ECON675 – Assignment 5

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Contents

1 Many instruments asymptotics

1.1 Some moments

First,

$$\mathbb{E}[\boldsymbol{u}'\boldsymbol{u}/n] = \frac{1}{n}\mathbb{E}[\boldsymbol{u}'\boldsymbol{u}] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[u_i^2] = \sigma_u^2.$$

An analogous derivation shows that $\mathbb{E}[\boldsymbol{v}'\boldsymbol{v}/n] = \sigma_v^2$.

Next,

$$\begin{aligned}
\mathbb{E}[\boldsymbol{x}'\boldsymbol{u}/n] = \frac{1}{n}\mathbb{E}[\boldsymbol{x}'\boldsymbol{u}] &= \frac{1}{n}\mathbb{E}[(\boldsymbol{\pi}'\boldsymbol{Z}' + \boldsymbol{v}')\boldsymbol{u}] \\
&= \frac{1}{n}\boldsymbol{\pi}'\boldsymbol{Z}'\mathbb{E}[\boldsymbol{u}] + \frac{1}{n}\mathbb{E}[\boldsymbol{v}'\boldsymbol{u}] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[v_i u_i] \\
&= \sigma_{uv}^2,
\end{aligned}$$

where I used the assumptions that $\boldsymbol{Z}$ and $\boldsymbol{\pi}$ are nonrandom and $\mathbb{E}[\boldsymbol{u}] = \boldsymbol{0}$.

Now,

$$\begin{aligned}
\mathbb{E}[\boldsymbol{x}'\boldsymbol{P}\boldsymbol{u}/n] &= \frac{1}{n}\mathbb{E}[(\boldsymbol{\pi}'\boldsymbol{Z}' + \boldsymbol{v}')\boldsymbol{P}\boldsymbol{u}] \\
&= \frac{1}{n}\mathbb{E}[\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{P}\boldsymbol{u}] + \frac{1}{n}\mathbb{E}[\boldsymbol{v}'\boldsymbol{P}\boldsymbol{u}] \\
&= \frac{1}{n}\mathbb{E}[\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{u}] + \frac{1}{n}\mathbb{E}[\boldsymbol{v}'\boldsymbol{P}\boldsymbol{u}] \\
&= \frac{1}{n}\mathbb{E}[\boldsymbol{v}'\boldsymbol{P}\boldsymbol{u}] \\
&= \frac{K}{n}\sigma_{uv}^2
\end{aligned}$$

since $\mathbb{E}[v_i u_j] = 0$ for all $i \neq j$ and $\sum_{i=1}^{n} P_{ii} = K$. An analogous derivation proves the last result $\mathbb{E}[\boldsymbol{u}'\boldsymbol{P}\boldsymbol{u}/n] = K/n\sigma_u^2$.

1.2 Some probability limits

First,

$$\begin{aligned}
\boldsymbol{x}'\boldsymbol{x}/n &= (\boldsymbol{\pi}'\boldsymbol{Z}' + \boldsymbol{v}')(\boldsymbol{Z}\boldsymbol{\pi} + \boldsymbol{v})/n \\
&= \frac{\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\pi}}{n} + \frac{\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{v}}{n} + \frac{\boldsymbol{v}'\boldsymbol{Z}\boldsymbol{\pi}}{n} + \frac{\boldsymbol{v}'\boldsymbol{v}}{n} \\
&\to_p \mu + \mathbb{E}[\boldsymbol{\pi}'\boldsymbol{z}_i v_i] + \mathbb{E}[\boldsymbol{z}_i'\boldsymbol{\pi} v_i] + \mathbb{E}[v_i^2] \\
&= \mu + \sigma_v^2
\end{aligned}$$

Next,

$$
\begin{aligned}
\boldsymbol{x}'\boldsymbol{P}\boldsymbol{x}/n &= (\boldsymbol{\pi}'\boldsymbol{Z}' + \boldsymbol{v}')\boldsymbol{P}(\boldsymbol{Z}\boldsymbol{\pi} + \boldsymbol{v})/n \\
&= \frac{\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\pi}}{n} + \frac{\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{v}}{n} + \frac{\boldsymbol{v}'\boldsymbol{Z}\boldsymbol{\pi}}{n} + \frac{\boldsymbol{v}'\boldsymbol{P}\boldsymbol{v}}{n} \\
&\to_p \mu + \rho\sigma_v^2
\end{aligned}
$$

since $K/n \to \rho$. An analogous derivation proves the last result, $\boldsymbol{x}'\boldsymbol{P}\boldsymbol{u}/n \to_p \rho\sigma_u^2$.

1.3 plim of the classical 2SLS estimator

The classical 2SLS estimator is

$$
\begin{aligned}
\hat{\beta}_{\text{2SLS}} &= (\boldsymbol{x}'\boldsymbol{P}\boldsymbol{x})^{-1}(\boldsymbol{x}'\boldsymbol{P}\boldsymbol{y}) \\
&= (\boldsymbol{x}'\boldsymbol{P}\boldsymbol{x})^{-1}\boldsymbol{x}'\boldsymbol{P}(\boldsymbol{x}\boldsymbol{\beta} + \boldsymbol{u}) \\
&= \beta + (\boldsymbol{x}'\boldsymbol{P}\boldsymbol{x})^{-1}(\boldsymbol{x}'\boldsymbol{P}\boldsymbol{u}) \\
&= \beta + (\boldsymbol{x}'\boldsymbol{P}\boldsymbol{x}/n)^{-1}(\boldsymbol{x}'\boldsymbol{P}\boldsymbol{u}/n) \\
&\to_p \beta + \frac{\rho\sigma_u^2}{\mu + \rho\sigma_v^2},
\end{aligned}
$$

using the CMT and the above results. Thus, $\hat{\beta}_{\text{2SLS}} = \beta + \frac{\rho\sigma_u^2}{\mu+\rho\sigma_v^2} + o_p(1)$.

1.4 plim of the bias-corrected 2SLS estimator