Observability:

- A LTI sys is obs if its initial condition $x(t_0)=X_0$ can be determined uniquely from the input signal $u(t)$ & measurement signal $y(t)$ given over finite time interval $t\in[t_0,t_e]$

Obs & unobs system

$x'=TX$, $A'=TAT^{-1}$, $C'=CT^{-1}$

T is obtained from O by changing row

Hautus:

$O_{h,i}=\begin{pmatrix}\lambda I-A\\ C\end{pmatrix}$ If $O_{h,i}.v_i=0$ then v_i is unobservable direction

1. find eigenvalue λ → $(\lambda I-A).V=0$ then unobs
2. find eigenvector V
3. O_{hi}; 4. $O_{hi}.V_i\neq 0$ or $CV_i=0$?

Hautus criterion:

A LTI sys is observable iff O_H has full rank n for all complex values of λ

Structural observability:

$\begin{pmatrix}* & 0 & 0\\ 0 & * & *\\ * & * & *\end{pmatrix}$

Kalman:

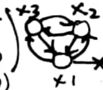
① make path tree
② build structural observability matrix $[O]$
③ check independent paths
④ no of independent path = rank$[O]$

Structural Hautus Criteria

Lunze:

① sys must be output connected in digraph
② rank$\left[\begin{pmatrix}A\\ C\end{pmatrix}\right]=n$

$\dot{x}=\begin{pmatrix}0 & * & *\\ * & 0 & *\\ * & * & 0\end{pmatrix}$, $y=(*\ 0\ 0)$



Wend:

① sys must be output connected
② digraph free of contractions i.e $N=T[N]$ {if $N>T[N]$ diagraph has contractions}

Infinite observability Gramian

→ $A^TM+MA+C^TC=0$ $M=\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}$

→ $\det(M)\neq 0$ → observable

Luenberger observer:

sys: $\dot{x}=AX+BU$, $y=CX$

obs: $\dot{\hat{x}}=A\hat{x}+BU+L(y-\hat{y})$, $\hat{y}=C\hat{x}$

estimation error: $\tilde{x}=\hat{x}-x$

$\dot{\tilde{x}}=\dot{\hat{x}}-\dot{x}=A(\hat{x}-x)+L(Cx-C\hat{x})$

$\dot{\tilde{x}}=(A-LC)\tilde{x}$

Observer canonical form

① Find O ② Find O^{-1}
③ $\bar{A}=OAO^{-1}$, ④ $\dot{\bar{x}}=\bar{A}\bar{x}$

$\bar{C}=CO^{-1}$

$\bar{A}=\begin{pmatrix}0 & 1 & 0 & \dots\\ 0 & 0 & 1 & \dots\\ a_0 & -a_1 & -a_2 & \dots\end{pmatrix}$ $\bar{C}=(1\ 0\ 0\ ..)$

Def^n: Observer canonical form

$\dot{x}^*=\underbrace{\begin{pmatrix}-a_{n-1} & 1 & \dots & 0\\ \vdots & & & \vdots\\ -a_1 & 0 & \dots & 1\\ -a_0 & 0 & \dots & 0\end{pmatrix}}_{A^*}x^*$ $y=(1\ 0..0)x^*$

Resulting Transformation matrix

$x^*=\underbrace{\begin{pmatrix}1 & 0 & 0\\ a_{n-1} & 1 & 0\\ \vdots & & \\ a_2 & a_3 & \dots\\ a_1 & a_2 & \end{pmatrix}}_{\bar{T}}\bar{x}$ $T=\bar{T}.O$ T in physical coordinats

Luenberger coordinats in canonical

$\dot{\hat{x}}^*=A^*\hat{x}^*+L^*(y-y^*)$

$L=T^{-1}L^*$; Ackermann

$L=(s\ \ As\ \ A^2s\ \dots\ A^ns)\begin{pmatrix}\tilde{a}_0\\ \tilde{a}_1\\ \vdots\end{pmatrix}$

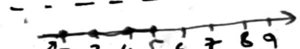
$1\lambda^3+\tilde{\alpha}_1\lambda^2+\tilde{\alpha}_2\lambda+\tilde{a}_3$

LQ optimal observer design

$L=PC^TR^{-1}$; find P from eq $P=\begin{pmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{pmatrix}$

$AP+PA^T-PC^TR^{-1}CP+Q=0$

Kalman:



smoothed $\hat{x}_{2|5}$ $\hat{x}_{6|5}$ prediction

a posteriori $\hat{x}_5^+$ a priori $\hat{x}_5^-$

→ $\hat{x}_k^-=\varphi_{k-1}\hat{x}_{k-1}^{(+)}$ ← (**)

$\hat{x}_{k-1}^+$: a posteriori of previous time

$\hat{x}_k^+$: a priori estimate of current

→ sys: $\dot{x}_1=x_2$, $\dot{x}_2=x_3$, $\dot{x}_3=f_3(x)$, $y=x_1$

obs = $\dot{\hat{x}}_1=\hat{x}_2+h_2.(x_1-\hat{x}_1)$ (High gain observer)

$\dot{\hat{x}}_2=\hat{x}_3+1.(x_1-\hat{x}_1)$

$\dot{\hat{x}}_3=\hat{f}_3(\hat{x})+h_0.(x_1-\hat{x}_1)$

$\hat{x}_1=(h_2s^2+h_1s+h_0/s^3+h_2s^2+h_1s+h_0)x_1$

$\hat{x}_2=(h_1s^2+h_0s/s^3+h_2s^2+h_1s+h_0)x_1$

$\hat{x}_3=(h_0s^2/s^3+h_2s^2+h_1s+h_0)x_1+(s^2+h_2s+h_1/s^3+h_2s^2+h_1s+h_0)\hat{F}_3$

$h_0=\frac{\tilde{a}_0}{\varepsilon^3}$, $h_1=\frac{\tilde{a}_1}{\varepsilon^2}$, $h_2=\frac{\tilde{a}_2}{\varepsilon}$

$\lambda^3+\tilde{\alpha}_2\lambda^2+\tilde{\alpha}_1\lambda+\tilde{a}_0=0$

Kalman Filter Assumptions:

1. w_k, v_k are random variables independent of x
2. w_k, v_k are randomly distributed around 0 with variances Q_k, R_k
3. system is assumed to be linear

* Approach:

1. Prediction Step (**) ← backside
2. update step: $x_k^{(+)} = K_k' \hat{x}_k^{(+)} + K_k y_k$

* Filter design problem: choice of K_k', K_k

* Demands on Filter:

1. update step should be unbiased $E[\hat{x}_k^+] \overset{!}{=} E[x_k]$ if $E[\hat{x}_k^-] = E[x_k]$
2. quadratic estimation error after update ⇝ minimized $P_k^+ = E[(\hat{x}_k^+ - x_k)^2] \overset{!}{=} \min$

K.F (Discrete - Discrete)

$x_k = \phi_{k-1} x_{k-1} + \Gamma_{k-1} w_{k-1}$ $\quad E[w_k] = 0 \quad E[w_k w_k^T] = Q_k$

$y_k = H_k x_k + v_k$ $\quad E[v_k] = 0 \quad E[v_k v_k^T] = R_k$

Eq: filter Prediction

$\hat{x}_k^{(-)} = \phi_{k-1} \hat{x}_{k-1}^{(+)}$ | $P_k^{(-)} = \phi_{k-1} P_{k-1}^{(+)} \phi_{k-1}^T + \Gamma_{k-1} Q_{k-1} \Gamma_{k-1}^T$

update:

$K_k = P_k^{(-)} H_k^T (H_k P_k^{(-)} H_k^T + R_k)^{-1}$

$\hat{x}_k^{(+)} = \hat{x}_k^{(-)} + K_k (y_k - \hat{y}_k)$

$P_k^{(+)} = (I - K_k H_k) P_k^{(-)}$

$\dot{x}(t) = A x(t) + G w(t)$

$y(t_k) = H_k x(t_k) + v_k$

K.F (cont - [illegible])

Filter eq: diff eqs

$\dot{\hat{x}}(t) = A \hat{x}(t)$ $\quad$ i.c. $\hat{x}(t_{k-1}) = \hat{x}_{k-1}^{(+)}$

$\dot{P} = AP + PA^T + GQG^T$ $\quad$ i.c. $P(t_{k-1}) = P_{k-1}^{(+)}$

resulting filter prediction:

$\hat{x}_k^{(-)} = \hat{x}(t_k)$, $P_k^{(-)} = P(t_k)$

filter Prediction: Same as above

$\hat{x}_k^- = e^{A\Delta T} \hat{x}_{k-1}^{(+)}$ $\quad P_k^{(-)} = \begin{bmatrix} e^{2A\Delta T} P_{11} & 0 \\ 0 & R_{22} e^{2A\Delta T} \end{bmatrix}$

$\hat{P}_k^- = e^{A\Delta T} P_{k-1}^+ e^{A\Delta T'} + \int e^{A\Delta T} GQG^T e^{A\Delta T'}$

Flow dia

KF (cont - cont) Kalman-Bucy Filter

1. differential eq:

$\dot{\hat{x}} = A\hat{x} + K(y - H\hat{x})$

$\dot{P} = AP + PA^T - KRK^T + GQG^T$ $\quad P = \begin{bmatrix} P_{11} & P_{21} & \cdots \\ P_{21} & \cdots & \\ P_{31} & \cdots & \end{bmatrix}$

2. Gain: $K = PH^T R^{-1}$

EKF (discrete)

sys eq: $x_k = \varphi(x_{k-1}, u_{k-1}) + \Gamma_{k-1} w_{k-1}$

$y_k = h(x_k, u_k) + v_k$

$E[w_k^2] = Q$, $E[v_k^2] = R_k$

Filter Pred: $\hat{x}_k^{(-)} = \varphi(\hat{x}_{k-1}^+, u_{k-1})$

$P_k^{(-)} = \frac{\partial \varphi}{\partial x}\Big|_{\hat{x}_{k-1}^{(+)}, u_{k-1}} P_{k-1}^{(+)} \frac{\partial \varphi}{\partial x}\Big|_{\hat{x}_{k-1}^+, u_{k-1}}^T + \Gamma_{k-1} Q_{k-1} \Gamma_{k-1}^T$

Filter Update:

$P_{xy} = P_k^{(-)} \cdot J^T$ $\quad J = \frac{\partial h}{\partial x}\Big|_{\hat{x}_k^{(-)}, u_k}$

$P_{yy} = J \cdot P_k^{(-)} \cdot J^T + R_k$

$K = P_{xy} \cdot P_{yy}^{-1}$

$\hat{x}_k^{(+)} = \hat{x}_k^{(-)} + K_k (y_k - \hat{y}_k^{(-)})$

$\hat{P}_k^{(+)} = P_k^{(-)} - K_k P_{yy} K_k^T$

EKF (Conti - Disc)

Sys eq:

$\dot{x}(t) = f(x(t), u(t)) + G w(t)$

$y_k = h(x_k, u(t_k)) + v_k$

Prediction:

$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t))$, i.c: $\hat{x}(t_{k-1}) = \hat{x}_{k-1}^{(+)}$

$\dot{P} = \frac{\partial f}{\partial x}\Big|_{\hat{x}(t), u(t)} P + P \frac{\partial f}{\partial x}\Big|_{\hat{x}(t), u(t)}^T + GQG^T$

with i.c. $P(t_{k-1}) = P_{k-1}^{(+)}$ $\quad P_k^{(-)} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$

update: same as discrete (above)

UKF: Sys Eq: $x_k = \varphi(x_{k-1}) + \Gamma_{k-1} w_{k-1}$

$y_k = h(x_k) + v_k$, $E[w_k^2] = Q_k$, $E[v_k^2] = R_k$

Prediction: $x_{k-1,0}^{(+)} = \hat{x}_{k-1}^{(+)}$ $\quad (K = 3 - n)$

1. Calc of Σ points $\quad x_{k-1,1,3}^{(+)} = \hat{x}_{k-1}^{(+)} \pm \sqrt{n+K}\,(\sqrt{P_{k-1}^{(+)}})^T$

$i = 1, \ldots n$

$i = n+1, \ldots, 2n$ $\quad x_{k-1,2,4}^{(+)} =$ " $\pm$ "

$P_{k-1}^{(+)} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ $\quad \sqrt{P_{k-1}^{(+)}} = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} = \sqrt{P_{k-1}^{(+)}}^T \sqrt{P_{k-1}^{(+)}}$

solution model for each sigma pt

$x_{k-1}^{(-)} = \varphi(x_{k-1}^{(+)}, i)$

prediction: $\hat{x}_k^{(-)} = \sum_{i=0}^{2n} W_i x_{k,i}^{(-)}$

$\hat{y}_k^{(-)} = \sum_{i=0}^{2n} W_i h(x_{k-i}^{(-)})$

$P_{xy} = \sum_{i=0}^{2n} W_i (x_{k,i}^{(-)} - \hat{x}_k^{(-)})(h(x_{k,i}^{(-)}) - \hat{y}_k^{(-)})^T$

$P_{yy} = \sum_{i=0}^{2n} W_i (h(x_{k,i}^{(-)}) - \hat{y}_k^{(-)})(h(x_{k,i}^{(-)}) - \hat{y}_k^{(-)})^T + R$

$P_k^{(-)} = \sum_{i=0}^{2n} W_i (x_{k,i}^{(-)} - \hat{x}_k^{(-)})(x_{k,i}^{(-)} - \hat{x}_k^{(-)})^T + \Gamma_{k-1} Q_{k-1} \Gamma_{k-1}^T$

choice of weights: $W_0 = \frac{K}{n+K}$, $W_i = \frac{1}{2(n+K)}$

Filter Update:

$K_k = P_{xy} P_{yy}^{-1}$; $\hat{x}_k^{(+)} = \hat{x}_k^{(-)} + K_k (y_k - \hat{y}_k^{(-)})$

$P_k^{(+)} = P_k^{(-)} - K_k P_{yy} K_k^T$

constrained K.F update

obj func: $J(K_k) = Tr(P_k^{(-)}) + Tr(K_k P_{yy} K_k^T) - 2 \cdot Tr(P_{xy} K_k^T) \overset{!}{=} \min$

eq. constr: $d_i(K_k) \overset{!}{=} 0$, $i = 1, \ldots, i_{max}$

ineq ": $g_j(K_k) \overset{!}{\leq} 0$, $j = 1, \ldots, j_{max}$

spl case: $i_{max} = 1$, $d_1 = D^T \hat{x}_k^{(+)} - d \overset{!}{=} 0$

min cond: $\frac{\partial \Lambda}{\partial K_k}$, $\frac{\partial \Lambda}{\partial \lambda}$ ⇝ Find λ

Resulting: $x_k^{(+)} = x_k^{(-)} + P_{xy} P_{yy}^{-1} (y_k - \hat{y}_k)$

nonlinear Observes:

Def^n: Sys Σ is locally observable if for every open neighborhood N of x_0 & every sol^n $x(t)$ completely in N $I_N(x_0) = x_0$ → $I_N(x) = x \;\forall x \in M$

Sys Σ is weakly observable at x_0, if there is some neighborhood V of x_0 where $I(x_0) \cap V = x_0$

Sys Σ is locally weakly obs at x_0 if there is some neighborhood V of x_0 where $I_N(x_0) \cap V = x_0$ for all solutions $x(t)$ completely in any neighborhood N of x_0

$i: \begin{pmatrix} S_{11} \\ S_{12} \end{pmatrix}$

$i-n: \begin{pmatrix} 0 \\ S_{22} \end{pmatrix}$