ECON675 – Assignment 3

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Contents

1 Non-linear least squares

1.1 Identifiability

This is a standard M-estimation problem. The parameter vector $\boldsymbol{\beta}_0$ is assumed to solve the population problem

$$\boldsymbol{\beta}_0 = \arg\min_{\beta \in \mathbb{R}^d} \mathbb{E}[(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))^2].$$

For $\boldsymbol{\beta}_0$ to be identified, it must be the *unique* solution to the above population problem (i.e. the unique minimizer). In math, this means for all $\epsilon > 0$ and for some $\delta > 0$:

$$\sup_{||\beta - \beta_0|| > \epsilon} M(\boldsymbol{\beta}) \geq M(\boldsymbol{\beta}_0) + \delta$$

where $M(\boldsymbol{\beta}) = \mathbb{E}[(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))^2]$. Of course $\boldsymbol{\beta}_0$ can be written in closed form if $\mu(\cdot)$ is linear. In this case, we know that

$$\boldsymbol{\beta}_0 = \mathbb{E}[\boldsymbol{x}_i\boldsymbol{x}_i']^{-1}\mathbb{E}[\boldsymbol{x}_i y_i].$$

1.2 Asymptotic normality

The M-estimator is asymptotically normal if:

1. $\hat{\boldsymbol{\beta}} \to_p \boldsymbol{\beta}_0$
2. $\boldsymbol{\beta}_0 \in int(B)$ and $m(\boldsymbol{x}_i, \boldsymbol{\beta}) \equiv (y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))^2$ is 3 times continuously differentiable.
3. $\Sigma_0 = \mathbb{V}[\frac{\partial}{\partial \beta} m(\boldsymbol{x}_i; \beta_0)] < \infty$ and $H_0 = \mathbb{E}[\frac{\partial^2}{\partial \beta \partial \beta'} m(\boldsymbol{x}_i; \beta_0)]$ is full rank (and therefore invertible).

Now, the FOC for the M-estimation problem is

$$0 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}))\boldsymbol{x}_i \tag{1}$$

where $\dot{\mu} = \frac{\partial}{\partial \beta}\mu(\boldsymbol{x}_i'\boldsymbol{\beta})$. So, we've converted the M-estimation problem into a Z-estimation problem. Then we can use the standard asymptotic normality result to arrive at a precise form of the asymptotic variance:

$$\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \to_d \mathcal{N}(0, H_0^{-1}\Sigma_0 H_0^{-1}).$$

Now, taking the second derivative gives the Hessian

$$\begin{aligned}
H_0 &= \mathbb{E}[\frac{\partial^2}{\partial \beta \partial \beta'} m(\boldsymbol{x}_i; \beta_0)] \\
&= \mathbb{E}\left[-\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))\boldsymbol{x}_i\boldsymbol{x}_i' + (y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))\ddot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))\boldsymbol{x}_i\boldsymbol{x}_i'\right] \\
&= -\mathbb{E}[\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i']
\end{aligned}$$

by LIE. And, the variance of the score is

$$\begin{aligned}\Sigma_0 &= \mathbb{V}[\frac{\partial}{\partial\beta}m(\boldsymbol{x}_i;\beta_0)] \\ &= \mathbb{E}\left[(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i'\right] \\ &= \mathbb{E}[\sigma^2(\boldsymbol{x}_i)\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i']\end{aligned}$$

again by LIE. Then we have the asymptotic variance

$$\boldsymbol{V}_0 = H_0^{-1}\Sigma_0 H_0^{-1}.$$

1.3 Variance estimator under heteroskedasticity

Under heteroskedasticity we can use the sandwich variance estimator

$$\widehat{\boldsymbol{V}}_{HC} = \hat{H}^{-1}\hat{\Sigma}\hat{H}^{-1},$$

where

$$\begin{aligned}\hat{H} &= \frac{1}{n}\sum_{i=1}^{n}\dot{\mu}(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}))^2\boldsymbol{x}_i\boldsymbol{x}_i' \\ \hat{\Sigma} &= \frac{1}{n}\sum_{i=1}^{n}\hat{e}_i^2\dot{\mu}(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}))^2\boldsymbol{x}_i\boldsymbol{x}_i'\end{aligned}$$

Now, to get an asymptotically valid CI for $||\boldsymbol{\beta}_0||^2$ we need to use the Delta Method. First, note that:

$$\begin{aligned}||\boldsymbol{\beta}_0||^2 &= \boldsymbol{\beta}_0'\boldsymbol{\beta}_0 \\ \implies \frac{\partial}{\partial\beta}||\boldsymbol{\beta}_0||^2 &= 2\boldsymbol{\beta}_0\end{aligned}$$

Then, using the Delta Method

$$\begin{aligned}\sqrt{n}(||\hat{\boldsymbol{\beta}}||^2 - ||\boldsymbol{\beta}_0||^2) &\to_d 2\boldsymbol{\beta}_0\mathcal{N}(0, \boldsymbol{V}_0) \\ &= \mathcal{N}(0, 4\boldsymbol{\beta}_0'\boldsymbol{V}_0\boldsymbol{\beta}_0)\end{aligned}$$

Thus, an asymptotically valid 95% CI for $||\boldsymbol{\beta}_0||^2$ is

$$CI_{95} = \left[\hat{\boldsymbol{\beta}} - 1.96\sqrt{\frac{4\hat{\boldsymbol{\beta}}'\widehat{\boldsymbol{V}}_{HC}\hat{\boldsymbol{\beta}}}{n}}, \hat{\boldsymbol{\beta}} + 1.96\sqrt{\frac{4\hat{\boldsymbol{\beta}}'\widehat{\boldsymbol{V}}_{HC}\hat{\boldsymbol{\beta}}}{n}}\right]$$

1.4 Variance estimator under homoskedasticity

Using the above results, under homoskedasticity, the asymptotic variance collapses to

$$\begin{aligned}\boldsymbol{V}_0 &= \mathbb{E}[\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i']^{-1}\sigma^2\mathbb{E}[\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}))^2\boldsymbol{x}_i\boldsymbol{x}_i']\mathbb{E}[\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i']^{-1}\\ &= \sigma^2\mathbb{E}[\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\boldsymbol{x}_i\boldsymbol{x}_i']^{-1}\end{aligned}$$

The variance estimator is now takes a simpler form

$$\widehat{\boldsymbol{V}}_{HO} = \hat{\sigma}^2\hat{H}^{-1}$$

where $\hat{H}$ is the same as above and

$$\hat{\sigma}^2 = \frac{1}{n-d}\sum_{i=1}^{n}(y_i - \mu(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}))^2$$

Then, as above, the asymptotically valid 95% CI for $||\boldsymbol{\beta}_0||^2$ is

$$CI_{95} = \left[\hat{\boldsymbol{\beta}} - 1.96\sqrt{\frac{4\hat{\boldsymbol{\beta}}'\widehat{\boldsymbol{V}}_{HO}\hat{\boldsymbol{\beta}}}{n}}, \hat{\boldsymbol{\beta}} + 1.96\sqrt{\frac{4\hat{\boldsymbol{\beta}}'\widehat{\boldsymbol{V}}_{HO}\hat{\boldsymbol{\beta}}}{n}}\right].$$

1.5 MLE

Given the assumption of a normal DGP we have the conditional density

$$f(y_i|\boldsymbol{x}_i) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\Big(-\frac{1}{2\sigma^2}(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\Big).$$

Then, the sample log-likelihood function is

$$\ell(\boldsymbol{\beta}, \sigma^2; \boldsymbol{X}) = n\log\Big(\frac{1}{\sqrt{2\pi\sigma^2}}\Big) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))^2$$

Dividing by n gives

$$\ell(\boldsymbol{\beta}, \sigma^2; \boldsymbol{X}) = \log\Big(\frac{1}{\sqrt{2\pi\sigma^2}}\Big) - \frac{1}{n2\sigma^2}\sum_{i=1}^{n}(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))^2$$

The FOC wrt $\boldsymbol{\beta}$ is

$$0 = \frac{1}{n\sigma^2}\sum_{i=1}^{n}(y_i - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}))\dot{\mu}(\boldsymbol{x}_i'\boldsymbol{\beta}))\boldsymbol{x}_i,$$

which is equivalent to the FOC for the M-estimation problem (1) (since σ^2 just scales the FOC, it does not affect the solution). Thus,

$$\hat{\boldsymbol{\beta}}_{MLE} = \hat{\boldsymbol{\beta}}_{M.est}.$$

Now, the FOC of the log-likelihood wrt σ^2 is

$$0 = -\frac{1}{2}(2\pi\sigma^2)^{-1}2\pi + \frac{1}{2n}(\sigma^2)^{-2}\sum_{i=1}^{n}(y_i - \mu(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}))^2$$

Solving for σ^2 gives the MLE:

$$\hat{\sigma}^2_{ML} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}))^2,$$

which is not the same as the estimator proposed in [4], since it does not adjust for the number of regressors.

1.6 When the link function is unknown

Suppose the link function is unknown, and consider two pairs of true parameters, $(\mu_1, \boldsymbol{\beta}_1)$ and $(\mu_2, \boldsymbol{\beta}_2)$ where $\mu_2(u) = \mu_1(u/c)$ and $\boldsymbol{\beta}_2 = c\boldsymbol{\beta}_1$ for some $c \neq 0$. Then the parameters are clearly different, but $\mu_1(\boldsymbol{x}_i'\boldsymbol{\beta}_1) = \mu_2(\boldsymbol{x}_i'\boldsymbol{\beta}_2)$.

1.7 Logistic link function

The link function is

$$\begin{aligned}
\mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0) &= \mathbb{E}[y_i|\boldsymbol{x}_i] \\
&= \mathbb{E}[\mathbf{1}(\boldsymbol{x}_i'\boldsymbol{\beta}_0 \geq \epsilon_i)|\boldsymbol{x}_i] \\
&= \Pr[\boldsymbol{x}_i'\boldsymbol{\beta}_0 \geq \epsilon_i|\boldsymbol{x}_i] \\
&= F(\boldsymbol{x}_i'\boldsymbol{\beta}_0) \\
&= \frac{1}{1+\exp(-\boldsymbol{x}_i'\boldsymbol{\beta}_0)}, \text{ if } s_0 = 1.
\end{aligned}$$

The conditional variance of y_i is

$$\sigma^2(\boldsymbol{x}_i)\mathbb{V}[y_i|\boldsymbol{x}_i]$$

Now, note that $y_i|\boldsymbol{x}_i$ is a Bernoulli random variable, with $\Pr[y_i = 1|\boldsymbol{x}_i] = F(\boldsymbol{x}_i'\boldsymbol{\beta}_0)$. Then

$$\begin{aligned}
\sigma^2(\boldsymbol{x}_i) &= F(\boldsymbol{x}_i'\boldsymbol{\beta}_0)(1 - F(\boldsymbol{x}_i'\boldsymbol{\beta}_0)) \\
&= \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0)(1 - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))
\end{aligned}$$

To derive an expression for the asymptotic variance, first note that for the logistic cdf: $\dot{\mu}(u) = (1 - \mu(u))\mu(u)$. Then, the asymptotic variance is

$$\boldsymbol{V}_0 = H_0^{-1}\Sigma_0 H_0^{-1}.$$

where

$$H_0 = \mathbb{E}[(1 - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^2\mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0)^2\boldsymbol{x}_i\boldsymbol{x}_i']$$

and

$$\Sigma_0 = \mathbb{E}[(1 - \mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0))^3\mu(\boldsymbol{x}_i'\boldsymbol{\beta}_0)^3\boldsymbol{x}_i\boldsymbol{x}_i']$$

1.8 Logistic link function, MLE

MLE gives the same point estimator as NLS (i.e. they have the same FOC; we did this in 672), but MLE is asymptotically efficient, so $\boldsymbol{V}_0^{ML} \leq \boldsymbol{V}_0^{NLS}$.

1.9 Some data work

(a) I estimated the logistic model with robust (HC1) standard errors in both `R` and `Stata`. The results from `R` are presented in Table 1. The standard errors from `Stata` are very slightly different, but I'm not sure why.

Table 1: **Logistic Regression Estimates for** $s = 1-$`dmissing`

	Coef.	Std. Err.	t-stat	p-val	CI.lower	CI.upper
Const.	1.755	0.335	5.245	0.000	1.099	2.411
S_age	1.333	0.123	10.826	0.000	1.092	1.575
S_HHpeople	-0.067	0.023	-2.871	0.004	-0.112	-0.021
$\log(\text{inc}+1)$	-0.119	0.044	-2.707	0.007	-0.205	-0.033

(b) Table 2 presents the 95% confidence interval and p-values for each coefficient derived from 999 bootstrap replications of the t-statistic: $t^* = (\beta^* - \hat{\beta}_{obs})/se^*$. The statistics are very similar to those in Table 1, which rely on large sample approximations.

The idea for computing bootstrapped CIs is simple: for each bootstrap replication, compute t^* for each coefficient; this gives an empirical distribution for t^*; then extract the desired quantiles from the empirical distribution, and compute the confidence intervals as

$$CI_{95}^{boot}(\beta) = \left[\hat{\beta}_{obs} + q^*_{0.025} \times \hat{se}_{obs},\ \hat{\beta}_{obs} + q^*_{0.975} \times \hat{se}_{obs}\right]$$

I computed the bootstrapped p-values as

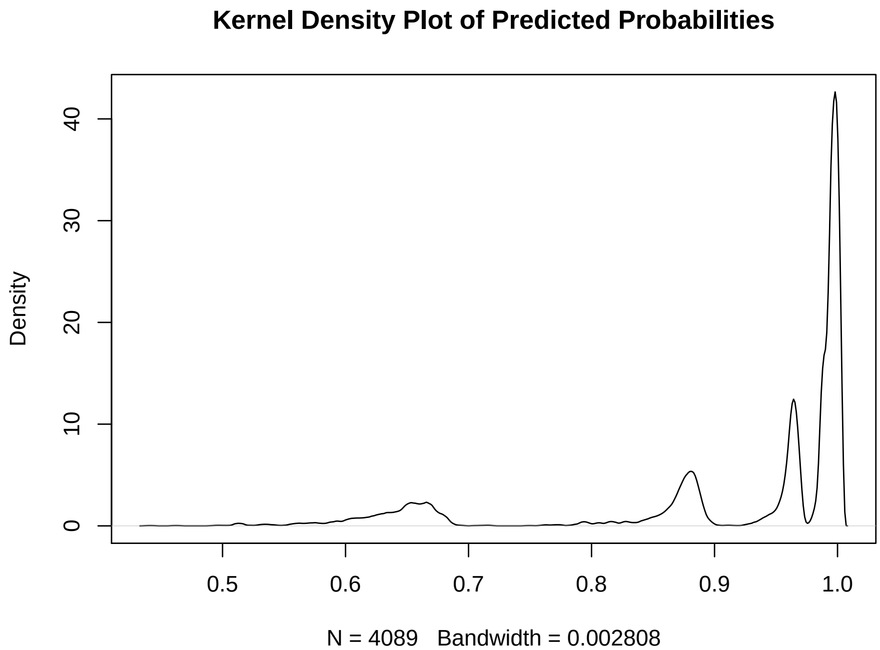
$$p^{boot} = \frac{1}{M}\sum_{i=1}^{M} \mathbf{1}[t^* \geq t_{obs}]$$

where M is the number of bootstrap replications.

Table 2: **Bootstrap Statistics for the Logistic Model of** $s = 1-$`dmissing`

	Coef.	CI.lower	CI.upper	p-val
Const.	1.755	1.157	2.471	0.000
S_age	1.333	1.142	1.609	0.000
S_HHpeople	-0.067	-0.112	-0.020	0.001
$\log(\text{inc}+1)$	-0.119	-0.216	-0.042	0.001

(c) I plot the kernel density estimate of the predicted probabilities of reporting data, $\hat{\mu}(\boldsymbol{x}_i'\hat{\boldsymbol{\beta}})$, using an Epanechnikov kernel with `R`'s unbiased cross-validation bandwidth.



2 Semiparametric GMM with missing data

2.1 An optimal instrument

We have the conditional moment condition:

$$\mathbb{E}[m(y_i^*, t_i, \boldsymbol{x}_i; \boldsymbol{\beta}_0)|t_i, \boldsymbol{x}_i] = 0$$

By LIE

$$\mathbb{E}\big[\boldsymbol{g}(t_i, \boldsymbol{x}_i)\mathbb{E}[m(y_i^*, t_i, \boldsymbol{x}_i; \boldsymbol{\beta}_0)|t_i, \boldsymbol{x}_i]\big] = 0$$

for any $\boldsymbol{g}(\cdot)$. Thus,

$$\begin{aligned}\mathbb{E}\big[\mathbb{E}[\boldsymbol{g}(t_i, \boldsymbol{x}_i)m(y_i^*, t_i, \boldsymbol{x}_i; \boldsymbol{\beta}_0)|t_i, \boldsymbol{x}_i]\big] &= 0 \\ \implies \mathbb{E}[\boldsymbol{g}(t_i, \boldsymbol{x}_i)m(y_i^*, t_i, \boldsymbol{x}_i; \boldsymbol{\beta}_0)] &= 0\end{aligned}$$

Now, we want to find the optimal $\boldsymbol{g}$ that minimizes AsyVar$(\hat{\boldsymbol{\beta}})$; call it $\boldsymbol{g}_0$. Let $\boldsymbol{z}_i = (t_i, \boldsymbol{x}_i)$, $\boldsymbol{w}_i = (y_i^*, t_i, \boldsymbol{x}_i)$. The GMM estimator is

$$\hat{\boldsymbol{\beta}} = \arg\min_{\beta\in\mathbb{R}^d} \left(\frac{1}{n}\sum_{i=1}^n \boldsymbol{g}(\boldsymbol{z}_i)m(\boldsymbol{w}_i, \boldsymbol{\beta})\right)' \boldsymbol{W} \left(\frac{1}{n}\sum_{i=1}^n \boldsymbol{g}(\boldsymbol{z}_i)m(\boldsymbol{w}_i, \boldsymbol{\beta})\right)$$

The FOC wrt $\boldsymbol{\beta}$ is

$$0 = \left(\frac{1}{n}\sum_{i=1}^n \frac{\partial}{\partial\beta}\boldsymbol{g}(\boldsymbol{z}_i)m(\boldsymbol{w}_i, \boldsymbol{\beta})\right)' \boldsymbol{W} \left(\frac{1}{n}\sum_{i=1}^n \boldsymbol{g}(\boldsymbol{z}_i)m(\boldsymbol{w}_i, \boldsymbol{\beta})\right)$$

We can write the FOC plugging in $\hat{\boldsymbol{\beta}}$:

$$0 = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\beta}\boldsymbol{g}(\boldsymbol{z}_i)m(\boldsymbol{w}_i,\hat{\boldsymbol{\beta}})\right)'\boldsymbol{W}\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{g}(\boldsymbol{z}_i)m(\boldsymbol{w}_i,\hat{\boldsymbol{\beta}})\right).$$

Then, if we take a mean value Taylor expansion of the last term in parentheses around the true parameter $\boldsymbol{\beta}_0$ and rearrange in the usual way, we get

$$\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0) = -\left[\boldsymbol{G}_n(\hat{\boldsymbol{\beta}})'\,\boldsymbol{W}\,\boldsymbol{G}_n(\tilde{\boldsymbol{\beta}})\right]^{-1}\boldsymbol{G}_n(\hat{\boldsymbol{\beta}})'\,\boldsymbol{W}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{g}(\boldsymbol{z}_i)m(\boldsymbol{w}_i,\boldsymbol{\beta}_0)\right)$$

where

$$\boldsymbol{G}_n(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\beta}\boldsymbol{g}(\boldsymbol{z}_i)m(\boldsymbol{w}_i,\boldsymbol{\beta})$$

Now we just need to let things converge via LLN and CLT to get

$$\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0) \to_d \mathcal{N}\left(0, [\boldsymbol{G}(\boldsymbol{\beta}_0)'\,\boldsymbol{W}\,\boldsymbol{G}(\boldsymbol{\beta}_0)]^{-1}\,\boldsymbol{G}(\boldsymbol{\beta}_0)'\,\boldsymbol{W}\mathbb{V}[\boldsymbol{g}(\boldsymbol{z}_i)m(\boldsymbol{w}_i,\boldsymbol{\beta}_0)]\,\boldsymbol{W}\,\boldsymbol{G}(\boldsymbol{\beta}_0\,[\boldsymbol{G}(\boldsymbol{\beta}_0)'\,\boldsymbol{W}\,\boldsymbol{G}(\boldsymbol{\beta}_0)]^{-1}\right)$$

Then with the optimal weight matrix $\boldsymbol{W} = \mathbb{V}[\boldsymbol{g}(\boldsymbol{z}_i)m(\boldsymbol{w}_i,\boldsymbol{\beta}_0)]^{-1}$ the asymptotic variance collapses to

$$\boldsymbol{V}_0 = \left[\boldsymbol{G}(\boldsymbol{\beta}_0)'\mathbb{V}[\boldsymbol{g}(\boldsymbol{z}_i)m(\boldsymbol{w}_i,\boldsymbol{\beta}_0)]^{-1}\,\boldsymbol{G}(\boldsymbol{\beta}_0)\right]^{-1}$$

And, as shown in class, this leads to the optimal instrument

$$\boldsymbol{g}_0(\boldsymbol{z}_i) = \mathbb{E}\left[\frac{\partial}{\partial\beta}m(\boldsymbol{w}_i,\boldsymbol{\beta})|\boldsymbol{z}_i\right]\mathbb{V}[m(\boldsymbol{w}_i,\boldsymbol{\beta}_0)|\boldsymbol{z}_i]^{-1}$$

Now, in the given case, since y_i is Bernoulli we know that

$$\mathbb{V}[m(\boldsymbol{w}_i,\boldsymbol{\beta}_0)|\boldsymbol{z}_i] = F(t_i\theta_0+\boldsymbol{x}_i'\boldsymbol{\gamma}_0)[1-F(t_i\theta_0+\boldsymbol{x}_i'\boldsymbol{\gamma}_0)]$$

And

$$\mathbb{E}\left[\frac{\partial}{\partial\beta}m(\boldsymbol{w}_i,\boldsymbol{\beta})|\boldsymbol{z}_i\right] = f(t_i\theta_0+\boldsymbol{x}_i'\boldsymbol{\gamma}_0)[t_i,\boldsymbol{x}_i]',$$

which gives the desired result. When $F(\cdot)$ is the logistic cdf we know that

$$f(u) = F(u)[1-F(u)],$$

so that

$$\boldsymbol{g}_0(t_i,\boldsymbol{x}_i) = [t_i,\boldsymbol{x}_i]'.$$

2.2 Missing completely at random

(a) The optimal unconditional moment condition is:

$$\mathbb{E}[\boldsymbol{g}_0(t_i, \boldsymbol{x}_i) m(y_i^*, t_i, \boldsymbol{x}_i; \boldsymbol{\beta}_0)] = 0$$

Now, $y_i = s_i y_i^*$. Thus,

$$\mathbb{E}[\boldsymbol{g}_0(t_i, \boldsymbol{x}_i) m(y_i, t_i, \boldsymbol{x}_i; \boldsymbol{\beta}_0) | s_i = 1] = \mathbb{E}[\boldsymbol{g}_0(t_i, \boldsymbol{x}_i) m(y_i^*, t_i, \boldsymbol{x}_i; \boldsymbol{\beta}_0) | s_i = 1]$$

And, with the MCAR assumption, the above moment condition is identical to the optimal one. Accordingly, a feasible estimator is simply

$$\hat{\boldsymbol{\beta}}_{\texttt{MCAR,feasible}} = \arg\min_{\theta, \gamma} \frac{1}{n} \sum_{i=1}^{n} [s_i \boldsymbol{g}_0(t_i, \boldsymbol{x}_i)(y_i - F(t_i\theta + \boldsymbol{x}_i' \boldsymbol{\gamma}))].$$

(b) I report the the results from using the feasible estimator below.

	Estimate	Std.Error	t	p-value	CI.lower	CI.upper
dpisofirme	-0.325	0.105	-3.106	0.032	-0.518	-0.167
S_age	-0.226	0.024	-9.326	0.000	-0.275	-0.177
S_HHpeople	0.027	0.018	1.563	0.140	-0.006	0.061
log_inc	0.023	0.017	1.341	0.184	-0.010	0.060

2.3 Missing at random

Start with the optimal moment condition

$$\mathbb{E}[\boldsymbol{g}_0(t_i, \boldsymbol{x}_i) m(y_i^*, t_i, \boldsymbol{x}_i; \boldsymbol{\beta}_0)] = 0.$$

Since $y_i = s_i * y_i^*$ and $s_i \perp y_i^* | (t_i, x_i)$,

$$\mathbb{E}[s_i m(y_i^*, t_i, \boldsymbol{x}_i; \boldsymbol{\beta}_0) | t_i, x_i] = \mathbb{E}[\boldsymbol{g}(t_i, \boldsymbol{x}_i) m(y_i^*, t_i, \boldsymbol{x}_i; \boldsymbol{\beta}_0)]$$

Thus, $\mathbb{E}[s_i m(y_i^*, t_i, x_i; \beta_0) | t_i, x_i] = 0$ is equivalent to the optimal unconditional moment restriction.

(b)

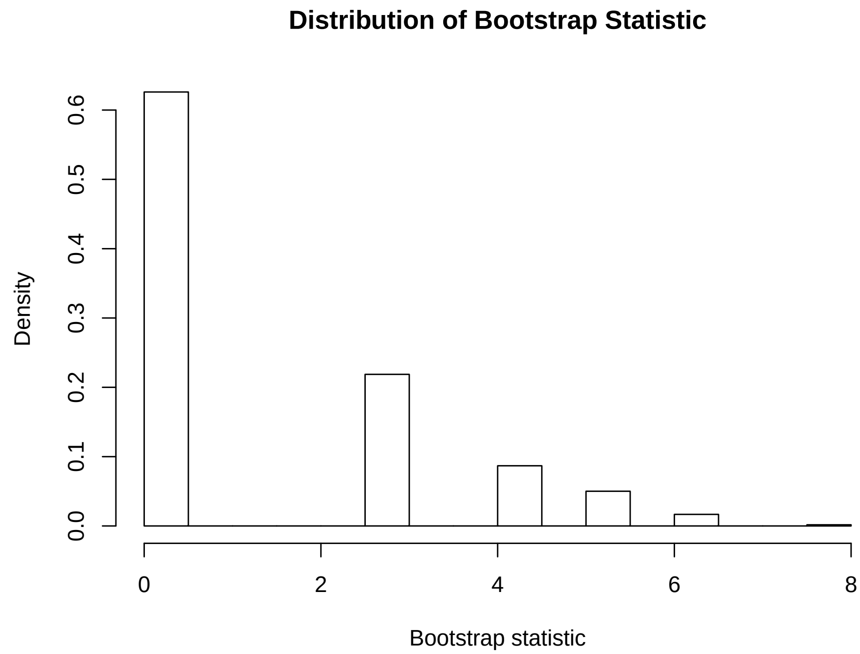
(c)

(d)

3 When bootstrap fails

3.1 Nonparametric bootstrap fail

I plot the empirical distribution of the bootstrap statistic, $n(\max\{x_i\} - \max\{x_i^*\})$, below. Clearly, the empirical distribution does not coincide with the theoretical Exponential (1) distribution.



3.2 Parametric bootstrap to the rescue

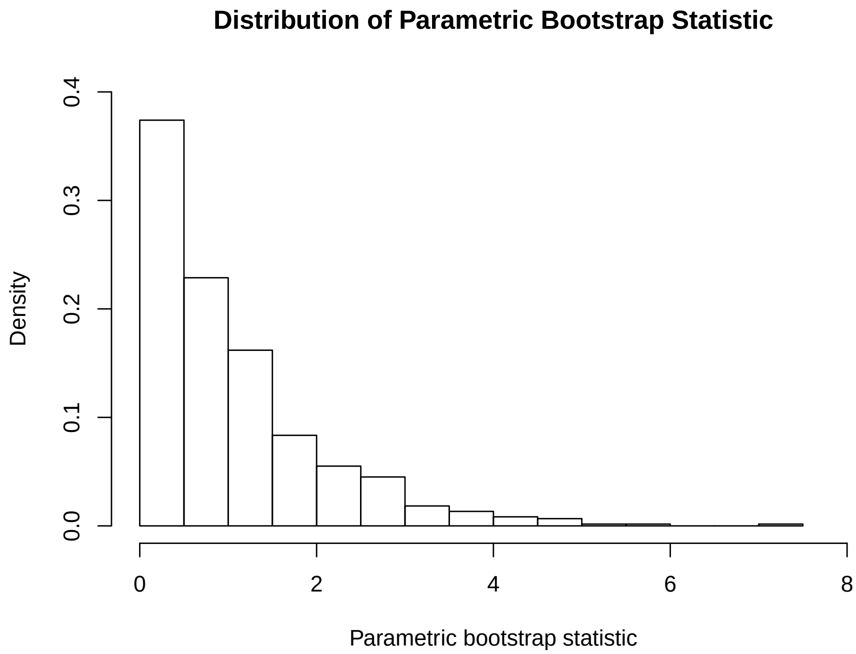
Now, consider the parametric bootstrap statistic, $t_{\mathtt{P}}^* = n(\max\{x_i\} - \max\{x_i^*\})$, where

$$x_i^* \sim_{iid} \mathtt{Uniform}[0, \max\{x_i\}].$$

I plot the empirical distribution of $t_{\mathtt{P}}^*$ below. Now, the empirical distribution *does* seem to coincide with the theoretical Exponential (1) distribution.

3.3 Intuition

In the nonparametric case, the bootstrap statistic has a mass point at zero since $\Pr[\max\{x_i\} = \max\{x_i^*\}]$ converges to 1. However, the parametric bootstrap corrects for this "bias": in this case $\Pr[\max\{x_i\} = \max\{x_i^*\}] = 0$, since $x_i^* \sim_{iid} \mathtt{Uniform}[0, \max\{x_i\}]$.



4 Appendix

4.1 `R` code

4.2 `STATA` code