ECON611 – Homework # 2

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October 17, 2018

1 Maximum likelihood estimation of a simple DSGE model

1.1

See attached code.

1.2

(a) I plot the recovered structural shocks below.

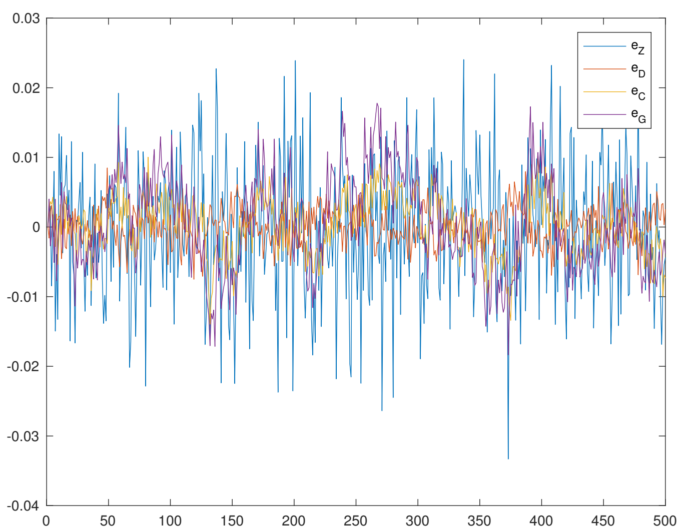

Figure 1: Recovered Structural Shocks from the Brock-Mirman Model

To check the procedure I simulate shock processes for $\epsilon_t = [\epsilon_t^\delta, \epsilon_t^c, \epsilon_t^z, \epsilon_t^g]$ and recover them. The recovered shocks are the same as the simulated shocks.

(b) I get $L = -6270.22$ (see attached code for details).

(c) extra question. I plot the surface plot below.

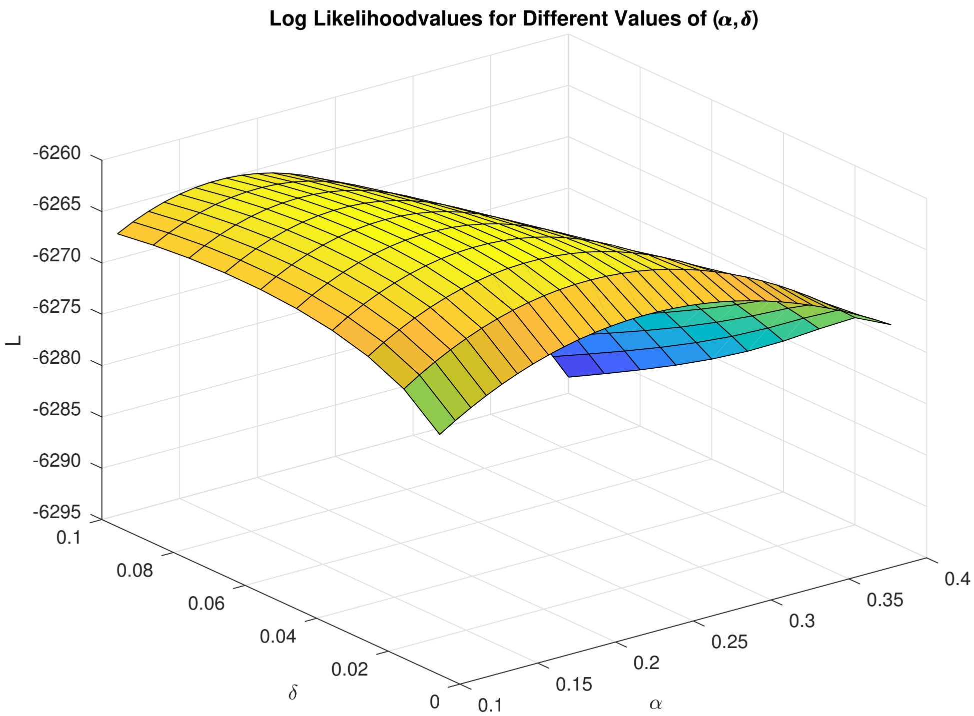

Figure 2: Log-Likelihood Values for Different (α, δ) Parameters

The likelihood is maximized at $(\alpha_{MLE}, \delta_{MLE}) = (0.2, 0.05)$, which doesn't correspond to our usual calibration. Interestingly, the likelihood seems quite flat, which we talked about in class.

2 An endowment economy

2.1

Both types of agents face identical problems. They solve

$$\max_{c,s} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$
$$\text{s.t } c_t + s_t = y_t + (1+r_{t-1})s_{t-1}$$
$$\& \ s_t^A + s_t^B = 0.$$

Since there is no uncertainty, the Bellman is

$$V(s_{-1}, y) = \max_s \{\ln(y + (1+r_{-1})s_{-1} - s) + \beta V(s, y')\}.$$

Combining the FOC wrt s, with the B-S condition gives the following optimality condition

$$\frac{1}{c_t} = (1+r_t)\beta \frac{1}{c_{t+1}} \tag{1}$$

Now, a competitive equilibrium is a sequence of allocations $\{c_t^i, s_t^i\}_{t,i}$ such that each agent's optimality condition (1) and budget constraint hold each period, given a sequence of interest rates $\{1+r_t\}_t$, and the bond market clears each period. It is easy to guess and verify that an equilibrium exists where for each agent, $c_t = 1$, $s_t = 0$ and $(1+r_t) = 1/\beta$ for all t. Clearly, this allocation satisfies the optimality condition and the budget constraint (since $c_t = y_t$ each period) and bond market clearing (since neither agent saves/borrows). Thus, in the competitive equilibrium, both agents consume their endowments each period, and there are no gains from trade.

2.2

Neither agent will react to this news in period 0. The only plausible reaction to the news would be for agents to borrow against potentially higher income next period (there is no way that this news could induce a savings response). But both agents want to make the same trade, since they are identical before the shock. Obviously, both agents cannot increase their borrowing simultaneously in period zero because this would contradict bond market clearing. Thus, in equilibrium, neither agent reacts to the news in period zero.

2.3

I didn't submit this, but having seen other people's solutions here we go.

Solve backwards. Start from period 2. We must have:

$$\frac{1}{c_2^i} = (1+r_2)\beta \frac{1}{c_3^i}$$
$$c_3^i = (1+r_2)\beta c_2^i$$

Summing over i:

$$\begin{aligned} c_3^A + c_3^B &= (1+r_2)\beta(c_2^A + c_2^B) \\ 2 &= (1+r_2)\beta 2 \\ \therefore (1+r_2)\beta &= 1. \end{aligned}$$

Thus, for $t \geq 2$, $(1+r_t)$ returns to its steady state value. And then the Euler equations imply that c_t^A and c_t^B are constant for $t \geq 2$. Following this logic:

$$\begin{aligned} c_2^A + c_2^B &= (1+r_1)\beta(c_1^A + c_1^B) \\ 2 &= 3(1+r_1)\beta \\ \therefore 1 + r_1 &= \frac{2}{3}\frac{1}{\beta} < 1 + \bar{r} = 1/\beta \end{aligned}$$

Thus, the interest rate falls in period 1. Now, we need the time paths for consumption...

2.4

The equilibrium conditions of the model are

$$\frac{1}{c_t^i} = (1+r_t)\beta\frac{1}{c_{t+1}^i} \tag{2}$$

$$c_t^i + s_t^i = y_t^i + (1+r_{t-1})s_{t-1}^i \tag{3}$$

$$y_t^i = 1 + e_t^i \tag{4}$$

$$s_t^A + s_t^B = 0 \tag{5}$$

for $i \in \{A, B\}$.

The non-stochastic steady state is given by

$$\begin{aligned} \bar{c}_A = \bar{c}_B = \bar{y}_A &= \bar{y}_B = 1 \\ \bar{s}_A = \bar{s}_B &= 0 \\ 1 + \bar{r} &= 1/\beta \end{aligned}$$

Log-linearizing around the steady state gives the following system of 7 equations with 7 variables:

$$-\tilde{c}_t^i = \beta\tilde{r}_t - \tilde{c}_{t+1}^i (\text{where } \tilde{r}_t = r_t - \bar{r}) \qquad \text{(‘euler’)} \tag{6}$$

$$\tilde{c}_t^i + s_t = \tilde{y}_t^i + (1+\bar{r})s_{t-1}^i \qquad \text{(‘budget constraint’)} \tag{7}$$

$$\tilde{y}_t^i = e_t^i \qquad \text{(‘income process’)} \tag{8}$$

$$s_t^A + s_t^B = 0 \qquad \text{(‘bond mkt clearing’)} \tag{9}$$

We consider a one-time shock to A's income in period 1. I plot the time paths for $\tilde{c}_t^A, \tilde{c}_t^B, s_t^A, s_t^B$ and r_t overleaf.

2.5

I'm guessing that $(1 + r_0)$ increases at the time of the announcement in period 0 because both agents want to borrow.

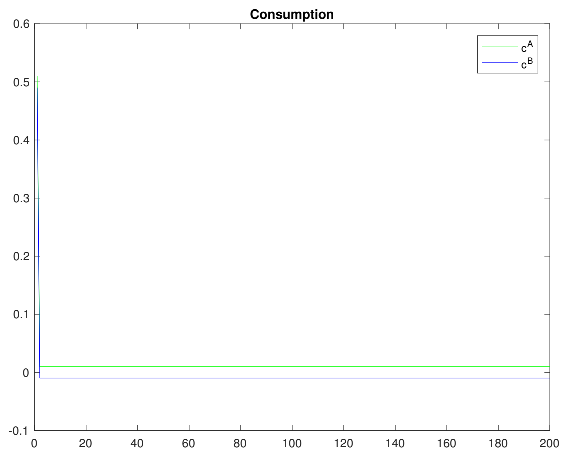

Figure 3: Consumption

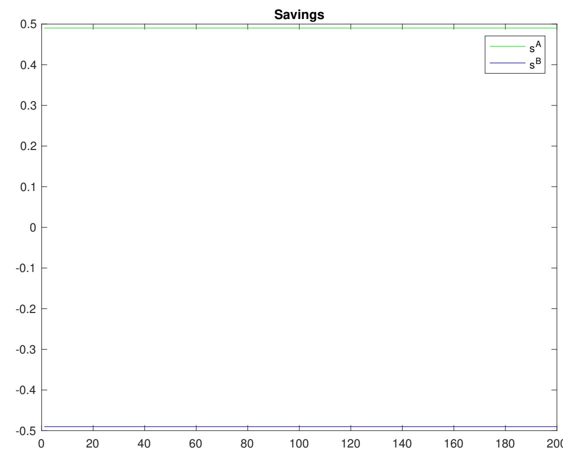

Figure 4: Savings

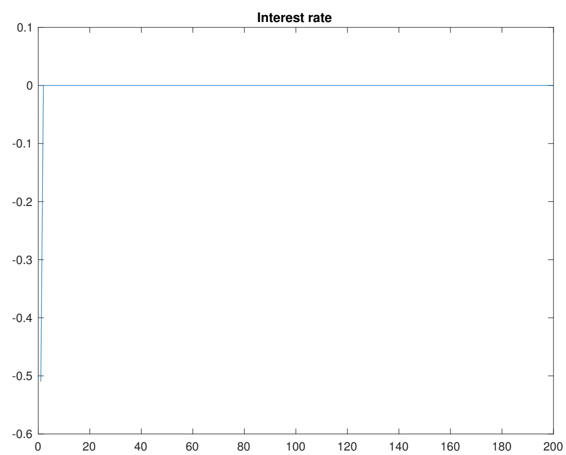

Figure 5: r_t

3 The Kalman Filter

My implementation of the Kalman Filter/Smoother is definitely wrong (given my weird results), and I'm not sure why. Nonetheless, I present my results below.

3.1

The Kalman Filter works as follows

1. *Initialization:* with $\rho = 0.8$ our initial assesment is $\hat{y}_{1|0} = 0$ and $V_{1|0} = \frac{\sigma_e^2}{1-\rho^2}$.

2. *Forecast of x_t:* the optimal date t forecast of the observable is $\hat{x}_{t|t-1} = \hat{y}_{t|t-1}$. Pick up the

associated MSE

$$\mathbb{E}[(x_t - \hat{x}_{t|t-1})^2] = V_{t|t-1} + \sigma_m^2 \equiv V_{t|t-1}^x$$

3. *Update forecast of y_t:* the updated forecast of y_t given observed data x_t is

$$\hat{y}_{t|t} = \hat{y}_{t|t-1} + V_{t|t-1}[V_{t|t-1} + \sigma_m^2]^{-1}(x_t - \hat{y}_{t|t-1})$$

   with updated MSE

$$V_{t|t} = V_{t|t-1} - \frac{V_{t|t-1}^2}{V_{t|t-1}^x}$$

4. *Compute forecast of y_{t+1} given x_t*:

$$\hat{y}_{t+1|t} = \rho\hat{y}_{t|t} = \rho\hat{y}_{t|t-1} + \rho V_{t|t-1}[V_{t|t-1} + \sigma_m^2]^{-1}(x_t - \hat{y}_{t|t-1})$$

   and pick up the associated MSE

$$V_{t+1|t} = \rho^2 V_{t|t} + \sigma_e^2$$

5. *Iterate.*

To get the log-likelihood of the observed data we exploit the Gaussian nature of the system. In particular, set $\hat{x}_{t|t-1} = \hat{y}_{t|t-1}$, then

$$f(x_t|X_{t-1}) = \phi(\hat{x}_{t|t-1}, V_{t|t-1}^x)$$

The log likelihood is simply

$$\ell(\rho) = \sum_{t=1}^{T} \ln \phi(\hat{x}_{t|t-1}, V_{t|t-1}^x)$$

With $\rho = 0.8$ I get

$$\ell(0.8) = -872.2657.$$

3.2

I present the log-likelihood values for different values of ρ below. This is how I know my recursion must be wrong: $\ell(p)$ is monotonically decreasing in ρ, which doesn't make sense.

3.3

I plot the smoothed series $\{\hat{y}_t\}$ on the next page. Again, the results don't make sense: $\hat{y}_t$ is more volatile than the observed data!

ρ	$\ell(\rho)$
0.75	-810.38
0.76	-822.3
0.77	-834.45
0.78	-846.83
0.79	-859.43
0.8	-872.27
0.81	-885.33
0.82	-898.64
0.83	-912.18
0.84	-925.96
0.85	-939.99
0.86	-954.26
0.87	-968.78
0.88	-983.54
0.89	-998.56
0.9	-1013.8
0.91	-1029.4
0.92	-1045.2
0.93	-1061.2
0.94	-1077.5
0.95	-1094.1

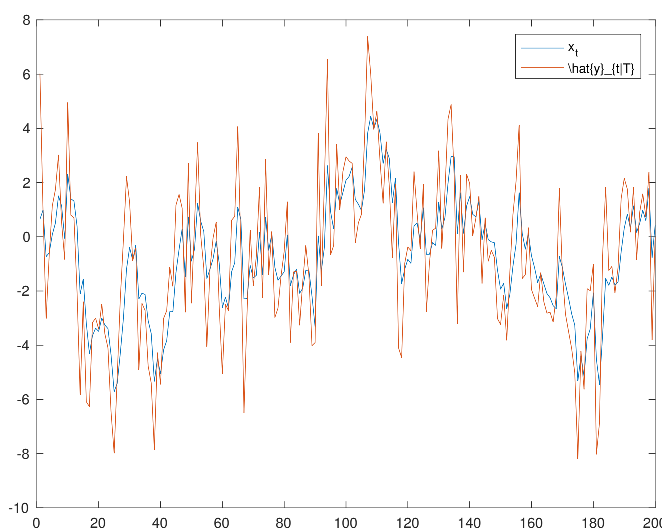

Figure 6: Observed Noisy Data and Estimate of the State