ECON675: Assignment 2

Anirudh Yadav

October 1, 2018

1 Question 1: Kernel Density Estimation

1.1 Density derivatives

I follow the derivation in Hansen's notes. We are interested in estimating

$$f^{(s)}(x) = \frac{d^s}{dx^s} f(x).$$

The natural estimator is

$$\hat{f}^{(s)}(x) = \frac{d^s}{dx^s} \hat{f}(x)$$

Now, we know that $\hat{f}(x) = \frac{1}{nh} \sum_i K\left(\frac{X_i - x}{h}\right)$. Thus,

$$\begin{aligned}
\hat{f}^{(1)}(x) &= \frac{-1}{nh^2} \sum_{i=1}^{n} K^{(1)}\left(\frac{X_i - x}{h}\right), \\
\hat{f}^{(2)}(x) &= \frac{1}{nh^3} \sum_{i=1}^{n} K^{(2)}\left(\frac{X_i - x}{h}\right), \\
&\vdots \\
\hat{f}^{(s)}(x) &= \frac{(-1)^s}{nh^{1+s}} \sum_{i=1}^{n} K^{(s)}\left(\frac{X_i - x}{h}\right).
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{E}[\hat{f}^{(s)}(x)] &= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\frac{(-1)^s}{h^{1+s}} K^{(s)}\left(\frac{X_i - x}{h}\right)\right] \\
&= \mathbb{E}\left[\frac{(-1)^s}{h^{1+s}} K^{(s)}\left(\frac{X_i - x}{h}\right)\right], \text{ since } X_i \text{ are iid.} \\
&= \int_{-\infty}^{\infty} \frac{(-1)^s}{h^{1+s}} K^{(s)}\left(\frac{z - x}{h}\right) f(z) dz
\end{aligned}$$

Next, we want to use integration by parts: $\int u dv = uv - \int v du$. Define

$$dv = \frac{(-1)^s}{h^s}\frac{1}{h}K^{(s)}\left(\frac{z-x}{h}\right) \implies v = \frac{(-1)^s}{h^s}K^{(s-1)}\left(\frac{z-x}{h}\right)$$

And

$$u = f(z) \implies du = f^{(1)}(z).$$

Thus,

$$\begin{aligned}\mathbb{E}[\hat{f}^{(s)}(x)] &= \left[\frac{(-1)^s}{h^s}K^{(s-1)}\left(\frac{z-x}{h}\right)f^{(1)}(z)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{(-1)^s}{h^s}K^{(s-1)}\left(\frac{z-x}{h}\right)f^{(1)}(z)dz. \\ &= -\int_{-\infty}^{\infty}\frac{(-1)^s}{h^s}K^{(s-1)}\left(\frac{z-x}{h}\right)f^{(1)}(z)dz\end{aligned}$$

Repeating this s times give

$$\begin{aligned}\mathbb{E}[\hat{f}^{(s)}(x)] &= (-1)^s\int_{-\infty}^{\infty}\frac{(-1)^s}{h}K\left(\frac{z-x}{h}\right)f^{(s)}(z)dz \\ &= \int_{-\infty}^{\infty}\frac{1}{h}K\left(\frac{z-x}{h}\right)f^{(s)}(z)dz\end{aligned}$$

Next, use the following change of variables: $u = \frac{z-x}{h}$, which implies $z = x + hu \implies dz = hdu$. Thus,

$$\mathbb{E}[\hat{f}^{(s)}(x)] = \int_{-\infty}^{\infty}K(u)f^{(s)}(x+hu)du \tag{1}$$

The next step is to take a Taylor expansion of $f^{(s)}(x+hu)$ around $x+hu = x$, which is valid if $h \to 0$. We get

$$f^{(s)}(x+hu) = f^{(s)}(x) + f^{(s+1)}(x)hu + \frac{1}{2}f^{(s+2)}(x)h^2u^2 + ... + \frac{1}{P!}f^{(s+P)}(x)h^Pu^P + o(h^P).$$

Substituting this expression back into (1), integrating over each term, and using the fact that $\int_{-\infty}^{\infty}K(u)du = 1$ and the notation

$$\mu_\ell(K) = u^\ell K(u)$$

gives

$$\mathbb{E}[\hat{f}^{(s)}(x)] = f^{(s)}(x) + f^{(s+1)}(x)h\mu_1(K) + \frac{1}{2}f^{(s+2)}(x)h^2\mu_2(K) + ... + \frac{1}{P!}f^{(s+P)}(x)h^P\mu_P(K) + o(h^P).$$

Finally, noting that since K is a P-order kernel, $\mu_\ell(K) = 0$ for all $\ell < P$, gives the desired result

$$\mathbb{E}[\hat{f}^{(s)}(x)] = f^{(s)}(x) + \frac{1}{P!}f^{(s+P)}(x)h^P\mu_P(K) + o(h^P).$$