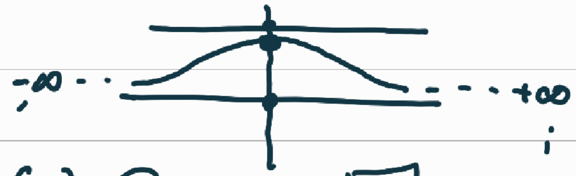
Cauchy Problems

21 / 01 / 2023

HOW TO APPROACH:

1. DOMAIN OF f
2. C^1 IN $\mathrm{Dom}(f)$ → LOCAL SOL^N
3. CHECK GLOBAL IF $\mathrm{dom}\, f \Rightarrow \underset{x}{\square} \times \underset{y}{\mathbb{R}}$
4. $\lim_{y\to\infty} f_y$ = FINITE → GLOBAL SOL^N.
5. CONST. SOL^N
6. MONOTONICITY
7. ASYMPTOTES. →



$\lim_{x\to-\infty} f(x)$ @ $y \to \square$

$\lim_{x\to+\infty} f(x)$

8. EXPLICIT SOL^N.

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JAN 2022

1. Let $k \in \mathbf{R}$, consider the following Cauchy Problem

$$\begin{cases} y'(x) = y(x)(y(x)-1)^{1/3} \\ y(0) = k. \end{cases}$$

a) Discuss local and global existence and uniqueness of solutions, depending on k.

b) Draw the graph of the solutions, defining the domain, studying the monotonicity, the convexity, and limits at the extrema of the domain for

b1) $k < 0$,

b2) $k \in (0,1)$,

b3) $k > 1$,

b4) $k = 1$.

SOL^N

LET $\left.\begin{array}{l} f(x) = y(y-1)^{1/3} \\ y(0) = k \end{array}\right\}$

- $\text{dom}(f) = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

- $f \in C^1(\text{dom } f)$

  By Local Cauchy - Lipz $\exists!$ local sol^n $y : \mathcal{U}(0) \to \mathbb{R}$

- CHECK FOR GLOBAL AS $\text{dom } f = \mathbb{R}^2$

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{3} \frac{y}{(y-1)^{2/3}} + (y-1)^{1/3}$$

$$\lim_{y \to \infty} f_y = \infty$$ SO, NO GLOBAL SOL^N.

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- FOR CONSTANT SOL^N

$$f(k) = 0$$

$$y(y-1)^{1/3} = 0$$

$$y \equiv 0 \quad \text{OR} \quad y \equiv 1$$

- MATCHING CONSTANT SOL^N WITH I.C.

GIVEN $y(0) = k$

SO, $y \equiv 0 \rightarrow k \equiv 0$

$y \equiv 1 \rightarrow k \equiv 1$

$y < 0 \rightarrow k < 0$

$0 < y < 1 \rightarrow 0 < k < 1$

$y > 1 \rightarrow k > 1$

IN QUEST^N

BCOZ WE HAVE 2 CONDN AND 2 SOL^N WONT INTERSECT b1) $k < 0$,

b2) $k \in (0,1)$,

b3) $k > 1$,

b4) $k = 1$.

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MONOTONICITY

→ FOR $K<0$ OR $y<0$

$f(x) = y(y-1)^{1/3}$

$= (-)(-)$

$= +VE$

→ FOR $K \in (0,1)$

$f(x) = \frac{1}{2}\left(\frac{1}{2}-1\right)^{1/3}$.

$= (+)(-)$

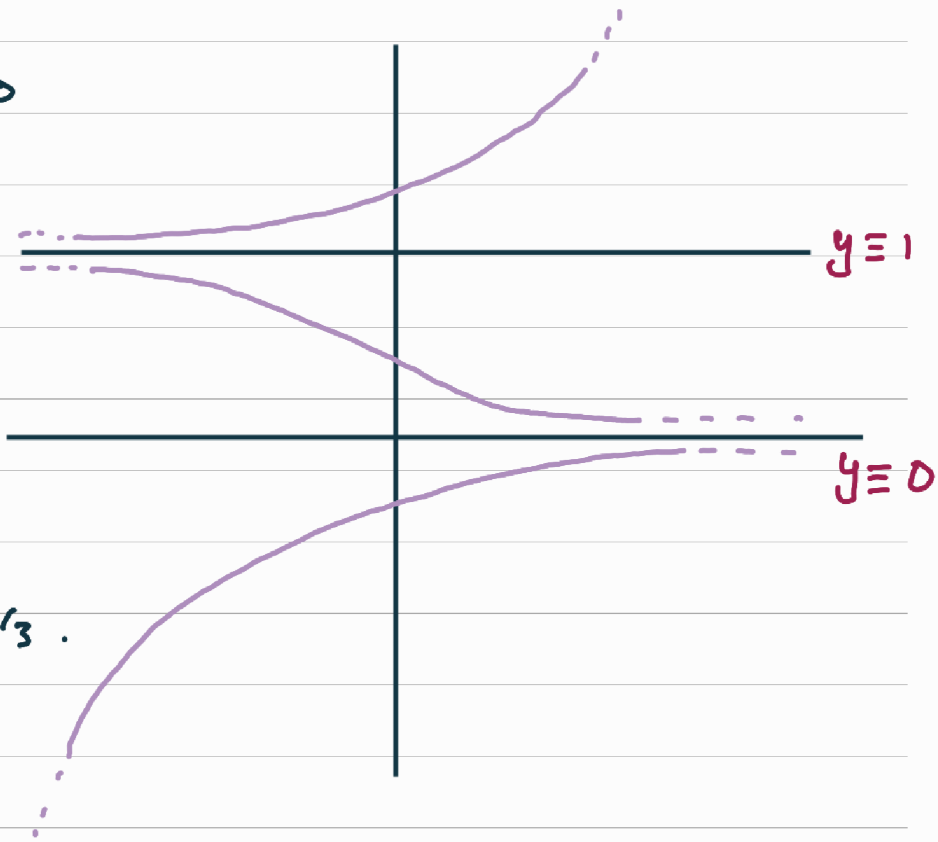
$= -VE$

→ FOR $K > 1$

$f(x) = 2(2-1)^{1/3}$

$= (+)(+)$

$= +VE$



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ASYMPTOTES

- FOR $K > 1$

$\rightarrow \lim_{x \to T_3} f(x) = \infty$

$\rightarrow \lim_{x \to -\infty} f(x) = T_1$

- FOR $K \in (0,1)$

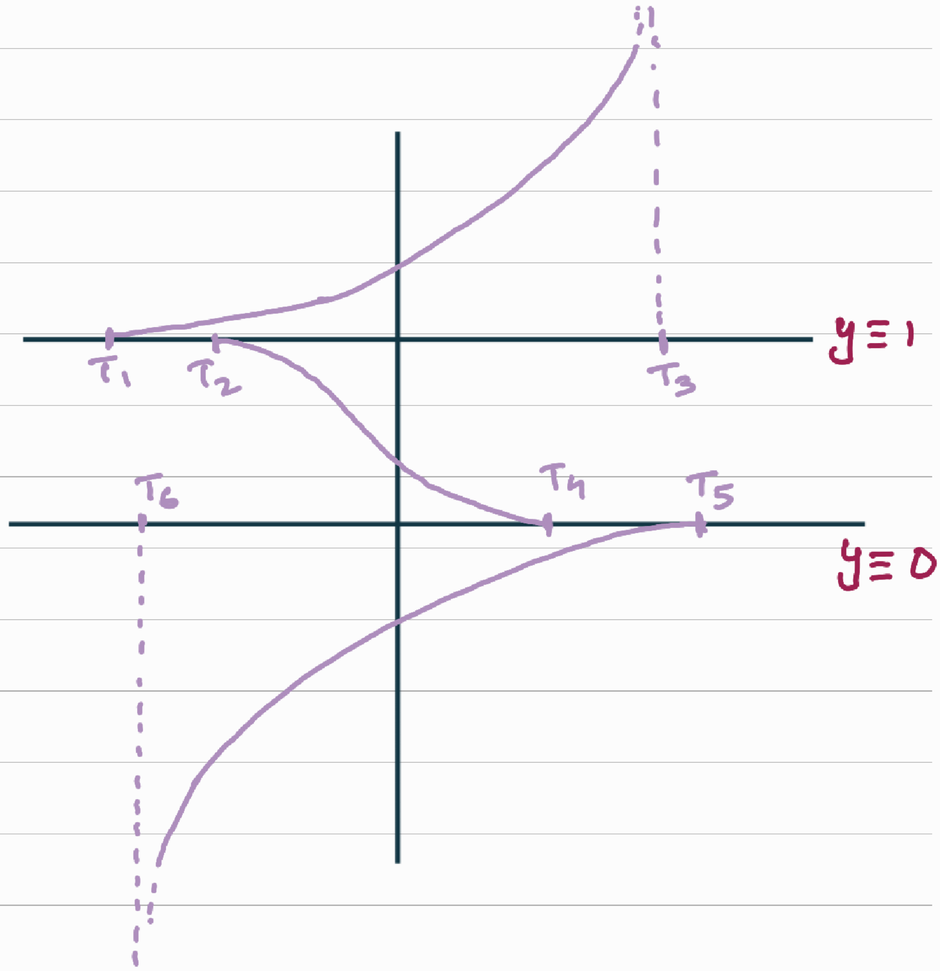
$\rightarrow \lim_{x \to +\infty} f(x) = T_4$

$\rightarrow \lim_{x \to -\infty} f(x) = T_2$

- FOR $K < 0$

$\rightarrow \lim_{x \to +\infty} f(x) = T_5$

$\rightarrow \lim_{x \to T_6} f(x) = -\infty$



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FEB 2022

1. Let $k \in \mathbf{R}$, consider the following Cauchy Problem

$$\begin{cases} y'(x) = \arctan[(2-y^2)(x^2+xy)] \\ y(0) = k. \end{cases}$$

a) Discuss local and global existence and uniqueness of solutions, depending on k.

b) Draw the graph of the solutions, defining the domain, studying the monotonicity, and limits at the extrema of the domain for

b1) $k = 0$,

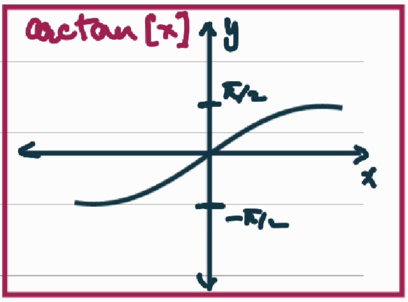
b2) $k = 1$,

b3) $k = -1$.

SOLN

$$f(x) = \arctan[(2-y^2)(x^2+xy)]$$
$$y(0) = K$$

- $\text{dom} f = \mathbb{R} \times \mathbb{R}$



- $f \in C^1(\text{dom} f)$

so, By local cauchy - Lipz

$\exists!$ local soln

- CHECK FOR GLOBAL (BCOZ $\text{dom} f = \mathbb{R} \times \mathbb{R}$)

$$f_y = \frac{(-2y)(x^2+xy) + x(2-y^2)}{[(2-y^2)(x^2+xy)]^2 + 1}$$

$$\simeq \frac{y^2 + y^2}{(y^3)^2} \simeq \frac{y^2}{y^6} \simeq \frac{1}{y^4}$$

$$\lim_{y \to \infty} |f_y| = 0 \rightarrow \text{FINITE.}$$

HENCE BY CAUCHY-LIPS. $\exists!$ GLOBAL SOL^N.

$$\text{dom}(f_y) = \mathbb{R} \times \mathbb{R}$$

0 CONST. SOL^N.

$$\arctan\left[(2-y^2)(x^2+xy)\right] = 0$$

$$(2-y^2)(x^2+xy) = 0$$

$y^2 = 2$ $\qquad$ $x^2 + xy = 0$

$y = \pm\sqrt{2}$ $\qquad$ $\cancel{x}y = -x^{\cancel{2}}$

$y = -x$

IF $\quad y = \sqrt{2}$ $\qquad$ $k = \sqrt{2}$

$y = -\sqrt{2}$ $\qquad$ $k = -\sqrt{2}$

$y > \sqrt{2}$ $\qquad$ $k > \sqrt{2}$

$y < \sqrt{2}$ $\qquad$ $k < \sqrt{2}$

$\arctan\left(\underbrace{(2-y^2)(x^2+xy)}_{A}\right)$

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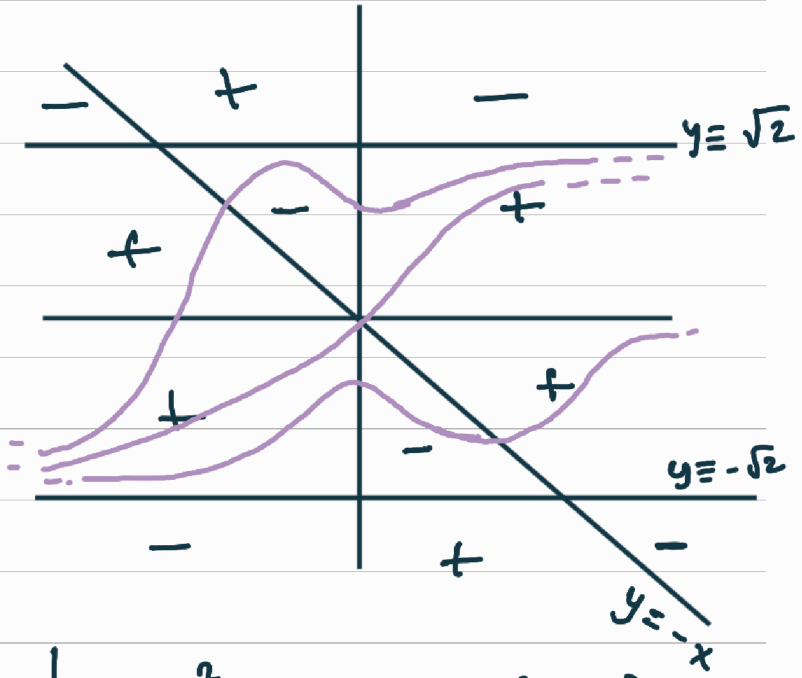
o MONOTONICITY

$k=0 \quad k=1 \quad k=-1$



$y' > 0$ IF $A > 0$

IF $\left.\begin{aligned}(2-y^2) &> 0 \\ (x^2+xy) &> 0\end{aligned}\right\}$

OR $\left.\begin{aligned}(2-y^2) &< 0 \\ (x^2+xy) &< 0\end{aligned}\right\}$

$y' < 0$ IF $A < 0$

IF $\left.\begin{aligned}(2-y^2) &> 0 \\ (x^2+xy) &< 0\end{aligned}\right\}$

OR $\left.\begin{aligned}(2-y^2) &< 0 \\ (x^2+xy) &> 0\end{aligned}\right\}$

$(2-y^2) > 0$ (+)

$y^2 < 2$

$y < \pm\sqrt{2} \Rightarrow -\sqrt{2} < y < \sqrt{2}$

$-\sqrt{2} \quad 0 \quad \sqrt{2}$ $\Rightarrow y < \sqrt{2}$

$(2-y^2) < 0$ (−)

$y^2 > 2$

$y > \pm\sqrt{2}$

$\sqrt{2} < y < -\sqrt{2}$

$-\sqrt{2} \quad 0 \quad +\sqrt{2}$

$x^2+xy > 0$ (+ve)

$x(x+y) > 0$

$x > 0 \quad x+y > 0 \rightarrow *$

$x^2+xy < 0$ (−ve)

$x(x+y) < 0$

$\left.\begin{aligned}x < 0 &\text{ OR } x+y > 0 \\ x > 0 &\text{ OR } x+y < 0\end{aligned}\right\}$

$x < 0 \quad x+y < 0$ +ve

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