

# ECON675 – Assignment 5

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# 1 Many instruments asymptotics

## 1.1 Some moments

First,

$$\mathbb{E}[\mathbf{u}'\mathbf{u}/n] = \frac{1}{n}\mathbb{E}[\mathbf{u}'\mathbf{u}] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[u_i^2] = \sigma_u^2.$$

An analogous derivation shows that  $\mathbb{E}[\mathbf{v}'\mathbf{v}/n] = \sigma_v^2$ .

Next,

$$\begin{aligned}\mathbb{E}[\mathbf{x}'\mathbf{u}/n] &= \frac{1}{n}\mathbb{E}[\mathbf{x}'\mathbf{u}] = \frac{1}{n}\mathbb{E}[(\boldsymbol{\pi}'\mathbf{Z}' + \mathbf{v}')\mathbf{u}] \\ &= \frac{1}{n}\boldsymbol{\pi}'\mathbf{Z}'\mathbb{E}[\mathbf{u}] + \frac{1}{n}\mathbb{E}[\mathbf{v}'\mathbf{u}] \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[v_i u_i] \\ &= \sigma_{uv}^2,\end{aligned}$$

where I used the assumptions that  $\mathbf{Z}$  and  $\boldsymbol{\pi}$  are nonrandom and  $\mathbb{E}[\mathbf{u}] = \mathbf{0}$ .

Now,

$$\begin{aligned}\mathbb{E}[\mathbf{x}'\mathbf{P}\mathbf{u}/n] &= \frac{1}{n}\mathbb{E}[(\boldsymbol{\pi}'\mathbf{Z}' + \mathbf{v}')\mathbf{P}\mathbf{u}] \\ &= \frac{1}{n}\mathbb{E}[\boldsymbol{\pi}'\mathbf{Z}'\mathbf{P}\mathbf{u}] + \frac{1}{n}\mathbb{E}[\mathbf{v}'\mathbf{P}\mathbf{u}] \\ &= \frac{1}{n}\mathbb{E}[\boldsymbol{\pi}'\mathbf{Z}'\mathbf{u}] + \frac{1}{n}\mathbb{E}[\mathbf{v}'\mathbf{P}\mathbf{u}] \\ &= \frac{1}{n}\mathbb{E}[\mathbf{v}'\mathbf{P}\mathbf{u}] \\ &= \frac{K}{n}\sigma_{uv}^2\end{aligned}$$

since  $\mathbb{E}[v_i u_j] = 0$  for all  $i \neq j$  and  $\sum_{i=1}^n P_{ii} = K$ . An analogous derivation proves the last result  $\mathbb{E}[\mathbf{u}'\mathbf{P}\mathbf{u}/n] = K/n\sigma_u^2$ .

## 1.2 Some probability limits

First,

$$\begin{aligned}\mathbf{x}'\mathbf{x}/n &= (\boldsymbol{\pi}'\mathbf{Z}' + \mathbf{v}')(\mathbf{Z}\boldsymbol{\pi} + \mathbf{v})/n \\ &= \frac{\boldsymbol{\pi}'\mathbf{Z}'\mathbf{Z}\boldsymbol{\pi}}{n} + \frac{\boldsymbol{\pi}'\mathbf{Z}'\mathbf{v}}{n} + \frac{\mathbf{v}'\mathbf{Z}\boldsymbol{\pi}}{n} + \frac{\mathbf{v}'\mathbf{v}}{n} \\ &\rightarrow_p \mu + \mathbb{E}[\boldsymbol{\pi}'\mathbf{z}_i v_i] + \mathbb{E}[\mathbf{z}_i' \boldsymbol{\pi} v_i] + \mathbb{E}[v_i^2] \\ &= \mu + \sigma_v^2\end{aligned}$$

Next,

$$\begin{aligned}
\mathbf{x}'\mathbf{P}\mathbf{x}/n &= (\boldsymbol{\pi}'\mathbf{Z}' + \mathbf{v}')\mathbf{P}(\mathbf{Z}\boldsymbol{\pi} + \mathbf{v})/n \\
&= \frac{\boldsymbol{\pi}'\mathbf{Z}'\mathbf{Z}\boldsymbol{\pi}}{n} + \frac{\boldsymbol{\pi}'\mathbf{Z}'\mathbf{v}}{n} + \frac{\mathbf{v}'\mathbf{Z}\boldsymbol{\pi}}{n} + \frac{\mathbf{v}'\mathbf{P}\mathbf{v}}{n} \\
&\rightarrow_p \mu + \rho\sigma_v^2.
\end{aligned}$$

The above convergence result involves a few steps, which I've suppressed for brevity. First, it uses the assumption that  $\mathbf{Z}$  and  $\boldsymbol{\pi}$  are nonrandom. More importantly, it uses the result that

$$\frac{\mathbf{v}'\mathbf{P}\mathbf{v}}{n} \rightarrow_p \mathbb{E}[\mathbf{v}'\mathbf{P}\mathbf{v}/n] = \rho\sigma_v^2$$

since  $K/n \rightarrow \rho$ . Note that this is not just a direct application of the WLLN, since we're not dealing with a sum of iid random variables. Rather, you can show that  $\mathbb{V}[\mathbf{v}'\mathbf{P}\mathbf{v}/n]$  is bounded in probability (i.e. it goes to zero at some rate), and then use the Markov/Chebyshev inequality to get the desired convergence result. This type of result will be used a lot in the following questions too.

An analogous derivation proves the last result,  $\mathbf{x}'\mathbf{P}\mathbf{u}/n \rightarrow_p \rho\sigma_u^2$ .

### 1.3 plim of the classical 2SLS estimator

The classical 2SLS estimator is

$$\begin{aligned}
\hat{\beta}_{2SLS} &= (\mathbf{x}'\mathbf{P}\mathbf{x})^{-1}(\mathbf{x}'\mathbf{P}\mathbf{y}) \\
&= (\mathbf{x}'\mathbf{P}\mathbf{x})^{-1}\mathbf{x}'\mathbf{P}(\mathbf{x}\beta + \mathbf{u}) \\
&= \beta + (\mathbf{x}'\mathbf{P}\mathbf{x})^{-1}(\mathbf{x}'\mathbf{P}\mathbf{u}) \\
&= \beta + (\mathbf{x}'\mathbf{P}\mathbf{x}/n)^{-1}(\mathbf{x}'\mathbf{P}\mathbf{u}/n) \\
&\rightarrow_p \beta + \frac{\rho\sigma_u^2}{\mu + \rho\sigma_v^2},
\end{aligned}$$

using the CMT and the above results. Thus,  $\hat{\beta}_{2SLS} = \beta + \frac{\rho\sigma_u^2}{\mu + \rho\sigma_v^2} + o_p(1)$ .

### 1.4 plim of the bias-corrected 2SLS estimator

The bias-corrected 2SLS estimator is

$$\begin{aligned}
\hat{\beta}_{2SLS} &= (\mathbf{x}'\check{\mathbf{P}}\mathbf{x})^{-1}(\mathbf{x}'\check{\mathbf{P}}\mathbf{y}) \\
&= \beta + (\mathbf{x}'\check{\mathbf{P}}\mathbf{x}/n)^{-1}(\mathbf{x}'\check{\mathbf{P}}\mathbf{u}/n)
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbf{x}'\check{\mathbf{P}}\mathbf{u}/n &= \frac{1}{n}(\boldsymbol{\pi}'\mathbf{Z}' + \mathbf{v}')(\mathbf{P} - \frac{K}{n}\mathbf{I}_n)\mathbf{u} \\
&= \frac{\boldsymbol{\pi}'\mathbf{Z}'\mathbf{u}}{n} - \frac{\frac{K}{n}\boldsymbol{\pi}'\mathbf{Z}'\mathbf{u}}{n} + \frac{\mathbf{v}'\mathbf{P}\mathbf{u}}{n} - \frac{\frac{K}{n}\mathbf{v}'\mathbf{u}}{n} \\
&\rightarrow_p 0 - 0 + \rho\sigma_{uv}^2 + \rho\sigma_{uv}^2 \\
&= 0.
\end{aligned}$$

Thus,  $\hat{\beta}_{2SLS} \rightarrow_p \beta$ .

## 1.5 Asymptotic normality of the bias-corrected 2SLS estimator

### 1.5.1

First note that

$$\begin{aligned}
x' \tilde{P} u &= (\pi' Z' + v')(P - \frac{K}{n} I_n) u \\
&= \pi' Z' (P - \frac{K}{n} I_n) u + v' (P - \frac{K}{n} I_n) u \\
&= \pi' Z' (P - \frac{K}{n} I_n) u + \left( \tilde{v}' + \frac{\sigma_{uv}^2}{\sigma_u^2} u' \right) (P - \frac{K}{n} I_n) u \\
&= \pi' Z' (P - \frac{K}{n} I_n) u + \tilde{v}' (P - \frac{K}{n} I_n) u + \frac{\sigma_{uv}^2}{\sigma_u^2} u' (P - \frac{K}{n} I_n) u,
\end{aligned}$$

as required.

### 1.5.2

Next, note that

$$\mathbb{E}[\pi' Z' (P - \frac{K}{n} I_n) u] = \pi' Z' \mathbb{E}[u] - \frac{K}{n} \pi' Z' \mathbb{E}[u] = 0,$$

since  $Z$  is nonrandom. Accordingly, the CLT implies that

$$\frac{1}{\sqrt{n}} \pi' Z' (P - \frac{K}{n} I_n) u \rightarrow_d \mathcal{N}(0, V_1(\rho)),$$

where

$$\begin{aligned}
V_1(\rho) &= \lim_{n \rightarrow \infty} \mathbb{V}[1/\sqrt{n} \pi' Z' (P - \frac{K}{n} I_n) u] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\pi' Z' (P - \frac{K}{n} I_n) u u' (P - \frac{K}{n} I_n) Z \pi] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \sigma_u^2 \left[ \pi' Z' (P - \frac{K}{n} I_n) (P - \frac{K}{n} I_n) Z \pi \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \sigma_u^2 \left[ \pi' Z' Z \pi - 2 \frac{K}{n} \pi' Z' Z \pi + \frac{K^2}{n^2} Z \pi' Z' Z \pi \right] \\
&= \sigma_u^2 (1 - \rho^2).
\end{aligned}$$

### 1.5.3

Now,

$$\begin{aligned}
\mathbb{E}[\tilde{v}' (P - K/n I_n) u] &= \mathbb{E} \left[ \left( v' - \frac{\sigma_{uv}^2}{\sigma_u^2} u' \right) P u - \frac{K}{n} \left( v' - \frac{\sigma_{uv}^2}{\sigma_u^2} u' \right) u \right] \\
&= \mathbb{E}[v' P u] - \frac{\sigma_{uv}^2}{\sigma_u^2} \mathbb{E}[u' P u] - \frac{K}{n} \mathbb{E}[v' u] + \frac{K}{n} \frac{\sigma_{uv}^2}{\sigma_u^2} \mathbb{E}[u' u]
\end{aligned}$$

Then, plugging in the results from part 1 gives

$$\mathbb{E}[\check{\mathbf{v}}'(\mathbf{P} - K/n\mathbf{I}_n)\mathbf{u}] = K\sigma_{uv}^2 - \frac{\sigma_{uv}^2}{\sigma_u^2}K\sigma_u^2 - \frac{K}{n} \cdot n\sigma_{uv}^2 + \frac{K}{n} \frac{\sigma_{uv}^2}{\sigma_u^2} \cdot n\sigma_u^2 = 0,$$

as required.

To get the convergence result we would do the following. Compute  $\mathbb{V}[\check{\mathbf{v}}'(\mathbf{P} - K/n\mathbf{I}_n)\mathbf{u}]$ . Using the assumption  $\mathbb{V}[\mathbf{u}|\check{\mathbf{v}}] = \sigma_u^2\mathbf{I}_n$ , it can be shown that

$$\lim_{n \rightarrow \infty} \mathbb{V}[\check{\mathbf{v}}'(\mathbf{P} - K/n\mathbf{I}_n)\mathbf{u}] = O(K).$$

Then, we can somehow use the Markov inequality to get the desired convergence result.

#### 1.5.4

Analogous derivations to the above question give the desired results.

#### 1.5.5

Now,

$$\begin{aligned} \mathbb{E}[\mathbf{x}'\check{\mathbf{P}}\mathbf{u}] &= \mathbb{E}[(\boldsymbol{\pi}'\mathbf{Z}' + \mathbf{v}')(\mathbf{P} - K/n\mathbf{I}_n)\mathbf{u}] \\ &= \mathbb{E}[\boldsymbol{\pi}'\mathbf{Z}'(\mathbf{P} - K/n\mathbf{I}_n)\mathbf{u}] + \mathbb{E}[\mathbf{v}'(\mathbf{P} - K/n\mathbf{I}_n)\mathbf{u}] \\ &= 0 + \mathbb{E}[\mathbf{v}'\mathbf{P}\mathbf{u}] - K/n\mathbb{E}[\mathbf{v}'\mathbf{u}] \\ &= K\sigma_{uv}^2 - K/n \cdot n\sigma_{uv}^2 \\ &= 0. \end{aligned}$$

And

$$\begin{aligned} \vartheta^2 &= \mathbb{V}[\mathbf{x}'\check{\mathbf{P}}\mathbf{u}/\sqrt{n}] = \frac{1}{n}\mathbb{E}[\mathbf{x}'\check{\mathbf{P}}\mathbf{u}\mathbf{u}'\check{\mathbf{P}}\mathbf{x}] \\ &= \frac{1}{n}\mathbb{E}[\mathbf{x}'(\mathbf{P} - K/n\mathbf{I}_n)\mathbf{u}\mathbf{u}'(\mathbf{P} - K/n\mathbf{I}_n)\mathbf{x}] \\ &= \frac{1}{n}\mathbb{E}[(\mathbf{x}'\mathbf{P}\mathbf{u} - K/n\mathbf{x}'\mathbf{u})(\mathbf{u}'\mathbf{P}\mathbf{x} - K/n\mathbf{u}'\mathbf{x})] \end{aligned}$$

#### 1.5.6

Note that

$$\sqrt{n}(\hat{\beta}_{2\text{SLS}} - \beta) = (\mathbf{x}'\check{\mathbf{P}}\mathbf{x}/n)^{-1}(\frac{1}{\sqrt{n}}\mathbf{x}'\check{\mathbf{P}}\mathbf{u})$$

And we assume that

$$\frac{1}{\sqrt{n}}\mathbf{x}'\check{\mathbf{P}}\mathbf{u} \rightarrow_d \mathcal{N}(0, \vartheta^2)$$

Thus,

$$\sqrt{n}(\hat{\beta}_{2\text{SLS}} - \beta) \rightarrow_d \mathcal{N}(0, \mathbb{E}[\mathbf{x}'\check{\mathbf{P}}\mathbf{x}]^{-1}\vartheta^2\mathbb{E}[\mathbf{x}'\check{\mathbf{P}}\mathbf{x}]^{-1})$$

Intuitively, I think that when  $K/n \rightarrow \rho = 0$ , then the many instruments problem dissipates, so that the bias-corrected 2SLS estimator and the classical 2SLS estimator are asymptotically equivalent.

## 2 Weak instruments – simulations

Table 1 (overleaf) presents the simulation results, which are a nice illustration of the weak instruments problem. The following results are worth noting:

- For all values of  $\gamma$ , the OLS estimators of  $\beta$  are very bad: recall that the true value of  $\beta$  is zero and for literally every one of the 20,000 OLS regressions, we reject the null that  $\beta = 0$  at the 95% level! This is unsurprising, given that  $x_i$  is endogenous.
- For lower values of  $\gamma$  (i.e. when  $z_i$  is a “weak” instrument for  $x_i$ ) the 2SLS estimators of  $\beta$  are also very bad. For higher values of  $\gamma$ , the 2SLS estimator is clearly consistent for  $\beta$ ; and for  $\gamma = \sqrt{99/n}$ , the 2SLS estimator is very precise.

Table 1: Weak Instrument Summary Statistics

(a) $\gamma^2 = 0/n$ ( $F \approx 1$ )						(b) $\gamma^2 = 0.25/n$ ( $F \approx 1.25$ )					
	mean	st.dev.	quantiles				mean	st.dev.	quantiles		
			0.1	0.5	0.9				0.1	0.5	0.9
<b>OLS</b>						<b>OLS</b>					
$\hat{\beta}$	0.99	0.010	0.977	0.99	1.003	$\hat{\beta}$	0.989	0.010	0.975	0.989	1.002
$SE(\hat{\beta})$	0.01	0.001	0.009	0.01	0.011	$SE(\hat{\beta})$	0.010	0.001	0.009	0.010	0.011
$\mathbf{1}_{\text{rej}}$	1.00	0.000	1.000	1.00	1.000	$\mathbf{1}_{\text{rej}}$	1.00	0.000	1.000	1.00	1.000
<b>2SLS</b>						<b>2SLS</b>					
$\hat{\beta}$	0.985	0.491	0.798	0.991	1.175	$\hat{\beta}$	0.875	1.207	0.411	0.826	1.392
$SE(\hat{\beta})$	1.153	40.031	0.070	0.144	0.610	$SE(\hat{\beta})$	3.506	105.963	0.085	0.228	1.328
$\mathbf{1}_{\text{rej}}$	0.873	0.333	0.000	1.000	1.000	$\mathbf{1}_{\text{rej}}$	0.697	0.460	0.000	1.000	1.000
$\hat{F}$	1.027	1.442	0.017	0.468	2.805	$\hat{F}$	1.273	1.770	0.023	0.584	3.467
(c) $\gamma^2 = 9/n$ ( $F \approx 10$ )						(d) $\gamma^2 = 99/n$ ( $F \approx 100$ )					
	mean	st.dev.	quantiles				mean	st.dev.	quantiles		
			0.1	0.5	0.9				0.1	0.5	0.9
<b>OLS</b>						<b>OLS</b>					
$\hat{\beta}$	0.947	0.018	0.924	0.947	0.970	$\hat{\beta}$	0.662	0.035	0.619	0.661	0.707
$SE(\hat{\beta})$	0.017	0.001	0.016	0.017	0.019	$SE(\hat{\beta})$	0.034	0.002	0.031	0.034	0.037
$\mathbf{1}_{\text{rej}}$	1.00	0.000	1.000	1.00	1.000	$\mathbf{1}_{\text{rej}}$	1.00	0.000	1.000	1.00	1.000
<b>2SLS</b>						<b>2SLS</b>					
$\hat{\beta}$	-0.002	0.614	-0.440	0.104	0.363	$\hat{\beta}$	-0.002	0.104	-0.140	0.009	0.121
$SE(\hat{\beta})$	0.502	2.888	0.147	0.283	0.743	$SE(\hat{\beta})$	0.103	0.023	0.077	0.099	0.134
$\mathbf{1}_{\text{rej}}$	0.150	0.357	0.000	0.000	1.000	$\mathbf{1}_{\text{rej}}$	0.057	0.232	0.000	0.000	0.000
$\hat{F}$	10.073	6.470	2.760	9.004	19.087	$\hat{F}$	100.842	25.222	70.239	98.868	133.923

### **3 Weak instruments – empirical studies**

**3.1 Angrist and Krueger (1991)**

**3.2 Bound, Jaeger and Baker (1995)**