Complex Numbers : (Complex Analytic Functions)

- $z = x + iy$; $i^2 = -1$; x = real, y = img
- $\bar{z} = x - iy$
- $|z| = \sqrt{x^2 + y^2}$
- $\theta = \tan^{-1}(y/x)$

- Complex variable in polar form:

$$z = r(\cos\theta + i\sin\theta)$$

- Complex variable in exponential form:

$$z = re^{i\theta} \; ; \; e^{i\theta} = \cos\theta + i\sin\theta$$

- Trigonometric ratios in terms of complex variable:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

- De Moivre's theorm:

If 'n' is any +ve integer then,

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

- Hyperbolic trigonometric ratios for complex variables:

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

- Relation between hyperbolic and ~~complex~~ (complex) trigonometric ratios:

$\sin(iz) = i\sinh z$	$\sinh(iz) = i\sin z$
$\cos(iz) = \cosh z$	$\cosh(iz) = \cos z$

$\sin(iz) = \dfrac{e^{i(iz)} - e^{-i(iz)}}{2i}$ $= \dfrac{e^{-z} - e^{z}}{2i}$ $= \dfrac{-1\,(e^{z} - e^{-z})}{2i}$ $= \dfrac{i^2\,(e^{z} - e^{-z})}{2i}$ $= i\sinh z$	$\sinh(iz) = \dfrac{e^{iz} - e^{-iz}}{2}$ $= \dfrac{i\,(e^{iz} - e^{-iz})}{2i}$ $= i\sin z$	$\cos(iz) = \dfrac{e^{i(iz)} + e^{-i(iz)}}{2}$ $= \dfrac{e^{-z} + e^{z}}{2}$ $= \cosh z$	$\cosh(iz) = \dfrac{e^{iz} + e^{-iz}}{2}$ $= \cos z$

Ex 2:

Q1) Find the real and imaginary parts of the following:

(i) e^z

Soln;

we have,

$f(z) = e^z$

$= e^{x+iy}$

$= e^x . e^{iy}$

$= e^x(\cos y + i\sin y)$

$= e^x\cos y + i e^x \sin y$

$\therefore$ Real part $= e^x\cos y$

Img part $= e^x \sin y$

(ii) $\cos z$

Soln;

we have,

$f(z) = \cos z$

$= \cos(x+iy)$

$= \cos x \cos iy - \sin x \sin iy$

$= \cos x \cosh y - i \sin x \sinh y$

$\therefore$ Real $= \cos x \cosh y$

Img $= -\sin x \sinh y$

Differentiability of complex fxn

A complex function $W = f(z) = u(x,y) + iv(x,y)$ is differentiable for the variable $z = x+iy$, if for $z = z_0$,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

It is denoted by,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

It is also written as;

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Analytic fxn: ($u_x, u_y, v_x, v_y \to$ partial derivatives wrt x, y)

A function is analytic if it is differentiable everywhere in the given domain.

imp Necessary condition for the function to be analytic:

(Cauchy - Reimann eqn / CR eqn)

Statement: If a complex function $W = f(z) = u(x,y) + iv(x,y)$ is analytic then, u_x, u_y, v_x, v_y exists and are continuous such that,

$u_x = v_y$ & $u_y = -v_x$. It is known as C-R eqn.

proof: A function,

$W = f(z) = u(x,y) + iv(x,y)$ is analytic so $f'(z)$ at the point $z = x+iy$ exists.

i.e $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists.

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$$= \lim_{(\Delta x + i\Delta y) \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

or, $$f'(z) = \lim_{(\Delta x + i\Delta y) \to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + i[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i\Delta y}$$

Let us suppose that it is differentiable wholly by x i.e, $\Delta x \to 0$ & $\Delta y = 0$

$$\therefore f'(z) = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right]$$

$$= u_x + i v_x \quad \text{--- (i)}$$

Again,
let $f(z)$ be differentiable wholly by y i.e $\Delta y \to 0$ & $\Delta x = 0$

$$\therefore f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y) + i[v(x, y+\Delta y) - v(x,y)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \left[-i \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} \right]$$

$$= -i u_y + v_y \quad \text{--- (ii)}$$

$\because f(z)$ is analytic,
$\therefore$ from (i) & (ii);

$$u_x + i v_x = -i u_y + v_y$$

$\therefore \boxed{(u_x = v_y) \ \&, \ (u_y = -v_x)}$ is the req. CR eqn.

CR eqn in polar form:

let $W = f(z) = u(x,y) + iv(x,y)$ be a given fxn.

let the polar form be, $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

Diff both sides wrt 'r';

$$f'(re^{i\theta})\, e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \quad \text{—— (I)}$$

Also, wrt 'θ';

$$f'(re^{i\theta})\, \cancel{a}rie^{i\theta} = \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$$

~~or~~ $\cancel{f'(re^{i\theta}).e^{i\theta} = }$

$$\text{or, } f'(re^{i\theta}).e^{i\theta} = \frac{1}{ir}\frac{\partial u}{\partial \theta} + \frac{i}{ir}\frac{\partial v}{\partial \theta}$$

$$\text{or, } f'(re^{i\theta}).e^{i\theta} = -\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{—— (II)}$$

from (I) & (II);

$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ are the req CR eqn in

$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$

~~polar form.~~

$\cancel{\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}}$

polar form

Q2)

Soln;

we have,

$f(z) = e^z$

$= e^{x+iy}$

$= e^{x} \cdot e^{iy}$

$= e^x(\cos y + i\sin y)$

$= e^x\cos y + i e^x \sin y$

$= u(x,y) + i\, v(x,y)$

$\therefore u(x,y) = e^x\cos y$, $v(x,y) = e^x\sin y$

$\therefore u_x = e^x\cos y$ $\therefore v_x = e^x\sin y$

$u_y = -e^x\sin y$ $v_y = e^x\cos y$

As, $u_x = v_y$ & $v_x = -u_y$. Hence, analytical fnn.
(CR eqn satisfied.)

OR (By defn)

we know,

At $z = z_0$,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{e^z - e^{z_0}}{z - z_0}$$

$$= e^{z_0} \lim_{(z-z_0) \to 0} \frac{e^{(z-z_0)} - 1}{(z - z_0)}$$

$$= e^{z_0} \times 1$$

$\therefore f'(z_0) = e^{z_0}$ is defined for any value of z_0.

Harmonic function:

A real valued function $\phi(x,y)$ is harmonic if it satisfies laplace eqn i.e;

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

If $f(z) = u + iv$ is analytic then u & v are harmonic fnns.

i.e $u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$

Then u & v are said to be harmonic conjugate of each other.

Ex 3

Sol>

we have;

$u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ —— (1)

$\therefore u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$u_{xx} = -\sin x \cosh y - 2\cos x \sinh y + 2$

&

$u_y = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$

$u_{yy} = \sin \cosh y + 2\cos x \sinh y - 2$

$\therefore u_{xx} + u_{yy} = 0$

$\therefore u$ is harmonic

let $f(z) = u + iv$ be an analytic fnn.

$\therefore$ By CR eqn;

$u_x = v_y$ & $u_y = -v_x$

So, $v_y = \cos x \cosh y - 2\sin x \cosh y + 2x + 4y$

Integrating both sides wrt y;

$v = \cos x \sinh y - 2\sin x \cosh y + 2xy + 2y^2 + h(x)$ —— (I)

Diff both sides wrt x;

$v_x = -\sin x \sinh y - 2\cos x \cosh y + 2y + h'(x)$

we know

$v_x = -u_y$.

So, $-\cancel{\sin x \sinh y} - \cancel{2\cos x \cosh y} + \cancel{2y} + h'(x) = -\cancel{\sin x \sinh y} - \cancel{2\cos x \cosh y} + \cancel{2y} - 4x$

$\therefore h'(x) = -4x$

$\therefore h(x) = -2x^2 + C$

So,

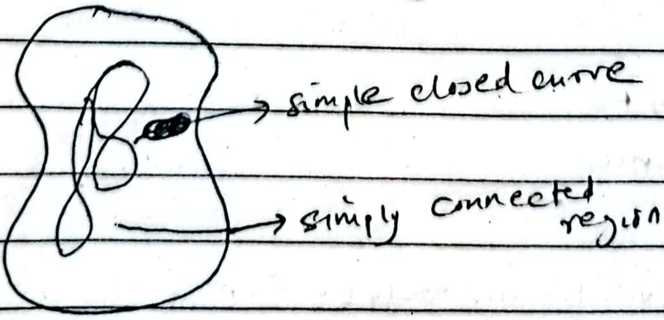
$v = \cos x \sinh y - 2\sin x \cosh y + 2xy + 2y^2 - 2x^2 + C$ —— (II)

Hence,

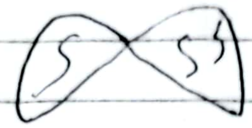
$u, v \to f(z) = u + iv$ (from (I) & (II))

Complex integration:

- Simple closed curve & simply connected region:



- Multiple connected region & multiple connected curve.

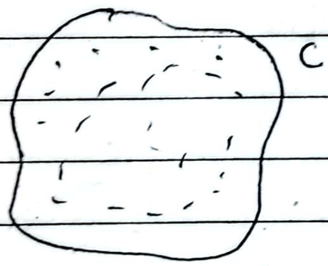


Theorm: Cauchy Integral Theorm.

Statement ~~Theorm:~~ ~~Cauchy~~. If $f(z)$ is a function analytic in and on the region enclosed by simple closed curve C and $f'(z)$ is continuous in C. Then,

$$\oint_C f(z)\,dz = 0$$

Proof:



let $f(z) = u(x,y) + iv(x,y)$ be fn defined for $z = x+iy$

$$\therefore \oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$

$$= \oint_C (u\,dx + iu\,dy + iv\,dx - v\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy) \quad \text{--- (i)}$$

$\because f'(z)$ is continuous i.e $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist such that;

From (i);

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \quad \text{--- (ii)}$$

∴ f(z) is analytic,

∴ u and v must satisfy CR eqn.

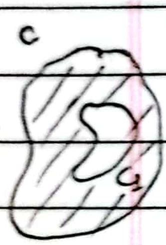
i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

∴ from (ii), we get;

$$\oint_C f(z)\,dz = 0$$

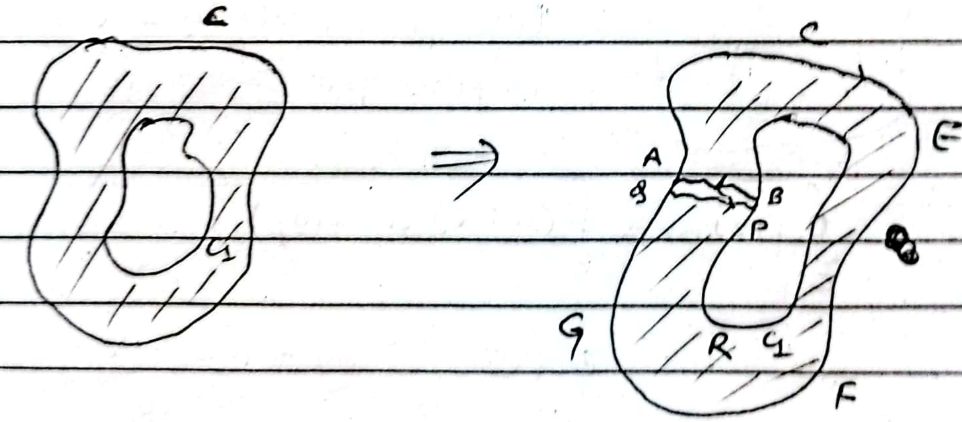
Extension of Cauchy integral theorm:

Statement: If f(z) is a fxn analytic in the region between two simple closed curve C and C_1. Then,



$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$

Proof→



Let C & C_1 be a simple closed curve and the region shown in the diagram is the region lying between C & C_1.

let us give a very minor cut AB, PQ, so ~~such~~ that BAEFGQPRB is the simple closed ~~curb~~ curve, such that f(z) is analytic.

∴ By cauchy integral theorm;

$$\oint_{BAEFGQPRB} f(z)\,dz = 0$$

or, $\int_{BA} f(z)\,dz + \oint_{C} f(z)\,dz + \int_{GP} f(z)\,dz + \oint_{C_1} f(z)\,dz = 0$

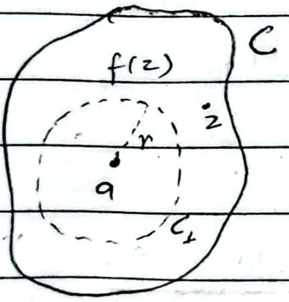
$$\therefore \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$

Cauchy Integral formula:

Statement: If $f(z)$ be a function analytic in a simply connected region C and 'a' is any point inside C, then,

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a).$$

Proof:



let $f(z)$ be analytic in the region shown in the figure.

Here, $f(z)$ is analytic in the given region 'C', but $\frac{f(z)}{z-a}$ is not analytic in the given region (discont. at 'a' भएर $\rightarrow \frac{1}{0}$)

Draw a circle 'C_1' with center at 'a' and radius 'r'

Hence, the fxn $\frac{f(z)}{(z-a)}$ is analytic in the region between C & C_1.

Hence, by extension of cauchy integral theorm; $\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(z)}{z-a}\,dz$ — (1)

Let us consider the point lying inside C_1 be $(z-a) = re^{i\theta}$

$\therefore$ From (i),

$$\oint_C \frac{f(z)}{z-a}\, dz = \oint_{C_1} \frac{f(a+re^{i\theta})}{\cancel{r}e^{i\theta}}\, i\cancel{r}e^{i\theta}\, d\theta$$

$$= i\oint_{C_1} f(a+re^{i\theta})\, d\theta$$

Now, as $r \to 0$, then θ tends to 0 to 2π.
(whole region C को भित्र, C_1 लाई सानो बनाको)

From (ii), we get;

$$\oint_C \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a)\, d\theta$$

$$\therefore \oint_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$$

Extension of Cauchy integral formula:

$$\oint_C \frac{f(z)}{z-a}\, dz = \frac{2\pi i}{0!} f(a)$$

$\longrightarrow$ Diff both sides,

$$\oint \frac{f(z)}{(z-a)^2}\, dz = \frac{2\pi i}{1!} f'(a)$$

$$\oint \frac{f(z)}{(z-a)^3}\, dz = \frac{2\pi i}{2!} f''(a)$$

$$\oint \frac{f(z)}{(z-a)^4}\, dz = \frac{2\pi i}{3!} f'''(a)$$

$$\vdots$$

$$\boxed{\oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^n(a)}$$

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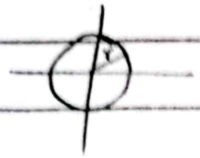
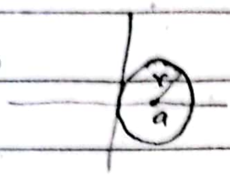
$|z| = r \Rightarrow$ 

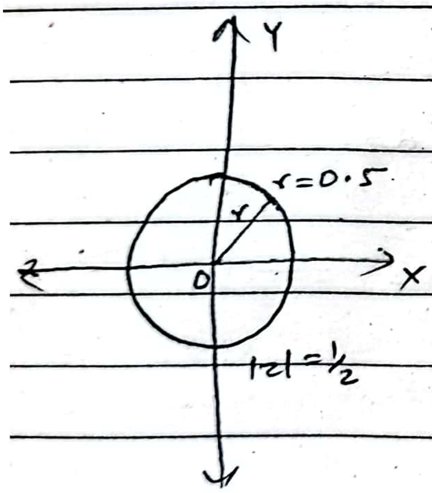
$|z-a| = r \Rightarrow$ 

we have,

given C is a circle.

$|z| = \frac{1}{2}$.

To evaluate : $\int_C \frac{3z^2+7z+1}{z+1}\, dz = ?$



Here, The given fxn is not analytic at/i.e. $z+1=0$

i.e $z=-1$.

Since, $z=-1$ doesnot lie in the simply connected region, $|z| = \frac{1}{2}$,

$\therefore f(z) = \frac{3z^2+7z+1}{z+1}$ is analytic in the simply connected region $|z| = \frac{1}{2}$

So,

$\therefore \int_C f(z)\, dz = 0$, By Cauchy integral theorm.

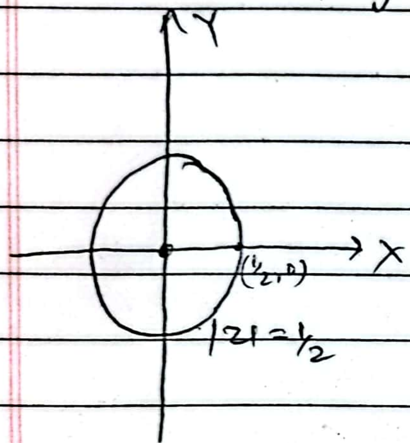
$\therefore \int_C \frac{3z^2+7z+1}{z+1}\, dz = 0$

(ii) we have,

The curve $C : |z| = \frac{1}{2}$

To evaluate : $\displaystyle\int_C \frac{2z+1}{z^2+z}\, dz = ?$

or, $\displaystyle\int \frac{2z+1}{z(z+1)}\, dz = ?$



Here,
The integral fnn, is not defined for $z=0$ or $z=-1$.

Here, $z = 0$ lies in $C : |z| = \frac{1}{2}$

Also, $z=-1$ doesnot lie in $C : |z| = \frac{1}{2}$.

Hence, $\therefore \displaystyle\int_C \frac{2z+1}{z(z+1)}\, dz = \int_C \frac{\frac{2z+1}{z+1}}{z}\, dz$

$\displaystyle = \int_C \frac{f(z)}{z-0}\, dz$ ——— (i)

is analytic everywhere in C

not analytic in C

⇓ we use Cauchy int. formula.

$\because f(z) = \frac{2z+1}{z+1}$ is fnn analytic in C,

$\therefore$ By Cauchy int. formula,

$\displaystyle\int_C \frac{2z+1}{z^2+z}\, dz = 2\pi i\, f(0)$

$= 2\pi i \times 1$

$= 2\pi i$

Singularity :

The point where fxn is not defined

$$f(z) = \frac{1}{z-2}$$

$z = 2$

Types of singularity :

1) **Isolated singularity :**

$z = a$

A singular point is said to be isolated singularity if its neighborhood doesnot contain any other singular point.

2) **Removable singularity :**

If the singular points of a fxn can be removed, then it is called removable singularity.

Eg :

$$f(z) = \frac{\sin(z-2)}{z-2}$$

$\lim_{z \to 2} f(z) = 1$ is defined and is finite.

3) **Essential singularity :**

point $z = a$ is essential singularity, if $\lim_{z \to a} f(z)$ doesnot exist.

Eg :

$$f(z) = \frac{\sin(z-2)}{(z-2)^2}$$

lim

* **Pole :** $f(z) = \dfrac{\sin(z-2)}{(z-2)^4}$

$$= \frac{1}{(z-2)^4}\left[(z-2) - \frac{(z-2)^3}{3!} + \frac{(z-2)^5}{5!} - \frac{(z-2)^7}{7!} + \cdots\right]$$

$$= \left(\frac{1}{(z-2)^3} - \frac{1}{3!(z-2)} + \frac{(z-2)}{5!} - \frac{(z-2)^3}{7!} + \cdots\right)$$

→ 2 is pole of order 3 and 1

~~has two poles~~

A singular point, $z=a$ is said to be pole of order 'm' if $\lim\limits_{z\to a} (z-a)^m f(z)$ exists.

* **Residue :**

Let $f(z)$ is a fun analytic everywhere except $z=a$. Then residue of $f(z)$ at $z=a$, is defined as:

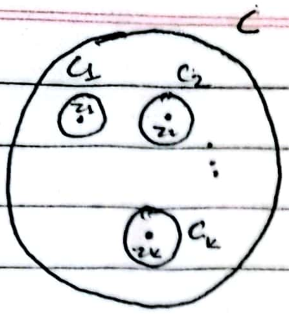
$$\left(\underset{z=a}{\mathrm{Re}}\, f\right) = \frac{1}{2\pi i}\int_C f(z)\,dz$$

Cauchy Residue theorm:

$$\int_C f(z)\,dz = 2\pi i\, \mathrm{Re}\left(\sum_{k=1}^{n} \underset{z=z_k}{f(z)}\right), \quad k=1,2,\ldots n \text{ (no. of singular points)}$$

Statement: Let $f(z)$ be a function analytic in simply connected region C, except some singular points $z_1, z_2, \ldots, z_k$. Then,

$$\int_C f(z)\,dz = 2\pi i\, \mathrm{Re}\left(\sum_{k=1}^{n} \underset{z=z_k}{f(z)}\right)$$



$f(z)$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_K} f(z)\,dz$$

$$= 2\pi i \left[\frac{1}{2\pi i}\int_{C_1} f(z)\,dz + \frac{1}{2\pi i}\int_{C_2} f(z)\,dz + \cdots + \frac{1}{2\pi i}\int_{C_K} f(z)\,dz \right]$$

$$\therefore \int_C f(z)\,dz = 2\pi i \left(\mathrm{Re} \sum_{k=1}^{n} \underset{z=z_k}{f(z)} \right)$$

Finding the residues:

(i) If $z=a$ is simple pole, then,

$$\left(\underset{z=a}{\mathrm{Re}\ f}\right) = \lim_{z\to a} (z-a)\cdot f(z)$$

(ii) If $z=a$ is a multiple pole of order m, then,

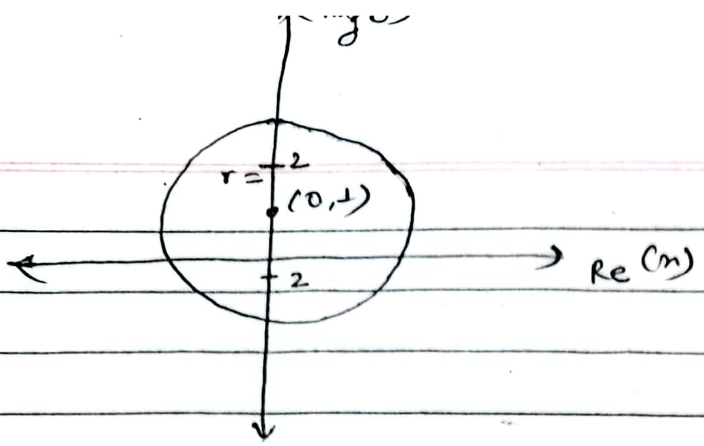
$$\left(\underset{z=a}{\mathrm{Re}\ f}\right) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}(z-a)^m f(z)}{dz^{m-1}}$$

Ex-8

Qsn 7) Soln; we have,

C is the circle $|z-i|=2$

To evaluate: $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = ?$



Here, the integral fxn is not analytical at $z+1=0$ & $z-2=0$

$\Rightarrow z=-1 \quad \Rightarrow z=2$

then,

$z=-1$ lies in C. Then,

$z=2$ doesnot lie in C.

Here $z=1$ is a pole of order 2.

$$\therefore \operatorname{Res}\left(\underset{z=-1}{f(z)}\right) = \frac{1}{(2-1)!} \lim_{z\to -1} \frac{d\left[(z+1)^2 . f(z)\right]}{dz}$$

$$= \lim_{z\to -1} \frac{d}{dz}\left[\cancel{(z+1)^2} \frac{(z-1)}{\cancel{(z+1)^2}(z-2)}\right]$$

$$= \lim_{z\to -1} \frac{d}{dz}\left(\frac{z-1}{z-2}\right)$$

$$= \lim_{z\to -1} \frac{(z-2)-(z-1)}{(z-2)^2}$$

$$= \lim_{z\to -1} \frac{-1}{(z-2)^2}$$

$$= \frac{-1}{9}$$

Hence, $\displaystyle\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = 2\pi i\left(\frac{-1}{9}\right) = \frac{-2\pi i}{9}$

Contour Integration:

* Real integration of type:

$$\int_C f(\cos\theta, \sin\theta)\, d\theta$$ by contour integration

STEP:

1. let C be a unit circle $|z| = 1$
2. Put $z = e^{i\theta}$

$$\Rightarrow dz = i e^{i\theta}\, d\theta$$

$$\Rightarrow \frac{dz}{iz} = d\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$= \frac{z + z^{-1}}{2}$$

$$= \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$= \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

3. Use Cauchy Residue theorm to solve the complex integration.

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Q1) we have,

$$I = \int_0^{2\pi} \frac{d\theta}{1+\frac{1}{2}\cos\theta}$$

$$= \int_0^{2\pi} \frac{2}{2+\cos\theta}\, d\theta \quad \text{—— (1)}$$

let c be a unit circle $|z|=1$

put $z = e^{i\theta}$

or, $dz = i.e^{i\theta}\, d\theta$.

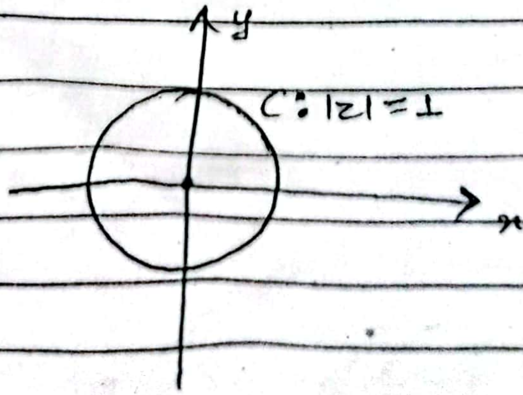
or $\frac{dz}{iz} = d\theta$

$\therefore \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$

From (1), we get;

$$I = \int_c \frac{2}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz} \; ; \text{ where } c : |z|=1$$

$$= \frac{4}{i}\int \frac{1}{z^2+4z+1}\, dz \quad \text{—— (II)}$$



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Here, $f(z) = \dfrac{1}{z^2+4z+1}$ is not analytic if:

$z^2 + 4z + 1 = 0$

i.e $z = -2 \pm \sqrt{3}$

$\therefore z = -2+\sqrt{3}$ & $-2-\sqrt{3}$ are the poles.

Here, $z = -2+\sqrt{3}$ lies in C

$z = -2-\sqrt{3}$ doesnot lie in C

$\therefore z = -2+\sqrt{3}$ is a simple pole

Hence,

By Cauchy Residue theorm:

From (ii),

$$I = \frac{4}{i} \times 2\pi i \times \left(\underset{z=-2+\sqrt{3}}{\operatorname{Res}} f(z) \right) \longrightarrow (iii)$$

Now,

$$\text{Residue } f(z) \text{ At } z=-2+\sqrt{3} = \underset{z=-2+\sqrt{3}}{\operatorname{Res}} f(z) = \lim_{z\to -2+\sqrt{3}} (z+2-\sqrt{3})\, f(z)$$

$$= \lim_{z \to (-2+\sqrt{3})} \cancel{(z+2-\sqrt{3})} \frac{1}{\cancel{(z+2-\sqrt{3})}(z+2+\sqrt{3})}$$

$$= \lim_{z\to(-2+\sqrt{3})} \frac{1}{(z+2+\sqrt{3})}$$

$$= \frac{1}{-\cancel{2}+\sqrt{3}+\cancel{2}+\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

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$\therefore$ From (iii), we get;

$$\int_0^{2\pi} \frac{1}{1+\frac{1}{2}\cos\theta}\, d\theta = \frac{4}{\cancel{i}} \times \cancel{2\pi i} \times \frac{1}{\cancel{2}\sqrt{3}}$$

$$= \frac{4\pi}{\sqrt{3}} \,//$$

* Real integration of the type:

$\int_{-\infty}^{\infty} f(x)\,dx$ by contour integration

$0 - 2\pi$

$0 - \pi$

$-\infty$ to ∞

0 to ∞

Theorm:

let $f(z)$ be a fnn defined on upper half of the plane and the poles of $f(z)$ and no poles lie on the real axis. Then,

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i . \Sigma R^+$$

where, ΣR^+ is the sum of the residues in upper half of the plane.

Provided that $zf(z) \longrightarrow 0$ as $|z| \longrightarrow \infty$

contour integration

* Improper integrals of the type:

$\int_{-\infty}^{\infty} f(x)dx \Big/ \int_0^{\infty} f(x)dx$ by contour integration

Q7) Soln;

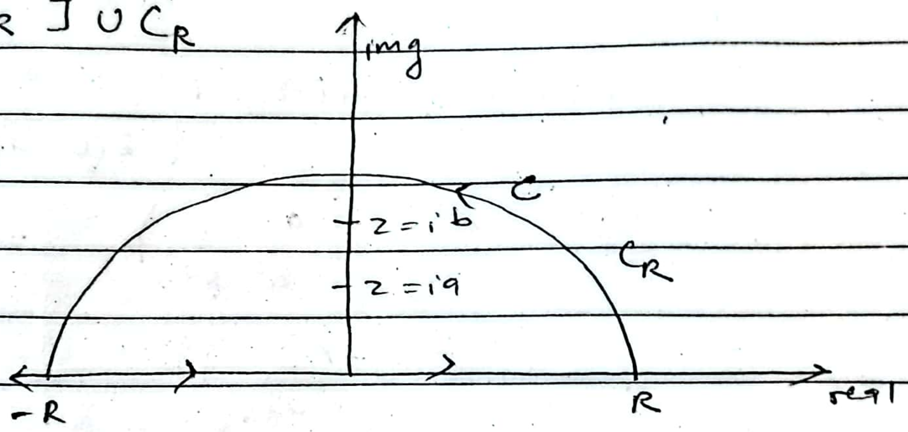
we have,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx \quad \text{---(i)}$$

let $\displaystyle\int_C f(z)\,dz = \int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz$

where', C is a contour which is made of semicircle C_R on upper ~~body~~ half of radius R and $[-R,R]$ in real axis be;

$C = [-R,R] \cup C_R$



Here, $f(z) = \dfrac{z^2}{(z^2+a^2)(z^2+b^2)}$ is not analytic if $(z^2+a^2)(z^2+b^2)$

$\Rightarrow z = \pm ia$ OR $z = \pm ib$

Here ia & ib lies in C

~~But z = -ia & -ib doesn~~ lie in C

simple pole

∴ By Cauchy residue theorem;

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residue at } ia \;\&\; ib \quad \text{---(ii)}$$

$$\therefore \left(\operatorname*{Res}_{z=ia} f(z)\right) = \lim_{z\to ia} (z-ia) f(z)$$

$$= \lim_{z\to ia} \cancel{(z-ia)} \frac{z^2}{(z+ia)\cancel{(z-ia)}(z^2+b^2)}$$

$$= \frac{(ia)^2}{2ia(b^2-a^2)} = \frac{ia}{2(b^2-a^2)}$$

$$\operatorname*{Res}_{z=ib} f(z) = \lim_{z \to ib} (z-ib)\, f(z)$$

$$= \lim_{z \to ib} (z-ib) \frac{z^2}{(z^2+a^2)(z+ib)(z-ib)}$$

$$= \frac{(ib)^2}{(a^2-b^2)\, 2ib}$$

$$= \frac{ib}{2(a^2-b^2)}$$

$\therefore$ from (ii),

we get,

$$\int_C f(z)\,dz = 2\pi i \times i \left(\frac{a}{2(b^2-a^2)} + \frac{b}{2(a^2-b^2)} \right)$$

$$= \frac{2\pi i^2}{2} \left(\frac{a-b}{b^2-a^2} \right)$$

$C = [-R, R] \cup C_R$

$$= (-\pi) \times \left(-\left(\frac{1}{a+b} \right) \right)$$

$$= \frac{\pi}{a+b}$$

or, $$\int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz = \frac{\pi}{a+b} \text{ ——— (iii)}$$

we know, $z f(z) \to 0$ as $|z| \to \infty$

$\therefore$ for $R \to \infty$, $\int_{C_R} f(z)\,dz = 0$

(यो हुँदा circle becomes st. line. So, integration of circle C_R = area = 0

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$\therefore$ Taking $\lim R \to \infty$ both sides of (iii);

$$\lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx = \frac{\pi}{a+b} //$$

{ Only for $-\infty$ to ∞ we change x to -

* $\int_0^{2\pi} \frac{\text{Trigonometric}}{\text{Algebraic}}$ / $\int_{-\infty}^{\infty} \frac{\text{Trigonometric}}{\text{Algebraic}}$ by contour integration

$e^{ix} = \cos x + i\sin x$

~~Trigonometric~~ → $\sin x \to$ img part of e^{ix}

$\cos x \to$ real part of e^{ix}

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+a^2}\,dx$$

Let $\int_C f(z)\,dz = \int_C \frac{\cos 2z}{z^2+a^2}\,dz$ = Real part of $\int_C \frac{e^{2iz}}{z^2+a^2}\,dz$

Transformation:

→ i) Translation
ii) Rotation
iii) Magnification
iv) Inverse
v) Magnification & rotation

I) Translation: $W = z + \alpha$ ($z = x+iy$; α — real or img or complex)

(Q) Using transformation, $w = z + 2 + i$ ($z = x+iy$), transform the rectangle $x=0$, $y=0$, $x=1$, $y=1$.

Soln;

we have, the rectangle:

$x=0,\ y=0,\ x=1,\ y=1.$

Then, we have the transformation

$$W = z + 2 + i$$
$$= x + iy + 2 + i$$
$$= x + 2 + i(y+1)$$
$$w = u + iv$$

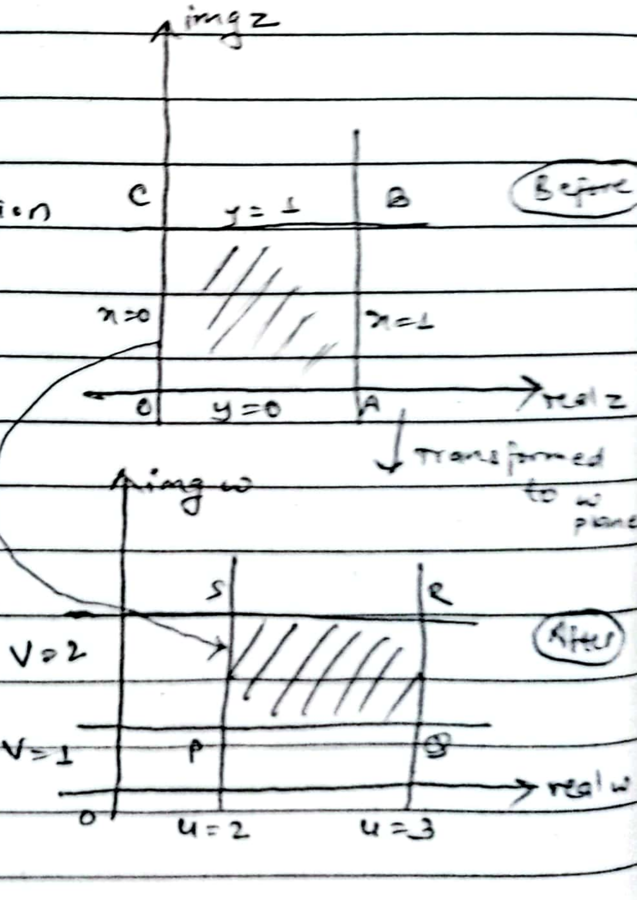
where, $u = x+2$; $v = y+1$

when, $x=0$, then $u=2$

when, $x=1$, then $u=3$

when, $y=0$, then $v=1$

when, $y=1$, then, $v=2$



$\rightarrow$ angle of rotation

Rotation : $w = e^{i\theta} . z$

Using transformation, $w = e^{i\pi/4} z$ (z rotated by $45°$)
Transform the triangle $x = 0$, $y = 0$, $x + y = 1$.

Sol^n,

we have, the triangle, $x = 0$, $y = 0$, $x + y = 1$.

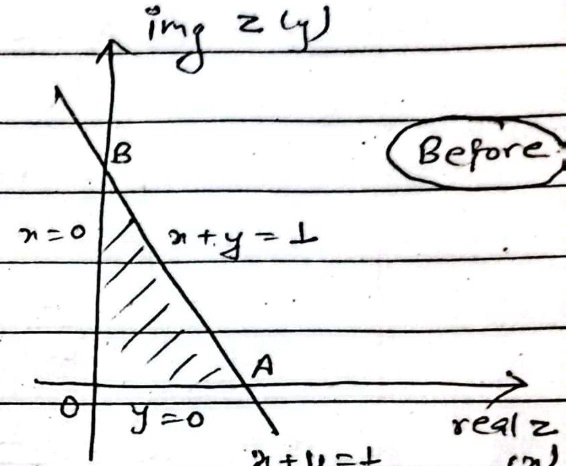
Then,

we have the transformation:

$$w = e^{i\pi/4} . z$$

or, $w = \left(\cos \pi/4 + i \sin \pi/4\right)(x + iy)$

$$= \left(\frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} y\right) + i \, \frac{1}{\sqrt{2}}(x + y)$$



$$w = u + vi$$

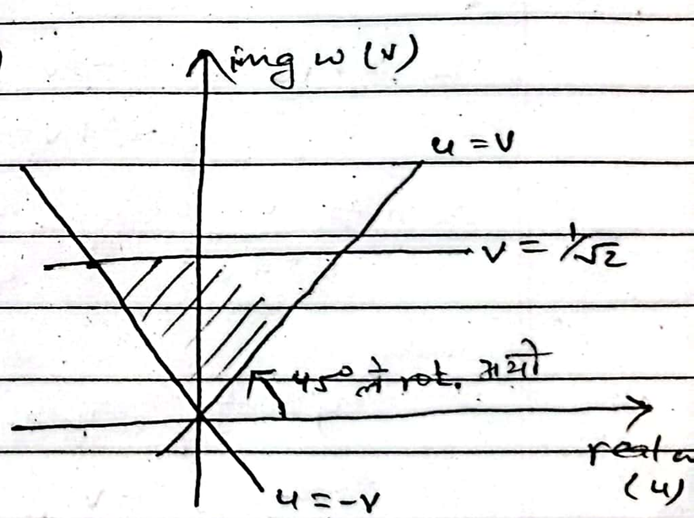
where, $u = \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} y$, $V = \frac{1}{\sqrt{2}}(x + y)$

$= \frac{1}{\sqrt{2}}(x - y)$

When, $x = 0$, $u = -y/2$, $v = y/\sqrt{2}$

when, $y = 0$, $u = x/\sqrt{2}$, $v = x/\sqrt{2}$

when, $x + y = 1$; then, $v = \frac{1}{\sqrt{2}}$



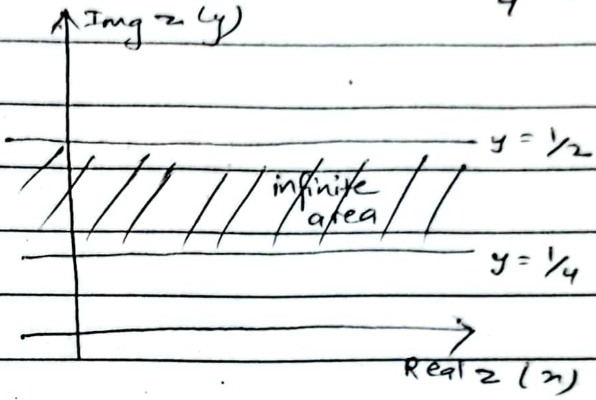
const.

Magnification : $w = cz$

Magnification & Rotation : $w = ce^{i\theta} . z$

5) Inverse : $w = \frac{1}{z}$

(9) Using transformation, $w = \frac{1}{z}$. Transform the strip: $\frac{1}{4} < y < \frac{1}{2}$



Before

Sol^n,

we have, the strip:

$$\frac{1}{4} < y < \frac{1}{2}$$

and,

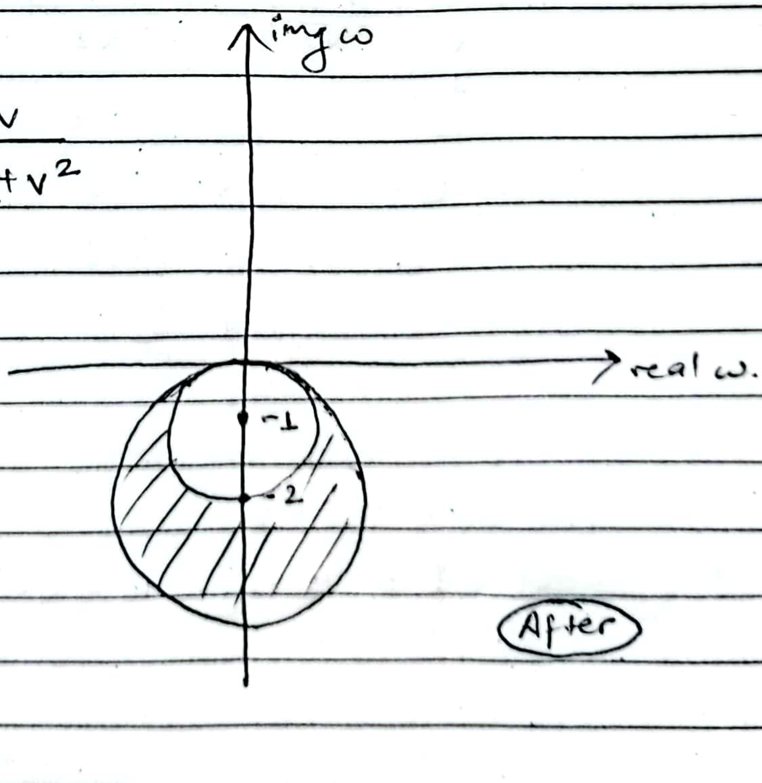
transformation: $w = \frac{1}{z}$.

or, $z = \frac{1}{w}$

or, $x + iy = \frac{1}{u + iv}$

or, $x + iy = \frac{u - iv}{u^2 + v^2}$

$\therefore x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$



After

when,

$y = \frac{1}{4}$. Then,

$$\frac{1}{4} = \frac{-v}{u^2+v^2}$$

or, $u^2 + v^2 + 4v = 0$

or, $u^2 + (v+2)^2 = 4$

$\Rightarrow$ radius = 2

Center : (0, 2)

when,

$y = \frac{1}{2}$. Then, $\frac{1}{2} = \frac{-v}{u^2+v^2} \Rightarrow u^2 + (v+1)^2 = 1$

$\Rightarrow$ radius = 1

center = (...

* conformal ~~unconformed~~ mapping preserves magnitude & direction

✓ imp # Linear Fractional Transformation:

↳ (Bilinear transformation)

→ A transformation $w = \dfrac{az+b}{cz+d}$; $ad - bc \neq 0$ and a, b, c, d are complex constants, is a ~~lin~~ bilinear transformation.

→ Preserves ratio of 4 points:

$$z, z_1, z_2, z_3 \xrightarrow{\text{image}} w, w_1, w_2, w_3$$

$$\frac{z - z_1}{z_1 - z_2} \times \frac{z_2 - z_3}{z_3 - z} = \frac{w - w_1}{w_1 - w_2} \times \frac{w_2 - w_3}{w_3 - w}$$

(Q) Find linear transformation which maps the points of the following:

1) $z_1 = 2, z_2 = i, z_3 = -2$ into $w_1 = 1, w_2 = i, w_3 = (-1)$

we know,

$$\frac{z - z_1}{z_1 - z_2} \times \frac{z_2 - z_3}{z_3 - z} = \frac{w - w_1}{w_1 - w_2} \times \frac{w_2 - w_3}{w_3 - w}$$

or, $$\frac{z-2}{2-i} \times \frac{i+2}{-2-z} = \frac{w-1}{1-i} \times \frac{i+1}{-1-w}$$

or, $$\frac{\quad}{4 - i^2} \times (i+2)^2 -$$

Taylor & Laurent Series :

Q) Find the Taylor series & Laurent series of

$$f(z) = \frac{1}{(z+1)(z+3)}$$ in the region

i) $|z| < 1$

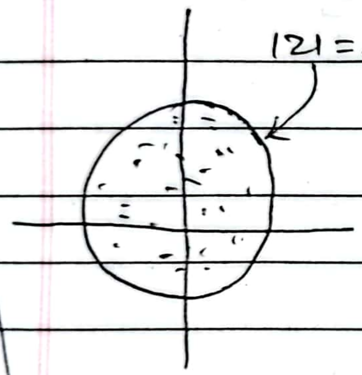
ii) $1 < |z| < 2$

iii) $|z| > 3$

iv) $0 < |z+1| < 2$

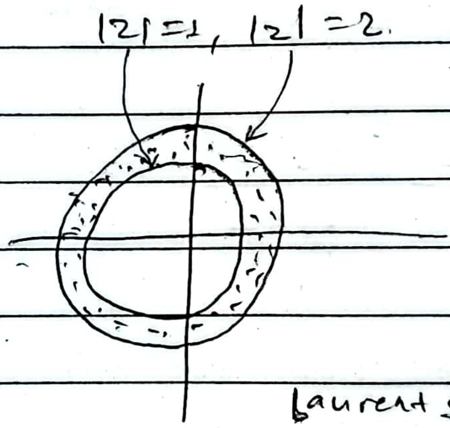
Sol^n;

(i) $|z| < 1$



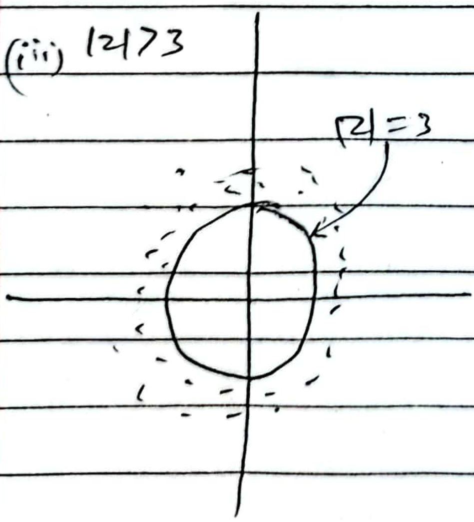
Taylor's series

(ii) $1 < |z| < 2$



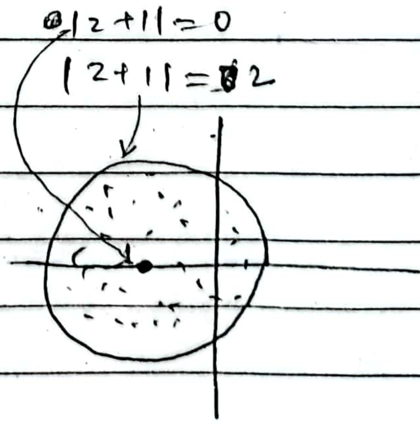
Laurent series

(iii) $|z| > 3$



Laurent series

(iv) $0 < |z+1| < 2$



~~Taylor~~ Laurent series.

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

Taylor series में never -ve power of variable

(i) we have,

~~taylor~~ $f(z) = \dfrac{1}{(z+1)(z+3)}$ in the region $|z|<1$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+z} - \frac{1}{3(1+z/3)}\right]$$

$$= \frac{1}{2}\left[(1+z)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$\frac{1}{z+1} \Rightarrow (z+1)^{-1}$

$z+3 \Rightarrow 3\left(1+\frac{z}{3}\right)$

$\Rightarrow 2\left(1+\frac{3}{2}\right)$

$$\therefore f(z) = \frac{1}{2}\left[(1-z+z^2-z^3+\dots) - \frac{1}{3}\left(1 - z/3 + \frac{z^2}{9} - \frac{z^3}{27} + \dots\right)\right]$$

(ii) we have,

$f(z) = \dfrac{1}{(z+1)(z+3)}$ in region $1<|z|<2$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{3(1+z/3)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$\therefore f(z) = \frac{1}{2}\left[\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}+\dots\right) - \frac{1}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\dots\right)\right]$$

(iii) we have,

$$f(z) = \frac{1}{(z+1)(z+3)} \quad \text{in region } |z|>3$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{z\left(1+\frac{3}{z}\right)}\right]$$

$$\therefore f(z) = \frac{1}{2z}\left[\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{3}{z}\right)^{-1}\right] //$$

(iv) we have,

$$f(z) = \frac{1}{(z+1)(z+3)} \quad \text{in region } 0<|z+1|<2$$

put $(z+1)=t$

Then, $0<|z+1|<2$

$\Rightarrow 0<|t|<2$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{t} - \frac{1}{t+2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{t} - \frac{1}{2\left(1+t/2\right)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{t} - \frac{1}{2}\left(1+\frac{t}{2}\right)^{-1}\right]$$