ECON641 – Problem Set 1

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Contents

1 Warmup: factor intensity reversals

First, I outline the small open economy environment of the 2×2 HO model (for my own purposes).

- Two goods, 1 and 2.

- Two factors, L and K; with endogenous factor prices w and r, respectively.

- Production technology is the same in both industries, but they may differ in their relative factor intensities.

- Exogenously given goods prices, p_1 and p_2 (i.e. the demand side of the economy is pinned down).

Roughly speaking, 'no factor intensity reversals' (NFIR) means the following: for any vector of factor prices (w, r), the ordering of relative factor intensities in both industries is always the same. For example, in equilibrium the production of good 1 may be more capital intensive than production of good 2; NFIR implies that at any other vector of factor prices, the production of good 1 must always be more capital intensive compared to good 2. We can show that production technology exhibits NFIR if, given p_1 and p_2, equilibrium factor prices are uniquely pinned down.

1.1 Cobb Douglas

Cobb Douglas production clearly satisfies NFIR. To see this, suppose that $F_1(K_1, L_1) = AK_1^\alpha L_1^{1-\alpha}$ and $F_2(K_2, L_2) = AK_2^\beta L_2^{1-\beta}$. The first order conditions for the profit maximization problem for industry 1 are standard:

$$p_1 \alpha A K_1^{\alpha-1} L_1^{1-\alpha} = r, \tag{1}$$
$$p_1(1-\alpha) A K_1^\alpha L_1^{-\alpha} = w. \tag{2}$$

Dividing (2) by (1) gives

$$\frac{1-\alpha}{\alpha} k_1 = \frac{w}{r} \implies k_1 = \frac{\alpha}{1-\alpha} \frac{w}{r}, \text{ where } k_1 = K_1/L_1 \tag{3}$$

Now, the zero profit condition in industry 1 is

$$rK_1 + wL_1 = p_1 A K_1^\alpha L_1^{1-\alpha}$$
$$\implies rk_1 + w = p_1 A k_1^\alpha \tag{4}$$

Plugging (3) into (4) and rearranging gives

$$p_1 = C_\alpha r^\alpha w^{1-\alpha} \tag{5}$$

where $C_\alpha = \frac{1}{A(1-\alpha)} \left(\frac{1-\alpha}{\alpha}\right)^\alpha$. An analogous derivation for industry 2 gives

$$p_2 = C_\beta r^\beta w^{1-\beta} \tag{6}$$

where $C_\beta = \frac{1}{A(1-\beta)} \left(\frac{1-\beta}{\beta} \right)^\beta$. Clearly, given p_1 and p_2, there is a unique solution to (5) and (6), (w^*, r^*), (unless $\alpha = \beta$).

Another (perhaps more intuitive) way to establish NFIR would be to use equation (3) and the equivalent expression for industry 2. These expressions imply that in equilibrium:

$$\frac{k_1}{k_2} = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}.$$

That is, the relative factor intensities between the two industries is independent of factor prices.

1.2 CES

CES production *does not* exhibit NFIR. To see this, suppose $F_i(K_i, L_i) = \left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}} \right]^{\frac{\sigma_i}{\sigma_i-1}}$ for $i = 1, 2$. The FOCs for industry i are

$$p_i \left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}} \right]^{\frac{1}{\sigma_i-1}} K_i^{-1/\sigma_i} = r \qquad (7)$$

$$p_i \left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}} \right]^{\frac{1}{\sigma_i-1}} L_i^{-1/\sigma_i} = w \qquad (8)$$

Combining these expressions gives

$$k_i^{-1/\sigma_i} = \frac{r}{w}$$

$$\implies k_i = \left(\frac{r}{w} \right)^{-\sigma_i}.$$

Thus, in equilibrium, the relative factor intensities between the two industries is

$$\frac{k_1}{k_2} = \left(\frac{r}{w} \right)^{\sigma_2-\sigma_1},$$

which clearly depends on factor prices (unless $\sigma_1 = \sigma_2$).

1.3 Leontief

Clearly the Leontief production function exhibits NFIR. Suppose both industries have the same production function $F(K, L) = \min\{K, L\}$. Then in equilibrium, both industries must have $k_i = 1$. Then, relative factor intensities do not depend on factor prices. More generally, suppose $F_i(K_i, L_i) = \min\{\alpha_i K, \beta_i L\}$. Then in equilibrium, each industry's capital-labor ratio will be $k_i = \beta_i/\alpha_i$. Again, relative factor intensities are independent of factor prices.

2 $2 \times 2 \times 2$ HO Model

2.1 Integrated equilibrium

Integrated equilibrium is the competitive equilibrium that would prevail if both factors and goods were freely traded across countries. Thus, to solve for integrated equilibrium quantities and prices I simply assume that there is a representative consumer that maximizes utility subject to their (world) income, both firms maximize profits, and markets clear. I want to get expressions for equilibrium quantities $\{C_A, C_B, X_A, X_B, K_A, K_B, L_A, L_B\}$ and prices $\{p_A, p_B, w, r\}$, so I need to derive 12 equations that represent the equilibrium, make a suitable normalization and then do a lot of algebra.

Denote world endowments of labor and capital as $\bar{L}$ and $\bar{K}$, respectively. Start with the consumer's problem because it is easy. The consumer solves

$$\max_{C_A, C_B} U(C_A, C_B) \text{ s.t. } p_A C_A + p_B C_B = w\bar{L} + r\bar{K}.$$

Given Cobb-Douglas preferences, it is easy to arrive at the demand functions:

$$C_A = \frac{1}{p_A} \alpha (w\bar{L} + r\bar{K}) \tag{9}$$

$$C_B = \frac{1}{p_B} (1 - \alpha)(w\bar{L} + r\bar{K}). \tag{10}$$

Next, consider the firms' profit maximization problems. I can skip a few steps here by using the results form Question 1 [1]. Combining the FOCs from the firms' maximization problems with the zero profit conditions gives expressions for goods prices in terms of factor prices:

$$p_A = \lambda_A r^{\beta_A} w^{1-\beta_A} \tag{11}$$

$$p_B = \lambda_B r^{\beta_B} w^{1-\beta_B} \tag{12}$$

where $\lambda_i = \frac{1}{(1-\beta_i)} \left(\frac{1-\beta_i}{\beta_i} \right)^{\beta_i}$ for $i = A, B$.

Next, I derive the unit factor requirements for both goods, which I'll use later in the market clearing conditions. To do so, I solve the firms' cost minimization problems. Firm i solves

$$\min_{K_i, L_i} rK_i + wL_i \text{ s.t } K_i^{\beta_i} L_i^{1-\beta_i} \geq 1$$

Again, I can skip a lot of the math because it is almost identical to that in Question 1 [1]. Each firm chooses its capital/labor ratio such that

$$\frac{K_i}{L_i} = \frac{\beta_i}{1 - \beta_i} \left(\frac{w}{r} \right).$$

Plugging this expression into the production function gives unit factor requirements for each sector. For labor:

$$\left[\frac{\beta_i}{1 - \beta_i} \left(\frac{w}{r} \right) \right]^{\beta_i} L_i = 1$$

$$\implies L_i \equiv a_{L,i}(w, r) = \left[\frac{\beta_i}{1 - \beta_i} \left(\frac{w}{r} \right) \right]^{-\beta_i} \tag{13}$$

Similarly, the unit capital requirement for sector i is

$$a_{K,i} = \left[\frac{\beta_i}{1-\beta_i} \left(\frac{w}{r} \right) \right]^{1-\beta_i} \tag{14}$$

Then, factor demands are given by

$$L_i = a_{L,i}(w,r)X_i \tag{15}$$
$$K_i = a_{K,i}(w,r)X_i \tag{16}$$

for $i = A, B$. Finally the market clearing conditions are:

$$X_i = C_i, \text{ for } i = A, B \tag{17}$$
$$L_A + L_B = \bar{L} \tag{18}$$
$$K_A + K_B = \bar{K}. \tag{19}$$

Equations (9)–(19) fully characterize the integrated equilibrium. Next, I need to do a lot of algebra to get expressions for equilibrium quantities and prices. Start with the labor market clearing condition

$$a_{L,A}(w,r)X_A + a_{L,B}(w,r)X_B = \bar{L}$$

Then, use the goods market clearing conditions, and plug in the demand functions (9) and (10) to get

$$\bar{L} = a_{L,A}(w,r)\frac{\alpha}{p_A}(w\bar{L} + r\bar{K}) + a_{L,B}(w,r)\frac{1-\alpha}{p_B}(w\bar{L} + r\bar{K})$$
$$= (w\bar{L} + r\bar{K}) \left[a_{L,A}(w,r)\frac{\alpha}{p_A} + a_{L,B}(w,r)\frac{1-\alpha}{p_B} \right].$$

Next, plug in the unit labor requirements (13) and goods prices (11) and (12):

$$\bar{L} = (w\bar{L} + r\bar{K}) \left[\left[\frac{\beta_A}{1-\beta_A} \left(\frac{w}{r} \right) \right]^{-\beta_A} \frac{\alpha}{\lambda_A r^{\beta_A} w^{1-\beta_A}} + \left[\frac{\beta_B}{1-\beta_B} \left(\frac{w}{r} \right) \right]^{-\beta_B} \frac{1-\alpha}{\lambda_B r^{\beta_B} w^{1-\beta_B}} \right].$$

This is an ugly expression, so it seems like the appropriate time to make the normalization $w = 1$. Plugging in the constants, λ_A and λ_B also helps to simplify the above expression; I get:

$$\bar{L} = (\bar{L} + r\bar{K}) \left[\alpha(1-\beta_A) + (1-\alpha)(1-\beta_B) \right],$$

which looks much better (phew!). Now, solving for r gives

$$r^* = \left[\frac{\alpha\beta_A + (1-\alpha)\beta_B}{\alpha(1-\beta_A) + (1-\alpha)(1-\beta_B)} \right] \frac{\bar{L}}{\bar{K}} \tag{20}$$

and I'm essentially done. By Walras' law the capital market also clears. To get equilibrium goods prices, p_A^* and p_B^*, plug r^* into (11) and (12). Then plug these equilibrium prices into (9) and (10) to get equilibrium consumption C_A^* and C_B^*, which in turn give X_A^* and X_B^* via goods market clearing. Finally, plug in r^* and X_i^* into (15) and (16) to get the equilibrium quantities of labor and capital used in each sector.

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2.2 FPE set

The FPE set is the set of country factor endowment pairs $((L_H, K_H), (L_F, K_F))$ such that every country can fully employ its factors and $w_H = w_F, r_H = r_F$. The idea is that factor prices will be equalized under free trade of goods (not factors) if the countries' endowments are not too dissimilar.

The integrated equilibrium factor demands for each sector are

$$L_A^* = \left[\frac{\alpha(1 - \beta_A)}{\alpha(1 - \beta_A) + (1 - \alpha)(1 - \beta_B)} \right] \bar{L}$$
$$L_B^* = \bar{L} - L_A^*$$
$$K_A^* = \left[\frac{\alpha\beta_A}{\alpha\beta_A + (1 - \alpha)\beta_B} \right] \bar{K}$$
$$K_B^* = \bar{K} - K_A^*$$

I plot the FPE set for the given parameters below; Home's 'origin' is at the lower left corner. The endowment vector is represented by the red dot; clearly, it is not in the FPE set.

FPE Set
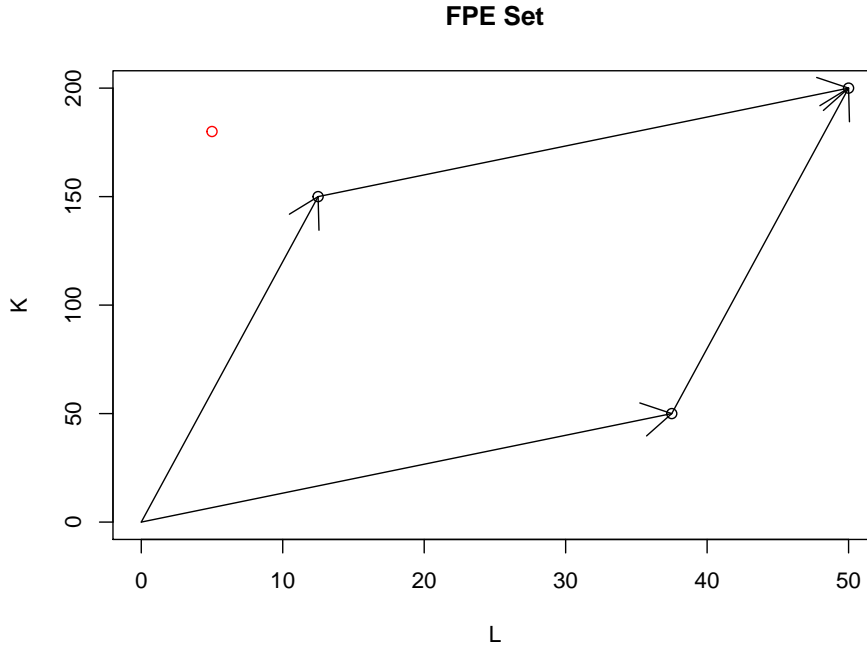
2.3 Autarky

Autarkic equilibrium prices and quanties in both countries look very similar to their integrated equilibrium counterparts, except with different factor endowments. With the normalization $w^j = 1$ for $j = H, F$, the same algebra as in the integrated equilibrium case gives the autarkic rental rates:

$$r^j = \left[\frac{\alpha\beta_A + (1 - \alpha)\beta_B}{\alpha(1 - \beta_A) + (1 - \alpha)(1 - \beta_B)} \right] \frac{L^j}{K^j}.$$

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Again, for brevity, I won't derive expressions for autarkic goods prices and quantities; but to do so, I would simply use the same procedure as noted in integrated equilibrium case.

Now, consider the case where trade in goods is not allowed but labor is fully mobile across countries (i.e. free immigration). In this case, there is a single 'world' labor market clearing condition:

$$a_{L,A}(w^H, r^H)X_A^H + a_{L,B}(w^H, r^H)X_B^H + a_{L,A}(w^F, r^F)X_A^F + a_{L,B}(w^F, r^F)X_B^F = \bar{L} \qquad (21)$$

Now, autarky equilibrium is a set of quantities $\{C_A^j, C_B^j, X_A^j, X_B^j, K_A^j, K_B^j, L_A^j, L_B^j\}_j$ and prices $\{p_A^j, p_B^j, w^j, r^j\}_j$ that solve the following system of equations along with the labor market clearing condition (21)

$$\alpha = \frac{p_A^j X_A^j}{p_A^j X_A^j + p_B^j X_B^j}$$
$$p_A^j = \lambda_A r_j^{\beta_A} w_j^{1-\beta_A}$$
$$p_B^j = \lambda_B r_j^{\beta_B} w_j^{1-\beta_B}$$
$$K_i^j = a_{K,i}(w^j, r^j)X_i^j \text{ for } i = A, B$$
$$X_i^j = C_i^j, \text{ for } i = A, B$$
$$K_A^j + K_B^j = K^j$$
$$L_i^j = a_{L,i}(w^j, r^j)X_i^j + a_{L,i}(w^j, r^j)X_i^j \text{ for } i = A, B$$

for $j = H, F$ (note that this is a system of 21 equations in 21 endogenous variables). This looks like a tough problem to solve, but note that with free immigration, in equilibrium the real wage will be equalized across both countries; that is

$$\frac{w^H}{P^H} = \frac{w^F}{P^F} \qquad (22)$$

where P^j is the ideal aggregate price index

$$P^j = \frac{(p_A^j)^\alpha (p_B^j)^{1-\alpha}}{\alpha(1-\alpha)} \text{ for } j = H, F.$$

Next, I use the fact that in equilibrium, goods prices equal the unit cost of production. Plugging these expressions for goods prices into (22) with the given parameters gives

$$\left(\frac{w^H}{r^H}\right)^{1/2} = \left(\frac{w^F}{r^F}\right)^{1/2},$$

which says that factor prices are equalized with free immigration. Then, using equation (3) from Question 1, I know that factor intensities for each industry are also equalized across countries in equilibrium. Now, I'm done. Since factor intensities are equalized in equilibrium, I can easily back out the required level of immigration from Foreign to Home:

$$\frac{K^H}{L^H + L^{\text{migrate}}} = \frac{K^F}{L^F - L^{\text{migrate}}} \implies L^{\text{migrate}} = 40.$$

2.4 Free trade

Start by noting that with free trade in goods, goods prices will be equalized across countries. Furthermore, since Home specializes in producing good B, I know from previous results that

$$\frac{\beta_B}{1-\beta_B}\frac{w^H}{r^H} = \frac{K^H}{L^H}.$$

With the given parameter values, we get

$$\frac{w^H}{r^H} = 12.$$

Next, note that Cobb Douglas preferences imply that in equilibrium

$$\frac{\alpha}{p_A}(Y^H + Y^F) = X_A^F$$

$$\frac{1-\alpha}{p_B}(Y^H + Y^F) = X_B^H + X_B^F,$$

where $Y^j = w^j L^j + r^j K^j$. Using the fact that $Y^H = p_B X_B^H$, I can rearrange the second condition to get

$$X_B^F = (1-\alpha)Y^F - \alpha Y^H.$$

Now, the labor market clearing condition in Foreign is

$$a_{L,A}(w^F, r^F)X_A^F + a_{L,B}(w^F, r^F)X_B^F = L^F$$

Plugging in unit labor requirements, using the goods market clearing conditions above and simplifying gives

$$\alpha(Y^H + Y^F)(1-\beta_A) + [(1-\alpha)Y^F - \alpha Y^H](1-\beta_B) = w^F L^F \tag{23}$$

Next, note that since the price of good B is equalized across countries

$$(r^H)^{\beta_B}(w^H)^{1-\beta_B} = (r^F)^{\beta_B}(w^F)^{1-\beta_B}$$

Normalizing $w^F = 1$ and plugging in $(w^H/r^H) = 12$ gives

$$r^H(12)^{1-\beta_B} = r^F \tag{24}$$

Finally I need to solve for r^F. To do so, I plug in (24) into (23), noting that

$$Y^H = r^H\left[\frac{w^H}{r^H}L^H + K^H\right] \text{ and } Y^F = L^F + r^F K^F.$$

Then I'm left with a single equation in one unknown, r^F, which can be solved easily. All other equilibrium quantities and prices are pinned down by r^F.

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3 Monopolistic competition with variable markups

3.1 Consumer's problem

The consumer's problem is

$$\max_{q_i} \int_0^N (1 - \exp(-\alpha q_i)) di$$

$$\text{s.t.} \int_0^N p_i q_i di = E.$$

The Lagrangian is

$$\mathcal{L} = \int_0^N (1 - \exp(-\alpha q_i)) di + \lambda \left[E - \int_0^N p_i q_i di \right].$$

The FOC wrt variety i is

$$\alpha \exp(-\alpha q_i) = \lambda p_i$$

Since this FOC holds for all varieties, for any pair of varieties i, j I can take the ratio

$$\frac{\alpha \exp(-\alpha q_i)}{\alpha \exp(-\alpha q_j)} = \frac{\lambda p_i}{\lambda p_j}$$

Rearranging this expression gives demand for good i as a function of demand for good j and their respective prices:

$$q_i = q_j - \frac{1}{\alpha} \log \left(\frac{p_i}{p_j} \right). \tag{25}$$

To derive the demand for good i as a function of E and goods prices only, multiply both sides of the above expression by p_j and then integrate over j:

$$\int_0^N q_i p_j dj = E - \frac{1}{\alpha} \int_0^N \log \left(\frac{p_i}{p_j} \right) p_j dj$$

$$\therefore q_i = \frac{E - \frac{1}{\alpha} \int_0^N \log \left(\frac{p_i}{p_j} \right) p_j dj}{\int_0^N p_j dj} \tag{26}$$

3.2 Firm's problem

[1] Firm i's profit maximization problem is

$$\max_{p_i, q_i} L q_i (p_i - mw)$$

$$\text{s.t. } (26)$$

Plugging the firm's demand constraint into it's profit function allows me to rewrite the objective as

$$\max_{p_i} L \left[\frac{E - \frac{1}{\alpha} \int_0^N \log\left(\frac{p_i}{p_j}\right) p_j dj}{\int_0^N p_j dj} \right] (p_i - mw).$$

The FOC is

$$\frac{E}{\int_0^N p_j dj} - \frac{1}{\alpha \int_0^N p_j dj} \frac{\partial}{\partial p_i} \left\{ (p_i - mw) \int_0^N \log\left(\frac{p_i}{p_j}\right) p_j dj \right\} = 0$$

$$\implies \frac{\partial}{\partial p_i} \left\{ (p_i - mw) \int_0^N \log\left(\frac{p_i}{p_j}\right) p_j dj \right\} = \alpha E$$

Now,

$$\frac{\partial}{\partial p_i} \left\{ (p_i - mw) \int_0^N \log\left(\frac{p_i}{p_j}\right) p_j dj \right\} = \frac{\partial}{\partial p_i} \left\{ (p_i - mw) \int_0^N \log(p_i) p_j - \log(p_j) p_j dj \right\}$$

$$= 1 \cdot \log(p_i) \int_0^N p_j dj + \frac{p_i - mw}{p_i} \int_0^N p_j dj - \int_0^N \log(p_j) p_j dj$$

$$= \int_0^N \log\left(\frac{p_i}{p_j}\right) p_j dj + \frac{p_i - mw}{p_i} \int_0^N p_j dj$$

Thus, the FOC for the firm's problem is

$$\int_0^N \log\left(\frac{p_i}{p_j}\right) p_j dj + \frac{p_i - mw}{p_i} \int_0^N p_j dj = \alpha E \tag{27}$$

[2] Now, under symmetric equilibrium, $p_i = p_j \equiv p$ for all i, j. Thus, the FOC (27) simplifies to

$$\int_0^N \log\left(\frac{p}{p}\right) p dj + \frac{p - mw}{p} \int_0^N p dj = \alpha E$$

$$\implies \frac{p - mw}{p} N p = \alpha E$$

$$\therefore p = mw + \frac{\alpha E}{N}. \tag{28}$$

Thus, the optimal price features a markup over the marginal cost, which is decreasing in the number of firms. This result differs from the CES case, in which the optimal price features a constant markup over marginal costs (and does not depend on the number of firms).

[3] To get the equilibrium quantity of each good sold, plug the optimal price (28) into the consumer's demand function (26):

$$q = \frac{E}{Np} = \frac{E}{N} \left[mw + \frac{\alpha E}{N} \right]^{-1}$$

$$= \frac{E}{N} \frac{N}{mwN + \alpha E}$$

$$= \frac{E}{mwN + \alpha E}$$

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And then, each firm's variable profit is

$$
\begin{aligned}
\Pi_v &= Lq(p - mw) \\
&= L\frac{E}{mwN + \alpha E}\frac{\alpha E}{N} \\
&= \frac{L}{N}\frac{\alpha E^2}{mwN + \alpha E}
\end{aligned}
\tag{29}
$$

3.3   Equilibrium number of firms

[1] The free entry condition implies

$$
\Pi = \frac{L}{N}\frac{\alpha E^2}{mwN + \alpha E} - wF = 0
$$

Plugging in $E = w$ implies that in equilibrium

$$
w\left[\frac{\alpha L}{N(mN + \alpha)} - F\right] = 0
$$

Solving for N using the quadratic formula gives

$$
N = \frac{-\alpha F + \sqrt{\alpha^2 F^2 + 4(mF)(\alpha L)}}{2mF}.
$$

[2] Now, I can solve for welfare w/p by using (28) and plugging in $E = w$:

$$
p = mw + \frac{\alpha w}{N}
$$

$$
\implies \frac{w}{p} = \frac{N}{\alpha + mN}
$$

Then

$$
\frac{\partial}{\partial N}\frac{w}{p} = \frac{\alpha}{(\alpha + mN)^2} > 0,
$$

so welfare is increasing in the number of firms (for $\alpha > 0$).

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4 A simple model of institutional comparative advantage

4.1 Optimal team size?

I'm not sure I understand this question, but here's my interpretation. Final goods producers have Leontief production functions; accordingly, final good producer z should optimally choose:

$$q(1) = q(2) = ... = q(z)$$
$$\implies \phi L(1) = \phi L(2) = ... = \phi L(z)$$
$$\implies L(1) = L(2) = ... = L(z),$$

so that $L(s)$ does not vary across inputs $s = 1, ..., z$. I find this result confusing because, given the wording of the question, I thought that intermediate goods producers chose $L(s)$ rather than final good producers. Anyway, let's proceed with the result that $L(s)$ does not vary across inputs. Then we get the result that final good producer z allocates labor across inputs according to

$$L(s) = q_z/\phi \text{ for all } s = 1, ..., z.$$

4.2 Expected output per worker

Expected output in industry z is given by

$$\mathbb{E}[y_z] = q_z \times [\text{no reneges}]$$
$$= \phi L \rho^z$$

where the last equality uses the fact that $L(s)$ is the same for all $s = 1, ..., z$. Then expected output per worker in sector z is simply $\phi \rho^z$.

4.3 Efficient contracting and comparative advantage

From above, the unit labor requirement in sector z is $a(z) = 1/(\phi \rho^z)$. Thus, I can easily derive the DFS-type $A(z)$ curve:

$$A(z) \equiv \frac{a^S(z)}{a^N(z)}$$
$$= \left(\frac{\rho_N}{\rho_S} \right)^z.$$

Now, to derive the slope of the $A(z)$ curve, start by taking logs

$$\log A(z) = z \log \left(\frac{\rho_N}{\rho_S} \right)$$
$$\implies \frac{d \log A(z)}{dz} = \log \left(\frac{\rho_N}{\rho_S} \right).$$

Then, I can use the fact that $\frac{d \log A(z)}{dz} = A'(z)/A(z)$ to get

$$A'(z) = \left(\frac{\rho_N}{\rho_S}\right)^z \log\left(\frac{\rho_N}{\rho_S}\right) > 0$$

since $\rho_N > \rho_S$. This result implies that for more complex goods, the ratio of North's productivity to South's is larger – that is, more efficient contracting creates comparative advantage.

4.4 Equilibrium conditions

Following DFS, market clearing in North requires

$$w^N L^N = (1 - S(\tilde{z}))[w^N L^N + w^S L^S],$$

where $\tilde{z}$ is the cutoff good (i.e. North produces goods $[\tilde{z}, \bar{z}]$ domestically and imports goods $[0, \tilde{z}]$). Rearranging the market clearing condition gives

$$\omega = \frac{1 - S(\tilde{z})}{S(\tilde{z})} \frac{L^S}{L^N}$$

Now, with Cobb Douglas preferences and a continuum of goods $(0, \bar{z}]$, $S(\tilde{z}) = \tilde{z}/\bar{z}$. Thus, the market clearing condition can be written in terms of ω and $\tilde{z}$:

$$\omega = \frac{\bar{z} - \tilde{z}}{\tilde{z}} \frac{L^S}{L^N} \tag{30}$$

Next, note that consumers in North will purchase goods from the lowest cost supplier. Since North is relatively more productive at higher z's it follows that consumers in North purchase goods $[\tilde{z}, \bar{z}]$ domestically and import goods $[0, \tilde{z}]$. Then, the cutoff good's unit price must be equalized in North and South

$$w^N a^N(\tilde{z}) = w^S a^S(\tilde{z})$$
$$\implies \omega = A(\tilde{z}), \tag{31}$$

which is the second equilibrium condition.

4.5 Pinning down production and trade patterns

The equilibrium cutoff good solves (30) and (31):

$$\frac{\bar{z} - \tilde{z}}{\tilde{z}} \frac{L^S}{L^N} = A(\tilde{z})$$
$$\implies \frac{\bar{z} - \tilde{z}}{\tilde{z}} \frac{L^S}{L^N} = \left(\frac{\rho_N}{\rho_S}\right)^{\tilde{z}}$$

I'm not sure how to solve this equation by hand, but a solution must exist since the LHS is decreasing in $\tilde{z}$, the RHS is increasing in $\tilde{z}$. As mentioned above, North produces and exports more complex goods than South due its more efficient contracting environment.

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5 Technology growth in a parameterized version of DFS

5.1

I follow the derivation in EK (2005). We are given the distribution of efficiencies for producing goods j at Home and Foreign:

$$F_i(z) = \Pr[Z_i(j) \leq z] = \exp(-T_i z^{-\theta})$$

Now, we want to derive the DFS-type $A(j)$ curve, which is defined as the ratio of H's efficiency of producing j to F's corresponding efficiency.

In the EK setup the efficiencies are realizations of a random variable. Accordingly, we think of j as the *probability* that the H's relative efficiency of producing j is less than some number:

$$
\begin{aligned}
j &= \Pr\left[\frac{Z}{Z^*} \leq A\right] \\
&= \Pr\left[Z \leq A Z^*\right] \\
&= \int_0^\infty \exp(-T(Az_*)^{-\theta}) f(z_*) dz_*.
\end{aligned}
$$

Now,

$$f(z_*) = \frac{d}{dz}\exp(-T^* z^{-\theta}) = \theta T^* z^{-\theta-1}\exp(-T^* z^{-\theta})$$

Substituting into the above integral gives

$$
\begin{aligned}
j &= T^* \int_0^\infty \exp(-T(Az_*)^{-\theta}) \times \theta z_*^{-\theta-1}\exp(-T^* z_*^{-\theta}) dz_* \\
&= T^* \int_0^\infty \exp(-(TA^{-\theta}+T^*)z_*^{-\theta}) \times \theta z_*^{-\theta-1} dz_* \\
&= \frac{T^*}{(TA^{-\theta}+T^*)} \int_0^\infty \exp(-(TA^{-\theta}+T^*)z_*^{-\theta}) \times -\theta z_*^{-\theta-1}(TA^{-\theta}+T^*) dz_* \\
&= \frac{T^*}{(TA^{-\theta}+T^*)} \int_0^\infty \exp(-(TA^{-\theta}+T^*)z_*^{-\theta}) \times \theta z_*^{-\theta-1}(TA^{-\theta}+T^*) dz_* \\
&= \frac{T^*}{(TA^{-\theta}+T^*)},
\end{aligned}
$$

since $\int_0^\infty \exp(-(TA^{-\theta}+T^*)z_*^{-\theta}) \times \theta z_*^{-\theta-1}(TA^{-\theta}+T^*) dz_* = 1$ (because it is the integral of the Frechet pdf with scale $(T^*A^{-\theta}+T)$). Thus, rearranging to get an expression for $A(j)$ gives

$$A(j) = \left[\left(\frac{1-j}{j}\right)\frac{T^*}{T}\right]^{-1/\theta}. \tag{32}$$

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5.2

First note that there are no trade costs so that $d_{ni} = 1$ for all $n, i \in \{F, H\}$.

Now, we know that within a country, goods will be purchased from the lowest cost source. Since Home has a comparative advantage at lower values of j, we know that Home will produce the range of goods $[0, \bar{j}]$ where

$$\frac{w}{z(\bar{j})} = \frac{w^*}{z^*(\bar{j})}.$$

The LHS of the above expression is the unit cost of producing $\bar{j}$ at home, and the RHS is the cost of buying the good from Foreign. Rearranging the above expression gives

$$\frac{z(\bar{j})}{z^*(\bar{j})} = \frac{w}{w^*}$$
$$\implies A(\bar{j}) = \omega, \tag{33}$$

where $\omega = w/w^*$. Similarly, Foreign will produce a range of goods $[\underline{j}, 1]$ domestically, such that

$$\frac{w^*}{z^*(\underline{j})} = \frac{w}{z(\underline{j})}$$
$$\implies A(\underline{j}) = \omega. \tag{34}$$

Thus, there is a unique cuttoff good.

Next, we need to invoke market clearing. Here, we note that preferences are Cobb Douglas, with equal weights across each good. Thus, each country spends a constant share of its income on each good. We know that Home produces $[0, \bar{j}]$ goods domestically, and exports $[0, \underline{j}]$ goods to Foreign. Thus, market clearing at Home requires

$$wL = \bar{j}wL + \underline{j}w^*L^* \tag{35}$$

$$\tag{36}$$

Substituting (33) and (34) into (35) and rearranging gives

$$L = LA^{-1}(\omega) + \frac{1}{\omega}L^*A^{-1}(\omega)$$
$$\implies \frac{1}{A^{-1}(\omega)} = 1 + \frac{1}{\omega}\frac{L^*}{L}$$

And, from the above derivation we know: $A^{-1}(\omega) = \frac{T^*}{(T\omega^{-\theta}+T^*)}$. Substituting this into the above expression gives

$$\frac{T\omega^{-\theta} + T^*}{T^*} = 1 + \frac{1}{\omega}\frac{L^*}{L}$$
$$\implies \frac{T}{T^*}\omega^{-\theta} = \frac{1}{\omega}\frac{L^*}{L}$$
$$\implies \omega = \left[\frac{T^*}{T}\frac{L^*}{L}\right]^{1/(1-\theta)}.$$

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And the equilibrium cutoff good is

$$\bar{j} = A^{-1}(\omega)$$

5.3

With no trade costs goods prices are identical in H and F. Thus, ω measures H's welfare relative to F's. From the above expression for equilibrium ω, if $\theta > 1$, then an increase in T^* reduces relative welfare in H. The comparative static is

$$\frac{d\omega}{dT^*} = \left[\frac{1}{T}\frac{L^*}{L}\right]^{1/(1-\theta)} \frac{1}{1-\theta}(T^*)^{\theta/(1-\theta)}.$$

So, if $\theta > 1$, then the effect of an increase in T^* is decreasing in L^*/L, which is observable in the data.

6 Key implications of EK's Ricardian model

6.1

The unit cost of sending good j from i to n is given by

$$C_{ni}(j) = \frac{c_i}{Z_i(j)} d_{ni} \tag{37}$$

6.2

We want to compute the probability that i will sell good j to n. Since the derivation below is the same for all goods, I suppress the index j.

(a) From (37) note that

$$Z_i = \frac{c_i d_{ni}}{C_{ni}}$$

Now,

$$\Pr[Z_i \le p] = \Pr[c_i d_{ni}/C_{ni} \le p]$$

Thus,

$$
\begin{aligned}
\Pr[Z_i \le c_i d_{ni}/p] &= \Pr[c_i d_{ni}/C_{ni} \le c_i d_{ni}/p] \\
&= \Pr[p \le C_{ni}] \\
&= \Pr[C_{ni} \ge p] \\
&= 1 - \Pr[C_{ni} \le p] \\
\therefore \Pr[C_{ni} \le p] &= 1 - \Pr[Z_i \le c_i d_{ni}/p] \\
&= 1 - F_i(c_i d_{ni}/p) \\
&= 1 - \exp(-T_i(c_i d_{ni})^{-\theta} p^{\theta}),
\end{aligned}
$$

which is the probability distribution of C_{ni}. Denote this distribution as G_{ni} so that

$$G_{ni}(p) = 1 - \exp(-T_i(c_i d_{ni})^{-\theta} p^{\theta}) \tag{38}$$

(b) Next we want to compute the probability that i is the cheapest supplier for n. Denote this probability as π_{ni}. We have

$$
\begin{aligned}
\pi_{ni} \equiv \Pr[C_{ni} = \min\{C_{ns}; s \ne i\}] \\
= \Pr[C_{ns} \ge C_{ni} \text{ for all } s \ne i] \\
= \int_0^{\infty} \prod_{s \ne i} [1 - G_{ns}(C_{ni})] dG_{ni}(C_{ni}) \\
= \int_0^{\infty} \prod_{s \ne i} [1 - G_{ns}(p)] dG_{ni}(p), \tag{39}
\end{aligned}
$$

where I just re-denote the integration dummy as p to ease notation.

(c) Now,

$$dG_{ni}(p) = \frac{d}{dp}G_{ni}(p) = \exp(-T_i(c_id_{ni})^{-\theta}p^{\theta})T_i(c_id_{ni})^{-\theta}\theta p^{\theta-1} \tag{40}$$

Substituting (38) and (40) into (39) gives

$$\begin{aligned}
\pi_{ni} &= \int_0^{\infty}\left[\prod_{s\neq i}\exp(-T_s(c_sd_{ns})^{-\theta}p^{\theta})\right]\exp(-T_i(c_id_{ni})^{-\theta}p^{\theta})T_i(c_id_{ni})^{-\theta}\theta p^{\theta-1}dp \\
&= \int_0^{\infty}\exp(-\sum_{s\neq i}T_s(c_sd_{ns})^{-\theta}p^{\theta})\exp(-T_i(c_id_{ni})^{-\theta}p^{\theta})T_i(c_id_{ni})^{-\theta}\theta p^{\theta-1}dp \\
&= \int_0^{\infty}\exp(-\sum_{i=1}^{N}T_i(c_{ni}d_{ni})^{-\theta}p^{\theta})T_i(c_id_{ni})^{-\theta}\theta p^{\theta-1}dp \\
&= \int_0^{\infty}\exp(-\Phi_np^{\theta})T_i(c_id_{ni})\theta p^{\theta-1}dp \\
&= T_i(c_id_{ni})^{-\theta}\int_0^{\infty}\exp(-\Phi_np^{\theta})\theta p^{\theta-1}dp \\
&= \frac{T_i(c_id_{ni})^{-\theta}}{\Phi_n}\int_0^{\infty}\exp(-\Phi_np^{\theta})\Phi_n\theta p^{\theta-1}dp \\
&= \frac{T_i(c_id_{ni})^{-\theta}}{\Phi_n},
\end{aligned}$$

since the integral in the second last line evaluates to 1, because it is a pdf of a probability distribution of the form (38).

6.3

Next we want to compute the probability distribution of goods prices actually bought in market n. Call this price P_n and recall that n buys the lowest cost good

$$P_n = \min_{i=1,..,N}\{C_{ni}\}$$

Thus

$$\begin{aligned}
\Pr[P_n \leq p] &= \Pr[\min_{i=1,..,N}\{C_{ni}\} \leq p] \\
&= 1 - \Pr[C_{ni} > p \text{ for all } i] \\
&= 1 - \prod_{i=1}^{N}[1 - G_{ni}(p)] \\
&= 1 - \prod_{i=1}^{N}\exp(-T_i(c_id_{ni})^{-\theta}p^{\theta}) \text{ using (38)}, \\
&= 1 - \exp(-\Phi_np^{\theta}).
\end{aligned}$$

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Denote this distribution as $G_n(p)$, so that

$$G_n(p) \equiv \Pr[P_n \le p] = 1 - \exp(-\Phi_n p^\theta) \tag{41}$$

6.4

Next, we want to compute the probability distribution of goods prices that n *actually buys* from country i. That is, we want to compute the conditional probability distribution:

$$\Pr[P_n \le p | P_n = C_{ni}]$$

Now,

$$
\begin{aligned}
Pr[P_n \le p | P_n = C_{ni}] &= \Pr[P_n \le p | C_{ni} = \min\{C_{ns}; s \ne i\}] \\
&= \Pr[C_{ni} \le p | C_{ni} = \min\{C_{ns}; s \ne i\}] \\
&= \frac{1}{\pi_{ni}} \int_0^p \prod_{s \ne i}[1 - G_{ns}(q)] dG_{ni}(q) \\
&= \frac{1}{\pi_{ni}} \pi_{ni} \int_0^p \exp(-\Phi_n q^\theta) \Phi_n \theta q^{\theta-1} dq,
\end{aligned}
$$

using our previous derivations. Thus,

$$
\begin{aligned}
\Pr[P_n \le p | P_n = C_{ni}] &= \int_0^p \exp(-\Phi_n q^\theta) \Phi_n \theta q^{\theta-1} dq \\
&= \int_0^p dG_n(q) \\
&= G_n(p).
\end{aligned}
$$

So, for goods that are purchased in n, conditioning on the source does not affect the distribution of the good's price. This result seems at odds with reality, as we discussed in class. One would think that German cars bought in the US would have a different price distribution compared to Japanese cars bought in the US.

6.5

I'm not exactly sure how to do this problem, but here's my best shot.

(a) Given CES preferences over the unit mass of goods, n's demand for good j is given by

$$
\begin{aligned}
X_n(j) &= \left(\frac{P_n(j)}{P_n}\right)^{1-\sigma} X_n \\
&= \left(\frac{\min_{i=1,..,N}\{C_{ni}(j)\}}{P_n}\right)^{1-\sigma} X_n
\end{aligned}
$$

(b) Then, n's expected expenditure on good j, sourced from i is

$$X_{ni}(j) = \mathbb{E}[X_n(j)|i*(j) = i]\Pr[i^*(j) = i]$$

$$= \mathbb{E}\left[\left(\frac{\min_{i=1,..,N}\{C_{ni}(j)\}}{P_n}\right)^{1-\sigma} X_n | i^*(j) = i\right]\pi_{ni}$$

$$= \mathbb{E}\left[\left(\frac{C_{ni}(j)}{P_n}\right)^{1-\sigma} X_n | i^*(j) = i\right]\pi_{ni}$$

$$= \mathbb{E}\left[\left(\frac{P_n}{P_n}\right)^{1-\sigma} X_n | i^*(j) = i\right]\pi_{ni}$$

$$= \pi_{ni}X_n.$$

6.6

We've shown that π_{ni} is the probability that country n purchases good any good j from i. Since there is a unit measure of goods, it follows that π_{ni} is the total fraction of the $j \in [0,1]$ goods that are sourced from from i in country n. Then, we can split the the total unit measure $j \in [0,1]$ of goods purchased in n into the share supplied by each source country. Recall that conditioning on the source of a good does not affect the distribution of the good's price in n. Accordingly, the fraction of n's total expenditure that goes to country i is the same as the fraction of goods that n purchases from country i; namely, π_{ni}.

7 Quantitative analysis in the EK model

7.1 A peek at the data

See attached `R` code for the aggregation of the WIO data.

Table 1 (overleaf) shows the ratio of imports of intermediate goods to total imports for each country in the WIOD. For each country in the sample, imports of intermediate goods accounts for more than half of total imports; the mean intermediate import share is 0.64. These results imply that it is extremely important to incorporate and intermediate goods sector when modelling aggregate trade flows.

Table 2 presents the ratio of each country's trade deficit (imports minus exports) to total expenditure. For most countries, the trade deficit is tiny compared to total expenditure; the mean trade deficit to expenditure ratio is 0.002. These results suggest that the usual balanced trade assumption is a reasonable one.

Table 1: **Intermediate Goods Share of Total Imports by Country**

	Country	Intermediate trade share
1	AUS	0.55
2	AUT	0.60
3	BEL	0.67
4	BGR	0.67
5	BRA	0.68
6	CAN	0.60
7	CHN	0.75
8	CYP	0.46
9	CZE	0.69
10	DEU	0.57
11	DNK	0.61
12	ESP	0.62
13	EST	0.63
14	FIN	0.71
15	FRA	0.61
16	GBR	0.56
17	GRC	0.59
18	HUN	0.76
19	IDN	0.78
20	IND	0.81
21	IRL	0.76
22	ITA	0.63
23	JPN	0.61
24	KOR	0.76
25	LTU	0.58
26	LUX	0.78
27	LVA	0.48
28	MEX	0.70
29	MLT	0.64
30	NLD	0.68
31	POL	0.64
32	PRT	0.56
33	ROM	0.65
34	RUS	0.53
35	SVK	0.67
36	SVN	0.58
37	SWE	0.67
38	TUR	0.64
39	TWN	0.66
40	USA	0.53
41	RoW	0.70

Table 2: **Ratio of Trade Deficit to Total Expenditure by Country**

	Country	Total Imports ($ b)	Total Exports ($ b)	Total Expenditure ($ b)	Deficit Ratio*	Trade deficit ($ b)
1	AUS	82183	89119	773044	-0.009	-6936
2	AUT	74129	76573	326787	-0.007	-2444
3	BEL	145488	159808	476141	-0.030	-14320
4	BGR	5687	5496	25011	0.008	191
5	BRA	70230	63215	1102372	0.006	7015
6	CAN	262124	318038	1274587	-0.044	-55914
7	CHN	233466	278005	3079150	-0.014	-44539
8	CYP	3720	1846	15698	0.119	1874
9	CZE	34011	31977	140246	0.015	2034
10	DEU	546757	612622	3331636	-0.020	-65865
11	DNK	54834	66476	256046	-0.045	-11642
12	ESP	174061	139171	1136802	0.031	34890
13	EST	3395	3174	12286	0.018	221
14	FIN	37210	50286	221058	-0.059	-13076
15	FRA	327210	348182	2392238	-0.009	-20972
16	GBR	371112	376887	2679956	-0.002	-5775
17	GRC	39250	16839	223624	0.100	22411
18	HUN	31956	28746	106129	0.030	3210
19	IDN	46340	64460	322243	-0.056	-18120
20	IND	65785	66604	900499	-0.001	-819
21	IRL	71357	87495	188948	-0.085	-16138
22	ITA	263034	269745	2128874	-0.003	-6711
23	JPN	392472	511342	8569322	-0.014	-118870
24	KOR	171804	197894	1127198	-0.023	-26090
25	LTU	4512	3723	19312	0.041	789
26	LUX	22195	25249	44725	-0.068	-3054
27	LVA	2702	2540	14573	0.011	162
28	MEX	172069	169818	1088325	0.002	2251
29	MLT	3216	2494	7928	0.091	722
30	NLD	181619	212380	710528	-0.043	-30761
31	POL	53523	44796	341961	0.026	8727
32	PRT	42004	28517	225600	0.060	13487
33	ROM	12402	10830	73977	0.021	1572
34	RUS	48576	97684	396030	-0.124	-49108
35	SVK	12680	12514	48242	0.003	166
36	SVN	9557	8933	39114	0.016	624
37	SWE	88073	108985	443365	-0.047	-20912
38	TUR	57821	41445	518382	0.032	16376
39	TWN	154033	170777	650183	-0.026	-16744
40	USA	1314500	981035	18636780	0.018	333465
41	RoW	1178338	1079715	6694889	0.015	98623

*Note**: the deficit ratio is trade deficit divided by total expenditure.

7.2 Counterfactuals in the EK model

We are interested in how endogenous variables (such as trade shares and prices) respond to changes in exogenous variables (such as trade costs). To determine these responses we DEK's exact hat algebra approach. Denote the counterfactual value of an exogenous variable x as x'. We can represent the change in x as the ratio $\hat{x} = x'/x$. Then, the counterfactual value of x is simply $x' = \hat{x}x$.

7.2.1 Counterfactual trade shares

In the baseline version of the EK model, the counterfactual trade share is

$$
\begin{aligned}
\pi'_{ni} &= \frac{T'_i(c'_i d'_{ni})^{-\theta}}{\sum_{k=1}^{N} T'_k(c'_k d'_{nk})^{-\theta}} \\
&= \frac{\hat{T}_i T_i(\hat{c}_i c_i \hat{d}_{ni} d_{ni})^{-\theta}}{\sum_{k=1}^{N} \hat{T}_k T_k(\hat{c}_k c_k \hat{d}_{nk} d_{nk})^{-\theta}} \\
&= \frac{T_i(c_i d_{ni})^{-\theta} \hat{T}_i(\hat{c}_i \hat{d}_{ni})^{-\theta}}{\sum_{k=1}^{N} T_k(c_k d_{nk})^{-\theta} \hat{T}_k(\hat{c}_k \hat{d}_{nk})^{-\theta}} \\
&= \frac{\frac{1}{\Phi_n} T_i(c_i d_{ni})^{-\theta} \hat{T}_i(\hat{c}_i \hat{d}_{ni})^{-\theta}}{\frac{1}{\Phi_n} \sum_{k=1}^{N} T_k(c_k d_{nk})^{-\theta} \hat{T}_k(\hat{c}_k \hat{d}_{nk})^{-\theta}} \\
&= \frac{\pi_{ni} \hat{T}_i(\hat{c}_i \hat{d}_{ni})^{-\theta}}{\sum_{k=1}^{N} \pi_{nk} \hat{T}_k(\hat{c}_k \hat{d}_{nk})^{-\theta}},
\end{aligned}
$$

which is now a function of observable trade shares, changes in input costs, and changes in exogenous variables. Now, since labor is the only input $c_i = w_i$, so that

$$
\pi'_{ni} = \frac{\pi_{ni} \hat{T}_i(\hat{w}_i \hat{d}_{ni})^{-\theta}}{\sum_{k=1}^{N} \pi_{nk} \hat{T}_k(\hat{w}_k \hat{d}_{nk})^{-\theta}} \tag{42}
$$

7.2.2 Changes in wages

The labor market clearing condition for country i in the counterfactual scenario is

$$
w'_i L'_i = \sum_{n=1}^{N} \pi'_{ni} X'_n
$$

Note that in competitive equilibrium we also must have $w_i L_i = X_i$. Thus, developing the above expression:

$$\hat{w}_i w_i \hat{L}_i L_i = \sum_{n=1}^{N} \pi'_{ni} w'_n L'_n$$

$$\implies \hat{w}_i X_i \hat{L}_i = \sum_{n=1}^{N} \pi'_{ni} \hat{w}_n w_n \hat{L}_n L_n$$

$$= \sum_{n=1}^{N} \pi'_{ni} \hat{w}_n \hat{L}_n X_n$$

$$\therefore \hat{w}_i = \frac{1}{X_i \hat{L}_i} \sum_{n=1}^{N} \pi'_{ni} \hat{w}_n \hat{L}_n X_n \tag{43}$$

7.2.3 Normalization

Note that our goal is to solve (42) and (43) for counterfactual trade shares $\{\pi'_{ni}\}$ and changes in wages $\{\hat{w}_i\}$ for some given changes in exogenous variables $\{\hat{T}_i, \hat{L}_i, \hat{d}_{ni}\}$. To do so, I follow Alvarez and Lucas (2006) and make the normalization $\sum_{i=1}^{N} w_i L_i = 1$. To see how this will help solve the problem, note that this normalization implies

$$\sum_{i=1}^{N} w_i L_i = 1 \implies \sum_{i=1}^{N} w'_i L_i = 1 \implies \sum_{i=1}^{N} \hat{w}_i w_i L_i = 1 \implies \sum_{i=1}^{N} \hat{w}_i X_i = 1.$$

The last equality makes sense: if $\hat{w}_i = 1$ for all i, then counterfactual world GDP doesn't change, and is equal to $\sum_{i=1}^{N} X_i$, which I will normalize to be 1 in the first step of the solution algorithm.

7.2.4 System of excess demands for labor

To solve the system of equations (42) and (43) we need to define some iterative procedure that hones in on the solution. From the counterfactual labor market clearing condition, the "excess wage bill" for country i is

$$Z_i(\hat{w}_i) = \sum_{n=1}^{N} \pi'_{ni} X'_n - w'_i L'_i$$

Using the same algebra as above gives the excess demand function

$$Z_i(\hat{w}_i) = \sum_{n=1}^{N} \pi'_{ni} \hat{w}_n \hat{L}_n X_n - \hat{w}_i X_i \hat{L}_i$$

We're looking for a vector of wage changes such that $Z(\hat{w}_i) = 0$ for all i. Note that in my data it turns out that $X_i \neq \sum_{n=1}^{N} \pi_{ni} X_n$. I think this has something to do with not accounting for value

added in the intermediate goods sector when I constructed my trade shares. Nonetheless, I can account for this discrepancy by redefining the excess demand equation as

$$Z_i(\hat{w}_i) = \sum_{n=1}^{N} \pi'_{ni} \hat{w}_n \hat{L}_n X_n - \hat{w}_i X_i \hat{L}_i - D_i \tag{44}$$

where $D_i = X_i - \sum_{n=1}^{N} \pi_{ni} X_n$.

7.2.5 An iterative solution algorithm

The solution algorithm proceeds as follows for a given set of changes in exogenous parameters:

1. Pick an initial guess for wage changes $\hat{\boldsymbol{w}}^0 = 1$. Normalize each country's GDP's so that they sum to 1; that is compute $\tilde{X}_i = X_i / \sum_{k=1}^{N} X_k$.

2. Compute counterfactual trade shares $\boldsymbol{\pi}'$ using (42).

3. Compute the excess demand vector $\boldsymbol{Z}(\hat{\boldsymbol{w}}^0)$ using (44).

4. Update the predicted wage change according to:

$$\hat{w}^0_{\texttt{update},i} = \hat{w}^0_i (1 + \nu Z(\hat{w}^0_i)/\tilde{X}_i), \text{ with } \nu \in (0, 1].$$

5. Renormalize the wage changes by computing

$$\hat{w}^1_i = \frac{1}{\tilde{X}_i} \frac{\hat{w}^0_{\texttt{update},i} \tilde{X}_i}{\sum_{k=1}^{N} \hat{w}^0_{\texttt{update},k} \tilde{X}_k}$$

6. Iterate until convergence, $\hat{\boldsymbol{w}}^t \approx \hat{\boldsymbol{w}}^{t-1}$ (according to some tolerance level).

Step 4 ensures that wage changes adjust in the correct direction, and step 5 ensures that world GDP sums to 1 at each iteration.

7.3 Simulation results

The table overleaf presents the simulation results for a 10% drop in bilateral trade costs: $\hat{d}_{ni} = 0.9$ for all $n \neq i$. The wage changes are all suspiciously close to 1 which makes me question my implementation of the solution algorithm, but perhaps this has something to do with my normalization. Nonetheless, these results show that wages fall in some countries and rise in others. Using the formula for the change in welfare from the slides:

$$W'/W = (\pi_{ii}/\pi'_{ii})^{1/\theta}$$

shows that all countries benefit from the reduction in trade costs, which thankfully makes sense.

Table 3: **Simulation Results for a 10% Reduction in Trade Costs in the EK Model**

	Country	$\hat{w}_i$	$W'/W = (\pi_{ii}/\pi'_{ii})^{1/\theta}$
1	AUS	1.005	1.017
2	AUT	1.012	1.034
3	BEL	1.020	1.044
4	BGR	1.009	1.034
5	BRA	0.993	1.010
6	CAN	1.011	1.031
7	CHN	1.005	1.012
8	CYP	0.985	1.035
9	CZE	1.011	1.036
10	DEU	1.009	1.025
11	DNK	1.017	1.032
12	ESP	0.997	1.024
13	EST	1.016	1.040
14	FIN	1.017	1.026
15	FRA	1.005	1.021
16	GBR	1.003	1.022
17	GRC	0.978	1.027
18	HUN	1.013	1.043
19	IDN	1.017	1.022
20	IND	0.999	1.012
21	IRL	1.029	1.052
22	ITA	1.003	1.019
23	JPN	1.004	1.007
24	KOR	1.009	1.024
25	LTU	1.007	1.035
26	LUX	1.037	1.066
27	LVA	1.007	1.028
28	MEX	0.999	1.024
29	MLT	1.015	1.056
30	NLD	1.018	1.037
31	POL	1.001	1.024
32	PRT	0.993	1.028
33	ROM	1.001	1.026
34	RUS	1.024	1.019
35	SVK	1.014	1.038
36	SVN	1.010	1.036
37	SWE	1.015	1.030
38	TUR	0.991	1.018
39	TWN	1.015	1.035
40	USA	0.989	1.011
41	RoW	1.003	1.027

8 Gravity

8.1

The gravity specification from AvW's (2003) Armington model (with symmetric trade costs) is given by

$$X_{ni} = \frac{Y_n Y_i}{Y_w} \left(\frac{d_{ni}}{P_n P_i} \right)^{1-\sigma} \tag{45}$$

Taking logs

$$\log X_{ni} = -\log Y_w + \log Y_n + \log Y_i + (1-\sigma)\log d_{ni} - (1-\sigma)P_n - (1-\sigma)P_i$$

Given the parametric form for trade costs, the estimating equation becomes

$$\log \left(\frac{X_{ni}}{Y_n Y_i} \right) = \alpha_0 + \alpha_1(1-\sigma)\log \text{distance}_{ni} + \alpha_2(1-\sigma)\text{no contiguity}_{ni}$$

$$+ \alpha_3(1-\sigma)\text{no common language}_{ni} + \alpha_4(1-\sigma)\text{no colonial ties}_{ni}$$

$$- (1-\sigma)m_n - (1-\sigma)m_i + \nu_{ni}, \tag{46}$$

where m_n and m_i are importer and exporter fixed effects, respectively. Note that AvW impose unit income elasticities by using $\log \left(\frac{X_{ni}}{Y_n Y_i} \right)$ as the dependent variable. Santos-Silva and Tenreyro (SST) (2006) note that this is probably not the best approach (see fn 29), but I'll stick with it anyway. For simplicity, I estimate (46) only for the subsample of observations where $X_{ni} > 0$ (I discuss the implications of this below). Table 1 (column (1)) presents the OLS estimates of (46).

Table 4: **Estimated Gravity Parameters Using Different Estimators**

Estimator	OLS	PPML	PPML
Dependent variable	$\ln(X_{ni}/Y_n Y_i)$	$X_{ni}/Y_n Y_i > 0$	$X_{ni}/Y_n Y_i$
log distance	-1.607	-1.115	-1.409
	(0.0235)	(0.0448)	(0.0412)
no contigity	-0.918	-0.596	-0.398
	(0.103)	(0.110)	(0.111)
no common language	-0.733	-0.805	-0.810
	(0.0515)	(0.103)	(0.116)
no colonial ties	-1.055	-0.894	-0.854
	(0.0960)	(0.160)	(0.169)
Fixed effects	Yes	Yes	Yes
Observations	19997	19997	26967

Notes: Eiker-White standard errors in parentheses

8.2

(a) Clearly a zero trade flow for a country-pair observation poses a problem for the estimation of (46) because the dependent variable would be equal to $-\infty$. As SST (2006) note, most studies

simply drop the zero observations when estimating gravity equations of the form (46), but Head and Meyer (HM) show that this results in selection bias (because zeros account for a large share of total country-pair observations). Another approach is to estimate the model using $X_{ni}+1$ instead of X_{ni}, but SST and HM note that this procedure is bad: it yields inconsistent estimators of the gravity parameters and the results depend on the units of measurement of X_{ni}. Another approach is to assume that the observed X_{ni} is truncated at its lowest observed value and estimate (46) using Tobit regression. SST's proposed PPML method (discussed below) allows for zeros.

(b) As SST (2006) mention, for the estimated parameters in (46) to be unbiased, ν_{ni} must be statistically independent of the regressors.

(c) The estimated coefficient on log distance is $\widehat{\alpha_1(1-\sigma)}$. That is, we cannot separately identify α_1 and $(1-\sigma)$.

(d) Comparing (46) and the equation just above it we can see that the fixed effects parameters are given by:

$$m_i = (1-\sigma)P_i$$
$$m_n = (1-\sigma)P_n$$

P_i and P_n are the ideal price indicies in country i and n, respectively. With $\sigma > 1$ the Armington model predicts that bilateral trade flows should be increasing in P_i and P_n. The idea is simple: if P_n is high, then goods are relatively expensive in the destination country, which makes it easier for i's imports to penetrate the market. Similarly, if P_i is high, then it's hard for i to export its good to all destinations, so it's relatively easier for i to sell in n (I'm not sure about this).

(e) Table 3 also presents the PPML estimates of the gravity equation, following SST (column (2) presents results for the positive trade sample and (3) presents results for the full sample). Consistent with SST, the PPML estimates of the coefficient on log distance are lower than the corresponding OLS estimate.

9 Appendix

9.1 R code for Q7

```
## ECON641: PROBLEM SET 1
## Q7: QUANTITATIVE ANALYSIS IN EK MODEL
## Anirudh Yadav
## 10/18/2018


######################################################################
# Load packages, clear workspace
######################################################################
rm(list = ls())              #clear workspace
library(foreach)             #for looping
library(data.table)          #for data manipulation
library(Matrix)              #fast matrix calcs
library(xtable)
options(scipen = 999)        #forces R to use normal numbers instead of scientific notation


######################################################################
# Construct bilateral intermeidate trade by country pair
######################################################################
data <- read.csv('PhD_Coursework/ECON641/HW1/wiot00_row_apr12.csv',stringsAsFactors=FALSE)

# Remove uneeded rows and columns
data <- data[-c(1,2,3,5,1441:nrow(data)),-c(1,2,4,1440:ncol(data))]

# Get vector of country names
country.names  <- as.character(data[,1])
country.names  <- country.names[2:length(country.names)]
country.unique <- unique(country.names)

# Remove country names from rows
data  <- data[-1,]

# Remove commas, convert to numbers
data  <- apply(data[,-1], 2, function(y) as.numeric(gsub(",", "", y)))
data  <- as.data.frame(data)

# Add country back as a grouping variable
data  <- cbind(country.names,data)


# Add column names
colnames(data) <- c("country",country.names)

# Convert to data table for easy manipulation
dt             <- as.data.table(data)

# Make vector of industry indicies
ind <- c(paste0("c", c(1:35)))

# Rename cols with industry suffixes
colnames(dt) <- c(colnames(dt)[1], paste0(colnames(dt)[2:ncol(dt)], "_", ind))


# Sum columns by country
col.sum <- dt[, lapply(.SD, sum), by = "country"]

# Sum rows, by country
# NOTES:
for (i in 1:length(country.unique)){
  col.sum[, temp := rowSums(.SD), .SDcols = grep(paste0(country.unique[i],"_"), colnames(col.sum))]
  setnames(col.sum, "temp", country.unique[i])
}
```

```
# subset the data table
intermediate.trade = col.sum[,country.unique,with=FALSE]
intermediate.trade[,supplier:=country.unique]

# reorder the cols
setcolorder(intermediate.trade,c("supplier",country.unique))

# Remove large data
rm(data,dt,col.sum)


########################################################################
# Construct bilateral final goods trade by country pair
########################################################################
data <- read.csv('PhD_Coursework/ECON641/HW1/wiot00_row_apr12.csv',stringsAsFactors=FALSE)

# Remove uneeded rows and columns
data <- data[-c(1,2,3,5,1441:nrow(data)),-c(1,2,4:1439,ncol(data))]

# Get vector of country names
country.names.l  <- as.character(data[,1])
country.names.l  <- country.names.l[2:length(country.names.l)]
country.unique   <- unique(country.names.l)

# Get a short vector of country names
country.names.s  <- as.character(data[1,])
country.names.s  <- country.names.s[2:length(country.names.s)]

# Remove country names row
data   <- data[-1,]

# Remove commas, convert to numbers
data   <- apply(data[,-1], 2, function(y) as.numeric(gsub(",", "", y)))
data   <- as.data.frame(data)

# Add country back as a grouping variable
data   <- cbind(country.names.l,data)


# Add column names
colnames(data) <- c("country",country.names.s)

# Convert to data table for easy manipulation
dt             <- as.data.table(data)

# Make vector of final good indicies
ind <- c(paste0("c", c(1:5)))

# Rename cols with industry suffixes
colnames(dt) <- c(colnames(dt)[1], paste0(colnames(dt)[2:ncol(dt)], "_", ind))


# Sum columns by country
col.sum <- dt[, lapply(.SD, sum), by = "country"]

# Sum rows, by country
for (i in 1:length(country.unique)){
  col.sum[, temp := rowSums(.SD), .SDcols = grep(paste0(country.unique[i],"_"), colnames(col.sum))]
  setnames(col.sum, "temp", country.unique[i])
}


# subset the data table
final.trade = col.sum[,country.unique,with=FALSE]
final.trade[,supplier:=country.unique]

# reorder the cols
```

```
setcolorder(final.trade,c("supplier",country.unique))

# Remove large data
rm(data,dt,col.sum)

########################################################################
# Construct total trade flows by country pair
########################################################################
total.trade = intermediate.trade[,-1]+final.trade[,-1]
total.trade[,supplier:=country.unique]
setcolorder(total.trade,c("supplier",country.unique))
total.trade[,total.sp:=colSums(.SD),.SDcols = country.unique]

########################################################################
# Construct ratio of intermediate imports to total imports, by country
########################################################################

N = length(country.unique)

int.share <- function(i){
  out = intermediate.trade[supplier!=country.unique[i],sum(get(country.unique[i]))]/+
    (total.trade[supplier!=country.unique[i],sum(get(country.unique[i]))])
}

intermediate.share = sapply(1:N,function(i) int.share(i))

mean.int.share = mean(intermediate.share)

intermediate.share = as.data.table(round(intermediate.share,2))
intermediate.share[,country:=country.unique]
colnames(intermediate.share)=c("intermediate.share",colnames(intermediate.share)[2])
setcolorder(intermediate.share,c("country","intermediate.share"))


########################################################################
# Construct ratio of trade deficit to total expenditure, by country
########################################################################

total.exp.f <- function(i){
  out = total.trade[supplier==country.unique[i],rowSums(.SD),.SDcols=country.unique[-i]]
}

total.imp.f <- function(i){
  out = total.trade[supplier!=country.unique[i],sum(get(country.unique[i]))]
}

total.exp = sapply(1:N, function(i) total.exp.f(i))
total.imp = sapply(1:N, function(i) total.imp.f(i))
total.sp  = total.trade[,total.sp]

trade.deficits = as.data.table(cbind(total.exp,total.imp,total.sp))
trade.deficits[,deficit:=total.imp-total.exp]
trade.deficits[,ratio:=round(deficit/total.sp,3)]
trade.deficits[,country:=country.unique]
setcolorder(trade.deficits,c("country","total.imp","total.exp","total.sp","ratio"))

mean.def.ratio = trade.deficits[,mean(ratio)]

########################################################################
# Construct bilateral trade shares
########################################################################

# Make matrix of total spending by country
m1=t(matrix(rep(total.trade[,total.sp],2),41,41))

# Get total bilateral trade values as a matrix
m2=as.matrix(total.trade[,country.unique,with=FALSE])
```

```r
# Get bilateral trade shares
bilateral.shares = as.data.table(m2/m1)

# Round shares to 3 decimal places
bilateral.shares.r = round(bilateral.shares,3)

bilateral.shares.r[,supplier:=country.unique]
setcolorder(bilateral.shares.r,c("supplier",country.unique))

# Remove intermediate matricies
rm(m2)


########################################################################
# 7.2 Counterfactuals
########################################################################
theta        = 8.28
T.hat        = 1
d.hat        = 0.9
d.hat.mat    = matrix(d.hat,41,41)+diag(rep(1-d.hat,41))
nu           = 0.2

# Initial guess of no change
w.hat0       = rep(1,N)

# Get vector of data discrepencies (probably due to value added)
D      =     rowSums(total.trade[,-c(1,43)])-total.trade[,total.sp]

# Enforce initial world GDP normalization
X               = total.trade[,total.sp]/sum(total.trade[,total.sp])

# Write function to update guess of w.hat given an initial guess
update.guess    = function(w.hat0,T.hat=1,d.hat=1){

    # Create matrix of initial guesses
    w.hat0.mat   = matrix(w.hat0,41,41)

    # Compute counterfactual trade shares
    pi.numerator   = bilateral.shares*(w.hat0.mat*d.hat.mat)^(-theta)
    pi.denominator = t(matrix(colSums(bilateral.shares*(w.hat0.mat*d.hat.mat)^(-theta)),41,41))
    pi.dash        = pi.numerator/pi.denominator

    # COMPUTE EXCESS DEMAND VECTOR
    Z               = rowSums(pi.dash*t(w.hat0.mat)*m1) - w.hat0*total.trade[,total.sp] - D

    # UPDATE GUESS USING EXCESS DEMAND
    w.hat.update    = w.hat0*(1+nu*Z/total.trade[,total.sp])

    # Renormalize the new guess
    T.w.x           = w.hat.update*X/sum(w.hat.update*X)

    # Compute updated guess
    w.hat1          = T.w.x/X

    return(w.hat1)
}

# LOOOOOOOOOOOOOOP until convergence
tol        = 10^(-4)
maxiter    = 100
converged  = FALSE
iter       = 0

while (converged == FALSE & iter < maxiter){

  w.hat1 = update.guess(w.hat0)

  diff   = w.hat1 - w.hat0
```

```
  converged = all(diff<tol)

  w.hat0 = w.hat1

  iter   = iter + 1
}

# COMPUTE WELFARE CHANGES
w.hat.mat       = matrix(w.hat1,41,41)
pi.numerator    = bilateral.shares*(w.hat0.mat*d.hat.mat)^(-theta)
pi.denominator  = t(matrix(colSums(bilateral.shares*(w.hat0.mat*d.hat.mat)^(-theta)),41,41))
pi.dash         = pi.numerator/pi.denominator

welfare         = (diag(as.matrix(bilateral.shares))/diag(as.matrix(pi.dash)))^(1/theta)

# PUT RESULTS IN A TABLE FOR LATEX
cf.results = as.data.table(cbind(w.hat1,welfare))
cf.results[,Country:=country.unique]
setcolorder(cf.results,c("Country","w.hat1","welfare"))
```

9.2   STATA code for Q8

```
********************************************************************************
* ECON641: PS1
* Q8: GRAVITY
* Anirudh Yadav
* 10/22/2018
********************************************************************************


********************************************************************************
* Preliminaries
********************************************************************************
clear all
set more off

* Set working directory
global dir "/Users/Anirudh/Desktop/GitHub//PhD_Coursework/ECON641/HW1"

********************************************************************************
* Import data create additional covariates
********************************************************************************

* Import gravity dataset
use "$dir/col_regfile09.dta"

* Create depvar as in AvW
gen   y   = flow/(gdp_o*gdp_d)
gen log_y = ln(y)

* Log distance
gen log_d = ln(distw)

* No common language
gen nocomml = 1-comlang_off

* No colonial ties
gen nocol = 1-col_hist

* No contiguity
gen nocontig = 1-contig

* Importer fixed effect
encode iso_d, gen(m_i)

* Exporter fixed effect
encode iso_o, gen(m_e)
```

```
********************************************************************************
* Run OLS for year 2000
********************************************************************************
eststo: reg log_y log_d nocontig nocomml nocol i.m_i i.m_e if log_y!=. & year==2000, robust


********************************************************************************
* Run PPML for year 2000
********************************************************************************

* Positive trade sample
eststo: poisson y log_d nocontig nocomml nocol i.m_i i.m_e if flow>0 & year==2000, robust


* Incl. zeros
eststo: poisson y log_d nocontig nocomml nocol i.m_i i.m_e if flow!=. & year==2000, robust


********************************************************************************
* Make table for tex file
********************************************************************************
esttab using Q8_results.tex, se nostar drop(*m_i *m_e) label replace booktabs
```