Notes on MNS (2016)

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1 Solving the households' problems

I solve the HH problem using the the endogenous grid point method (EGM) (which is a numerical method for implementing policy function iteration). I focus on the steady state solution for simplicity.

Each household faces the same problem, so I will suppress the h subscripts. They solve

$$\begin{aligned}
&\max_{c,\ell,b_{t+1}} \sum_t \beta^t \left(\frac{c_t^{1-\gamma}}{1-\gamma} - \frac{\ell_t^{1+\psi}}{1+\psi} \right) \\
&\text{s.t. } c_t + \frac{b_{t+1}}{1+r_t} = b_t + W_t z_t \ell_t - \tau_t \bar{\tau}(z_t) + D_t \\
&\& \ b_{t+1} \geq 0.
\end{aligned}$$

The HH's Euler equation is standard

$$c_t^{-\gamma} \geq \beta(1+r_t) \sum_{z_{t+1}} \Pr(z_{t+1}|z_t) c_{t+1}^{-\gamma}, \tag{1}$$

which holds with equality when the borrowing constraint is not binding. The intratemporal labor supply condition is also standard

$$\ell_t^{\psi} = W_t z_t c_t^{-\gamma} \tag{2}$$

$$\implies \ell_t = c_t^{-\gamma/\psi} (W_t z_t)^{1/\psi} \tag{3}$$

1.1 Initial guess of the household's policy function

For the EGM method, I need an initial guess of the policy function. Let's keep this as simple as possible: suppose we start in the steady state and that households consume all of their available resources and save nothing. That is

$$c = b + \bar{W} z \ell - \tau \bar{\tau}(z) + \bar{D}$$

Substituting the labor supply condition into the above expression, and rearranging a little gives

$$c = c^{-\gamma/\psi} (\bar{W} z)^{\frac{1+\psi}{\psi}} + b - \tau \bar{\tau}(z) + \bar{D} \tag{4}$$

Thus, the initial guess for the policy function, denoted by $g^0(z,b)$, is the level of c that solves (4). With MNS's calibration, this is just a quadratic equation in c (since $\gamma = \psi = 2$), which I can solve easily. So I have an initial consumption level for each (z,b) pair.

1.2 Iterating on the Euler equation

For starters, assume that the Euler equation holds with equality (I'll deal with the borrowing constraint properly below). Then, with my initial guess of the policy function, I can write the Euler as

$$c^{-\gamma} = \beta(1+r)\sum_{z'} \Pr(z'|z)(g^0(z',b'))^{-\gamma}.$$

Then, today's consumption is simply

$$c = \left(\beta(1+r)\sum_{z'} \Pr(z'|z)(g^0(z',b'))^{-\gamma}\right)^{-1/\gamma} \tag{5}$$

Now, recall that the HH optimally chooses b_{t+1}, while b_t is predetermined. Thus, (5) gives today's consumption for someone who drew productivity z today, and optimally chose b' for tomorow's bond holdings. So, we can write (5) as

$$\tilde{g}^0(z,b') = \left(\beta(1+r)\sum_{z'} \Pr(z'|z)(g^0(z',b'))^{-\gamma}\right)^{-1/\gamma} \tag{6}$$

Next, define a grid over values of *tomorrow's* bond holdings b' and call it $B = \{b_1, ..., b_{n_b}\}$, with $b_1 = 0$. Also suppose we have a grid of possible productivity draws, $Z = \{z_1, ..., z_{n_z}\}$, and an associated transition matrix Γ. Then we can write (6) as

$$\tilde{g}^0(z_i,b_j) = \left(\beta(1+r)\sum_{\ell=1}^{n_z} \Gamma_{i\ell}(g^0(z_\ell,b_j))^{-\gamma}\right)^{-1/\gamma}, \tag{7}$$

which gives the optimal policy c, for an agent with current productivity z_j, and who optimally chooses tomorrow's bond holdings b_j. Thus, for each (z_i, b_j) pair I can get the associated optimal consumption today using (7).

Note that I can easily stack (7) in matrix form. Let $\boldsymbol{G}^0$ be the $n_z \times n_b$ matrix containing our initial guesses of the optimal policy (so $\boldsymbol{G}^0_{ij}$ gives the optimal consumption for state pair (z_i, b_j)). Then, the updated guess can simply be computed as

$$\widetilde{\boldsymbol{G}}^0 = \left(\beta(1+r)\boldsymbol{\Gamma}(\boldsymbol{G}^0)^{-\gamma}\right)^{-1/\gamma},$$

which is essentially what my `Julia` code is doing (note that my `Julia` code actually uses a nested for loop, but I could probably change it to the above matrix code!)

Next, I can use the budget constraint to back out *today's* bond holdings b_t:

$$b_t = c_t + \frac{b_{t+1}}{1+r_t} - W_t z_t \ell_t + \tau_t \bar{\tau}(z_t) - D_t$$

Using the intratemporal labor supply condition (2) to substitute in for ℓ_t, gives

$$b_t = c_t + \frac{b_{t+1}}{1+r_t} - c^{-\gamma/\psi} (\bar{W} z)^{\frac{1+\psi}{\psi}} + \tau_t \bar{\tau}(z_t) - D_t$$

Using the grid indicies, and my guess for today's consumption, I can write:

$$b^*_{i,j} = \tilde{g}^0(z_i, b_j) + \frac{b_j}{1+r} - \left(\tilde{g}^0(z_i, b_j)\right)^{-\gamma/\psi} (\bar{W} z_i)^{\frac{1+\psi}{\psi}} + \tau \bar{\tau}(z_i) - \bar{D},$$

where $b^*_{i,j}$ denotes the bond holdings today for a household who gets a productivity draw z_i today, and who optimally chooses tomorrow's level of bond holdings, b_j. Then, we can finally define the 'endogenous grid' of today's bond holdings (for each level of productivity z_i):

$$B^*_i = \{b^*_{i,1}, ..., b^*_{i,n_b}\}.$$

Next, note that for each grid point in B^*_i, I have the updated guess of the optimal policy:

$$g^1(z_i, b^*_{i,j}) = \tilde{g}^0(z_i, b_j)$$

However, the grid points in B^*_i need not (and will not, in general) coincide with the points in the original grid B. Thus, to get the updated guesses $g^1(z_i, b_j)$ for each $b_i \in B$ I need to interpolate $g^1(z_i, b^*_{i,j})$, which is pretty straightforward in `Julia`. MNS use a shape-preserving cubic spline, but for now I'll stick with a simple linear interpolation.

Now, I need to deal with the borrowing constraint, $b_{t+1} \geq 0$. Recall that $b^*_{i,j}$ is today's bond holdings for someone with current productivity z_i and who optimally chooses b_j. Suppose the borrowing constraint binds; then agent optimally chooses $b_{t+1} = b_j = 0$. Thus, from the borrowing constraint, today's bond holdings are

$$b^*_{i,1} = \tilde{g}^0(z_i, b_j) - \left(\tilde{g}^0(z_i, b_j)\right)^{-\gamma/\psi} (\bar{W} z_j)^{\frac{1+\psi}{\psi}} + \tau \bar{\tau}(z_j) - \bar{D}.$$

Recall that $\tilde{g}_0(z_i, b_1)$ is our guess of optimal consumption today. Accordingly, $b^*_{i,1}$ represents the *highest* level of bond holdings *today* for someone with current productivity z_i, such that the borrowing constraint binds. Then, for all grid points $b_j \leq b^*_{i,1}$, when $z = z_i$, the borrowing constraint must bind (i.e. $b_{t+1} = 0$). Then, we can use the budget constraint to get the updated guess of today's optimal consumption for constrained agents:

$$g^{1,c}(z_i, b_j) = b_j + \left(g^{1,c}(z_i, b_j)\right)^{-\gamma/\psi} (\bar{W} z_i)^{\frac{1+\psi}{\psi}} - \tau \bar{\tau}(z_i) + \bar{D} \text{ for all } b_j \leq b^*_{i,1},$$

where c superscript denotes 'constrained'. Again, with MNS's calibration this is just a quadratic equation.

Putting all this together, the updated guess of the optimal policy function is

$$g^1(z_i, b_j) = \begin{cases} \tilde{g}^0(z_i, b_j) & \text{if } b_j > b^*_{i,1} \text{ (and appropriately interpolated)} \\ g^{1,c}(z_i, b_j) & \text{otherwise.} \end{cases}$$

1.3 Calibration

1.3.1 The Markov chain for productivity

I found the Markov chain for productivity in MNS's `initparams.m` file (in the '`parameters`' folder). It looks like the grid is given by

$$Z = \{0.493, 1, 2.031\}$$

with transition matrix

$$\mathbf{\Gamma} = \begin{bmatrix} 0.966 & 0.0338 & 0.00029 \\ 0.017 & 0.966 & 0.017 \\ 0.0003 & 0.0337 & 0.966 \end{bmatrix}$$

1.3.2 Grid of bond holdings

I use a grid of 200 equally spaced points ranging from $b_{min} = 0$ and $b_{max} = 75$. Note that MNS use an unequally spaced grid, with more points at lower asset levels, because the policy function has more curvature in this region.

1.3.3 Steady state prices

Steady state prices are

- $\bar{W} = 1/\mu$ (i.e. inverse over the intermediate goods firms' markup);
- $\bar{r} = 0.005$ (I just took this from MNS's `initparams.m` file; add some commentary on what this means and that it ensures $\beta(1+r) < 1$ – WHY IS THIS IMPORTANT IN HET AGENT MODELS?)
- $\bar{D} = \bar{Y}(1 - \bar{W}) =?$
- $\bar{\tau} =?$

1.4 Some analysis

See the QuantEcon lecture on occasionally binding borrowing constraint for some cool graphs I could replicate!

1.5 Simulating the distribution of bond holdings

The above QuantEcon lecture is also useful for this! Also see CH's notes!

2 Overview and results

Main results:

1. In standard NKM, the response of (current) output/consumption to forward guidance is too big.

2. In a NKM with idiosyncratic labor income shocks and borrowing constraints, the response of output/consumption to forward guidance is far lower.

3. Something about ZLB [fill in]

2.1 Intuition [for my own understanding]

3 Result #1

I've replicated this in AIM. Some notes on the AIM code

- Note that the variable r_t in the code is the deviation of the real rate from the natural rate: $\tilde{r}_t = i_t - \mathbb{E}_t \pi_{t+1} - r_t^n$. With this note, you can easily map my code back to the exposition in the paper.

4 Result # 2

This is the main result of the paper, and it requires solving MNS's heterogeneous agent model. Note this their model is essentially just a Bewley/Hugget/Aiygari model with sticky prices. This is a hard problem. Based MNS's online appendix, these are the main steps in solving for equilibrium:

1. Solve the households' problems using the endogenous grid point method (Carrol, 2006).

2. Simulate the distribution of the households' asset holdings using Young's (2006) nonstochastic histogram method.

3. Checking the equilibrium

4. Updating the initial guess using results from a 'simpler' economy.

Useful resources:

- `https://sites.google.com/a/nyu.edu/glviolante/teaching/quantmacro` [which points to some books too]

The following QuantEcon lectures are probably going to be useful

- Markov chains
- EGM
- Implementing a discrete state dynamic program
- Hugget model
- Aiyagari model

4.1 Solving the households' problems

I need to use the endogenous grid point method (EGM) (which is a numerical method for implementing policy function iteration). Note that value function iteration is too slow for this problem. Also note that this method requires approximating the policy function for consumption using a 'shape preserving cubic spline'.

A good place to start may be to write down and program the Bewley/Hugget model. Then extend to Aiygari, which has a firm. These models are obviously more basic than MNS, but at least I can get a decent handle on using grids, EGM and deriving the stationary distribution of asset holdings.

4.1.1 EGM

Here's roughly how EGM works. [I need to understand how exactly this relates to 'time iteration'; and the fact that MNS assume that the economy returns to steady state after 250 periods.]

Consider the simplified consumer's problem from MNS (i.e. I've assumed exogenous income rather than a wage and a labor choice)

$$\begin{aligned} &\max \sum_t \beta^t u(c_t) \\ &\text{s.t. } c_t + \frac{b_{t+1}}{1+r_t} = b_t + z_t \\ &\text{\& } b_{t+1} \geq 0. \end{aligned}$$

The Euler equation is (you can check handwritten notes for the details):

$$u'(c_t) \geq \beta(1+r_t) \sum_{z_{t+1}} \Pr(z_{t+1}|z_t) u'(c_{t+1})$$

For simplicity assume that the solution is interior (i.e. $b_{t+1} > 0$) so that the Euler holds with equality (note that I'll need to figure out how to deal with the constraint too!). I want to implement EGM, which is a policy function iteration method (c.f. value function iteration). Let $g_0(b, z)$ denote our initial guess of the optimal (consumption) policy for a given state pair (b, z). (Note that we are trying to solve for the true policy function $g(b, z)$). Then, I can write the Euler as:

$$u'(c) = \beta(1+r) \sum_{z'} \Pr(z'|z) u'(g_0(b', z')).$$

Then, today's consumption is simply

$$c = u'^{-1} \left(\beta(1+r) \sum_{z'} \Pr(z'|z) u'(g_0(b', z')) \right) \tag{8}$$

Now, recall that the HH optimally chooses b_{t+1}, which b_t is predetermined. Thus, (8) gives today's consumption for someone who drew income z today, and optimally chose b'. So, we can write (8) as

$$\tilde{g}_0(b', z) = u'^{-1}\left(\beta(1+r)\sum_{z'}\Pr(z'|z)u'(g_0(b', z'))\right) \tag{9}$$

Next, define a grid over values of *tomorrow's* bond holdings b' and call it $B = \{b_1, ..., b_{n_b}\}$, with $b_1 = 0$. Also suppose we have a grid of possible income draws, $Z = \{z_1, ..., z_{n_z}\}$, and an associated transition matrix Γ. Then we can write (9) as

$$\tilde{g}_0(b_i, z_j) = u'^{-1}\left(\beta(1+r)\sum_{\ell}\Gamma_{j\ell}u'(g_0(b_i, z_\ell))\right), \tag{10}$$

which gives the optimal policy c, for an agent with current income z_j, and who optimally chooses tomorrow's bond holdings b_i. Thus, for each (b_i, z_j) pair we can get the associated optimal consumption today using (10).

Next, we can use the budget constraint to back out *today's* bond holdings b_t:

$$b_t = c_t + \frac{b_{t+1}}{1+r_t} - z_t$$

Using the grid indicies, we can write:

$$b^*_{i,j} = \tilde{g}_0(b_i, z_j) + \frac{b_i}{1+r} - z_j,$$

where $b^*_{i,j}$ defines bond holdings today for a HH who gets an income draw z_j today, and who optimally chooses tomorrow's level of bond holdings, b_j. Then, we can finally define the 'endogenous grid' of today's bond holdings (for each level of income z_j):

$$B^*_j = \{b^*_{1,j}, ..., b^*_{n_b,j}\}.$$

Next, note that for each grid point in B^*_j we have the updated guess of the optimal policy:

$$g_1(b^*_{i,j}, z_j) = \tilde{g}_0(b_i, z_j)$$

Note that the grid points in B^*_j need not (and will not, in general) coincide with the points in the original grid B. Thus, to get the updated guesses $g_1(b_i, z_j)$ for each $b_i \in B$ we need to interpolate $g_1(b^*_{i,j}, z_j)$ (see below).

Now, let's try to deal with the borrowing constraint, $b_{t+1} \geq 0$. Recall that $b^*_{i,j}$ is today's bond holdings for someone with current income z_j and who optimally chooses b_i. Suppose the borrowing constraint binds; then agent optimally chooses $b_{t+1} = b_1 = 0$. Thus,

$$b^*_{1,j} = \tilde{g}_0(b_1, z_j) - z_j.$$

Recall that $\tilde{g}_0(b_1, z_j)$ is our guess of optimal consumption today. Thus, $b^*_{1,j}$ represents the *highest* level of bond holdings *today* for someone with current income z_j, such that the borrowing constraint binds. Then, for all grid points $b_i \leq b^*_{1,j}$, when $z = z_j$, the borrowing constraint must bind (i.e. $b_{t+1} = 0$). Then, we can use the budget constraint to get the updated guess of today's optimal consumption for constrained agents:

$$g_1^{constrained}(b_i, z_j) = b_i + z_j \text{ for all } b_i \leq b^*_{1,j}$$

Putting all this together, the updated guess of the optimal policy function is

$$g_1(b^*_{i,j}, z_j) = \begin{cases} \tilde{g}_0(b_i, z_j) & \text{if } b_i > b^*_{1,j} \\ b_i + z_j & \text{otherwise.} \end{cases}$$

Notes on EGM:

- QuantEcon (the 'coleman_egm' function computes the updated guess of the policy funtion, and the 'check_convergence' function implements the iteration).
- CEMFI (I summarized these above).

4.1.2 Interpolating the policy function

- QuantEcon recommends the Interpolations package in Julia. Not sure if it implements shape preserving cubic splines But maybe I can just start with their in-built linear interpolation?
- I think Rudd and MF have notes on approximations using splines.

4.1.3 The Markov Chain for idiosyncratic wage risk

MNS discretize an AR(1) process for z to a three-point Markov chain. I think it's OK to skip the details of this step given the time constraint. But I still need to get the actual transition matrix they used somehow!

- MNS use the `rouwen.m` function to get the transition matrix (in the `initparams.m` file) Link.
- QuantEcon has an inbuilt `rouwenhorst` function. Link.
- I think z is mean zero.

4.1.4 CH suggestions

- Focus on solving the HH's problems before even thinking about the other steps of the solution.
- For the initial guess of the policy function, suppose that all households consume their idiosyncratic wage income, steady state dividends and bond holdings (i.e. they save nothing).

4.2 Simulating the distribution of asset holdings

4.3 Checking equilibrium

4.4 Updating the guess