Vectors and Spaces

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1 Introduction

Consider a scenario where we have to shop apples and bananas,

We purchased,

Three apples and One bananas costing of total 10 rupees on Monday.
One apples and One banana costing of total 4 rupees on Tuesday.

Let us denote price of each apple by a and b for each bananas.

Now, we can represent these purchases made in these two days through these two equation,

$$3a + 1b = 10$$

$$1a + 1b = 4$$

By solving these Equations we get price of each apple as Rs.3 and each banana as Rs.1 .

Now if there are more items on our shopping list then solving these simultaneous equations may be complex.

So instead we may represent the prices of each item on a vector and number of items purchased as elements of a matrix,

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$$

Now ahead we look at these mathematical objects and understand what they are how they work.

2 What is Linear Algebra?

Linear algebra is the branch of mathematics concerning linear equations such as

$$a_1x_1 + ... + a_nx_n = b$$

linear functions such as

$$(x_1, ..., x_n) \rightarrow a_1x_1 + ... + a_nx_n$$

and their representations in vector spaces and through matrices.

3 What is a Vector?

Vectors are quantities which possess a magnitude and a direction. A vector is an object that moves in space. They don't have to be geometrical objects in space, they can be viewed as a list of attributes or space of data like,

In physics vectors can be imagined in space,
In data science, it is a list of numbers.
Vectors can be used in variety of fields such as alloys represented in vectors in metallurgy,etc.

For example, let us take a vector representation different attributes of a house.

$$\begin{bmatrix} Area \\ NumberofBedrooms \\ CostofHouse \end{bmatrix}$$

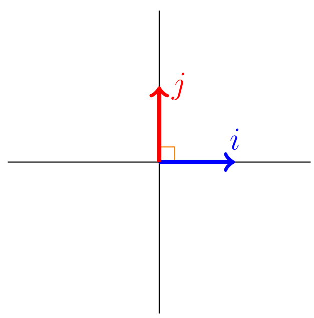
Suppose we buy a house having area of 150 Sq $\mathrm{m}^2, 3$ Bedrooms and a cost of $50, 00, 000$ rs, then it's vector will be represented as,

$$\begin{bmatrix} 150 \\ 3 \\ 50,00,000 \end{bmatrix}$$

4 Vector Operations

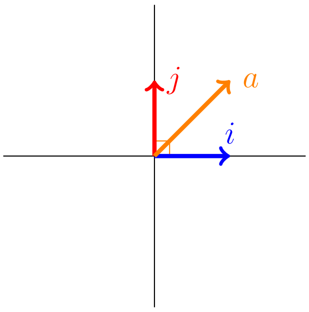
4.1 Unit Vectors

A unit vector is a vector of length 1.
Let $\vec{i}$, $\vec{j}$ be the unit vectors along positive x and y directions respectively.



Then a vector is represented by

$$\vec{a} = \begin{bmatrix} \text{Length of Vector along unit vector i} \\ \text{Length of Vector along unit vector j} \end{bmatrix}$$



Here $\vec{a} = x \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

where x and y are constant, both equal to 1.

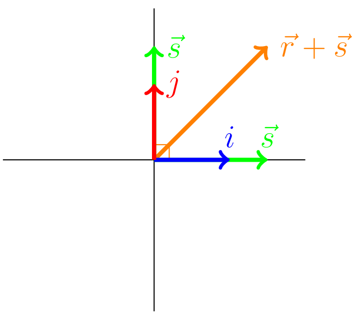
4.2 Vector Addition

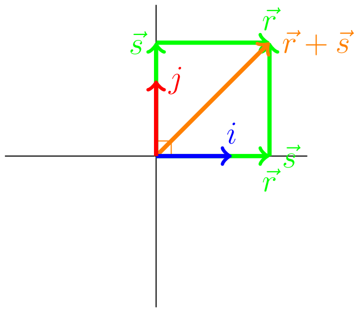
When we add Vectors $\vec{r}$ and $\vec{s}$

$\vec{r} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$ *and* $\vec{s} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$

We get the resultant vector $\vec{r} + \vec{s}$

$\vec{r} + \vec{s} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$

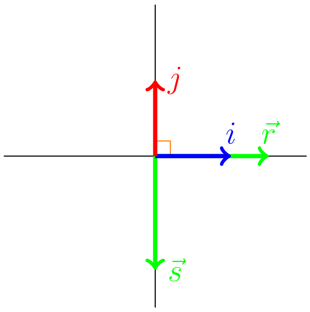




4.3 Vector Multiplication

In Vector Multiplication , we multiply a scalar value with a vector

In an example below $\vec{r} = 1.5 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$ *and* $\vec{s} = -1.5 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \text{-1.5} \end{bmatrix}$
Here, we multiplied scalar value of 1.5 to unit vector $\vec{i}$ and -1.5 to vector $\vec{j}$.



4.4 Vector Subtraction

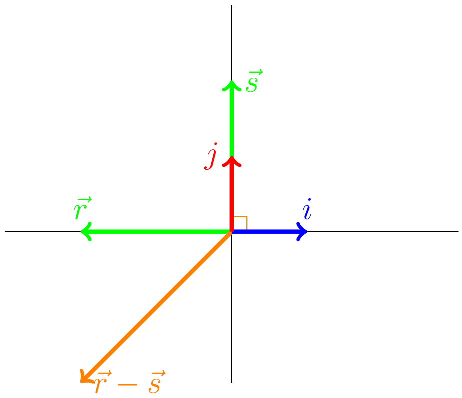
Vector Subtraction is addition of a vector having multiplied by -1 first.

$\vec{r} = \begin{bmatrix} \text{-2} \\ 0 \end{bmatrix}$ *and* $\vec{s} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

therefore

$\vec{r} - \vec{s} = \begin{bmatrix} \text{-2} \\ 0 \end{bmatrix} + (-1) \times \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \text{-2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \text{-2} \end{bmatrix} = \begin{bmatrix} \text{-2} \\ \text{-2} \end{bmatrix}$

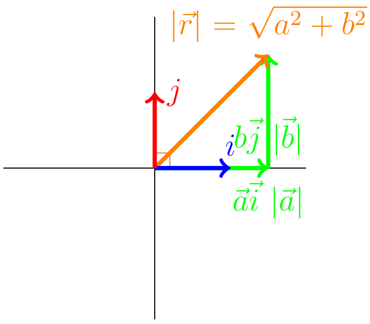
Here $\vec{s}$ is first multiplied by -1 and then added with $\vec{r}$.



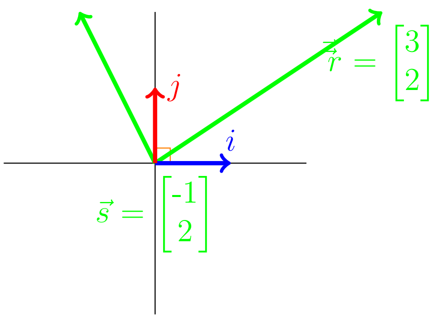
5 Modulus and Inner product

The magnitude is also called the modulus or the length of the vector.
For Example, modulus of a vector $\vec{r} = a\vec{i} + b\vec{j} = \begin{bmatrix} \text{a} \\ b \end{bmatrix}$ is given as,

$|\vec{r}| = \sqrt{a^2 + b^2}$



Inner product of vectors is multiplication of each component of vectors then addition of the multiplied product giving a scalar result.



5.1 Commutative Property

Inner product or Dot product follows commutative property.

$\vec{r} \cdot \vec{s}$
$= \mathrm{r}_i s_i + r_j s_j$
$= 3 \cdot -1 + 2 \cdot 2$
$= -3 + 4 = 1$
$= s \cdot r$

5.2 Distributive over Addition

r$\cdot(s + t) = r \cdot s + r \cdot t$

$$r = \begin{bmatrix} r_1 \\ r_2 \\ . \\ . \\ r_n \end{bmatrix}, \ s = \begin{bmatrix} s_1 \\ s_2 \\ . \\ . \\ s_n \end{bmatrix}, \ t = \begin{bmatrix} t_1 \\ t_2 \\ . \\ . \\ t_n \end{bmatrix}$$

r$\cdot(s + t)$
$= r_1 \cdot (s_1 + t_1) + r_2 \cdot (s_2 + t_2) + ... + r_n \cdot (s_n + t_n)$
$= r_1 \cdot s_1 + r_1 \cdot t_1 + r_2 \cdot s_2 + r_2 \cdot t_2 + ... + r_n \cdot s_n + r_n \cdot t_n$
$= (r_1 \cdot s_1 + r_2 \cdot s_2 + ... + r_n \cdot s_n) + (r_1 \cdot t_1 + r_2 \cdot t_2 + ... + r_n \cdot t_n)$
$= r \cdot s + r \cdot t$

Hence Proved

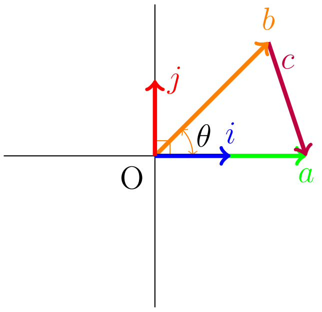
5.3 Associative over Scalar Multiplication

r$\cdot(a \cdot s) = a \cdot (r \cdot s)$

$LHS = r_1 \cdot (a_1 \cdot s_1) + r_2 \cdot (a_2 \cdot s_2)$
$= a \cdot (r_1 \cdot s_1 + r_2 \cdot s_2)$
$= a \cdot (r \cdot s)$

Hence Proved

6 Cosine and Dot Product



We know that by Cosine Rule,

$c^2 = a^2 + b^2 - 2ab\cos\theta$

$\therefore |\vec{r} - \vec{s}| = |\vec{r}| + |\vec{s}| - 2|\vec{r}||\vec{s}|\cos\theta$

$= (\vec{r} - \vec{s}) \cdot (\vec{r} - \vec{s}) = \vec{r} \cdot \vec{r} - \vec{s} \cdot \vec{r} - \vec{s} \cdot \vec{r} - \vec{s} \cdot \vec{s}$

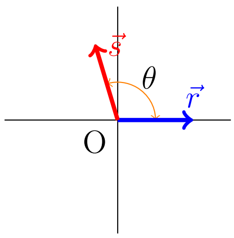
$= |\vec{r}^2| - 2\vec{s} \cdot \vec{r} + |\vec{s}^2|$

Now by Equation (1) and (2),

$\implies \vec{s} \cdot \vec{r} = -2|\vec{r}||\vec{s}|\cos\theta$

$\therefore \boxed{\vec{r} \cdot \vec{s} = |\vec{r}||\vec{s}|\cos\theta}$

Examples,

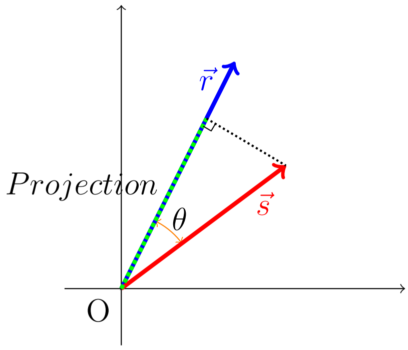


But when $\theta = 90°$, $\cos 90° = 0, \therefore \vec{r} \cdot \vec{s} = 0$

Also when $\theta = 180°$, $\cos 180° = -1, \therefore \vec{r} \cdot \vec{s} = -|\vec{r}||\vec{s}|$

And when $\theta = 0°$, $\cos 0° = 1, \therefore \vec{r} \cdot \vec{s} = |\vec{r}||\vec{s}|$

7 Projection of Vectors



$\cos\theta = \frac{Adjacent}{Hypotenuse} = \frac{Adjacent}{|\vec{s}|}$

$\therefore \vec{r}\cdot\vec{s} = |\vec{r}||s|\cos\theta$
Where $\rightarrow$ $|s|\cos\theta$

$\therefore \vec{r}\cdot\vec{s} = |\vec{r}|\cdot$ Projection

Scalar Projection $\rightarrow \frac{\vec{r}\cdot\vec{s}}{|\vec{r}|} = |\vec{s}|\cdot\cos\theta$

Vector Projection $\rightarrow \frac{\vec{r}\cdot\vec{s}\cdot\vec{r}}{|\vec{r}|\cdot|\vec{r}|}$

It's Testing Time 

1. A ship travels with velocity given by $\begin{bmatrix}1\\2\end{bmatrix}$, with current flowing in the direction given by $\begin{bmatrix}1\\1\end{bmatrix}$ with respect to some co-ordinate axes.

   What is the velocity of the ship in the direction of the current?
   (Hint: Apply Projection that you have studied now)

2. At 12:00 pm, a spaceship is at position $\begin{bmatrix}3\\2\\4\end{bmatrix}$ km away from the origin with respect to some 3 dimensional co ordinate system. The ship is travelling with velocity $\begin{bmatrix}-1\\2\\-3\end{bmatrix}$ km/h What is the location of the

spaceship after 2 hours have passed?

Note: *Solutions are given at the end of the document.*

8 Linear Dependence and Independence

8.1 What are Linear Dependence and Independence?

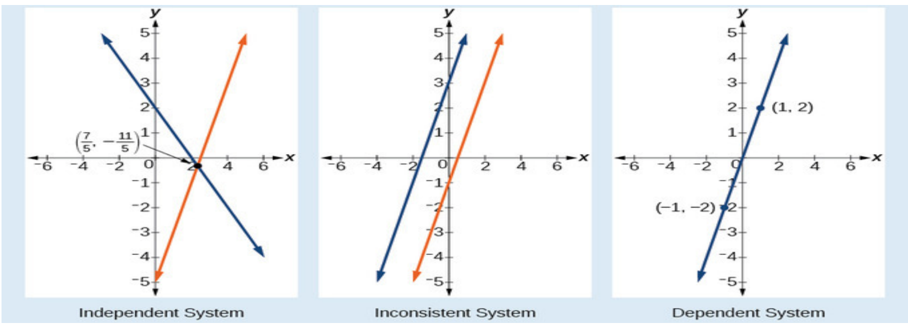
Linear Dependence in a system of linear equations having more than or two equations referring to the same line containing infinite number of solutions to satisfy the conditions of the equations.

Linear Independence in system of linear equations means that the two equations only meet at one point i.e.(the intersection between the two lines). This single point in the entire universe will solve both equations at the same time.

Testing Equations for Dependence and Independence:

1. If the slopes are different, then the system is independent.

2. If the slopes are the same, then the system is either dependent (same line) or inconsistent (parallel lines).



8.2 Linearly Dependent Vectors

Vectors are linearly dependent if there is a linear combination of them that equals the zero vector, without the coefficients of the linear combination being zero.

$$a_1\vec{v}_1 + a_3\vec{v}_2 + ... + a_n\vec{v}_n = \vec{0}$$

8.3 Linearly Independent Vectors

Several vectors are linearly independent if none of them can be expressed as a linear combination of the others.

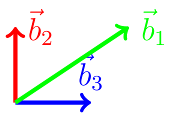
$$a_1\vec{v}_1 + a_3\vec{v}_2 + ... + a_n\vec{v}_n = \vec{0}$$

$$a_1 = a_2 = ... = a_n = 0$$

9 Basis

Basis is a set of n vectors that:

1. Are not linear combinations of each others(Linearly Independent).
2. Span the space
   The Space is then n dimensional.



Here $\vec{b}_3 \neq$a$_1\vec{b}_1$ + $a_2\vec{b}_2 \rightarrow$ *Linearly Independent*

10 Matrices

10.1 What are Matrices?

A Matrix is a rectangular array of numbers arranged in rows and columns.
For Example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{34} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

The number of rows and columns that a matrix has is called its **dimension** or its **order**. The dimension (or order) of the above matrix is 3 x 4, meaning that it has 3 rows and 4 columns.
Numbers that appear in the rows and columns of a matrix are called **elements** of the matrix.

Matrices can be used to solve Simultaneous Equations.

In the above apples and bananas problem, we have transformed a equation into the matrix form.

$$3a + 1b = 10, 1a + 1b = 4$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$$

Matrices can also be used to transform Space.
The above matrix form can be represented as:

$$A \cdot r = r'$$
$$A \cdot nr = r'$$

$Where, A \rightarrow Matrix, r \rightarrow$Original Vector, r' $\rightarrow$ Spanned Vector

$$A \cdot (n\vec{e}_1 + n\vec{e}_2) = nA\vec{e}_1 + nA\vec{e}_2$$

Where, $\vec{e}_1, \vec{e}_2 \rightarrow$ Unit Vector along X axis and Y axis respectively

Let us take a vector $r = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and span it using matrix $A = \begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix}$

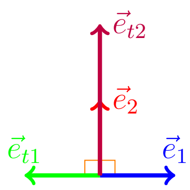
$$\therefore \begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \left(3* \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2* \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 12 \\ 32 \end{bmatrix}$$

The Vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is spanned into $\begin{bmatrix} 12 \\ 32 \end{bmatrix}$

10.2 Matrix Transformation

Let us apply Transformation Matrix $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ to unit vectors $\vec{e}_1$ and $\vec{e}_2$.



Where, $\vec{e}_{t1}, \vec{e}_{t2} \rightarrow$ Vector Transformed from $\vec{e}_1$, $\vec{e}_2$ respectively

These matrices can also be used to rotate the vectors along the plane.
For Example using,

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

and other matrices.

10.3 Composition or Combination

We can apply n transformations to a vector by combining these transformation matrix.
But the order in which we apply this transformation can also change the results.
For Example, Let us take two transformation matrix $\mathrm{A}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\mathrm{A}_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
Now, $\mathrm{A}_1 \cdot A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $A_2 \cdot A_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

From above we can conclude that,

$$\boxed{A_1 \cdot A_2 \neq A_2 \cdot A_1}$$

10.4 Determinant and Inverse

A determinant of a matrix represents a single number which is obtained by multiplying and adding its elements in a special way.
Determinant of 2 x 2 Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, |A| = ad - bc$$

Determinant of 3 x 3 Matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} |A| = a(ei - fh) - b(di - gf) + c(dh - eg)$$

For example, if we have the 2 × 2 square matrix:

$$\begin{bmatrix} 5 & 7 \\ 2 & -3 \end{bmatrix}$$

then the determinant of this matrix is written within vertical lines as follows:

$$\begin{vmatrix} 5 & 7 \\ 2 & -3 \end{vmatrix} = [(5)(3) + (7)(2)] = 29$$

The inverse of a square matrix A, sometimes called a reciprocal matrix, is a matrix A^{-1} such that
$AA^{-1} = I$,
where I is the identity matrix.

A square matrix A has an inverse if the determinant $|A|! = 0$. A matrix possessing an inverse is called non singular or invertible otherwise, matrix is a singular matrix.

For 2×2 matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

it's inverse is given as,

$$A^{-1} = 1/(A) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = 1/(ad - bc) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

A general n $\times$ n matrix can be inverted using methods such as the Gauss-Jordan elimination, Gaussian elimination or LU decomposition.

11 Solving using Elimination

For solving equation in matrix form, we have to reduce matrix.
There are various methods for reducing the matrices, some of theme are:

11.1 Echelon Method

In this method, the given matrix is reduced into echelon form.
Echelon form of any matrix is the one having following properties:

1. All zero rows (if any) belong at the bottom of the matrix.
2. A pivot in a non-zero row, which is the left-most non-zero value in the row, is always strictly to the right of the pivot of the row above it.

For Example,

$$\begin{bmatrix} 1 & 2 & 5 & 8 \\ 0 & 6 & 3 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 5 \\ 0 & 6 & 3 \\ 0 & 0 & 1 \end{bmatrix}, etc$$

11.2 Gauss Elimination

Important Conclusion: $\boxed{A \cdot A^{-1} = I}$

In Gauss Elimination, we reduce the $n \times n$ matrix into Identity matrix to solve the simultaneous equation or get the values of elements in the vector. For Example

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 6 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 5 \end{bmatrix}$$

We can represent the above expression as,
$AX = B$
Multiplying Both sides by $A^{-1}, weget$
$\therefore A^{-1}AX = A^{-1}B$
$\therefore IX = A^{-1}B$

$\boxed{X = A^{-1}B}$

12 Einstein Summation Convention

Einstein summation convention is a notational convention that implies summation over a set of indexed terms in a formula.

Suppose we have to multiple two matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{14} \\ a_{21} & a_{22} & a_{23} & ... & a_{34} \\ : & : & : & ... & : \\ : & : & : & ... & : \\ a_{n1} & a_{n2} & a_{n3} & ... & a_{n4} \end{bmatrix} andB = \begin{bmatrix} b_{11} & b_{12} & b_{13} & ... & b_{14} \\ b_{21} & b_{22} & b_{23} & ... & b_{34} \\ : & : & : & ... & : \\ : & : & : & ... & : \\ b_{n1} & b_{n2} & b_{n3} & ... & b_{n4} \end{bmatrix}$$

to give matrix AB.
If we want element ab_{13} of matrix AB, we have to do,
$ab_{13} = a_{11}b_{13} + a_{12}b_{23} + ... + a_{1n}b_{n3}$

Now Einstein Summation Convention says that, the element can be found using the expression,

$$\therefore \boxed{ab_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk} = a_{ij}b_{jk}}$$

If we do this for all the possible cases of i and k we get the entire AB matrix.

13 Solutions

1. Since the problem is using velocity as a vector. We want the projection of the ship velocity onto the current velocity.

   Let u = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ be the vector representing the ship's velocity and v = $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be the vector representing current direction.
   $\therefore v \cdot \frac{u \cdot v}{|v|^2} = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$

2. Initial position of the Spaceship was $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ km

   It's velocity is $\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$ km/h

   So, in 2 hours, it travels $2 * \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$ kms respectively in 3D co-ordinate system.

   Then it's final position willl be:

   $$\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} km$$