STA258H5

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One Sample Hypothesis Tests on a Population Proportion p

$H_0: p = p_0$ (or $p \leq p_0$) $\quad H_a: p > p_0$

$H_0: p = p_0$ (or $p \geq p_0$) $\quad H_a: p < p_0$

$H_0: p = p_0$ $\quad H_a: p \neq p_0$

Test Statistic

$$z^* = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

Reference distribution: standard normal

Review / Summary

One Sample Inference on a Population mean

σ known

CI

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

Hyp Test

$H_0: \mu = \mu_0$ $\quad H_a: \mu > \mu_0$

$\mu < \mu_0$

$\mu \neq \mu_0$

$$z^* = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

σ unknown

CI

$$\bar{x} \pm t_{(n-1, \alpha/2)} \frac{s}{\sqrt{n}}$$

Hyp Test

$H_0: \mu = \mu_0$ $\quad H_a: \mu > \mu_0$

$\mu < \mu_0$

$\mu \neq \mu_0$

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

New! One Sample Inference on a population proportion

CI

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Hyp Test

$H_0: p = p_0$ $\quad H_a: p > p_0$

$p < p_0$

$p \neq p_0$

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

SE depends on p_0

Test of Hypothesis for One Proportion

Example: 100-Cup Challenge

Successes

A YouTuber goes to her nearest Tim Hortons and buys 100 empty cups. After rolling up the rims, she ends up with 12 winning cups out of the 100 she bought, all of them were food prizes.
If the probability of winning a food prize is supposed to be $\frac{1}{6}$, does she have evidence to claim that the probability of winning a food prize is less than $\frac{1}{6}$?

n

$H_0: p = \frac{1}{6}$

$H_a: p < \frac{1}{6}$

Example (slide 4)

Let's choose level of significance to be $\alpha = 0.05$.

$H_0: p = \frac{1}{6}$ $\qquad$ $H_a: p < \frac{1}{6}$

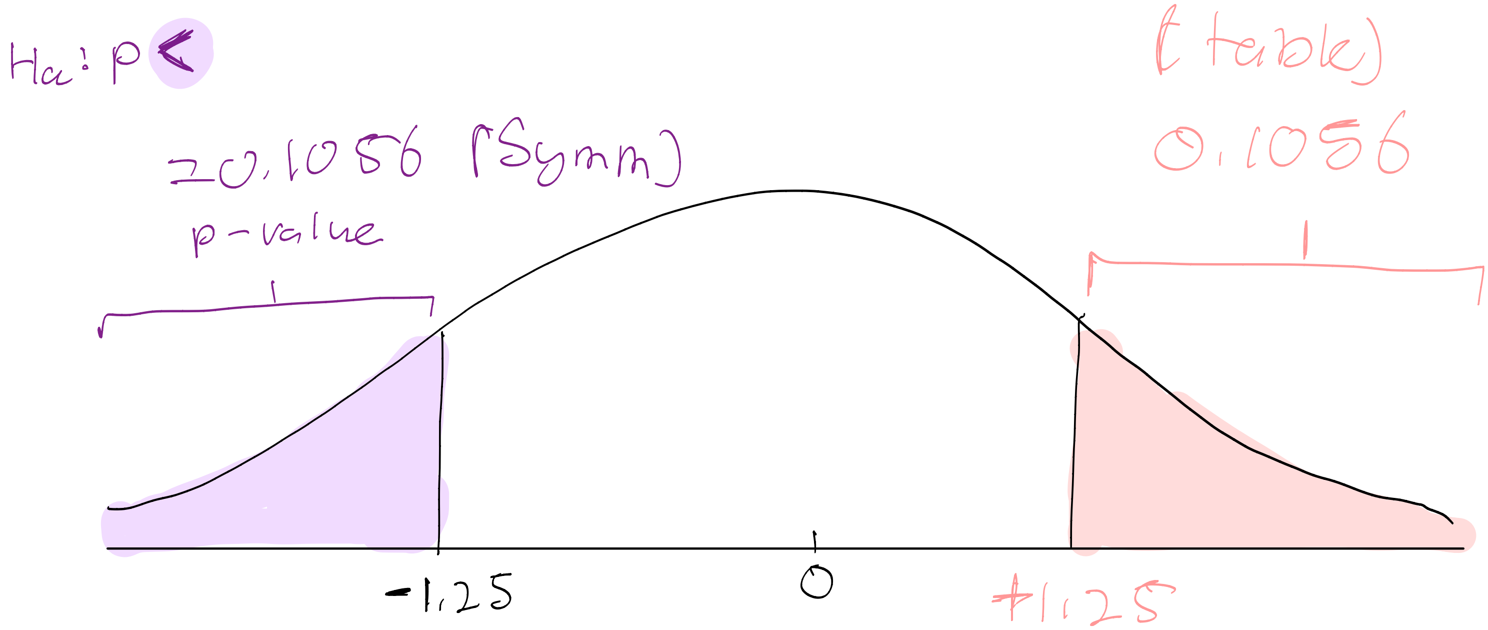
[or $H_0: p \geq \frac{1}{6}$ $\qquad$ $H_a: p < \frac{1}{6}$]

$n = 100$, $X = 12$, $\hat{p} = \frac{12}{100} = 0.12$

test statistic

$$z^* = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.12 - 1/6}{\sqrt{\frac{1/6(1-1/6)}{100}}} = -1.25$$

reference distribution: standard normal



$0.1056 > 0.05$

p-value $> \alpha$ → Evidence does not reject Ho

There is insufficient evidence to reject the null hypothesis that the proportion of winning cups is $\frac{1}{6}$

Assumptions / Requirements for One Sample hypothesis tests on a mean (σ known, σ unknown)

- Random Sample
- Independent Sample
- If n is small ($n < 30$), population should be normal.

Assumptions / Requirements for One Sample hypothesis tests on a proportion

- Random Sample
- Independent Sample
- Sample is sufficiently large
  $n\hat{p} \geq 10$ AND $n(1-\hat{p}) \geq 10$

Probability Models

The **sample space S** of a random phenomenon is the set of all possible outcomes.
An **event** is an outcome or a set of outcomes of a random phenomenon. That is, an event is a subset of the sample space.
A **probability model** is a mathematical description of a random phenomenon consisting of two parts: a sample space S and a way of assigning probabilities to events.

Example

Rolling a fair die (random phenomenon). There are 6 possible outcomes when we roll a die.
The sample space for rolling a die and counting the pips is

$$S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$$

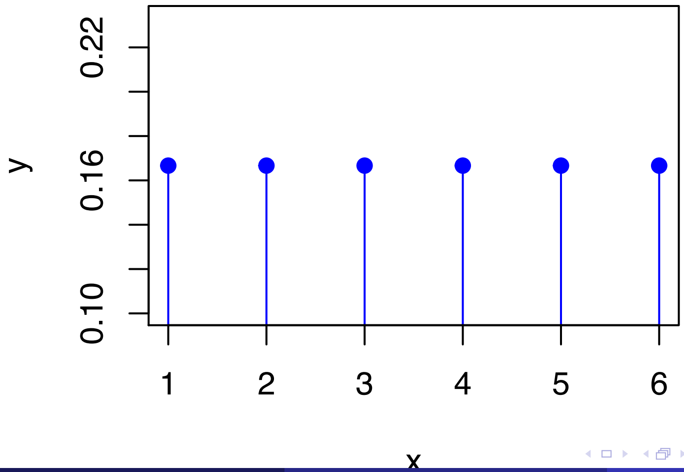
"Roll a 6" is an event, that contains one of these 6 outcomes.

Discrete Uniform Distribution

A random variable X has a **discrete uniform distribution** if each of the n values in its range, say, x_1, x_2, . . . , x_n has equal probability. Then,

$$f(x_i) = \frac{1}{n}$$

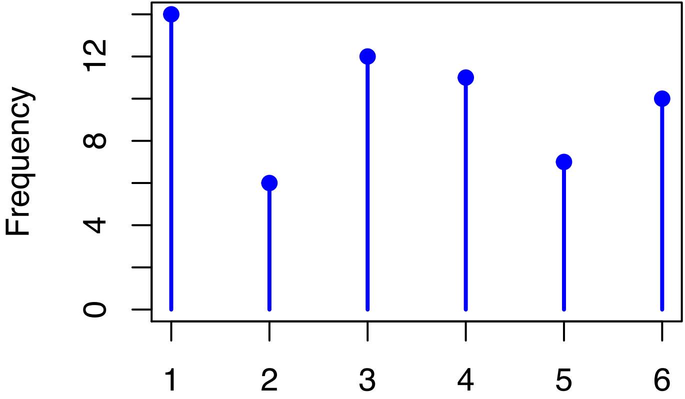
Probability mass function (pmf)



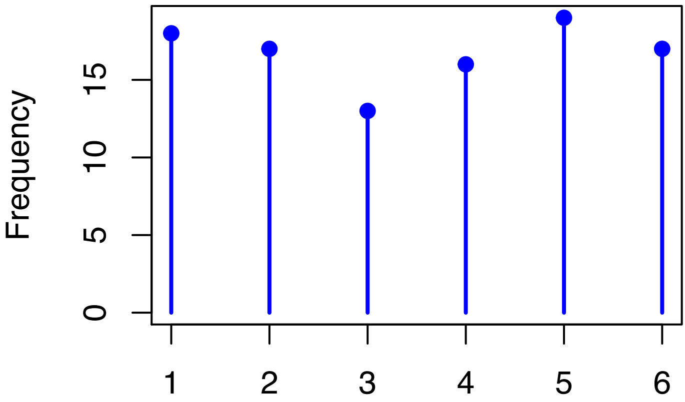
Six simulations

```
die=c(1,2,3,4,5,6);

sample(die,1,replace=TRUE);

## [1] 2

sample(die,6,replace=TRUE);

## [1] 1 3 2 6 3 1
```

60 simulations



100 simulations



Random Variable

A **random variable** is a variable whose value is a numerical outcome of a random phenomenon.
The **probability distribution** of a random variable X tells us what values X can take and how to assign probabilities to those values.

The Binomial setting

- There are a fixed number n of observations.
- The n observations are all **independent**. That is, knowing the result of one observation tells you nothing about the other obsevations.
- Each observation falls into one of just two categories, which for convenience we call "success" and "failure".
- The probability of a success, call it p, is the same for each observation.

Example

Think of rolling a die n times as an example of the binomial setting. Each roll gives either a six or a number different from six. Knowing the outcome of one roll doesn't tell us anything about other rolls, so the n rolls are independent. If we call six a success, then p is the probability of a six and remains the same as long as we roll the same die. The number of sixes we count is a random variable X. The distribution of X is called a **binomial distribution**.

Binomial Distribution

A random variable Y is said to have a **binomial distribution** based on n trials with success probability p if and only if

$$p(y) = \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}, \;\; y = 0, 1, 2, ..., n \text{ and } 0 \leq p \leq 1.$$

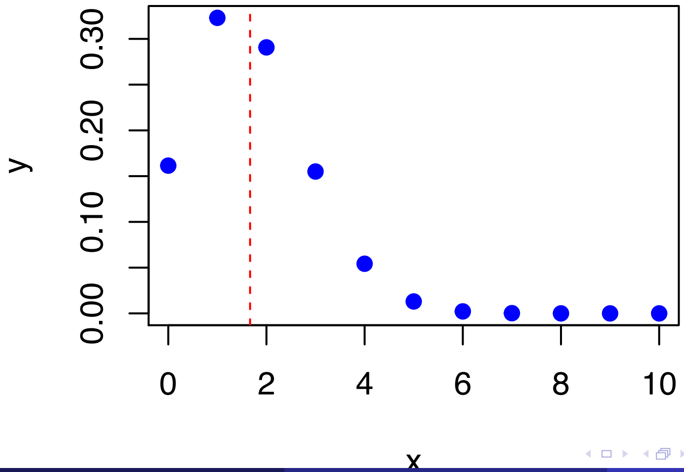
A few simulations

```
## Simulation: Binomial with n=10 and p=1/6.

rbinom(1,size=10,prob=1/6);

## [1] 3

rbinom(1,size=10,prob=1/6);

## [1] 1

rbinom(1,size=10,prob=1/6);

## [1] 0
```

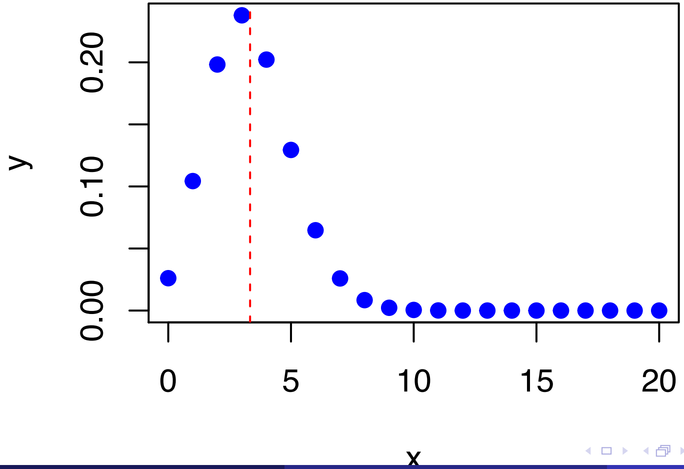
Probability Mass Function when n=10 and p=1/6

```
## Pmf: Binomial with n=10 and p=1/6.

x<-seq(0,10,by=1);

y<-dbinom(x,10,1/6);

plot(x,y,type="p",col="blue",pch=19);
```

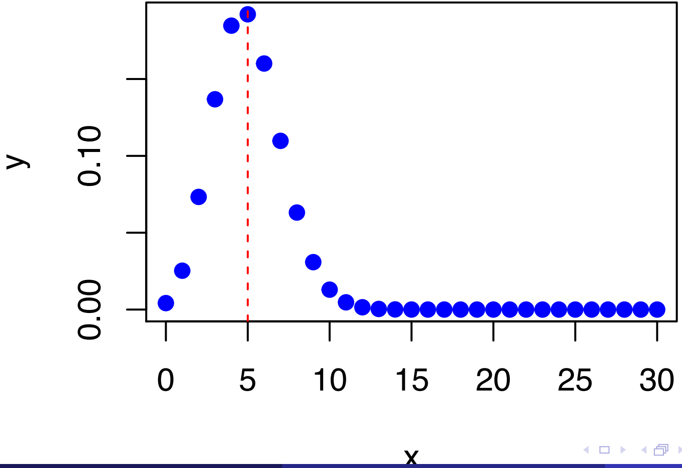
Probability Mass Function when n=10 and p=1/6



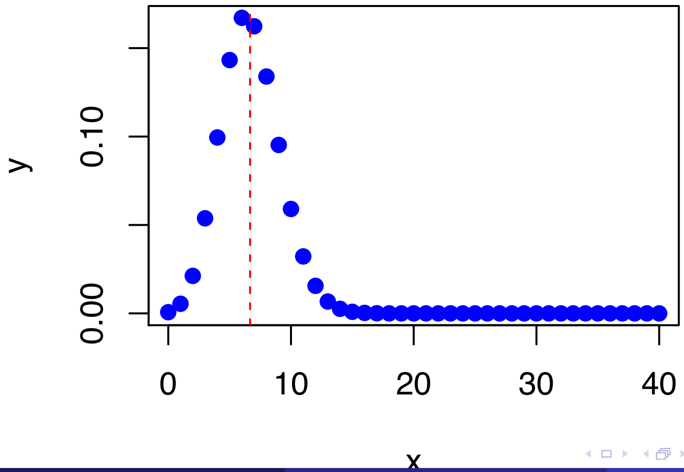
Pmf when n=20 and p=1/6



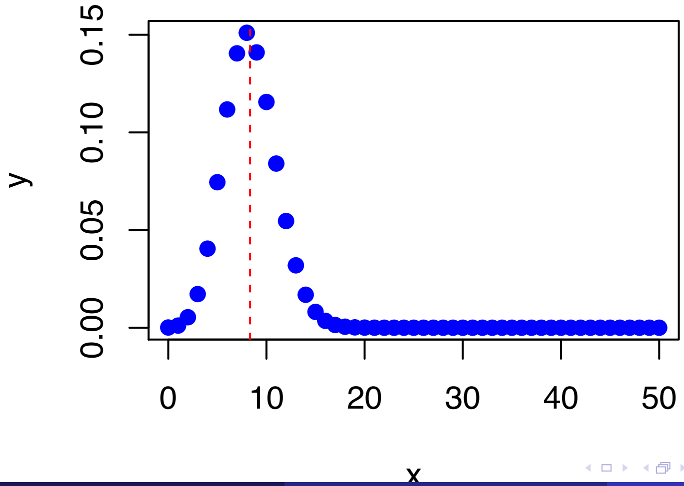
Pmf when n=30 and p=1/6



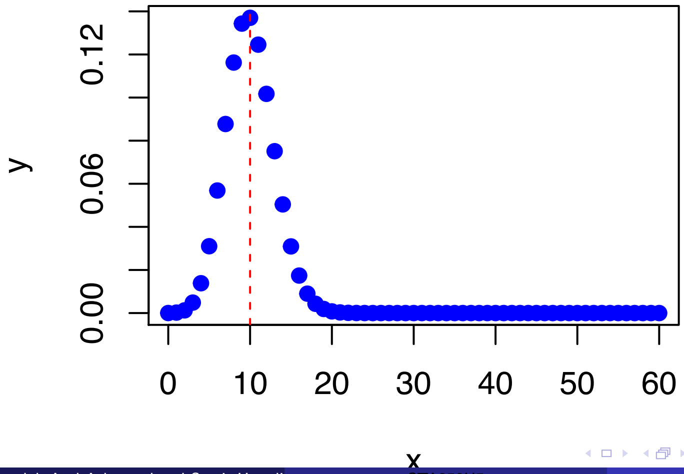
Pmf when n=40 and p=1/6



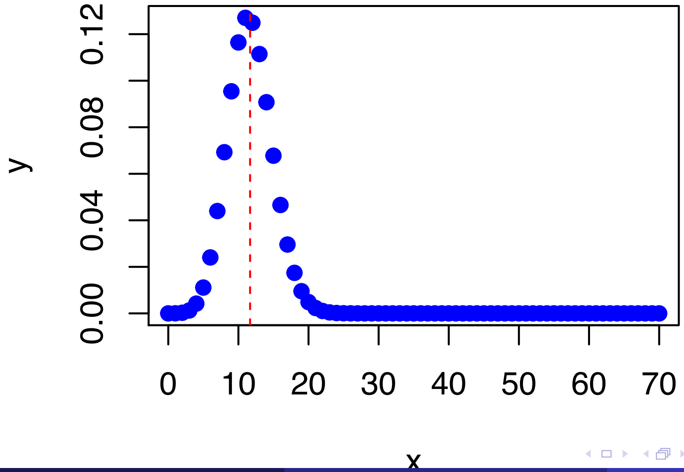
Pmf when n=50 and p=1/6



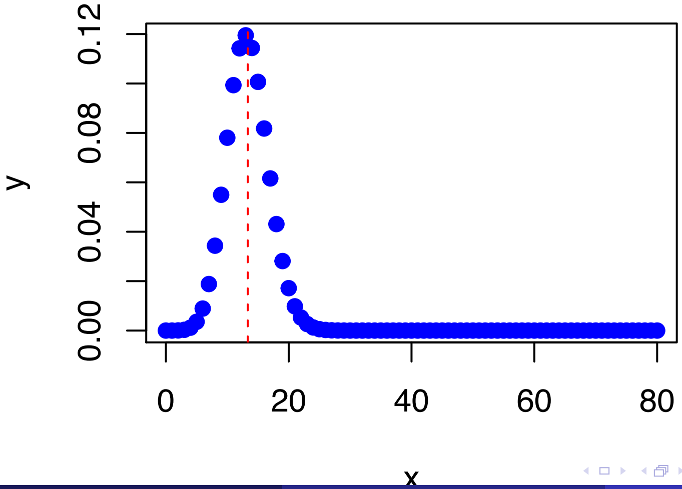
Pmf when n=60 and p=1/6



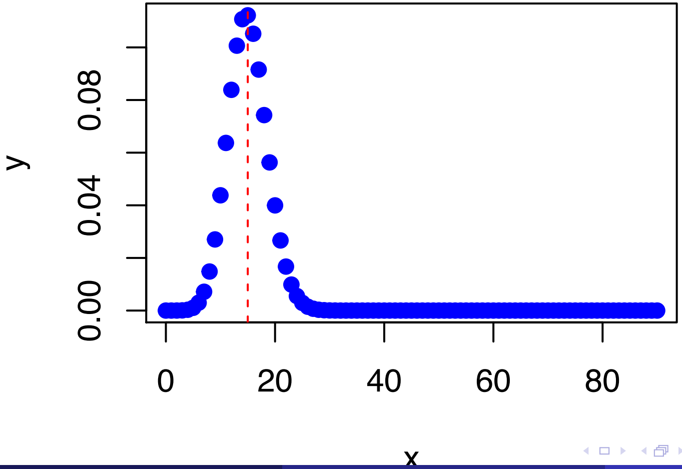
Pmf when n=70 and p=1/6



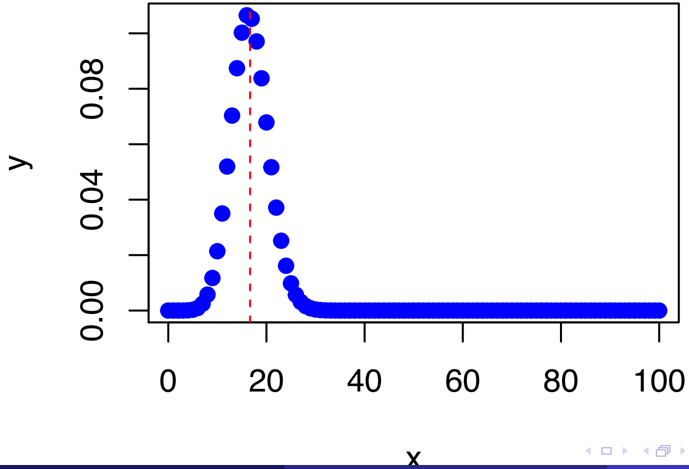
Pmf when n=80 and p=1/6



Pmf n=90 and p=1/6



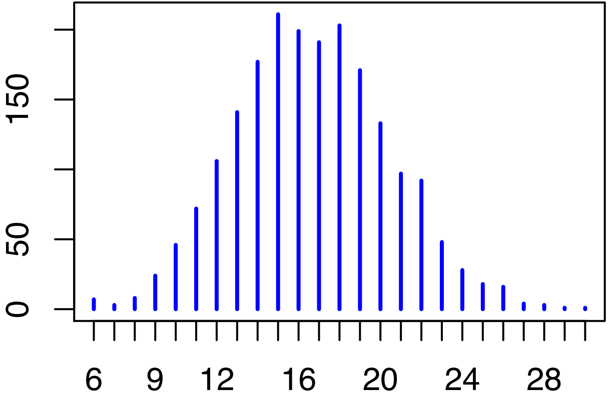
Pmf when n=100 and p=1/6



A few values from our pmf (n=100 and p=1/6)

```
dbinom(c(15,16,17,18),size=100,prob=1/6);

## [1] 0.10023663 0.10650142 0.10524847 0.09706247
```

Simulation: 2000 YouTuber, n=100, and p=1/6



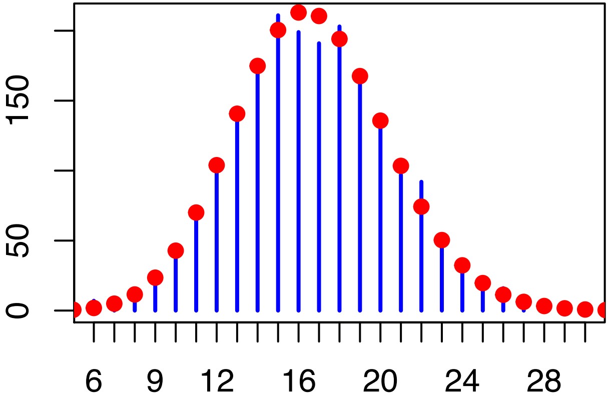
A few values from our simulation

```
## vec.prop
##   6   7   8   9  10  11  12
##   7   3   8  24  46  72 106
## [1] 266
## [1] 0.133
```

P-value

It turns out that our P-value for this simulation is:
0.133

Simulation vs Theoretical pmf



Sampling Distribution of a sample proportion

Draw an SRS of size n from a large population that contains proportion p of "successes". Let $\hat{p}$ be the **sample proportion** of successes,

$$\hat{p} = \frac{\text{number of successes in the sample}}{n}$$

Then:

- The **mean** of the sampling distribution of $\hat{p}$ is p.
- The **standard deviation** of the sampling distribution is

$$\sqrt{\frac{p(1-p)}{n}}.$$

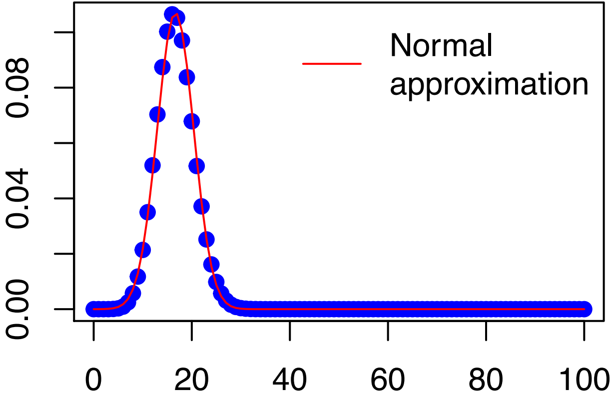
Draw an SRS of size n from a large population that contains proportion p of "successes". Let $\hat{p}$ be the **sample proportion** of successes,

$$\hat{p} = \frac{\text{number of successes in the sample}}{n}$$

Then:

- As the sample size increases, the sampling distribution of $\hat{p}$ becomes **approximately Normal.** That is, for large n, $\hat{p}$ has approximately the $N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$ distribution.

Binomial with Normal Approximation



Hypotheses Tests for a Proportion

To test the hypothesis $H_0 : p = p_0$, compute the z_* statistic,
$z_* = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$
In terms of a variable Z having the standard Normal distribution, the approximate P-value for a test of H_0 against
$H_a : p > p_0$: is : $P(Z > z_*)$
$H_a : p < p_0$: is : $P(Z < z_*)$
$H_a : p \neq p_0$: is : $2P(Z > |z_*|)$

Solution

Step 1. $H_0 : p = \frac{1}{6}$ vs $H_a : p < \frac{1}{6}$

$\hat{p} = \frac{12}{100} = 0.12$

Step 2. (Without continuity correction)

$$z_* = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.12 - 0.1667}{\sqrt{\frac{(0.1667)(1-0.1667)}{100}}} \approx -1.25$$

(WITH continuity correction)

$$P[X \le 12] \approx P[X \le 12.5] = P[\frac{X}{n} \le 0.125] = P\left[\frac{\bar{X} - \mu}{\sigma / n} \le \frac{0.125 - 0.1667}{0.0373}\right]$$
$$= P[Z \le -1.1179]$$

Step 3. Using Normal table, P-value $=$

$P(Z < z_*) = P(Z < -1.1179) \approx 0.1314$

P-value is not small enough to provide evidence against H_0, we can't reject H_0. We conclude that there is not evidence to claim that probability of winning a food prize is less than $\frac{1}{6}$.

R Code

```
prop.test(12,100,p=1/6,alternative="less");

##
##  1-sample proportions test with continuity correction
##
## data:  12 out of 100, null probability 1/6
## X-squared = 1.25, df = 1, p-value = 0.1318
## alternative hypothesis: true p is less than 0.1666667
## 95 percent confidence interval:
##  0.0000000 0.1894571
## sample estimates:
##    p
## 0.12
```

To test the hypothesis $H_0 : p = p_0$, compute the z_* statistic,
$z_* = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$
In terms of a variable Z having the standard Normal distribution, the approximate P-value for a test of H_0 against
$H_a : p > p_0$: is : $P(Z > z_*)$
$H_a : p < p_0$: is : $P(Z < z_*)$
$H_a : p \neq p_0$: is : $2P(Z > |z_*|)$

Introduction to Hypothesis Testing (Significance Test)

Consider the following problem: In 1980s, it was generally believed that congenital abnormalities affect 5% of the nation's children. Some people believe that the increase in the number of chemicals in the environment in recent years has led to an increase in the incidence of abnormalities. A recent study examined 384 children and found that 46 of them showed signs of abnormality. Is this strong evidence that the risk has increased?

- The above statement serves as a hypothesis, moreover it is a Research Hypothesis.

A hypothesis is:

- a statement about a population.
- a predication that a parameter describing some characteristics of a variable (e.g., true proportion, p) takes a particular numerical value or falls in a certain range of values.

Introduction to Hypothesis Testing (Significance Test)

For conducting a Significance Test:

- Researchers (you) use data to summarize the evidence about a hypothesis.
- With data, you can compare the point estimates of parameters to the values predicted by the hypothesis.

Important Ideas about Hypothesis Testing

- All the hypothesis tests boils down to the same question: "Is an observed difference or pattern too large to be attributed to chance?"
- We measure "how large" by putting our sample results in the context of a sampling distribution model (e.g., Normal model, t distribution).

Specify Statistical Model

- To plan a statistical hypothesis test, specify the model you will use to test the null hypothesis and the parameter of interest.
- All model require assumptions, so you will need to state them and check any corresponding conditions.
- For example, if the conditions are satisfied, we can model the sampling distribution of the proportion with a Normal model. Otherwise, we cannot proceed with the test (we need to stop and reconsider).

Steps in conducting Hypothesis Testing

1. State the null and the alternative hypothesis.
2. Check the necessary assumptions.
3. Identify the test-statistic. Find the value of the test-statistic.
4. Find the p-value of the test-statistic.
5. State (if any) a conclusion.

Example of Hypothesis Testing for a Proportion

In 1980s, it was generally believed that congenital abnormalities affect 5% of the nation's children. Some people believe that the increase in the number of chemicals in the environment in recent years has led to an increase in the incidence of abnormalities. A recent study examined 384 children and found that 46 of them showed signs of abnormality. Is this strong evidence that the risk has increased?

Example: Hypothesis Testing for a Proportion (One-sided Test) - Step 1

Step 1. Set up the null and alternative hypothesis:

- The null hypothesis is the current belief: $H_0 : p = p_0$

In our example it would have a form: $H_0 : p = 0.05$

- The Alternative hypothesis is what the researcher(s) [you] want to prove: $H_a : p > p_0$

In our example it would have a form: $H_a : p > 0.05$
This means a one-sided test

- The goal here is to provide evidence against H_0 (e.g., suggest H_a).

You want to conclude H_a.
Try a Proof by Contradiction Assume H_0 is true ... and hope your data contradicts it

Example: Hypothesis Testing for a Proportion (One-sided Test) - Step 2

Step 2. Check the Necessary Assumptions:

- Independence Assumption: There is no reason to think that one child having genetic abnormalities would affect the probability that other children have them.
- Randomization Condition: This sample may not be random, but genetic abnormalities are plausibly independent. The sample is probably representative of all children, with regards to genetic abnormalities.
- 10% Condition: The sample of 384 children is less than 10% of all children.
- Success/Failure Condition: $np = (384)(0.05) = 19.2$ and $n(1-p) = (384)(0.95) = 364.8$ are both greater than 10, so the sample is large enough.

Example: Hypothesis Testing for a Proportion (One-sided Test) - Step 3

Step 3. Identify the test-statistics. Find the value of the test-statistic:
Since the conditions are met, assume H_0 is true:
The sampling distribution of $\hat{p}$ becomes **approximately Normal.** That is, for large n, $\hat{p}$ has approximately the $N\left(p_0, \sqrt{\frac{p_0(1-p_0)}{n}}\right)$ distribution.

$$z_* = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.1198-0.05}{\sqrt{\frac{(0.05)(0.95)}{384}}} \approx 6.28$$

Recall that $\hat{p} = \frac{46}{384} = 0.1198$.
The value of z^* is approximately 6.28, meaning that the observed proportion of children with genetic abnormalities is over 6 standard deviations above the hypothesized proportion ($p_0 = 0.05$).

About the P-value of the Test-statistics

- P-value is a conditional probability.
- It is not the probability that H_0 (null hypothesis: current belief) is true.
- It is: P(observed statistic value [or even more extreme] — H_0]. Given H_0 (the null hypothesis), because H_0 gives the parameter values that we need to find required probability.
- P-value serves as a measure of the strength of the evidence against the null hypothesis (but it should not serve as a hard and fast rule for decision).
- If p-value $= 0.03$ (for example) all we can say is that there is 3% chance of observing the statistic value we actually observed (or one even more inconsistent with the null value).

P-value of the Test-statistics

- P-value is the chance (the proportion) of getting a, for instance, $\hat{p}$ as far as or further from H_0 than the value observed.
- P-value is the probability of getting at least something (e.g., sample proportion $\hat{p}$) more extreme (e.g., unusual, unlikely, or rare) than what we have already found (our observed value of $\hat{p}$) that provide even stronger evidence against H_0.
- The more extreme the z-score (large in absolute values) are the ones that denote farther departure of the observed value (e.g., our $\hat{p}$) from the parameter value (p_0) in H_0.
- In the one-sided test, e.g., $H_a : p > p_0$, p-value is one-tailed probability. This is the probability that sample proportion $\hat{p}$ falls at least as far from p_0 in one direction as the observed value of $\hat{p}$.
- In the two-sided test, e.g., $H_a : p \neq p_0$, p-value is two-tailed probability. This is the probability that sample proportion $\hat{p}$ falls at least as far from p_0 in either direction as the observed value of $\hat{p}$.

Example: Hypothesis Testing for a Proportion (One-sided Test) - Step 4

Step 4. Find the p-value of the test-statistic.
P-value $= P(Z > 6.28) \approx 0.000$ (better to report $p - value < 0.0001$)
Note: We find the area above Z of 6.28 since $H_a : p > 0.05$.
Meaning of this p-value:
If 5% of children have genetic abnormalities, the chance of observing 46 children with genetic abnormalities in a random sample of 384 children is almost 0.

P-values

The probability, computed assuming that H_0 is true, that the test statistic would take a value as extreme or more extreme than that actually observed is called the **P-value** of the test. The smaller the P-value, the stronger the evidence against H_0 provided by the data.

Small P-values are evidence against H_0, because they say that the observed result is unlikely to occur when H_0 is true. Large P-values fail to give evidence against H_0.

The P-value Scale

If P-value < 0.001, we have very strong evidence against H_0.
If $0.001 \leq$ P-value < 0.01, we have strong evidence against H_0.
If $0.01 \leq$ P-value < 0.05, we have evidence against H_0.
If $0.05 \leq$ P-value < 0.075, we have some evidence against H_0.
If $0.075 \leq$ P-value < 0.10, we have slight evidence against H_0.

Use p-value Method to Make a Decision (Reject or Fail to Reject Ho)

But how small is small p-value?
We would need to choose an α-level (significance-level): a number such that if:

- $P - value \leq \alpha$-level, we reject H_0; We can conclude H_a (we have evidence to support our claim). Often we phrase as a statistically significant result at that specified α-level.
- $P - value > \alpha$-level, we fail to reject H_0; We cannot conclude H_a (we have not enough evidence to support our claim; thus, H_0 is plausible - We do not accept H_0). Often we phrase as the result is not statistically significant at that specified α-level.
- The default α-level (significance-level) is typically $\alpha = 0.05$ (but it can be different based on the context of the study - it is usually not higher than 0.10).

Example: Hypothesis Testing for a Proportion (One-sided Test) - Step 5

Step 5. Give (if any) a conclusion.
p-value is less than 0.0001, which is less than $\alpha = 0.05$; We reject $H_0 : p = 0.05$, and conclude $H_a : p > 0.05$. Our result is statistically significant at $\alpha = 0.05$.
There is a very strong evidence that more than 5% of children have genetic abnormalities.

Example: Hypothesis Testing for a Proportion (One-sided Test) - All Steps

$H_0 : p = 0.05$ vs $H_a : p > 0.05$
$\hat{p} = \frac{46}{384} \approx 0.1198$
$z_* = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.1198 - 0.05}{\sqrt{\frac{(0.05)(0.95)}{384}}} \approx 6.28$
P-value $= P(Z > z_*) = P(Z > 6.28) \approx 1.747 \times 10^{-10}$
P-value $= < \alpha$, we reject $H_0 : p = 0.05$. Our result is statistically significant at $\alpha = 0.05$. There is a very strong evidence that more than 5% of children have genetic abnormalities.

R Code

```r
prop.test(x=46, n = 384 ,p=0.05,alternative="greater",
correct=FALSE);

##
##  1-sample proportions test without continuity correction
##
## data:  46 out of 384, null probability 0.05
## X-squared = 39.377, df = 1, p-value = 1.747e-10
## alternative hypothesis: true p is greater than 0.05
## 95 percent confidence interval:
##  0.09516097 1.00000000
## sample estimates:
##         p
## 0.1197917
```

95% CI for a Proportion

The p-value in the previous example was extremely small (less than 0.0001). That is a strong evidence to suggest that more than 5% of children have genetic abnormalities. However, it does not say that the percentage of sampled children with genetic abnormalities was "a lot more than 5%". That is, the p-value by itself says nothing about how much greater the percentage might be. The confidence interval provides that information.

95% CI for a Proportion

To assess the difference in practical terms, we should also construct a confidence interval:

$$
\begin{aligned}
0.1198 \pm (1.96 \times 0.0166) \\
0.1198 \pm 0.0324 \\
(0.0874, 0.1522)
\end{aligned}
$$

Interpretation: We are 95% Confident that the true percentage of children with genetic abnormalities is between 8.74% and 15.22%.

R Code

```r
prop.test(x=46, n = 384, correct=FALSE);

##
##  1-sample proportions test without continuity correction
##
## data:  46 out of 384, null probability 0.5
## X-squared = 222.04, df = 1, p-value < 2.2e-16
## alternative hypothesis: true p is not equal to 0.5
## 95 percent confidence interval:
##  0.09102214 0.15609290
## sample estimates:
##         p
## 0.1197917
```

95% CI for p: (9.1%, 15.6%) - We are 95% confident that the true percentage of all children that have genetic abnormalities is between approximately 9.1% and 15.6%. Since both values of this CI are more than the hypothesized value of $p = 0.05$ (5%), we can further infer that this true percentage is more than 5%.

Do environmental chemicals cause congenital abnormalities?

We do not know that environmental chemicals cause genetic abnormalities. We merely have evidence that suggests that a greater percentage of children are diagnosed with genetic abnormalities now, compared to the 1980s.

More About P-values

- Big p-values just mean that what we have observed is not surprising. It means that the results are in line with our assumption that the null hypothesis models the world, so we have no reason to reject it.
- A big p-value does not prove that the null hypothesis is true.
- When we see a big p-value, all we can say is: we cannot reject H_0 (we fail to reject H_0) - we cannot conclude H_a (We have no evidence to support H_a).

Additional Examples.

Example

Consider the following hypothesis test:
$H_0 : p = 0.75$
$H_a : p < 0.75$
A sample of 300 items was selected. Compute the p-value and state your conclusion for each of the following sample results. Use $\alpha = 0.05$.
a. $\hat{p} = 0.68$
b. $\hat{p} = 0.72$
c. $\hat{p} = 0.70$
d. $\hat{p} = 0.77$

Solution a.

$z_* = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{0.68 - 0.75}{\sqrt{0.75(1-0.75)/300}} = -2.80$
Using Normal table, P-value $= P(Z < z_*) = P(Z < -2.80) = 0.0026$
P-value$< \alpha = 0.05$, reject H_0.

Solution b.

$z_* = \frac{\hat{p}-p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{0.72-0.75}{\sqrt{0.75(1-0.75)/300}} = -1.20$
Using Normal table, P-value $= P(Z < z_*) = P(Z < -1.20) = 0.1151$
P-value$> \alpha = 0.05$, do not reject H_0.

Solution c.

$z_* = \frac{\hat{p}-p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{0.70-0.75}{\sqrt{0.75(1-0.75)/300}} = -2.00$
Using Normal table, P-value $= P(Z < z_*) = P(Z < -2.00) = 0.0228$
P-value$< \alpha = 0.05$, reject H_0.

Example

Consider the following hypothesis test:
$H_0 : p = 0.20$
$H_a : p \neq 0.20$
A sample of 400 provided a sample proportion $\hat{p} = 0.175$.
a. Compute the value of the test statistic.
b. What is the p-value?
c. At the $\alpha = 0.05$, what is your conclusion?
d. What is the rejection rule using the critical value? What is your conclusion?

Solution

a. $z_* = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.175-0.20}{\sqrt{\frac{(0.20)(0.80)}{400}}} = -1.25$

b. Using Normal table, P-value $=$
$2P(Z > |z_*|) = 2P(Z > |-1.25|) = 2P(Z > 1.25) = 2(0.1056) = 0.2112$

c. P-value $> \alpha = 0.05$, we CAN'T reject H_0.

Problem

A study found that, in 2005, 12.5% of U.S. workers belonged to unions. Suppose a sample of 400 U.S. workers is collected in 2006 to determine whether union efforts to organize have increased union membership.
a. Formulate the hypotheses that can be used to determine whether union membership increased in 2006.
b. If the sample results show that 52 of the workers belonged to unions, what is the p-value for your hypothesis test?
c. At $\alpha = 0.05$, what is your conclusion?

Solution

a. $H_0 : p = 0.125$ vs $H_a : p > 0.125$
b. $\hat{p} = \frac{52}{400} = 0.13$
$z_* = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.13-0.125}{\sqrt{\frac{(0.125)(0.875)}{400}}} = 0.30$
Using Normal table, P-value =
$P(Z > z_*) = P(Z > 0.30) = 1 - 0.6179 = 0.3821$
c. P-value $=> 0.05$, do not reject H_0. We cannot conclude that there has been an increase in union membership.

R Code

```
prop.test(52,400,p=0.125,alternative="greater",
correct=FALSE);

##
##  1-sample proportions test without continuity correction
##
## data:  52 out of 400, null probability 0.125
## X-squared = 0.091429, df = 1, p-value = 0.3812
## alternative hypothesis: true p is greater than 0.125
## 95 percent confidence interval:
##  0.1048085 1.0000000
## sample estimates:
##    p
## 0.13
```

Problem

A study by Consumer Reports showed that 64% of supermarket shoppers believe supermarket brands to be as good as national name brands. To investigate whether this result applies to its own product, the manufacturer of a national name-brand ketchup asked a sample of shoppers whether they believed that supermarket ketchup was as good as the national brand ketchup.

Problem (cont.)

a. Formulate the hypotheses that could be used to determine whether the percentage of supermarket shoppers who believe that the supermarket ketchup was as good as the national brand ketchup differed from 64%.
b. If a sample of 100 shoppers showed 52 stating that the supermarket brand was as good as the national brand, what is the p-value?
c. At $\alpha = 0.05$, what is your conclusion?

Solution

a. $H_0 : p = 0.64$ vs $H_a : p \neq 0.64$
b. $\hat{p} = \frac{52}{100} = 0.52$
$z_* = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.52 - 0.64}{\sqrt{\frac{(0.64)(0.36)}{100}}} = -2.50$
Using Normal table, P-value =
$2P(Z > |z_*|) = 2P(Z > |-2.50|) = 2P(Z > 2.50) = 2(0.0062) = 0.0124$
c. P-value $= < 0.05$, reject H_0. Proportion differs from the reported 0.64.

R Code

```
prop.test(52,100,p=0.64,alternative="two.sided",
correct=FALSE);

##
##  1-sample proportions test without continuity correction
##
## data:  52 out of 100, null probability 0.64
## X-squared = 6.25, df = 1, p-value = 0.01242
## alternative hypothesis: true p is not equal to 0.64
## 95 percent confidence interval:
##  0.4231658 0.6153545
## sample estimates:
##    p
## 0.52
```

Problem

The National Center for Health Statistics released a report that stated 70% of adults do not exercise regularly. A researcher decided to conduct a study to see whether the claim made by the National Center for Health Statistics differed on a state-by-state basis.
a. State the null and alternative hypotheses assuming the intent of the researcher is to identify states that differ from 70% reported by the National Center for Health Statistics.
b. At $\alpha = 0.05$, what is the research conclusion for the following state: Wisconsin: 252 of 350 adults did not exercise regularly.

Solution (Wisconsin)

a. $H_0 : p = 0.70$ vs $H_a : p \neq 0.70$

b. Wisconsin $\hat{p} = \frac{252}{350} = 0.72$

$z_* = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.72-0.70}{\sqrt{\frac{(0.70)(0.30)}{350}}} = 0.82$

Using Normal table, P-value $=$
$2P(Z > |z_*|) = 2P(Z > |0.82|) = 2P(Z > 0.82) = 2(0.2061) = 0.4122$

c. P-value > 0.05, we don't have enough evidence to reject H_0. There is not enough evidence against the claim made by the National Center for Health Statistics.

Solution

a. $t_* = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{17-18}{4.5/\sqrt{48}} = -1.54$

b. Degrees of freedom $= n - 1 = 47$.

P-value $= 2P(T > |t_*|) = 2P(T > |-1.54|) = 2P(T > 1.54)$

Using t-table, P-value is between 0.10 and 0.20.

Exact P-value $= 0.1303$ (using R).

c. Since P-value $> \alpha = 0.05$, we CAN'T reject H_0.

Test of Hypothesis for One Variance

Hypothesis Tests for One Variance

- Data from a single normal population Independent observations
- Variance unknown
- Large or small sample

Hypothesis Test
$H_0 : \sigma^2 = \sigma_0^2$ vs $H_a : \sigma^2 \neq \sigma_0^2$ (or $\sigma^2 > \sigma_0^2$ or $\sigma^2 < \sigma_0^2$).
Assume H_0 is true, then

$$\text{Test statistic: } \chi_*^2 = \frac{(n-1)s^2}{\sigma_0^2} \sim \chi_{n-1}^2$$

Decision rules:
$H_a: \sigma^2 \neq \sigma_0^2$.
Reject H_0 if $\chi^2_* > \chi^2_{n-1;\alpha/2}$ or if $\chi^2_* < \chi^2_{n-1;1-\alpha/2}$.
$H_a: \sigma^2 > \sigma_0^2$.
Reject H_0 if $\chi^2_* > \chi^2_{n-1;\alpha}$ or if $P[\chi^2_{n-1} > \chi^2_*]$ is too small.
$H_a: \sigma^2 < \sigma_0^2$.
Reject H_0 if $\chi^2_* < \chi^2_{n-1;1-\alpha}$ or if $P[\chi^2_{n-1} < \chi^2_*]$ is too small.
Note. This is NOT robust to departures from Normality.

Example

A company produces metal pipes of a standard length, and claims that the standard deviation of the length is at most 1.2 cm. One of its clients decides to test this claim by taking a sample of 25 pipes and checking their lengths. They found that the standard deviation of the sample is 1.5 cm. Does this undermine the company's claim? Use $\alpha = 0.05$.
Note. Assume length is Normally distributed.

Solution

$H_0 : \sigma^2 \leq 1.2^2$ vs $H_a : \sigma^2 > 1.2^2$.
$\chi^2_* = \frac{(n-1)s^2}{\sigma^2} = \frac{(25-1)1.5^2}{1.2^2} = 37.5$
P-value $= P[\chi^2_{24} > 37.5] \approx 0.0389$

```
1-pchisq(37.5,df=24);

## [1] 0.0389818
```

Conclusion

We reject $H_0 : \sigma^2 \leq 1.2^2$. We have evidence to indicate that the variance of the length of metal pipes is more than 1.2^2.

Test of Hypotheses concerning a Population Variance

Assumptions: $Y_1, Y_2, ..., Y_n$ constitute a random sample from a Normal distribution with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$.

$H_0 : \sigma^2 = \sigma_0^2$

$$H_a : \begin{cases} \sigma^2 > \sigma_0^2 & \text{upper-tailed alternative} \\ \sigma^2 < \sigma_0^2 & \text{lower-tailed alternative} \\ \sigma^2 \neq \sigma_0^2 & \text{two-tailed alternative} \end{cases}$$

Test of Hypotheses concerning a Population Variance

Test statistic: $\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$

$$\text{Rejection Region} \; : \begin{cases} \chi^2 > \chi^2_{\alpha} & \text{upper-tailed RR} \\ \chi^2 < \chi^2_{1-\alpha} & \text{lower-tailed RR} \\ \chi^2 > \chi^2_{\alpha/2} \; or \; \chi^2 < \chi^2_{1-\alpha/2} & \text{two-tailed RR} \end{cases}$$

Example

A manufacturer of car batteries claims that the life of his batteries is approximately Normally distributed with a standard deviation equal to 0.9 year. If a random sample of 10 of these batteries has a standard deviation of 1.2 years, do you think that $\sigma > 0.9$ year? Use a 0.05 level of significance.

Solution

Step 1. State hypotheses.

$H_0 : \sigma^2 = 0.81$

$H_a : \sigma^2 > 0.81$

Solution

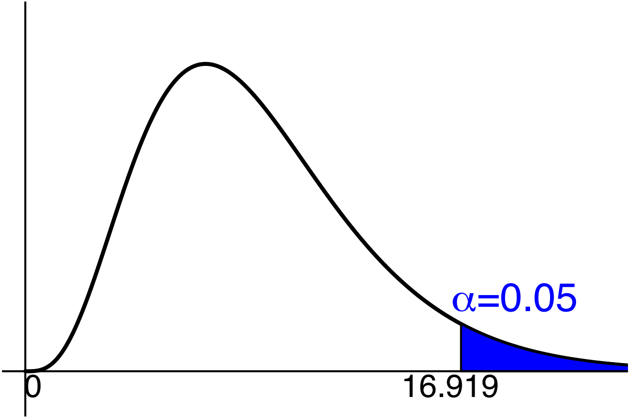
Step 2. Compute test statistic.
$S^2 = 1.44$, $n = 10$, and

$$\chi^2 = \frac{(9)(1.44)}{0.81} = 16$$

Solution

Step 3. Find Rejection Region.

From Figure and our table we see that the null hypothesis is rejected when $\chi^2 > 16.919$, where $\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$ with $\nu = 9$ degrees of freedom.


α=0.05
0
16.919

Solution

Step 4. Conclusion.
The χ^2 statistic is not significant at the 0.05 level. We conclude that there is insufficient evidence to claim that $\sigma > 0.9$ year.

Solution

Step 4. Conclusion.
The χ^2 statistic is not significant at the 0.05 level. We conclude that there is insufficient evidence to claim that $\sigma > 0.9$ year.