ECON641 – Problem Set 1

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October 16, 2018

Contents

1 Warmup: factor intensity reversals

First, I outline the small open economy environment of the 2×2 HO model (for my own purposes).

- Two goods, 1 and 2.
- Two factors, L and K; with endogenous factor prices w and r, respectively.
- Production technology is the same in both industries, but they may differ in their relative factor intensities.
- Exogenously given goods prices, p_1 and p_2 (i.e. the demand side of the economy is pinned down).

Roughly speaking, 'no factor intensity reversals' (NFIR) means the following: for any vector of factor prices (w, r), the ordering of relative factor intensities in both industries is always the same. For example, in equilibrium the production of good 1 may be more capital intensive than production of good 2; NFIR implies that at any other vector of factor prices, the production of good 1 must always be more capital intensive compared to good 2. We can show that production technology exhibits NFIR if, given p_1 and p_2, equilibrium factor prices are uniquely pinned down.

1.1 Cobb Douglas

Cobb Douglas production clearly satisfies NFIR. To see this, suppose that $F_1(K_1, L_1) = AK_1^{\alpha}L_1^{1-\alpha}$ and $F_2(K_2, L_2) = AK_2^{\beta}L_2^{1-\beta}$. The first order conditions for the profit maximization problem for industry 1 are standard:

$$p_1 \alpha A K_1^{\alpha-1} L_1^{1-\alpha} = r, \tag{1}$$
$$p_1(1-\alpha) A K_1^{\alpha} L_1^{-\alpha} = w. \tag{2}$$

Dividing (2) by (1) gives

$$\frac{1-\alpha}{\alpha} k_1 = \frac{w}{r} \implies k_1 = \frac{\alpha}{1-\alpha}\frac{w}{r}, \text{ where } k_1 = K_1/L_1 \tag{3}$$

Now, the zero profit condition in industry 1 is

$$\begin{aligned} rK_1 + wL_1 &= p_1 A K_1^{\alpha} L_1^{1-\alpha} \\ \implies rk_1 + w &= p_1 A k_1^{\alpha} \end{aligned} \tag{4}$$

Plugging (3) into (4) and rearranging gives

$$p_1 = C_\alpha r^\alpha w^{1-\alpha} \tag{5}$$

where $C_\alpha = \frac{1}{A(1-\alpha)}\left(\frac{1-\alpha}{\alpha}\right)^\alpha$. An analogous derivation for industry 2 gives

$$p_2 = C_\beta r^\beta w^{1-\beta} \tag{6}$$

where $C_\beta = \frac{1}{A(1-\beta)}\left(\frac{1-\beta}{\beta}\right)^\beta$. Clearly, given p_1 and p_2, there is a unique solution to (5) and (6), (w^*, r^*), (unless $\alpha = \beta$).

Another (perhaps more intuitive) way to establish NFIR would be to use equation (3) and the equivalent expression for industry 2. These expressions imply that in equilibrium:

$$\frac{k_1}{k_2} = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}.$$

That is, the relative factor intensities between the two industries is independent of factor prices.

1.2 CES

CES production *does not* exhibit NFIR. To see this, suppose $F_i(K_i, L_i) = \left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}}\right]^{\frac{\sigma_i}{\sigma_i-1}}$ for $i = 1, 2$. The FOCs for industry i are

$$p_i\left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}}\right]^{\frac{1}{\sigma_i-1}} K_i^{-1/\sigma_i} = r \tag{7}$$

$$p_i\left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}}\right]^{\frac{1}{\sigma_i-1}} L_i^{-1/\sigma_i} = w \tag{8}$$

Combining these expressions gives

$$\begin{aligned} k_i^{-1/\sigma_i} &= \frac{r}{w} \\ \implies k_i &= \left(\frac{r}{w}\right)^{-\sigma_i}. \end{aligned}$$

Thus, in equilibrium, the relative factor intensities between the two industries is

$$\frac{k_1}{k_2} = \left(\frac{r}{w}\right)^{\sigma_2-\sigma_1},$$

which clearly depends on factor prices (unless $\sigma_1 = \sigma_2$).

1.3 Leontief

Clearly the Leontief production function exhibits NFIR. Suppose both industries have the same production function $F(K, L) = \min\{K, L\}$. Then in equilibrium, both industries must have $k_i = 1$. Then, relative factor intensities do not depend on factor prices. More generally, suppose $F_i(K_i, L_i) = \min\{\alpha_i K, \beta_i L\}$. Then in equilibrium, each industry's capital-labor ratio will be $k_i = \beta_i/\alpha_i$. Again, relative factor intensities are independent of factor prices.

2 $2 \times 2 \times 2$ HO Model

3 Technology growth in a parameterized version of DFS

3.1

I follow the derivation in EK (2005). We are given the distribution of efficiencies for producing goods j at Home and Foreign:

$$F_i(z) = \Pr[Z_i(j) \leq z] = \exp(-T_i z^{-\theta})$$

Now, we want to derive the DFS-type $A(j)$ curve. I follow EK's derivation, which defines $A(j)$ as F's efficiency of producing j to H's corresponding efficiency.[1]

In the EK setup the efficiencies are realizations of a random variable. Accordingly, we think of j as the *probability* that the F's relative efficiency of producing j is less than some number:

$$\begin{aligned} j &= \Pr\left[\frac{Z^*}{Z} \leq A\right] \\ &= \Pr\left[Z^* \leq AZ\right] \\ &= \int_0^\infty \exp(-T^*(Az)^{-\theta}) f(z) dz. \end{aligned}$$

Now,

$$f(z) = \frac{d}{dz}\exp(-Tz^{-\theta}) = \theta T z^{-\theta-1}\exp(-Tz^{-\theta})$$

Substituting into the above integral gives

$$\begin{aligned} j &= T\int_0^\infty \exp(-T^*(Az)^{-\theta}) \times \theta z^{-\theta-1}\exp(-Tz^{-\theta}) dz \\ &= T\int_0^\infty \exp(-(T^*A^{-\theta}+T)z^{-\theta}) \times \theta z^{-\theta-1} dz \\ &= \frac{T}{(T^*A^{-\theta}+T)}\int_0^\infty \exp(-(T^*A^{-\theta}+T)z^{-\theta}) \times -\theta z^{-\theta-1}(T^*A^{-\theta}+T) dz \\ &= \frac{T}{(T^*A^{-\theta}+T)}\int_0^\infty \exp(-(T^*A^{-\theta}+T)z^{-\theta}) \times \theta z^{-\theta-1}(T^*A^{-\theta}+T) dz \\ &= \frac{T}{(T^*A^{-\theta}+T)}, \end{aligned}$$

since $\int_0^\infty \exp(-(T^*A^{-\theta}+T)z^{-\theta}) \times \theta z^{-\theta-1}(T^*A^{-\theta}+T)dz = 1$ (because it is the integral of the Frechet pdf with scale $(T^*A^{-\theta}+T)$). Thus, rearranging to get an expression for $A(j)$ gives

$$A(j) = \left[\left(\frac{1-j}{j}\right)\frac{T}{T^*}\right]^{-1/\theta}.$$

[1] Note that in DFS, $A(j)$ was the ratio of H's efficiency of producing j to F's corresponding efficiency.

3.2

First note that there are no trade costs so that $d_{ni} = 1$ for all $n, i \in \{F, H\}$.

Now, we know that within a country, goods will be purchased from the lowest cost source. Since Home has a comparative advantage at lower values of j, we know that Home will produce the range of goods $[0, \bar{j}]$ where

$$\frac{w}{z(\bar{j})} = \frac{w^*}{z^*(\bar{j})}.$$

The LHS of the above expression is the unit cost of producing $\bar{j}$ at home, and the RHS is the cost of buying the good from Foreign. Rearranging the above expression gives

$$\begin{aligned} \frac{z(\bar{j})}{z^*(\bar{j})} &= \frac{w}{w^*} \\ \implies A(\bar{j}) &= \omega, \end{aligned} \tag{9}$$

where $\omega = w/w^*$. Similarly, Foreign will produce a range of goods $[\underline{j}, 1]$ domestically, such that

$$\begin{aligned} \frac{w^*}{z^*(\underline{j})} &= \frac{w}{z(\underline{j})} \\ \implies A(\underline{j}) &= \omega. \end{aligned} \tag{10}$$

Thus, there is a unique cuttoff good.

Next, we need to invoke market clearing. Here, we note that preferences are Cobb Douglas, with equal weights across each good. Thus, each country spends a constant share of its income on each good. We know that Home produces $[0, \bar{j}]$ goods domestically, and exports $[0, \underline{j}]$ goods to Foreign. Thus, market clearing at Home requires

$$wL = \bar{j}wL + \underline{j}w^*L^* \tag{11}$$

$$\tag{12}$$

Substituting (9) and (10) into (11) and rearranging gives

$$L = LA^{-1}(\omega) + \frac{1}{\omega}L^*A^{-1}(\omega)$$