ECON641 – Problem Set 1

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1 Warmup: factor intensity reversals

First, I outline the small open economy environment of the 2×2 HO model (for my own purposes).

- Two goods, 1 and 2.
- Two factors, L and K; with endogenous factor prices w and r, respectively.
- Production technology is the same in both industries, but they may differ in their relative factor intensities.
- Exogenously given goods prices, p_1 and p_2 (i.e. the demand side of the economy is pinned down).

Roughly speaking, 'no factor intensity reversals' (NFIR) means the following: for any vector of factor prices (w, r), the ordering of relative factor intensities in both industries is always the same. For example, in equilibrium the production of good 1 may be more capital intensive than production of good 2; NFIR implies that at any other vector of factor prices, the production of good 1 must always be more capital intensive compared to good 2. We can show that production technology exhibits NFIR if, given p_1 and p_2, equilibrium factor prices are uniquely pinned down.

1.1 Cobb Douglas

Cobb Douglas production clearly satisfies NFIR. To see this, suppose that $F_1(K_1, L_1) = AK_1^{\alpha}L_1^{1-\alpha}$ and $F_2(K_2, L_2) = AK_2^{\beta}L_2^{1-\beta}$. The first order conditions for the profit maximization problem for industry 1 are standard:

$$p_1 \alpha A K_1^{\alpha-1} L_1^{1-\alpha} = r, \tag{1}$$
$$p_1 (1-\alpha) A K_1^{\alpha} L_1^{-\alpha} = w. \tag{2}$$

Dividing (2) by (1) gives

$$\frac{1-\alpha}{\alpha} k_1 = \frac{w}{r} \implies k_1 = \frac{\alpha}{1-\alpha}\frac{w}{r}, \text{ where } k_1 = K_1/L_1 \tag{3}$$

Now, the zero profit condition in industry 1 is

$$\begin{aligned} rK_1 + wL_1 &= p_1 A K_1^{\alpha} L_1^{1-\alpha} \\ \implies rk_1 + w &= p_1 A k_1^{\alpha} \end{aligned} \tag{4}$$

Plugging (3) into (4) and rearranging gives

$$p_1 = C_\alpha r^{\alpha} w^{1-\alpha} \tag{5}$$

where $C_\alpha = \frac{1}{A(1-\alpha)}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}$. An analogous derivation for industry 2 gives

$$p_2 = C_\beta r^{\beta} w^{1-\beta} \tag{6}$$

where $C_\beta = \frac{1}{A(1-\beta)}\left(\frac{1-\beta}{\beta}\right)^\beta$. Clearly, given p_1 and p_2, there is a unique solution to (5) and (6), (w^*, r^*), (unless $\alpha = \beta$).

Another (perhaps more intuitive) way to establish NFIR would be to use equation (3) and the equivalent expression for industry 2. These expressions imply that in equilibrium:

$$\frac{k_1}{k_2} = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}.$$

That is, the relative factor intensities between the two industries is independent of factor prices.

1.2 CES

CES production *does not* exhibit NFIR. To see this, suppose $F_i(K_i, L_i) = \left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}}\right]^{\frac{\sigma_i}{\sigma_i-1}}$ for $i = 1, 2$. The FOCs for industry i are

$$p_i \left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}}\right]^{\frac{1}{\sigma_i-1}} K_i^{-1/\sigma_i} = r \tag{7}$$

$$p_i \left[K_i^{\frac{\sigma_i-1}{\sigma_i}} + L_i^{\frac{\sigma_i-1}{\sigma_i}}\right]^{\frac{1}{\sigma_i-1}} L_i^{-1/\sigma_i} = w \tag{8}$$

Combining these expressions gives

$$\begin{aligned} k_i^{-1/\sigma_i} &= \frac{r}{w} \\ \implies k_i &= \left(\frac{r}{w}\right)^{-\sigma_i}. \end{aligned}$$

Thus, in equilibrium, the relative factor intensities between the two industries is

$$\frac{k_1}{k_2} = \left(\frac{r}{w}\right)^{\sigma_2-\sigma_1},$$

which clearly depends on factor prices (unless $\sigma_1 = \sigma_2$).

1.3 Leontief

Clearly the Leontief production function exhibits NFIR. Suppose both industries have the same production function $F(K, L) = \min\{K, L\}$. Then in equilibrium, both industries must have $k_i = 1$. Then, relative factor intensities do not depend on factor prices. More generally, suppose $F_i(K_i, L_i) = \min\{\alpha_i K, \beta_i L\}$. Then in equilibrium, each industry's capital-labor ratio will be $k_i = \beta_i/\alpha_i$. Again, relative factor intensities are independent of factor prices.

2 $2 \times 2 \times 2$ HO Model

3 Technology growth in a parameterized version of DFS

3.1

I follow the derivation in EK (2005). We are given the distribution of efficiencies for producing goods j at Home and Foreign:

$$F_i(z) = \Pr[Z_i(j) \leq z] = \exp(-T_i z^{-\theta})$$

Now, we want to derive the DFS-type $A(j)$ curve, which is defined as the ratio of H's efficiency of producing j to F's corresponding efficiency.

In the EK setup the efficiencies are realizations of a random variable. Accordingly, we think of j as the *probability* that the H's relative efficiency of producing j is less than some number:

$$\begin{aligned} j &= \Pr\left[\frac{Z}{Z^*} \leq A\right] \\ &= \Pr\left[Z \leq AZ^*\right] \\ &= \int_0^\infty \exp(-T(Az_*)^{-\theta}) f(z_*) dz_*. \end{aligned}$$

Now,

$$f(z_*) = \frac{d}{dz}\exp(-T^* z^{-\theta}) = \theta T^* z^{-\theta-1}\exp(-T^* z^{-\theta})$$

Substituting into the above integral gives

$$\begin{aligned} j &= T^* \int_0^\infty \exp(-T(Az_*)^{-\theta}) \times \theta z_*^{-\theta-1} \exp(-T^* z_*^{-\theta}) dz_* \\ &= T^* \int_0^\infty \exp(-(TA^{-\theta} + T^*) z_*^{-\theta}) \times \theta z_*^{-\theta-1} dz_* \\ &= \frac{T^*}{(TA^{-\theta} + T^*)} \int_0^\infty \exp(-(TA^{-\theta} + T^*) z_*^{-\theta}) \times -\theta z_*^{-\theta-1}(TA^{-\theta} + T^*) dz_* \\ &= \frac{T^*}{(TA^{-\theta} + T^*)} \int_0^\infty \exp(-(TA^{-\theta} + T^*) z_*^{-\theta}) \times \theta z_*^{-\theta-1}(TA^{-\theta} + T^*) dz_* \\ &= \frac{T^*}{(TA^{-\theta} + T^*)}, \end{aligned}$$

since $\int_0^\infty \exp(-(TA^{-\theta} + T^*) z_*^{-\theta}) \times \theta z_*^{-\theta-1}(TA^{-\theta} + T^*) dz_* = 1$ (because it is the integral of the Frechet pdf with scale $(T^* A^{-\theta} + T)$). Thus, rearranging to get an expression for $A(j)$ gives

$$A(j) = \left[\left(\frac{1-j}{j}\right)\frac{T^*}{T}\right]^{-1/\theta}. \tag{9}$$

3.2

First note that there are no trade costs so that $d_{ni} = 1$ for all $n, i \in \{F, H\}$.

Now, we know that within a country, goods will be purchased from the lowest cost source. Since Home has a comparative advantage at lower values of j, we know that Home will produce the range of goods $[0, \bar{j}]$ where

$$\frac{w}{z(\bar{j})} = \frac{w^*}{z^*(\bar{j})}.$$

The LHS of the above expression is the unit cost of producing $\bar{j}$ at home, and the RHS is the cost of buying the good from Foreign. Rearranging the above expression gives

$$\begin{aligned} \frac{z(\bar{j})}{z^*(\bar{j})} &= \frac{w}{w^*} \\ \implies A(\bar{j}) &= \omega, \end{aligned} \tag{10}$$

where $\omega = w/w^*$. Similarly, Foreign will produce a range of goods $[\underline{j}, 1]$ domestically, such that

$$\begin{aligned} \frac{w^*}{z^*(\underline{j})} &= \frac{w}{z(\underline{j})} \\ \implies A(\underline{j}) &= \omega. \end{aligned} \tag{11}$$

Thus, there is a unique cuttoff good.

Next, we need to invoke market clearing. Here, we note that preferences are Cobb Douglas, with equal weights across each good. Thus, each country spends a constant share of its income on each good. We know that Home produces $[0, \bar{j}]$ goods domestically, and exports $[0, \underline{j}]$ goods to Foreign. Thus, market clearing at Home requires

$$wL = \bar{j}wL + \underline{j}w^*L^* \tag{12}$$

$$\tag{13}$$

Substituting (10) and (11) into (12) and rearranging gives

$$\begin{aligned} L &= LA^{-1}(\omega) + \frac{1}{\omega}L^*A^{-1}(\omega) \\ \implies \frac{1}{A^{-1}(\omega)} &= 1 + \frac{1}{\omega}\frac{L^*}{L} \end{aligned}$$

And, from the above derivation we know: $A^{-1}(\omega) = \frac{T^*}{(T\omega^{-\theta} + T^*)}$. Substituting this into the above expression gives

$$\begin{aligned} \frac{T\omega^{-\theta} + T^*}{T^*} &= 1 + \frac{1}{\omega}\frac{L^*}{L} \\ \implies \frac{T}{T^*}\omega^{-\theta} &= \frac{1}{\omega}\frac{L^*}{L} \\ \implies \omega &= \left[\frac{T^*}{T}\frac{L^*}{L}\right]^{1/(1-\theta)}. \end{aligned}$$

And the equilibrium cutoff good is

$$\bar{j} = A^{-1}(\omega)$$

3.3

With no trade costs goods prices are identical in H and F. Thus, ω measures H's welfare relative to F's. From the above expression for equilibrium ω, if $\theta > 1$, then an increase in T^* reduces relative welfare in H. The comparative static is

$$\frac{d\omega}{dT^*} = \left[\frac{1}{T}\frac{L^*}{L}\right]^{1/(1-\theta)} \frac{1}{1-\theta}(T^*)^{\theta/(1-\theta)}.$$

So, if $\theta > 1$, then the effect of an increase in T^* is decreasing in L^*/L, which is observable in the data.

4 Key implications of EK's Ricardian model

4.1

The unit cost of sending good j from i to n is given by

$$C_{ni}(j) = \frac{c_i}{Z_i(j)} d_{ni} \tag{14}$$

4.2

We want to compute the probability that i will sell good j to n. Since the derivation below is the same for all goods, I suppress the index j.

(a) From (14) note that

$$Z_i = \frac{c_i d_{ni}}{C_{ni}}$$

Now,

$$\Pr[Z_i \leq p] = \Pr[c_i d_{ni}/C_{ni} \leq p]$$

Thus,

$$\begin{aligned}
\Pr[Z_i \leq c_i d_{ni}/p] &= \Pr[c_i d_{ni}/C_{ni} \leq c_i d_{ni}/p] \\
&= \Pr[p \leq C_{ni}] \\
&= \Pr[C_{ni} \geq p] \\
&= 1 - \Pr[C_{ni} \leq p] \\
\therefore \Pr[C_{ni} \leq p] &= 1 - \Pr[Z_i \leq c_i d_{ni}/p] \\
&= 1 - F_i(c_i d_{ni}/p) \\
&= 1 - \exp(-T_i (c_i d_{ni})^{-\theta} p^{\theta}),
\end{aligned}$$

which is the probability distribution of C_{ni}. Denote this distribution as G_{ni} so that

$$G_{ni}(p) = 1 - \exp(-T_i (c_i d_{ni})^{-\theta} p^{\theta}) \tag{15}$$

(b) Next we want to compute the probability that i is the cheapest supplier for n. Denote this probability as π_{ni}. We have

$$\begin{aligned}
\pi_{ni} \equiv \Pr[C_{ni} &= \min\{C_{ns}; s \neq i\}] \\
&= \Pr[C_{ns} \geq C_{ni} \text{ for all } s \neq i] \\
&= \int_0^{\infty} \prod_{s \neq i} [1 - G_{ns}(C_{ni})] dG_{ni}(C_{ni}) \\
&= \int_0^{\infty} \prod_{s \neq i} [1 - G_{ns}(p)] dG_{ni}(p), \qquad (16)
\end{aligned}$$

where I just re-denote the integration dummy as p to ease notation.

(c) Now,

$$dG_{ni}(p) = \frac{d}{dp} G_{ni}(p) = \exp(-T_i(c_i d_{ni})^{-\theta} p^{\theta}) T_i (c_i d_{ni})^{-\theta} \theta p^{\theta - 1} \tag{17}$$

Substituting (15) and (17) into (16) gives

$$\begin{aligned}
\pi_{ni} &= \int_0^{\infty} \left[\prod_{s \neq i} \exp(-T_s(c_s d_{ns})^{-\theta} p^{\theta}) \right] \exp(-T_i(c_i d_{ni})^{-\theta} p^{\theta}) T_i (c_i d_{ni})^{-\theta} \theta p^{\theta-1} dp \\
&= \int_0^{\infty} \exp(-\sum_{s \neq i} T_s (c_s d_{ns})^{-\theta} p^{\theta}) \exp(-T_i(c_i d_{ni})^{-\theta} p^{\theta}) T_i (c_i d_{ni})^{-\theta} \theta p^{\theta-1} dp \\
&= \int_0^{\infty} \exp(-\sum_{i=1}^{N} T_i (c_{ni} d_{ni})^{-\theta} p^{\theta}) T_i (c_i d_{ni})^{-\theta} \theta p^{\theta-1} dp \\
&= \int_0^{\infty} \exp(-\Phi_n p^{\theta}) T_i (c_i d_{ni}) \theta p^{\theta-1} dp \\
&= T_i (c_i d_{ni})^{-\theta} \int_0^{\infty} \exp(-\Phi_n p^{\theta}) \theta p^{\theta-1} dp \\
&= \frac{T_i (c_i d_{ni})^{-\theta}}{\Phi_n} \int_0^{\infty} \exp(-\Phi_n p^{\theta}) \Phi_n \theta p^{\theta-1} dp \\
&= \frac{T_i (c_i d_{ni})^{-\theta}}{\Phi_n},
\end{aligned}$$

since the integral in the second last line evaluates to 1, because it is a pdf of a probability distribution of the form (15).

4.3

Next we want to compute the probability distribution of goods prices actually bought in market n. Call this price P_n and recall that n buys the lowest cost good

$$P_n = \min_{i=1,..,N} \{C_{ni}\}$$

Thus

$$\begin{aligned}
\Pr[P_n \leq p] &= \Pr[\min_{i=1,..,N} \{C_{ni}\} \leq p] \\
&= 1 - \Pr[C_{ni} > p \text{ for all } i] \\
&= 1 - \prod_{i=1}^{N} [1 - G_{ni}(p)] \\
&= 1 - \prod_{i=1}^{N} \exp(-T_i(c_i d_{ni})^{-\theta} p^{\theta}) \text{ using (15)}, \\
&= 1 - \exp(-\Phi_n p^{\theta}).
\end{aligned}$$

Denote this distribution as $G_n(p)$, so that

$$G_n(p) \equiv \Pr[P_n \le p] = 1 - \exp(-\Phi_n p^\theta) \tag{18}$$

4.4

Next, we want to compute the probability distribution of goods prices that n *actually buys* from country i. That is, we want to compute the conditional probability distribution:

$$\Pr[P_n \le p | P_n = C_{ni}]$$

Now,

$$\begin{aligned}
Pr[P_n \le p | P_n = C_{ni}] &= \Pr[P_n \le p | C_{ni} = \min\{C_{ns}; s \ne i\}] \\
&= \Pr[C_{ni} \le p | C_{ni} = \min\{C_{ns}; s \ne i\}] \\
&= \frac{1}{\pi_{ni}} \int_0^p \prod_{s \ne i} [1 - G_{ns}(q)] dG_{ni}(q) \\
&= \frac{1}{\pi_{ni}} \pi_{ni} \int_0^p \exp(-\Phi_n q^\theta) \Phi_n \theta q^{\theta - 1} dq,
\end{aligned}$$

using our previous derivations. Thus,

$$\begin{aligned}
\Pr[P_n \le p | P_n = C_{ni}] &= \int_0^p \exp(-\Phi_n q^\theta) \Phi_n \theta q^{\theta - 1} dq \\
&= \int_0^p dG_n(q) \\
&= G_n(p).
\end{aligned}$$

So, for goods that are purchased in n, conditioning on the source does not affect the distribution of the good's price. This result seems at odds with reality, as we discussed in class. One would think that German cars bought in the US would have a different price distribution compared to Japanese cars bought in the US.

4.5

I'm not exactly sure how to do this problem, but here's my best shot.

(a) Given CES preferences over the unit mass of goods, n's demand for good j is given by

$$\begin{aligned}
X_n(j) &= \left(\frac{P_n(j)}{P_n} \right)^{1-\sigma} X_n \\
&= \left(\frac{\min_{i=1,..,N}\{C_{ni}(j)\}}{P_n} \right)^{1-\sigma} X_n
\end{aligned}$$

(b) Then, n's expected expenditure on good j, sourced from i is

$$\begin{aligned}X_{ni}(j) &= \mathbb{E}[X_n(j)|i*(j)=i]\Pr[i^*(j)=i] \\ &= \mathbb{E}\left[\left(\frac{\min_{i=1,..,N}\{C_{ni}(j)\}}{P_n}\right)^{1-\sigma} X_n|i^*(j)=i\right]\pi_{ni} \\ &= \mathbb{E}\left[\left(\frac{C_{ni}(j)}{P_n}\right)^{1-\sigma} X_n|i^*(j)=i\right]\pi_{ni} \\ &= \mathbb{E}\left[\left(\frac{P_n}{P_n}\right)^{1-\sigma} X_n|i^*(j)=i\right]\pi_{ni} \\ &= \pi_{ni}X_n.\end{aligned}$$

4.6

We've shown that π_{ni} is the probability that country n purchases good any good j from i. Since there is a unit measure of goods, it follows that π_{ni} is the total fraction of the $j \in [0,1]$ goods that are sourced from from i in country n. Then, we can split the the total unit measure $j \in [0,1]$ of goods purchased in n into the share supplied by each source country. Recall that conditioning on the source of a good does not affect the distribution of the good's price in n. Accordingly, the fraction of n's total expenditure that goes to country i is the same as the fraction of goods that n purchases from country i; namely, π_{ni}.