GRAVITATIONAL WAVE SIGNAL DETECTION AND SOURCE RECONSTRUCTION USING A NETWORK OF ADVANCED-LIGO DETECTORS

B-Tech Project

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November 21, 2013

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ABSTRACT

One of the outcomes of Einstein's General Theory of Relativity is that it predicted that if two binary stars spiral into each other, the energy released during the inward motion is in the form of gravitational waves (radiation) which travel outward from the source. These waves can be detected on Earth through LIGO but there has been no direct detection yet. The reason is a very low magnitude signal buried in noise which limits the sensitivity of the detector. This project aims at implementing matched filtering technique to extract the signal from the noise.

ACKNOWLEDGMENTS

I wish to acknowledge and express my sincere gratitude towards the efforts put in by my mentor Dr Anand Sengupta in helping me out with the formulation of this project and waiting patiently while I learnt about the subject which is an entirely new field to me. I was suitably helped by him at multiple times with theory and its successful implementation.

I would also like to thank the extended help offered by my batch mates Akash Bapat, Shashank Tyagi and Sudhamsu Krishna and for giving their valuable inputs whenever I was stuck at any point.

This is a preliminary stage in the project and I wish to continue this work for as long as possible.

Nishank Jain

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INTRODUCTION

What is a Gravitational Wave?

Gravitational Waves (GW) was one of the earliest predictions of the General Theory of Relativity by Einstein. They are ripples created in spacetime due to the motion of an object which changes the curvature of spacetime. If the object accelerates, the changes in curvature (ripples) propagate at the speed of light. These are known as Gravitational Waves. The strength of the waves decreases as the inverse of the distance to the source.

Why do we need to detect it?

According to theory, GW will be able to provide information about black holes and other interesting objects in the Universe which can shed light in the early stages of the development of the Universe. This is not possible with conventional astronomy because in the early stages (before cosmological recombination) the Universe was opaque to electromagnetic radiation, therefore setting the limits on observation by traditional means such as telescopes. If the GW are detected, it will also be a test of the General Theory of Relativity.

How to detect it?

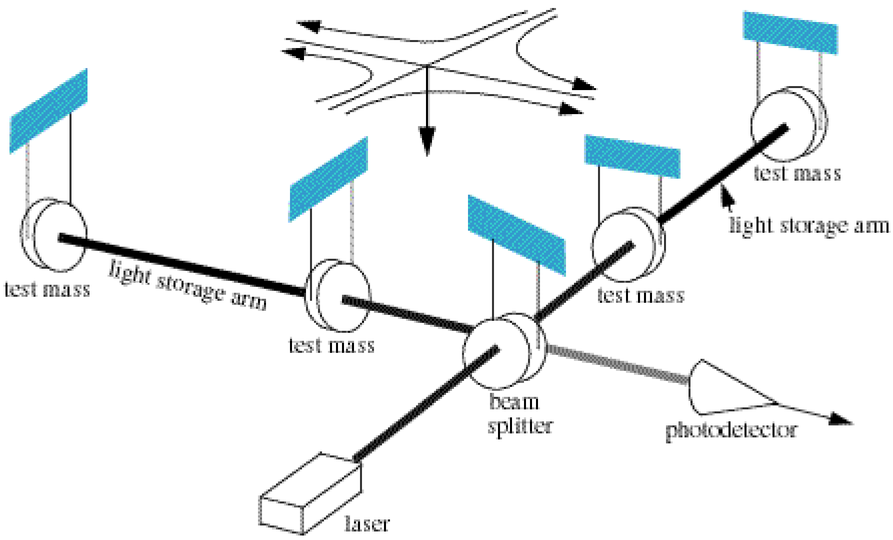
GW detectors have been developed and installed in 3 places:

1. LIGO Livingston Observatory (Louisiana)
2. LIGO Hanford Observatory (Washington) – two detectors
3. VIRGO (Cascina, Italy)

Sources of Gravitational Waves

GW can arrive from inspiralling binary star systems which can compose of white dwarfs, neutron stars, black holes etc. They can also arrive from non-symmetric supernovae explosions.

PRINCIPLE OF GW DETECTION



A power-recycled Michelson interferometer with Fabry-Perot arms

The laser beam of up to 200 W is passed through an optical mode cleaner. It is then split into two paths at the beam splitter into the Fabry-Perot arms which store the beams to increase the effective path length. Due to a GW, there is a change in the length of the cavities which causes the beams to become slightly out of phase with the incident beam. The system is set up such that the beams should destructively interfere at the photodetector. But due to this phase difference, there will a small interference which can be measured.

Light is also affected by the gravitational waves so the part of light which is not affected by the GW is returned to the interferometer using a power-recycling mirror. The detection process is affected by different types of noises which will be described later. These make the detection a challenging job. This is the reason for installing multiple detectors to compare the signals and reduce the impact of noise.

PROBLEMS IN DETECTION

Detecting a GW is a great engineering challenge. There are primarily two problems in the detection of the GW wave – Amplitude and Frequency. The frequency range of the GW arriving at Earth is 10^{-16} to 10^{4} Hz. It is desired to detect the GW at earlier stages of inspiral that is at low frequencies. But at low frequencies, noise can result in apparent phase shift of laser light in the detector. The strain produced in the interferometer is defined as:

$$s(t) = \frac{\Delta L_x - \Delta L_y}{L}$$

This signal has two major additive components: GW signal (h(t)) and noise sources (n(t)). The noise appearing here is of three fundamental types:

1. Seismic Noise: $f < 40$Hz
2. Suspension Thermal Noise: 40Hz $< f <$ 200Hz
3. Photon Shot Noise: $f >$ 200Hz

The magnitude of the noise is comparable to the signal so it observed that the signal is buried in the noise. To extract the signal from the noise, matched filtering is performed on the obtained signal. This is described in the next section.

MATCHED FILTERING

The basic feature of a matched filter is to correlate the unknown signal with a known signal (template) to detect whether the template is present in the unknown signal. This maximizes the Signal-to-Noise ratio as follows:

Let the template be h. Let the observed signal be x. Le the desirable signal be s. Let the additive noise be n. Le the output of the filter be y.

$$X = s + v$$

$$y = hx = hs + hv = y_s + y_v$$

$$SNR = \frac{|y_s|^2}{E|y_v|^2}$$

The numerator is the power of the output signal due to the desired signal and the denominator is the power of the output signal due to the noise signal. To maximize SNR we need to choose h. In the case of detection of GW signals, the template h chosen is a chirp waveform which evolves as follows:

$$\tilde{h}_c(f) = \frac{2GM_\odot}{(1\,\mathrm{Mpc})c^2}\left(\frac{5\mu}{96M_\odot}\right)^{\frac{1}{2}}\left(\frac{M}{\pi^2 M_\odot}\right)^{\frac{1}{3}} f^{-\frac{7}{6}}\left(\frac{GM_\odot}{c^3}\right)^{-\frac{1}{6}} e^{i\Psi(f;M,\eta)}$$

$$\tilde{h}_s(f) = i\tilde{h}_c(f)$$

$$\Psi(f;M,\eta) = 2\pi f t_c - 2\phi_0 - \pi/4 + \frac{3}{128\eta}\left[x^{-5} + \left(\frac{3715}{756} + \frac{55}{9}\eta\right)x^{-3} - 16\pi x^{-2} + \left(\frac{15\,293\,365}{508\,032} + \frac{27\,145}{504}\eta + \frac{3085}{72}\eta^2\right)x^{-1}\right]$$

$$x = (\pi M f G/c^3)^{1/3}$$

Where,

$h_c(f)$ = cosine chirp
$h_s(f)$ = sine chirp
G = Gravitational Constant
M_O = Solar Mass
c = Speed of Light
M = Total Mass
μ = Reduced Mass
f = Gravitational Wave Frequency
Ψ = Phase Evolution
t_c = End Time of the inspiral
η = μ/M
Φ(t) = Orbital Phase of the binary

The above equation is a Stationary Phase Approximation (SPA) used to express the chirp waveforms directly in the frequency domain. We need Fourier Transforms of the time-domain cosine and sine chirps to perform filtering. To reduce the complexity in computation, the SPA is taken. The waveform is terminated at the Gravitational Wave Frequency f_{isco} where

$$f_{isco} = \frac{c^3}{6\sqrt{6}\pi GM}$$

The above chirp is generated in MATLAB and it looks as in Figure *1*.

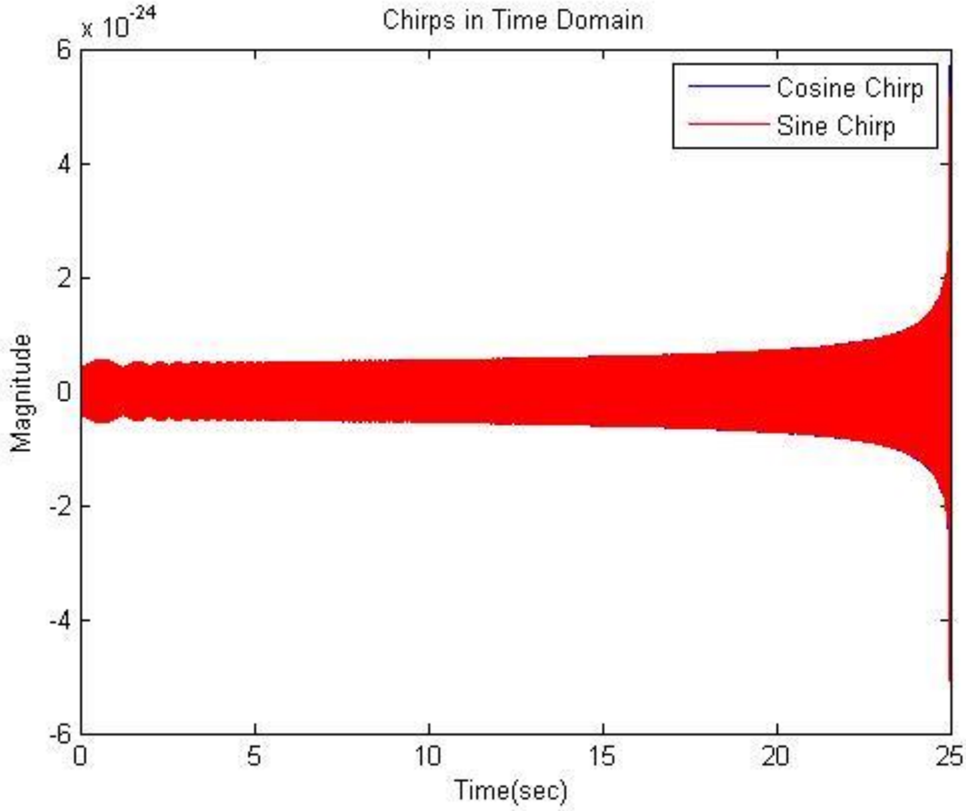


Figure 1: Time Domain representations of the cosine and sine chirps as generated from MATLAB using the chirp function.

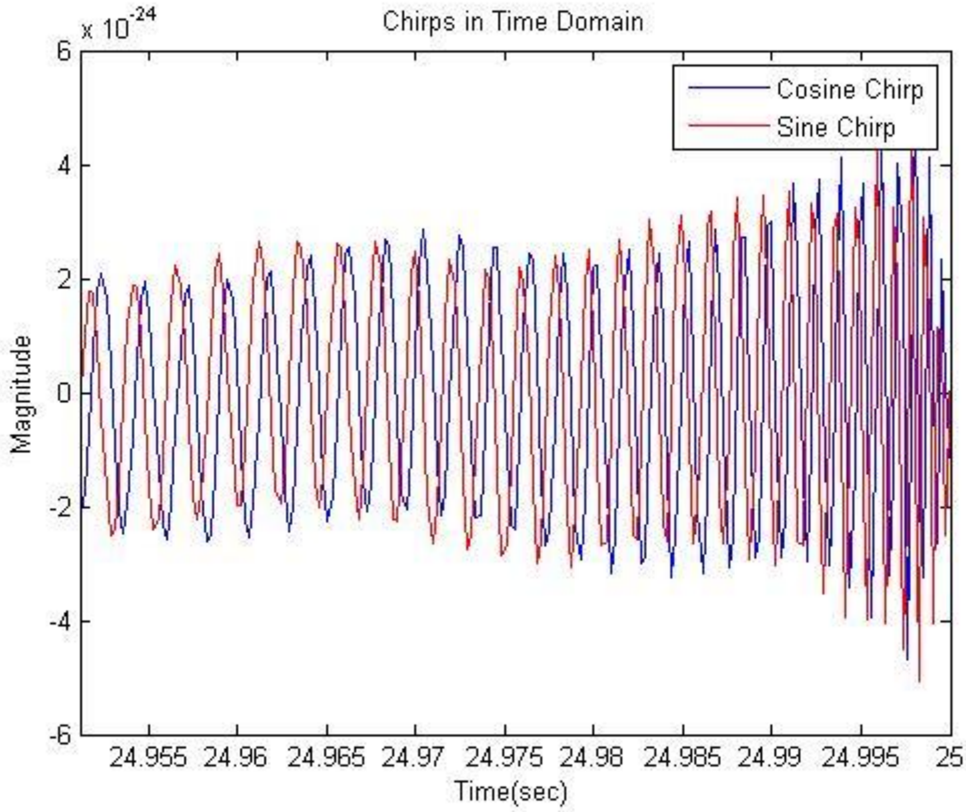


Figure 2: Magnified sine and cosine chirps showing 90^o phase difference

On magnification we can see both the chirp signals 90° out of phase as shown in Figure *2*. Once the chirps are generated, next we move onto the generation of noise.

GENERATION OF LIGO NOISE

There are three different noise curves that we can generate for Advanced LIGO, Initial LIGO and VIRGO. The amplitude spectral densities[1] of these are plotted in MATLAB and are shown in Figure *3*.

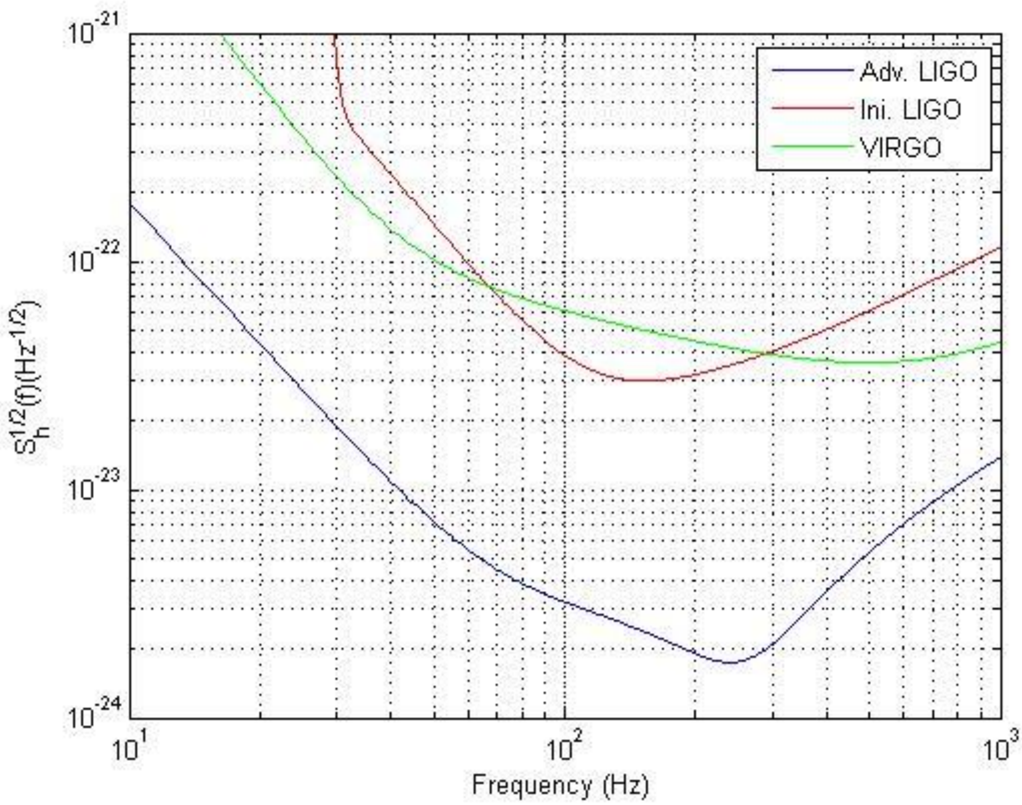


Figure 3: Amplitude Spectral Densities of the noise in three different detectors - Advanced LIGO Detector, Initial LIGO Detector and VIRGO

In this project we work with Initial LIGO, once we have the amplitude spectral density (ASD), we generate a random phase between ($-\pi$,π) and take the product of the amplitude and the phase to generate noise in frequency domain. The noise is then inverted to account for negative frequencies and then concatenated with the former noise to generate a symmetric noise distribution in frequency domain. This magnitude of this distribution is plotted and the ASD of Initial LIGO is overlaid onto it to generate the graph in Figure *4*. We can see that the noise follows the distribution of ASD as expected. We then take an Inverse Fourier Transform of this distribution to obtain the noise in time domain which is shown in Figure *5*.

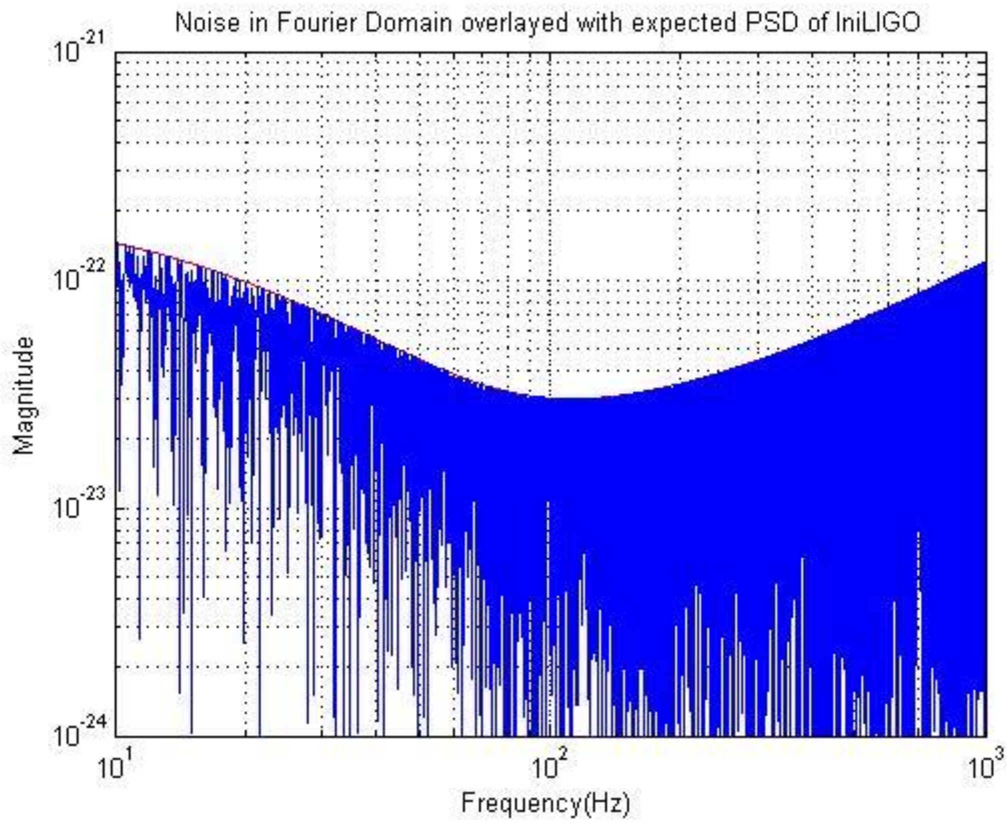


Figure 4: Frequency domain noise (amplitude) with the expected Amplitude Spectral Density (in red) of Initial LIGO

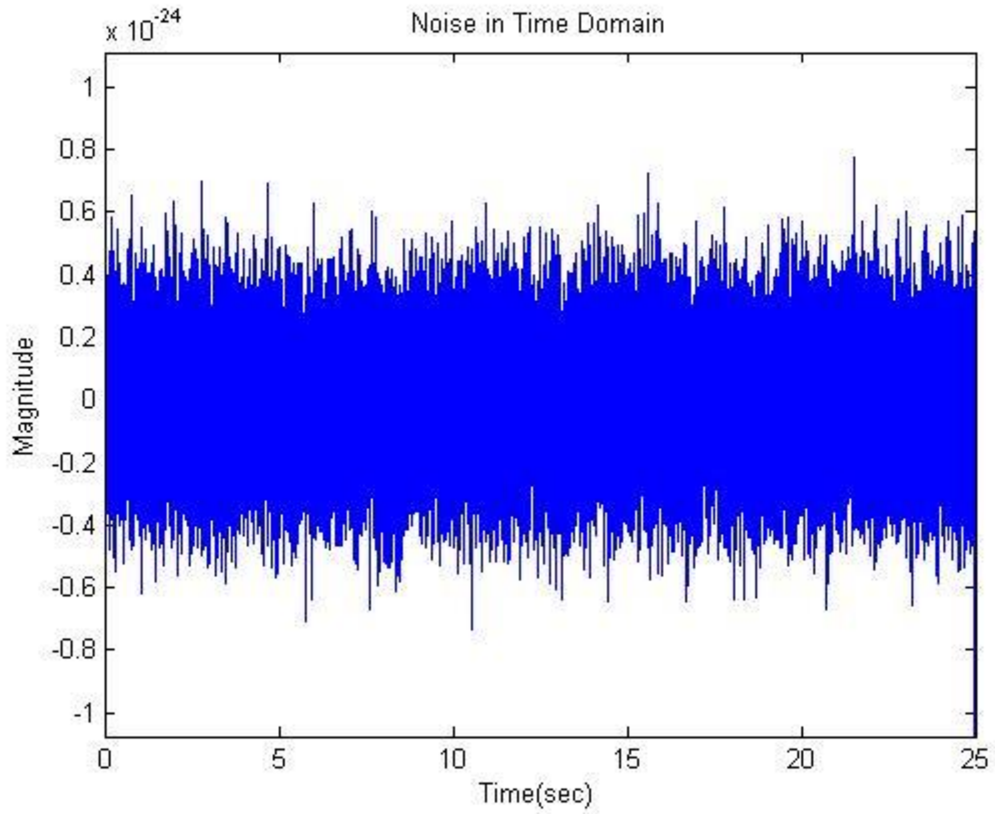


Figure 5: Initial LIGO Noise in time domain generated in MATLAB

Now that we have the noise as well as the chirp signals in both the frequency and time domains, we now move onto the next step i.e. designing a digital matched filter.

DESIGNING A DIGITAL MATCHED FILTER

The matched filter gives the maximized SNR ρ(t). ρ(t) is the sum of the squares of two Gaussian random variables x(t) and y(t) normalized by σ^2. To get ρ(t) from ρ(f), there can be two ways:

1. Compute two real inverse FFTs one each for x(t) and y(t).
2. A single complex inverse FFT.

It is found that doing a single complex inverse FFT is more efficient than performing two real FFTs.

So we first write the normalization constant as follows:

$$\begin{aligned}\sigma^2 &= 2\frac{1}{N\Delta t}\sum_{k=0}^{N-1}\frac{\tilde{h}_c(f_k)\tilde{h}_c^*(f_k)}{S_n(|f_k|)} \\ &= 2\frac{\Delta t}{N}\sum_{k=0}^{N-1}\frac{\tilde{h}_{ck}\tilde{h}_{ck}^*}{S_n(|f_k|)} \\ &= 2\frac{\Delta t}{N}\left(\frac{\tilde{h}_{c0}\tilde{h}_{c0}^*}{S_n(|f_k|)} + 2\sum_{k=1}^{N/2-1}\frac{\tilde{h}_{ck}\tilde{h}_{ck}^*}{S_n(|f_k|)} + \frac{\tilde{h}_{cN/2}\tilde{h}_{cN/2}^*}{S_n(|f_k|)}\right)\end{aligned}$$

σ^2, as we can see is the correlation of the chirp signal with itself divided by the power spectral density of the noise summed over k from (1, N/2-1). Here k serves the following role in relation to frequency:

When 0 < k < N/2, 0 < f < f_s and when N/2 < k < N, -f_s < f < 0. Since in the above equation, k is varying from 1 to N/2-1, the left hand side of the frequency is all taken to be zero. In addition to that, the k=0 term (the first term in the last equation) is set to zero as it is the DC term. Also, we assume that there is no power at the Nyquist Frequency, so the k=N/2 term is also set to zero (the right hand side term in the last equation). Finally we are left with only the central term in the last equation over which we perform summation to arrive at the normalization constant.

Next we write the matched filter time series by defining z_j of which x_j is the real part and y_j is the complex part. x_j and y_j are the discretized forms of their respective forms x(t) and y(t). From these discretized forms we obtain z_j to be as follows[2]:

$$z_j = 4\frac{\Delta t}{N}\sum_{k=1}^{N/2-1} e^{2\pi ijk/N}\,\frac{\tilde{s}_k\tilde{h}^*_{ck}}{S_n\left(|f_k|\right)}$$
$$= \frac{\Delta t}{N}\sum_{k=0}^{N-1} e^{2\pi ijk/N}\,\tilde{z}_k$$

$$\tilde{z}_k = \begin{cases} 4\frac{\tilde{s}_k\tilde{h}^*_{ck}}{S_n(|f_k|)} & 0 < k < \frac{N}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

From this we can now compute the square of the signal-to-noise ratio as follows:

$$\rho^2(t_j) = \frac{x_j^2 + y_j^2}{\sigma^2} = \frac{1}{\sigma^2}|z_j|^2$$

Using MATLAB, we can plot the time series of $\rho^2(t_j)$ which is described in the next section.

GENERATE TIME SERIES OF SNR SQUARED

Since we have the SNR^2, we can now use it to inject a signal and plot the time series of the SNR^2. In the first case, we take the input signal as just the noise that we generated in the section on the generation of LIGO noise. We get the plot as seen in Figure *6*. It is seen that the magnitude of SNR^2 is low which shows the fact that there is no correlation between the noise and the chirp signal which results in a low SNR.

In the second case, we take the input signal to be the sum of the noise as well as the chirp signal. It is expected that the SNR will blow up because there will be a high correlation between the chirp signal and itself. The observation is exactly what we get from the simulation shown in Figure *7*.

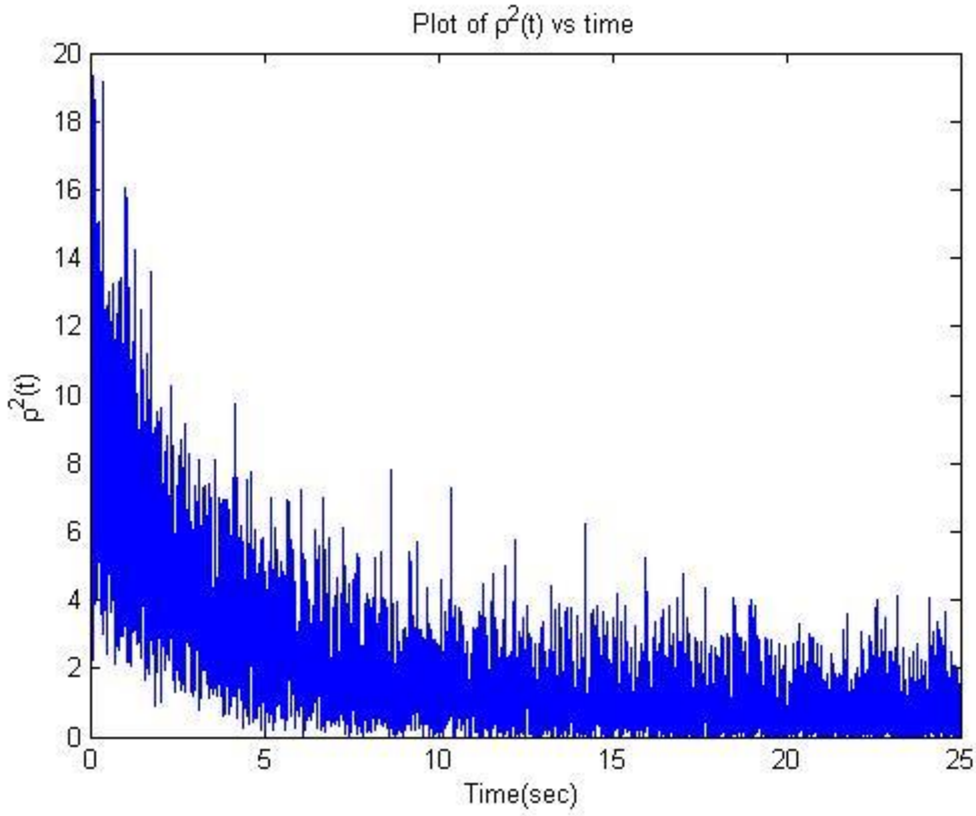


Figure 6: SNR^2 when the injected signal contains only noise

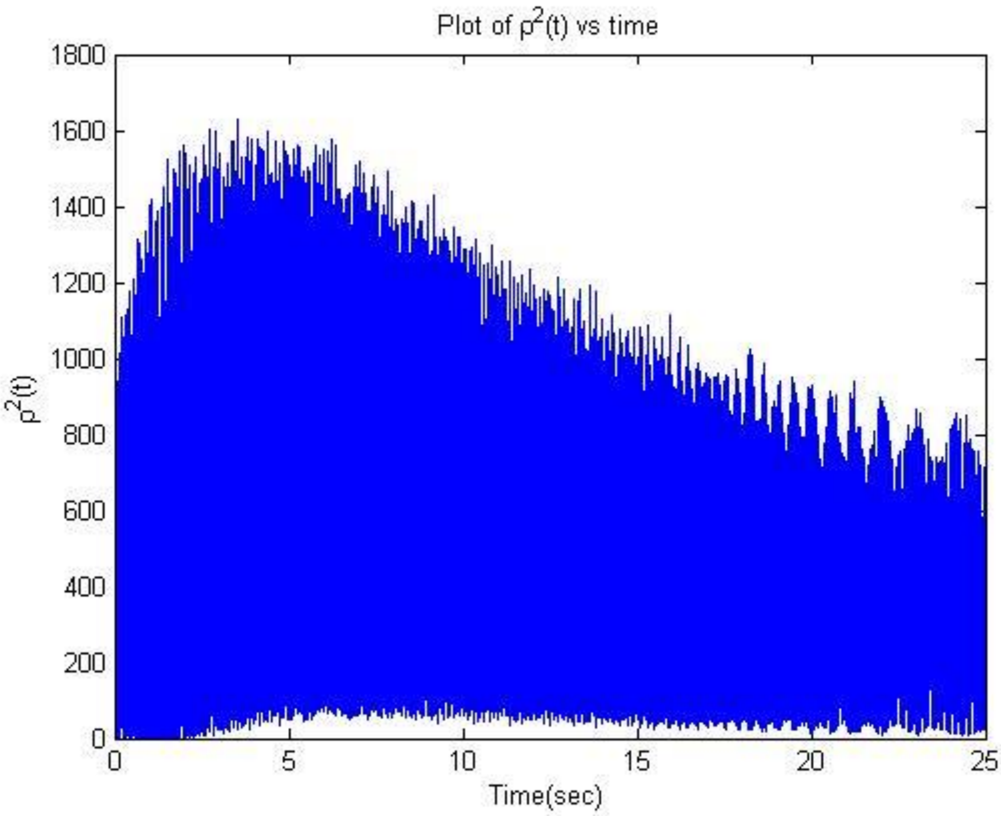


Figure 7: SNR^2 when the injected signal contains noise as well as the chirp

CUMULATIVE DISTRIBUTION FUNCTIONS

Since we have got the time series of the square of the signal-to-noise ratio, we now plot the cumulative distribution function of the same. It is expected that the curve generated will closely match a chi-squared cumulative distribution function with two degrees of freedom. The expected result from Duncan's Thesis is shown in Figure *8*, while the results from simulation are shown in Figure *9*. There is a slight deviation in the filtering code distribution from the expected chi-square distribution due to a scaling factor which can be managed in future analysis.

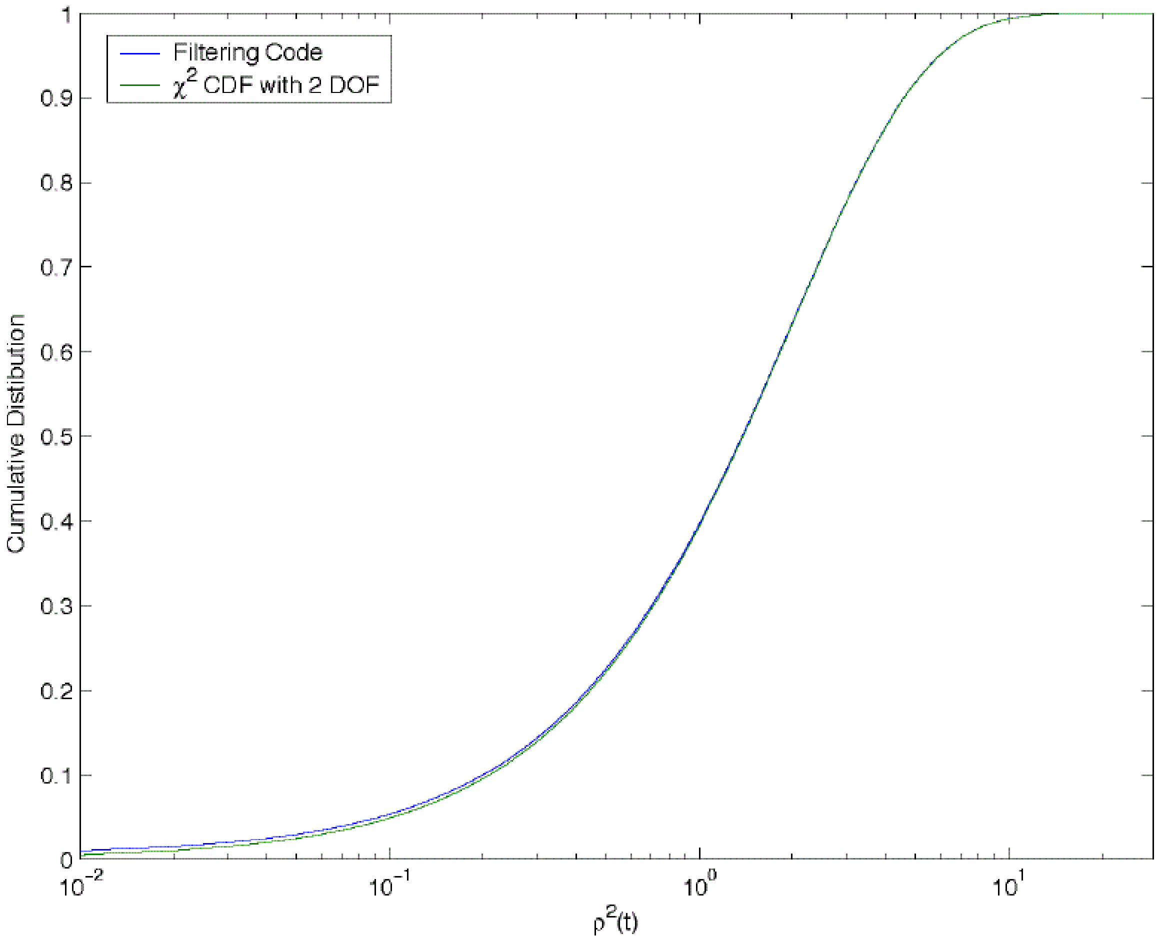


Figure 8: The figure shows the cumulative distribution function of the filtering code output i.e. $\rho^2(t)$ as well as the chi-squared distribution as given in Duncan's Thesis with two degrees of freedom. It is observed that the curves are in close agreement

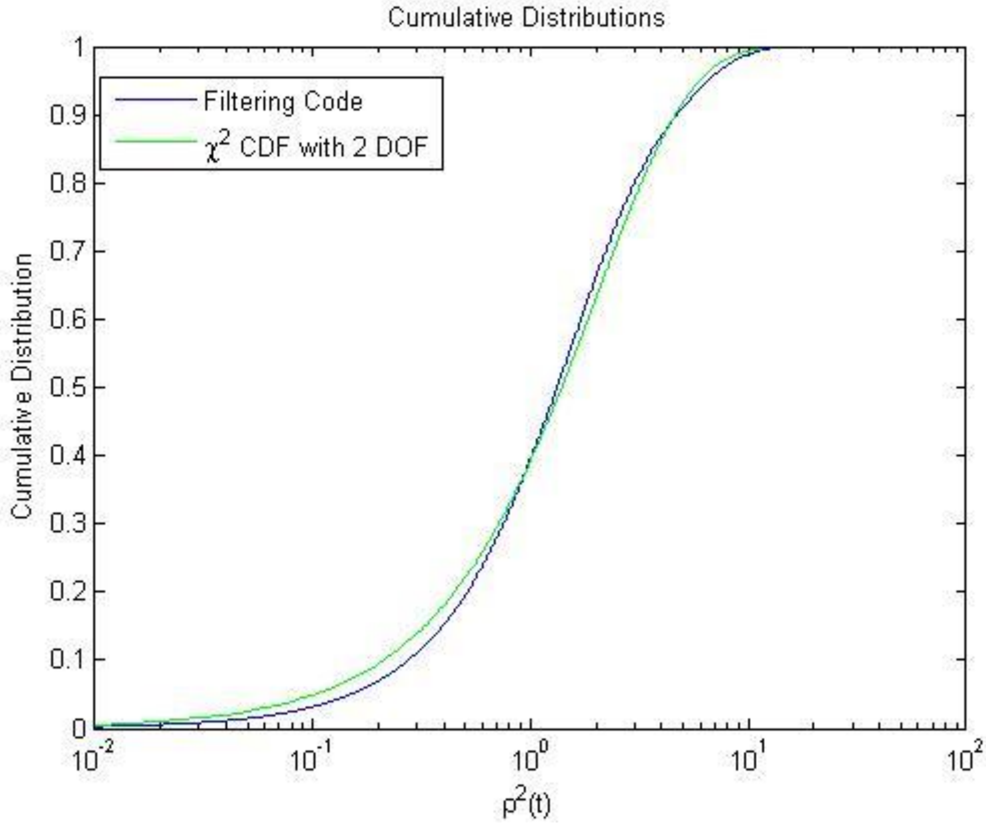


Figure 9: The figure shows the cumulative distribution function of the filtering code output i.e. $\rho^2(t)$ as well as the chi-squared distribution with two degrees as the results of MATLAB simulation. It is observed that the curves are in close agreement

CONCLUSIONS

The results obtained from the simulations of noise, chirp and SNR are close to the expected results as given in Duncan's Thesis. With more iterations of the code, they can be exactly matched with the expected results.

FUTURE WORK

1. Injecting a real signal corrupted with noise
2. Plot the SNR time series and the cumulative distribution functions for the same
3. Use Fisher matrix to determine the spread of maximum likelihood estimator for maximizing SNR
4. Contrast it with a novel technique by Michele Vallisneri

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[2]Searching for Gravitational Radiation from Binary Black Hole MACHOs in the Galactic Halo
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