ECON641 – Problem Set 1

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Contents

1 Warmup: factor intensity reversals

First, I outline the small open economy environment of the 2×2 HO model (for my own purposes).

- Two goods, 1 and 2.
- Two factors, L and K; with endogenous factor prices w and r, respectively.
- Production technology is the same in both industries, but they may differ in their relative factor intensities.
- Exogenously given goods prices, p_1 and p_2 (i.e. the demand side of the economy is pinned down).

Roughly speaking, 'no factor intensity reversals' (NFIR) means the following: for any vector of factor prices (w, r), the ordering of relative factor intensities in both industries is always the same. For example, in equilibrium the production of good 1 may be more capital intensive than production of good 2; NFIR implies that at any other vector of factor prices, the production of good 1 must always be more capital intensive compared to good 2. We can show that production technology exhibits NFIR if, given p_1 and p_2, equilibrium factor prices are uniquely pinned down.

1.1 Cobb Douglas

Cobb Douglas production clearly satisfies NFIR. To see this, suppose that $F_1(K_1, L_1) = AK_1^{\alpha}L_1^{1-\alpha}$ and $F_2(K_2, L_2) = AK_2^{\beta}L_2^{1-\beta}$. The first order conditions for the profit maximization problem for industry 1 are standard:

$$p_1 \alpha A K_1^{\alpha-1} L_1^{1-\alpha} = r, \tag{1}$$
$$p_1(1-\alpha) A K_1^{\alpha} L_1^{-\alpha} = w. \tag{2}$$

Dividing (2) by (1) gives

$$\frac{1-\alpha}{\alpha} k_1 = \frac{w}{r} \implies k_1 = \frac{\alpha}{1-\alpha}\frac{w}{r}, \text{ where } k_1 = K_1/L_1 \tag{3}$$

Now, the zero profit condition in industry 1 is

$$\begin{aligned} rK_1 + wL_1 &= p_1 A K_1^{\alpha} L_1^{1-\alpha} \\ \implies rk_1 + w &= p_1 A k_1^{\alpha} \end{aligned} \tag{4}$$

Plugging (3) into (4) and rearranging gives

$$p_1 = C_\alpha r^\alpha w^{1-\alpha} \tag{5}$$

where $C_\alpha = \frac{1}{A(1-\alpha)}\left(\frac{1-\alpha}{\alpha}\right)^\alpha$. An analogous derivation for industry 2 gives

$$p_2 = C_\beta r^\beta w^{1-\beta} \tag{6}$$

where $C_\beta = \frac{1}{A(1-\beta)}\left(\frac{1-\beta}{\beta}\right)^\beta$. Clearly, given p_1 and p_2, there is a unique solution to (5) and (6), (w^*, r^*), (unless $\alpha = \beta$).

Another (perhaps more intuitive) way to establish NFIR would be to use equation (3) and the equivalent expression for industry 2. These expressions imply that in equilibrium:

$$\frac{k_1}{k_2} = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}.$$

That is, the relative factor intensities between the two industries is independent of factor prices.

1.2 CES

CES production *does not* exhibit NFIR. To see this, suppose $F_i(K_i, L_i) = \left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}}\right]^{\frac{\sigma_i}{\sigma_i-1}}$ for $i = 1, 2$. The FOCs for industry i are

$$p_i\left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}}\right]^{\frac{1}{\sigma_i-1}} K_i^{-1/\sigma_i} = r \tag{7}$$

$$p_i\left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}}\right]^{\frac{1}{\sigma_i-1}} L_i^{-1/\sigma_i} = w \tag{8}$$

Combining these expressions gives

$$\begin{aligned} k_i^{-1/\sigma_i} &= \frac{r}{w} \\ \implies k_i &= \left(\frac{r}{w}\right)^{-\sigma_i}. \end{aligned}$$

Thus, in equilibrium, the relative factor intensities between the two industries is

$$\frac{k_1}{k_2} = \left(\frac{r}{w}\right)^{\sigma_2-\sigma_1},$$

which clearly depends on factor prices (unless $\sigma_1 = \sigma_2$).

1.3 Leontief

Clearly the Leontief production function exhibits NFIR. Suppose both industries have the same production function $F(K, L) = \min\{K, L\}$. Then in equilibrium, both industries must have $k_i = 1$. Then, relative factor intensities do not depend on factor prices. More generally, suppose $F_i(K_i, L_i) = \min\{\alpha_i K, \beta_i L\}$. Then in equilibrium, each industry's capital-labor ratio will be $k_i = \beta_i/\alpha_i$. Again, relative factor intensities are independent of factor prices.

2 $2 \times 2 \times 2$ HO Model

3 Technology growth in a parameterized version of DFS

3.1

I follow the derivation in EK (2005). We are given the distribution of efficiencies for producing goods j at Home and Foreign:

$$F_i(z) = \Pr[Z_i(j) \leq z] = \exp(-T_i z^{-\theta})$$

Now, we want to derive the DFS-type $A(j)$ curve, which is defined as the ratio of H's efficiency of producing j to F's corresponding efficiency.

In the EK setup the efficiencies are realizations of a random variable. Accordingly, we think of j as the *probability* that the H's relative efficiency of producing j is less than some number:

$$\begin{aligned} j &= \Pr\left[\frac{Z}{Z^*} \leq A\right] \\ &= \Pr\left[Z \leq AZ^*\right] \\ &= \int_0^\infty \exp(-T(Az_*)^{-\theta}) f(z_*) dz_*. \end{aligned}$$

Now,

$$f(z_*) = \frac{d}{dz}\exp(-T^* z^{-\theta}) = \theta T^* z^{-\theta-1}\exp(-T^* z^{-\theta})$$

Substituting into the above integral gives

$$\begin{aligned} j &= T^* \int_0^\infty \exp(-T(Az_*)^{-\theta}) \times \theta z_*^{-\theta-1} \exp(-T^* z_*^{-\theta}) dz_* \\ &= T^* \int_0^\infty \exp(-(TA^{-\theta} + T^*) z_*^{-\theta}) \times \theta z_*^{-\theta-1} dz_* \\ &= \frac{T^*}{(TA^{-\theta} + T^*)} \int_0^\infty \exp(-(TA^{-\theta} + T^*) z_*^{-\theta}) \times -\theta z_*^{-\theta-1} (TA^{-\theta} + T^*) dz_* \\ &= \frac{T^*}{(TA^{-\theta} + T^*)} \int_0^\infty \exp(-(TA^{-\theta} + T^*) z_*^{-\theta}) \times \theta z_*^{-\theta-1} (TA^{-\theta} + T^*) dz_* \\ &= \frac{T^*}{(TA^{-\theta} + T^*)}, \end{aligned}$$

since $\int_0^\infty \exp(-(TA^{-\theta} + T^*) z_*^{-\theta}) \times \theta z_*^{-\theta-1} (TA^{-\theta} + T^*) dz_* = 1$ (because it is the integral of the Frechet pdf with scale $(T^* A^{-\theta} + T)$). Thus, rearranging to get an expression for $A(j)$ gives

$$A(j) = \left[\left(\frac{1-j}{j}\right)\frac{T^*}{T}\right]^{-1/\theta}. \tag{9}$$

3.2

First note that there are no trade costs so that $d_{ni} = 1$ for all $n, i \in \{F, H\}$.

Now, we know that within a country, goods will be purchased from the lowest cost source. Since Home has a comparative advantage at lower values of j, we know that Home will produce the range of goods $[0, \bar{j}]$ where

$$\frac{w}{z(\bar{j})} = \frac{w^*}{z^*(\bar{j})}.$$

The LHS of the above expression is the unit cost of producing $\bar{j}$ at home, and the RHS is the cost of buying the good from Foreign. Rearranging the above expression gives

$$\begin{aligned} \frac{z(\bar{j})}{z^*(\bar{j})} &= \frac{w}{w^*} \\ \implies A(\bar{j}) &= \omega, \end{aligned} \tag{10}$$

where $\omega = w/w^*$. Similarly, Foreign will produce a range of goods $[\underline{j}, 1]$ domestically, such that

$$\begin{aligned} \frac{w^*}{z^*(\underline{j})} &= \frac{w}{z(\underline{j})} \\ \implies A(\underline{j}) &= \omega. \end{aligned} \tag{11}$$

Thus, there is a unique cuttoff good.

Next, we need to invoke market clearing. Here, we note that preferences are Cobb Douglas, with equal weights across each good. Thus, each country spends a constant share of its income on each good. We know that Home produces $[0, \bar{j}]$ goods domestically, and exports $[0, \underline{j}]$ goods to Foreign. Thus, market clearing at Home requires

$$wL = \bar{j}wL + \underline{j}w^*L^* \tag{12}$$

$$\tag{13}$$

Substituting (10) and (11) into (12) and rearranging gives

$$\begin{aligned} L &= LA^{-1}(\omega) + \frac{1}{\omega}L^*A^{-1}(\omega) \\ \implies \frac{1}{A^{-1}(\omega)} &= 1 + \frac{1}{\omega}\frac{L^*}{L} \end{aligned}$$

And, from the above derivation we know: $A^{-1}(\omega) = \frac{T^*}{(T\omega^{-\theta} + T^*)}$. Substituting this into the above expression gives

$$\begin{aligned} \frac{T\omega^{-\theta} + T^*}{T^*} &= 1 + \frac{1}{\omega}\frac{L^*}{L} \\ \implies \frac{T}{T^*}\omega^{-\theta} &= \frac{1}{\omega}\frac{L^*}{L} \\ \implies \omega &= \left[\frac{T^*}{T}\frac{L^*}{L}\right]^{1/(1-\theta)}. \end{aligned}$$

And the equilibrium cutoff good is

$$\bar{j} = A^{-1}(\omega)$$

3.3

With no trade costs goods prices are identical in H and F. Thus, ω measures H's welfare relative to F's. From the above expression for equilibrium ω, if $\theta > 1$, then an increase in T^* reduces relative welfare in H. The comparative static is

$$\frac{d\omega}{dT^*} = \left[\frac{1}{T}\frac{L^*}{L}\right]^{1/(1-\theta)} \frac{1}{1-\theta}(T^*)^{\theta/(1-\theta)}.$$

So, if $\theta > 1$, then the effect of an increase in T^* is decreasing in L^*/L, which is observable in the data.

4 Key implications of EK's Ricardian model

4.1

The unit cost of sending good j from i to n is given by

$$C_{ni}(j) = \frac{c_i}{Z_i(j)} d_{ni} \tag{14}$$

4.2

We want to compute the probability that i will sell good j to n. Since the derivation below is the same for all goods, I suppress the index j.

(a) From (14) note that

$$Z_i = \frac{c_i d_{ni}}{C_{ni}}$$

Now,

$$\Pr[Z_i \leq p] = \Pr[c_i d_{ni}/C_{ni} \leq p]$$

Thus,

$$\begin{aligned}
\Pr[Z_i \leq c_i d_{ni}/p] &= \Pr[c_i d_{ni}/C_{ni} \leq c_i d_{ni}/p] \\
&= \Pr[p \leq C_{ni}] \\
&= \Pr[C_{ni} \geq p] \\
&= 1 - \Pr[C_{ni} \leq p] \\
\therefore \Pr[C_{ni} \leq p] &= 1 - \Pr[Z_i \leq c_i d_{ni}/p] \\
&= 1 - F_i(c_i d_{ni}/p) \\
&= 1 - \exp(-T_i(c_i d_{ni})^{-\theta} p^{\theta}),
\end{aligned}$$

which is the probability distribution of C_{ni}. Denote this distribution as G_{ni} so that

$$G_{ni}(p) = 1 - \exp(-T_i(c_i d_{ni})^{-\theta} p^\theta) \tag{15}$$

(b) Next we want to compute the probability that i is the cheapest supplier for n. Denote this probability as π_{ni}. We have

$$\begin{aligned}
\pi_{ni} \equiv \Pr[C_{ni} &= \min\{C_{ns}; s \neq i\}] \\
&= \Pr[C_{ns} \geq C_{ni} \text{ for all } s \neq i] \\
&= \int_0^\infty \prod_{s\neq i}[1 - G_{ns}(C_{ni})]dG_{ni}(C_{ni}) \\
&= \int_0^\infty \prod_{s\neq i}[1 - G_{ns}(p)]dG_{ni}(p), \qquad (16)
\end{aligned}$$

where I just re-denote the integration dummy as p to ease notation.

(c) Now,

$$dG_{ni}(p) = \frac{d}{dp}G_{ni}(p) = \exp(-T_i(c_i d_{ni})^{-\theta} p^\theta) T_i(c_i d_{ni})^{-\theta} \theta p^{\theta-1} \tag{17}$$

Substituting (15) and (17) into (16) gives

$$\begin{aligned}
\pi_{ni} &= \int_0^\infty \left[\prod_{s\neq i} \exp(-T_s(c_s d_{ns})^{-\theta} p^\theta)\right] \exp(-T_i(c_i d_{ni})^{-\theta} p^\theta) T_i(c_i d_{ni})^{-\theta} \theta p^{\theta-1} dp \\
&= \int_0^\infty \exp(-\sum_{s\neq i} T_s(c_s d_{ns})^{-\theta} p^\theta) \exp(-T_i(c_i d_{ni})^{-\theta} p^\theta) T_i(c_i d_{ni})^{-\theta} \theta p^{\theta-1} dp \\
&= \int_0^\infty \exp(-\sum_{i=1}^{N} T_i(c_{ni} d_{ni})^{-\theta} p^\theta) T_i(c_i d_{ni})^{-\theta} \theta p^{\theta-1} dp \\
&= \int_0^\infty \exp(-\Phi_n p^\theta) T_i(c_i d_{ni}) \theta p^{\theta-1} dp \\
&= T_i(c_i d_{ni})^{-\theta} \int_0^\infty \exp(-\Phi_n p^\theta) \theta p^{\theta-1} dp \\
&= \frac{T_i(c_i d_{ni})^{-\theta}}{\Phi_n} \int_0^\infty \exp(-\Phi_n p^\theta) \Phi_n \theta p^{\theta-1} dp \\
&= \frac{T_i(c_i d_{ni})^{-\theta}}{\Phi_n},
\end{aligned}$$

since the integral in the second last line evaluates to 1, because it is a pdf of a probability distribution of the form (15).