ECON675 – Assignment 5

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Contents

1 Many instruments asymptotics

1.1 Some moments

First,

$$\mathbb{E}[\boldsymbol{u}'\boldsymbol{u}/n] = \frac{1}{n}\mathbb{E}[\boldsymbol{u}'\boldsymbol{u}] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[u_i^2] = \sigma_u^2.$$

An analogous derivation shows that $\mathbb{E}[\boldsymbol{v}'\boldsymbol{v}/n] = \sigma_v^2$.

Next,

$$\begin{aligned}\mathbb{E}[\boldsymbol{x}'\boldsymbol{u}/n] = \frac{1}{n}\mathbb{E}[\boldsymbol{x}'\boldsymbol{u}] &= \frac{1}{n}\mathbb{E}[(\boldsymbol{\pi}'\boldsymbol{Z}' + \boldsymbol{v}')\boldsymbol{u}] \\ &= \frac{1}{n}\boldsymbol{\pi}'\boldsymbol{Z}'\mathbb{E}[\boldsymbol{u}] + \frac{1}{n}\mathbb{E}[\boldsymbol{v}'\boldsymbol{u}] \\ &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[v_i u_i] \\ &= \sigma_{uv}^2,\end{aligned}$$

where I used the assumptions that $\boldsymbol{Z}$ and $\boldsymbol{\pi}$ are nonrandom and $\mathbb{E}[\boldsymbol{u}] = \boldsymbol{0}$.

Now,

$$\begin{aligned}\mathbb{E}[\boldsymbol{x}'\boldsymbol{P}\boldsymbol{u}/n] &= \frac{1}{n}\mathbb{E}[(\boldsymbol{\pi}'\boldsymbol{Z}' + \boldsymbol{v}')\boldsymbol{P}\boldsymbol{u}] \\ &= \frac{1}{n}\mathbb{E}[\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{P}\boldsymbol{u}] + \frac{1}{n}\mathbb{E}[\boldsymbol{v}'\boldsymbol{P}\boldsymbol{u}] \\ &= \frac{1}{n}\mathbb{E}[\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{u}] + \frac{1}{n}\mathbb{E}[\boldsymbol{v}'\boldsymbol{P}\boldsymbol{u}] \\ &= \frac{1}{n}\mathbb{E}[\boldsymbol{v}'\boldsymbol{P}\boldsymbol{u}] \\ &= \frac{K}{n}\sigma_{uv}^2\end{aligned}$$

since $\mathbb{E}[v_i u_j] = 0$ for all $i \neq j$ and $\sum_{i=1}^{n} P_{ii} = K$. An analogous derivation proves the last result $\mathbb{E}[\boldsymbol{u}'\boldsymbol{P}\boldsymbol{u}/n] = K/n\sigma_u^2$.

1.2 Some probability limits

First,

$$\begin{aligned}\boldsymbol{x}'\boldsymbol{x}/n &= (\boldsymbol{\pi}'\boldsymbol{Z}' + \boldsymbol{v}')(\boldsymbol{Z}\boldsymbol{\pi} + \boldsymbol{v})/n \\ &= \frac{\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\pi}}{n} + \frac{\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{v}}{n} + \frac{\boldsymbol{v}'\boldsymbol{Z}\boldsymbol{\pi}}{n} + \frac{\boldsymbol{v}'\boldsymbol{v}}{n} \\ &\to_p \mu + \mathbb{E}[\boldsymbol{\pi}'\boldsymbol{z}_i v_i] + \mathbb{E}[\boldsymbol{z}_i'\boldsymbol{\pi} v_i] + \mathbb{E}[v_i^2] \\ &= \mu + \sigma_v^2\end{aligned}$$

Next,

$$\begin{aligned}
\boldsymbol{x}'\boldsymbol{P}\boldsymbol{x}/n &= (\boldsymbol{\pi}'\boldsymbol{Z}' + \boldsymbol{v}')\boldsymbol{P}(\boldsymbol{Z}\boldsymbol{\pi} + \boldsymbol{v})/n \\
&= \frac{\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\pi}}{n} + \frac{\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{v}}{n} + \frac{\boldsymbol{v}'\boldsymbol{Z}\boldsymbol{\pi}}{n} + \frac{\boldsymbol{v}'\boldsymbol{P}\boldsymbol{v}}{n} \\
&\to_p \mu + \rho\sigma_v^2.
\end{aligned}$$

The above convergence result involves a few steps, which I've suppressed for brevity. First, it uses the assumption that $\boldsymbol{Z}$ and $\boldsymbol{\pi}$ are nonrandom. More importantly, it uses the result that

$$\frac{\boldsymbol{v}'\boldsymbol{P}\boldsymbol{v}}{n} \to_p \mathbb{E}[\boldsymbol{v}'\boldsymbol{P}\boldsymbol{v}/n] = \rho\sigma_v^2$$

since $K/n \to \rho$. Note that this is not just a direct application of the WLLN, since we're not dealing with a sum of iid random variables. Rather, you can show that $\mathbb{V}[\boldsymbol{v}'\boldsymbol{P}\boldsymbol{v}/n]$ is bounded in probability (i.e. it goes to zero at some rate), and then use the Markov/Chebyshev inequality to get the desired convergence result. This type of result will be used a lot in the following questions too.

An analogous derivation proves the last result, $\boldsymbol{x}'\boldsymbol{P}\boldsymbol{u}/n \to_p \rho\sigma_u^2$.

1.3 plim of the classical 2SLS estimator

The classical 2SLS estimator is

$$\begin{aligned}
\hat{\beta}_{\text{2SLS}} &= (\boldsymbol{x}'\boldsymbol{P}\boldsymbol{x})^{-1}(\boldsymbol{x}'\boldsymbol{P}\boldsymbol{y}) \\
&= (\boldsymbol{x}'\boldsymbol{P}\boldsymbol{x})^{-1}\boldsymbol{x}'\boldsymbol{P}(\boldsymbol{x}\boldsymbol{\beta} + \boldsymbol{u}) \\
&= \beta + (\boldsymbol{x}'\boldsymbol{P}\boldsymbol{x})^{-1}(\boldsymbol{x}'\boldsymbol{P}\boldsymbol{u}) \\
&= \beta + (\boldsymbol{x}'\boldsymbol{P}\boldsymbol{x}/n)^{-1}(\boldsymbol{x}'\boldsymbol{P}\boldsymbol{u}/n) \\
&\to_p \beta + \frac{\rho\sigma_u^2}{\mu + \rho\sigma_v^2},
\end{aligned}$$

using the CMT and the above results. Thus, $\hat{\beta}_{\text{2SLS}} = \beta + \frac{\rho\sigma_u^2}{\mu+\rho\sigma_v^2} + o_p(1)$.

1.4 plim of the bias-corrected 2SLS estimator

The bias-corrected 2SLS estimator is

$$\begin{aligned}
\hat{\beta}_{\text{2SLS}} &= (\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{x})^{-1}(\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{y}) \\
&= \beta + (\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{x}/n)^{-1}(\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{u}/n)
\end{aligned}$$

Now,

$$\begin{aligned}
\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{u}/n &= \frac{1}{n}(\boldsymbol{\pi}'\boldsymbol{Z}' + \boldsymbol{v}')(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u} \\
&= \frac{\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{u}}{n} - \frac{\frac{K}{n}\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{u}}{n} + \frac{\boldsymbol{v}'\boldsymbol{P}\boldsymbol{u}}{n} - \frac{\frac{K}{n}\boldsymbol{v}'\boldsymbol{u}}{n} \\
&\to_p 0 - 0 + \rho\sigma_{uv}^2 + \rho\sigma_{uv}^2 \\
&= 0.
\end{aligned}$$

Thus, $\hat{\beta}_{\texttt{2SLS}} \to_p \beta$.

1.5 Asymptotic normality of the bias-corrected 2SLS estimator

1.5.1

First note that

$$\begin{aligned}
\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{u} &= (\boldsymbol{\pi}'\boldsymbol{Z}' + \boldsymbol{v}')(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u} \\
&= \boldsymbol{\pi}'\boldsymbol{Z}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u} + \boldsymbol{v}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u} \\
&= \boldsymbol{\pi}'\boldsymbol{Z}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u} + \left(\check{\boldsymbol{v}}' + \frac{\sigma_{uv}^2}{\sigma_u^2}\boldsymbol{u}'\right)(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u} \\
&= \boldsymbol{\pi}'\boldsymbol{Z}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u} + \check{\boldsymbol{v}}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u} + \frac{\sigma_{uv}^2}{\sigma_u^2}\boldsymbol{u}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u},
\end{aligned}$$

as required.

1.5.2

Next, note that

$$\mathbb{E}[\boldsymbol{\pi}'\boldsymbol{Z}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u}] = \boldsymbol{\pi}'\boldsymbol{Z}'\mathbb{E}[\boldsymbol{u}] - \frac{K}{n}\boldsymbol{\pi}'\boldsymbol{Z}'\mathbb{E}[\boldsymbol{u}] = 0,$$

since $\boldsymbol{Z}$ is nonrandom. Accordingly, the CLT implies that

$$\frac{1}{\sqrt{n}}\boldsymbol{\pi}'\boldsymbol{Z}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u} \to_d \mathcal{N}(0, V_1(\rho)),$$

where

$$\begin{aligned}
V_1(\rho) &= \lim_{n\to\infty} \mathbb{V}[1/\sqrt{n}\boldsymbol{\pi}'\boldsymbol{Z}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u}] \\
&= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[\boldsymbol{\pi}'\boldsymbol{Z}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{u}\boldsymbol{u}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{Z}\boldsymbol{\pi}] \\
&= \lim_{n\to\infty} \frac{1}{n}\sigma_u^2\left[\boldsymbol{\pi}'\boldsymbol{Z}'(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)(\boldsymbol{P} - \frac{K}{n}\boldsymbol{I}_n)\boldsymbol{Z}\boldsymbol{\pi}\right] \\
&= \lim_{n\to\infty} \frac{1}{n}\sigma_u^2\left[\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\pi} - 2\frac{K}{n}\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\pi} + \frac{K^2}{n^2}\boldsymbol{Z}\boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\pi}\right] \\
&= \sigma_u^2(1-\rho^2).
\end{aligned}$$

1.5.3

Now,

$$\begin{aligned}
\mathbb{E}[\check{\boldsymbol{v}}'(\boldsymbol{P} - K/n\boldsymbol{I}_n)\boldsymbol{u}] &= \mathbb{E}\left[\left(\boldsymbol{v}' - \frac{\sigma_{uv}^2}{\sigma_u^2}\boldsymbol{u}'\right)\boldsymbol{P}\boldsymbol{u} - \frac{K}{n}\left(\boldsymbol{v}' - \frac{\sigma_{uv}^2}{\sigma_u^2}\boldsymbol{u}'\right)\boldsymbol{u}\right] \\
&= \mathbb{E}[\boldsymbol{v}'\boldsymbol{P}\boldsymbol{u}] - \frac{\sigma_{uv}^2}{\sigma_u^2}\mathbb{E}[\boldsymbol{u}'\boldsymbol{P}\boldsymbol{u}] - \frac{K}{n}\mathbb{E}[\boldsymbol{v}'\boldsymbol{u}] + \frac{K}{n}\frac{\sigma_{uv}^2}{\sigma_u^2}\mathbb{E}[\boldsymbol{u}'\boldsymbol{u}]
\end{aligned}$$

Then, plugging in the results from part 1 gives

$$\mathbb{E}[\check{\boldsymbol{v}}'(\boldsymbol{P} - K/n\boldsymbol{I}_n)\boldsymbol{u}] = K\sigma_{uv}^2 - \frac{\sigma_{uv}^2}{\sigma_u^2}K\sigma_u^2 - \frac{K}{n}\cdot n\sigma_{uv}^2 + \frac{K}{n}\frac{\sigma_{uv}^2}{\sigma_u^2}\cdot n\sigma_u^2 = 0,$$

as required.

To get the convergence result we would do the following. Compute $\mathbb{V}[\check{\boldsymbol{v}}'(\boldsymbol{P} - K/n\boldsymbol{I}_n)\boldsymbol{u}]$. Using the assumption $\mathbb{V}[\boldsymbol{u}|\check{\boldsymbol{v}}] = \sigma_u^2\boldsymbol{I}_n$, it can be shown that

$$\lim_{n\to\infty}\mathbb{V}[\check{\boldsymbol{v}}'(\boldsymbol{P} - K/n\boldsymbol{I}_n)\boldsymbol{u}] = O(K).$$

Then, we can somehow use the Markov inequality to get the desired convergence result.

1.5.4

Analogous derivations to the above question give the desired results.

1.5.5

Now,

$$\begin{aligned}
\mathbb{E}[\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{u}] &= \mathbb{E}[(\boldsymbol{\pi}'\boldsymbol{Z}' + \boldsymbol{v}')(\boldsymbol{P} - K/n\boldsymbol{I}_n)\boldsymbol{u}] \\
&= \mathbb{E}[\boldsymbol{\pi}'\boldsymbol{Z}'(\boldsymbol{P} - K/n\boldsymbol{I}_n)\boldsymbol{u}] + \mathbb{E}[\boldsymbol{v}'(\boldsymbol{P} - K/n\boldsymbol{I}_n)\boldsymbol{u}] \\
&= 0 + \mathbb{E}[\boldsymbol{v}'\boldsymbol{P}\boldsymbol{u}] - K/n\mathbb{E}[\boldsymbol{v}'\boldsymbol{u}] \\
&= K\sigma_{uv}^2 - K/n\cdot n\sigma_{uv}^2 \\
&= 0.
\end{aligned}$$

And

$$\begin{aligned}
\vartheta^2 = \mathbb{V}[\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{u}/\sqrt{n}] &= \frac{1}{n}\mathbb{E}[\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{u}\boldsymbol{u}'\check{\boldsymbol{P}}\boldsymbol{x}] \\
&= \frac{1}{n}\mathbb{E}[\boldsymbol{x}'(\boldsymbol{P} - K/n\boldsymbol{I}_n)\boldsymbol{u}\boldsymbol{u}'(\boldsymbol{P} - K/n\boldsymbol{I}_n)\boldsymbol{x}] \\
&= \frac{1}{n}\mathbb{E}[(\boldsymbol{x}'\boldsymbol{P}\boldsymbol{u} - K/n\boldsymbol{x}'\boldsymbol{u})(\boldsymbol{u}'\boldsymbol{P}\boldsymbol{x} - K/n\boldsymbol{u}'\boldsymbol{x})]
\end{aligned}$$

1.5.6

Note that

$$\sqrt{n}(\hat{\beta}_{\texttt{2SLS}} - \beta) = (\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{x}/n)^{-1}(\frac{1}{\sqrt{n}}\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{u})$$

And we assume that

$$\frac{1}{\sqrt{n}}\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{u} \to_d \mathcal{N}(0, \vartheta^2)$$

Thus,

$$\sqrt{n}(\hat{\beta}_{\texttt{2SLS}} - \beta) \to_d \mathcal{N}(0, \mathbb{E}[\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{x}]^{-1}\vartheta^2\mathbb{E}[\boldsymbol{x}'\check{\boldsymbol{P}}\boldsymbol{x}]^{-1})$$

Intuitively, I think that when $K/n \to \rho = 0$, then the many instruments problem dissipates, so that the bias-corrected 2SLS estimator and the classical 2SLS estimator are asymptotically equivalent.