathbb{E}[\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}))^2\boldsymbol{x}_i\boldsymbol{x}_i']\mathbb{E}[\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i']^{-1}$$
$$= \sigma^2\mathbb{E}[\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i']^{-1}$$

The variance estimator is now takes a simpler form

$$\widehat{\boldsymbol{V}}_{HO} = \hat{\sigma}^2\hat{H}^{-1}$$

where $\hat{H}$ is the same as above and

$$\hat{\sigma}^2 = \frac{1}{n-d}\sum_{i=1}^{n}(y_i - \mu(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}))^2$$

Then, as above, the asymptotically valid 95% CI for $||\boldsymbol{\beta}_0||^2$ is

$$CI_{95} = \left[\hat{\boldsymbol{\beta}} - 1.96\sqrt{\frac{4\hat{\boldsymbol{\beta}}'\widehat{\boldsymbol{V}}_{HO}\hat{\boldsymbol{\beta}}}{n}}, \hat{\boldsymbol{\beta}} + 1.96\sqrt{\frac{4\hat{\boldsymbol{\beta}}'\widehat{\boldsymbol{V}}_{HO}\hat{\boldsymbol{\beta}}}{n}}\right].$$

1.5   MLE

Given the assumption of a normal DGP we have the conditional density

$$f(y_i|\boldsymbol{x}_i) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\right).$$

Then, the sample log-likelihood function is

$$\ell(\boldsymbol{\beta}, \sigma^2; \boldsymbol{X}) = n \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))^2$$

Dividing by n gives

$$\ell(\boldsymbol{\beta}, \sigma^2; \boldsymbol{X}) = \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{n2\sigma^2} \sum_{i=1}^{n} (y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))^2$$

The FOC wrt $\boldsymbol{\beta}$ is

$$0 = \frac{1}{n\sigma^2} \sum_{i=1}^{n} (y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta})\boldsymbol{x}_i,$$

which is equivalent to the FOC for the M-estimation problem (1) (since σ^2 just scales the FOC, it does not affect the solution). Thus,

$$\hat{\boldsymbol{\beta}}_{MLE} = \hat{\boldsymbol{\beta}}_{M.est}.$$

Now, the FOC of the log-likelihood wrt σ^2 is

$$0 = -\frac{1}{2}(2\pi\sigma^2)^{-1}2\pi + \frac{1}{2n}(\sigma^2)^{-2} \sum_{i=1}^{n} (y_i - \mu(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}))^2$$

Solving for σ^2 gives the MLE:

$$\hat{\sigma}^2_{ML} = \frac{1}{n} \sum_{i=1}^{n} (y_i - \mu(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}))^2,$$

which is not the same as the estimator proposed in [4], since it does not adjust for the number of regressors.

1.6 When the link function is unknown

Suppose the link function is unknown, and consider two pairs of true parameters, $(\mu_1, \boldsymbol{\beta}_1)$ and $(\mu_2, \boldsymbol{\beta}_2)$ where $\mu_2(u) = \mu_1(u/c)$ and $\boldsymbol{\beta}_2 = c\boldsymbol{\beta}_1$ for some $c \neq 0$. Then the parameters are clearly different, but $\mu_1(\boldsymbol{x}_i'\boldsymbol{\beta}_1) = \mu_2(\boldsymbol{x}_i'\boldsymbol{\beta}_2)$.

1.7 Logistic link function

The link function is

$$\begin{aligned}
\mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0) &= \mathbb{E}[y_i | \boldsymbol{x}_i] \\
&= \mathbb{E}[\mathbf{1}(\boldsymbol{x}_i'\boldsymbol{\beta}_0 \geq \epsilon_i) | \boldsymbol{x}_i] \\
&= \Pr[\boldsymbol{x}_i'\boldsymbol{\beta}_0 \geq \epsilon_i | \boldsymbol{x}_i] \\
&= F(\boldsymbol{x}_i'\boldsymbol{\beta}_0) \\
&= \frac{1}{1 + \exp(-\boldsymbol{x}_i'\boldsymbol{\beta}_0)}, \quad \text{if } s_0 = 1.
\end{aligned}$$

The conditional variance of y_i is

$$\sigma^2(\boldsymbol{x}_i)\mathbb{V}[y_i|\boldsymbol{x}_i]$$

Now, note that $y_i|\boldsymbol{x}_i$ is a Bernoulli random variable, with $\Pr[y_i = 1|\boldsymbol{x}_i] = F(\boldsymbol{x}_i'\boldsymbol{\beta}_0)$. Then

$$\begin{aligned}\sigma^2(\boldsymbol{x}_i) &= F(\boldsymbol{x}_i'\boldsymbol{\beta}_0)(1 - F(\boldsymbol{x}_i'\boldsymbol{\beta}_0))\\ &= \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0)(1 - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))\end{aligned}$$

To derive an expression for the asymptotic variance, first note that for the logistic cdf: $\dot{\mu}(u) = (1 - \mu(u))\mu(u)$. Then, the asymptotic variance is

$$\boldsymbol{V}_0 = H_0^{-1}\Sigma_0 H_0^{-1}.$$

where

$$H_0 = \mathbb{E}[(1 - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0)^2\boldsymbol{x}_i\boldsymbol{x}_i']$$

and

$$\Sigma_0 = \mathbb{E}[(1 - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^3\mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0)^3\boldsymbol{x}_i\boldsymbol{x}_i']$$

1.8 Logistic link function, MLE

MLE gives the same point estimator as NLS, but is asymptotically efficient, so $\boldsymbol{V}_0^{ML} \leq \boldsymbol{V}_0^{NLS}$.