

ECON675: Assignment 2

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1 Question 1: Kernel Density Estimation

1.1 Density derivatives

I follow the derivation in Hansen's notes. We are interested in estimating

$$f^{(s)}(x) = \frac{d^s}{dx^s} f(x).$$

The natural estimator is

$$\hat{f}^{(s)}(x) = \frac{d^s}{dx^s} \hat{f}(x)$$

Now, we know that $\hat{f}(x) = \frac{1}{nh} \sum_i K\left(\frac{X_i - x}{h}\right)$. Thus,

$$\begin{aligned}\hat{f}^{(1)}(x) &= \frac{-1}{nh^2} \sum_{i=1}^n K^{(1)}\left(\frac{X_i - x}{h}\right), \\ \hat{f}^{(2)}(x) &= \frac{1}{nh^3} \sum_{i=1}^n K^{(2)}\left(\frac{X_i - x}{h}\right), \\ &\vdots \\ \hat{f}^{(s)}(x) &= \frac{(-1)^s}{nh^{1+s}} \sum_{i=1}^n K^{(s)}\left(\frac{X_i - x}{h}\right).\end{aligned}$$

Now,

$$\begin{aligned}\mathbb{E}[\hat{f}^{(s)}(x)] &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[\frac{(-1)^s}{h^{1+s}} K^{(s)}\left(\frac{X_i - x}{h}\right)\right] \\ &= \mathbb{E}\left[\frac{(-1)^s}{h^{1+s}} K^{(s)}\left(\frac{X_i - x}{h}\right)\right], \text{ since } X_i \text{ are iid.} \\ &= \int_{-\infty}^{\infty} \frac{(-1)^s}{h^{1+s}} K^{(s)}\left(\frac{z - x}{h}\right) f(z) dz\end{aligned}$$

Next, we want to use integration by parts: $\int u dv = uv - \int v du$. Define

$$dv = \frac{(-1)^s}{h^s} \frac{1}{h} K^{(s)} \left(\frac{z-x}{h} \right) \implies v = \frac{(-1)^s}{h^s} K^{(s-1)} \left(\frac{z-x}{h} \right)$$

And

$$u = f(z) \implies du = f^{(1)}(z) dz.$$

Thus,

$$\begin{aligned} \mathbb{E}[\hat{f}^{(s)}(x)] &= \left[\frac{(-1)^s}{h^s} K^{(s-1)} \left(\frac{z-x}{h} \right) f^{(1)}(z) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{(-1)^s}{h^s} K^{(s-1)} \left(\frac{z-x}{h} \right) f^{(1)}(z) dz \\ &= - \int_{-\infty}^{\infty} \frac{(-1)^s}{h^s} K^{(s-1)} \left(\frac{z-x}{h} \right) f^{(1)}(z) dz \end{aligned}$$

Repeating this s times give

$$\begin{aligned} \mathbb{E}[\hat{f}^{(s)}(x)] &= (-1)^s \int_{-\infty}^{\infty} \frac{(-1)^s}{h} K \left(\frac{z-x}{h} \right) f^{(s)}(z) dz \\ &= \int_{-\infty}^{\infty} \frac{1}{h} K \left(\frac{z-x}{h} \right) f^{(s)}(z) dz \end{aligned}$$

Next, use the following change of variables: $u = \frac{z-x}{h}$, which implies $z = x + hu \implies dz = h du$. Thus,

$$\mathbb{E}[\hat{f}^{(s)}(x)] = \int_{-\infty}^{\infty} K(u) f^{(s)}(x + hu) du \quad (1)$$

The next step is to take a Taylor expansion of $f^{(s)}(x + hu)$ around $x + hu = x$, which is valid if $h \rightarrow 0$. We get

$$f^{(s)}(x + hu) = f^{(s)}(x) + f^{(s+1)}(x)hu + \frac{1}{2}f^{(s+2)}(x)h^2u^2 + \dots + \frac{1}{P!}f^{(s+P)}(x)h^Pu^P + o(h^P).$$

Substituting this expression back into (1), integrating over each term, and using the fact that $\int_{-\infty}^{\infty} K(u) du = 1$ and the notation

$$\mu_\ell(K) = \int_{-\infty}^{\infty} u^\ell K(u) du$$

gives

$$\mathbb{E}[\hat{f}^{(s)}(x)] = f^{(s)}(x) + f^{(s+1)}(x)h\mu_1(K) + \frac{1}{2}f^{(s+2)}(x)h^2\mu_2(K) + \dots + \frac{1}{P!}f^{(s+P)}(x)h^P\mu_P(K) + o(h^P).$$

Finally, noting that since K is a P -order kernel, $\mu_\ell(K) = 0$ for all $\ell < P$, gives the desired result

$$\mathbb{E}[\hat{f}^{(s)}(x)] = f^{(s)}(x) + \frac{1}{P!}f^{(s+P)}(x)h^P\mu_P(K) + o(h^P). \quad (2)$$

Next we consider the variance of the derivative estimator.

$$\begin{aligned}\mathbb{V}[\hat{f}^{(s)}(x)] &= \mathbb{V}\left[\frac{(-1)^s}{nh^{1+s}} \sum_{i=1}^n K^{(s)}\left(\frac{X_i - x}{h}\right)\right] \\ &= \frac{1}{nh^{2+2s}} \mathbb{V}\left[K^{(s)}\left(\frac{X_i - x}{h}\right)\right],\end{aligned}$$

since $\{X_i\}$ are iid there are no covariance terms and each term has the same variance. Continuing,

$$\begin{aligned}\mathbb{V}[\hat{f}^{(s)}(x)] &= \frac{1}{nh^{2+2s}} \left\{ \mathbb{E}\left[K^{(s)}\left(\frac{X_i - x}{h}\right)^2\right] - \mathbb{E}\left[K^{(s)}\left(\frac{X_i - x}{h}\right)\right]^2 \right\} \\ &= \frac{1}{nh^{2+2s}} \mathbb{E}\left[K^{(s)}\left(\frac{X_i - x}{h}\right)^2\right] - \frac{1}{n} \mathbb{E}\left[\frac{1}{h^{1+s}} K^{(s)}\left(\frac{X_i - x}{h}\right)\right]^2 \\ &= \frac{1}{nh^{2+2s}} \mathbb{E}\left[K^{(s)}\left(\frac{X_i - x}{h}\right)^2\right] + o\left(\frac{1}{nh^{1+2s}}\right),\end{aligned}$$

since from above we know that the second term ultimately negligible compared to $\frac{1}{nh^{1+2s}}$ as $h \rightarrow \infty$. Thus,

$$\mathbb{V}[\hat{f}^{(s)}(x)] = \frac{1}{nh^{1+2s}} \int_{-\infty}^{\infty} \frac{1}{h} K^{(s)}\left(\frac{z - x}{h}\right)^2 f(z) dz + o\left(\frac{1}{nh^{1+2s}}\right)$$

Again we use the change of variables $u = \frac{z-x}{h}$ so that

$$\mathbb{V}[\hat{f}^{(s)}(x)] = \frac{1}{nh^{1+2s}} \int_{-\infty}^{\infty} K^{(s)}(u)^2 f(x + hu) du + o\left(\frac{1}{nh^{1+2s}}\right)$$