Recent progress in the monotiling conjecture

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Some notation: Convolutions and tilings

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$$f \star g(x) := \sum_{y \in \mathbb{Z}^d} f(x-y)g(y).$$

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Let $F \subset \mathbb{Z}^d$ be a finite set. We say that F tiles $\mathbb{Z}^d$ at level k if there exists a co-tiler $A \subset \mathbb{Z}^d$ such that

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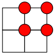
Tiling at level 1 is just called tiling.

Tiling by a box can is always periodic but can aperiodic in one direction: Aperiodic in vertical direction



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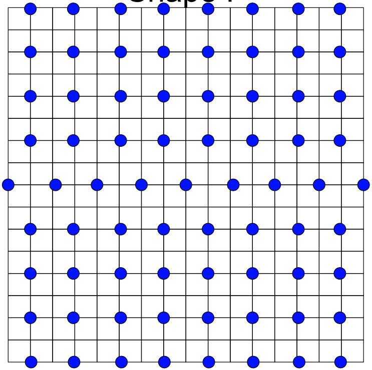

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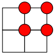

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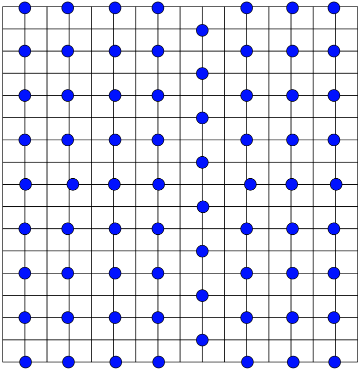

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Some notation: Convolutions and tilings

$\delta_y \star 1_F$ is just the translate of the set F.

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In other words, there are translates of the set F (given by elements of A) which cover the $\mathbb{Z}^d$ lattice k times.

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Let $y \in \mathbb{Z}^d \setminus \{0\}$ and consider the discrete derivative in the direction y by

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As with usual derivatives $\Delta_y^k(f) = 0$ if and only if $(f(x+ny); n \in \mathbb{N})$ is a polynomial in n for all $x \in \mathbb{Z}^d$.

Monotiling Conjecture

F tiles $\mathbb{Z}^d$ if there exists A such that

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4. Related to Golomb-Welch conjecture in linear codes.

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6. Wide open for higher dimensions.

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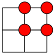
When all tilings are periodic?



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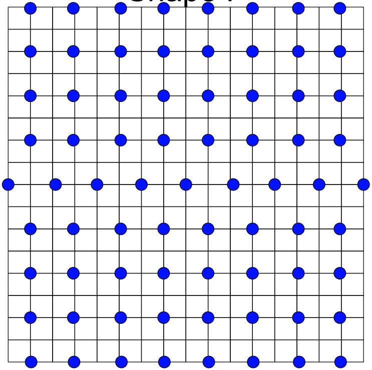

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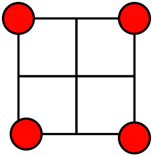
When some of them are weakly periodic but not periodic



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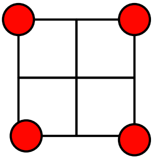

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The even sublattice can be tiled so that it is aperiodic in the horizontal direction while the odd sublattice can can be tiled so that it is aperiodic in the vertical direction.

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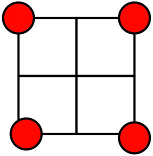

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The resulting configuration will be weakly periodic but not periodic.

What did Bhattacharya prove?

Note that the set of A such that $1_A \star 1_F = 1$ is invariant under translations by $\mathbb{Z}^d$. In fact it can be considered as a compact subset of the $\{0, 1\}^{\mathbb{Z}^2}$ and has the structure of a shift of finite type.

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For $d = 2$, Bhattacharya said that with respect to any invariant measure, every co-tiler A is weakly periodic.

From this result he used some standard ideas to show that a periodic tiling exists.

And what have Greenfeld and Tao done?

Theorem

Let $F \subset \mathbb{Z}^2$ be a finite set. Let $A \subset \mathbb{Z}^2$ be such that $1_A \star 1_F = 1$. Then A is weakly periodic.

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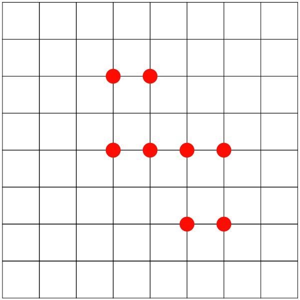
Fundamentally the ideas being used are not very different from that by Bhattacharya but the usage is much more elegant and clearer.

Level-4 tiling which is not weakly periodic

$$1_F := 1_{(0,0),(1,0)} \star 1_{(0,0),(0,2)} \star 1_{(0,0),(2,-2)}.$$



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4. The visit times to B, that is,

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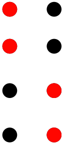
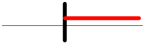
5. Because α is irrational, for most choices of B, the level 4 co-tiler A is not weakly periodic.

The set B is the union of

Z/4Z  times  R/Z

Z/2Z

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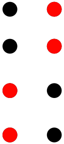
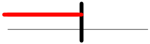
Z/4Z  times  R/Z

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I conjecture that all tilings of a certain level of a set F arise as visit times of a rotation of a compact group.

While it is false as stated, a precise conjecture close to it can be formulated along these lines (this is done at the end).

The main theorem

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We will concentrate on $d = 2$ and level 1 tilings though a lot of ideas go through to higher dimensions and higher level tilings.

Dilation lemma (starting point for Szegedy, Bhattacharya, Kari-Szabados)

Let $F \subset \mathbb{Z}^2$ be a finite set and $A \subset \mathbb{Z}^2$ be such that

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Finally $1_{pF} \star 1_A$ is bounded by $|pF| = |F|$ and hence $1_{pF} \star 1_A = 1$. The same fact for r now follows since the prime factors of r are greater than $|F|$.

We will always assume that $0 \in F$.

Periodic decomposition

We assume $0 \in F$. Let $F \subset \mathbb{Z}^2$ be a finite set and $A \subset \mathbb{Z}^2$ be such that

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If q is the product of all primes less than or equal to $|F|$ then

$$1_{(tq+1)F} * 1_A = 1.$$

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$$1_F \star 1_A = 1.$$

If q is the product of all primes less than or equal to $|F|$ then

$$1_{(tq+1)F} * 1_A = 1.$$

This implies that

$$1_A = 1 - \sum_{f \in F \setminus \{0\}} \delta_{(tq+1)f} \star 1_A = 1 - \sum_{f \in F \setminus \{0\}} \frac{1}{N} \sum_{t=1}^{N} \delta_{(tq+1)f} \star 1_A.$$

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where $\phi_f : \mathbb{Z}^2 \to [0,1]$ is periodic in the qf direction.

A similar step is taken by Kari-Szabados by taking an ultrafilter limit and by Bhattacharya by considering an ergodic invariant measure to start with. The dilation lemma also finds its presence in the work of Szegedy.

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Choose e such that it is linearly independent of all the h_i and

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Consider the function $P_{x,j} : \mathbb{Z} \to [0,1]$ given by

$$P_{x,j}(n) := \phi_j(x + ne).$$

Some intuition about the $P_{x,j}$

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$\phi_j : \mathbb{Z}^2 \to [0,1]$ is periodic in linearly independent directions h_j. e is linearly independent of all h_i and in $\langle h_i, h_j \rangle$ for all i, j. Let

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Let us see why they are polynomials.

$P_{x,j}$ is a nice polynomial

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(1) follows immediately from $\star$. For (2) we need to show $\Delta_e^{m-1} \phi_j = 0$ mod 1.

$\Delta_e^{m-1}\phi_j = 0 \bmod 1$

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We have $e = a_i h_i + b_i h_j$. Since ϕ_i is $\langle h_i \rangle$ periodic we have

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But since we have ϕ_j is $\langle h_j \rangle$-periodic we have

$$\Delta_e \phi_j = \Delta_{e - b_i h_j}\phi_j.$$

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Thus from $\star$ we have that

$$\Delta_e^{m-1}\phi_j \bmod 1 = \left(\prod_{i \neq j} \Delta_{e - b_i h_j}\right)(\phi_j) \bmod 1 = \left(\prod_{i \neq j} \Delta_{e - b_i h_j}\right)\left(\sum_{t=1}^{m} \phi_t\right) \bmod 1 = 0.$$

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We want to bound $P_{x,i}(n_1) + P_{x,j}(n_2) = \phi_i(x + n_1 e) + \phi_j(x + n_2 e)$ for part (3).

Bounding $P_{x,i}(n_1) + P_{x,j}(n_2)$

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So it is enough to show that there is some $\tilde{x}$ such that

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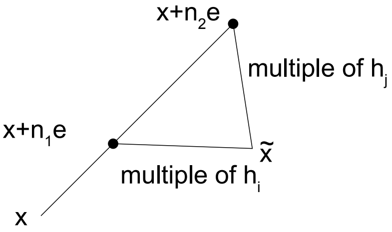

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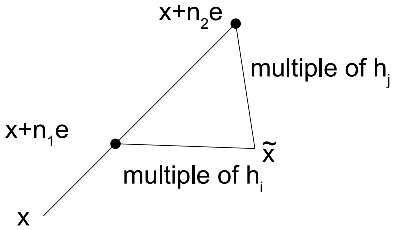

This follows because e is linearly independent of h_i, h_j and is in $\langle h_i, h_j \rangle$.

A set $A \subset \mathbb{Z}^d$ is called weakly-periodic if there is a lattice $\Lambda \subset \mathbb{Z}^d$ such that for all its cosets $x + \Lambda$,

$$(x + \Lambda) \cap A$$

is periodic.

Theorem

Let $F \subset \mathbb{Z}^2$ be a finite set. Let $A \subset \mathbb{Z}^2$ be such that $1_A \star 1_F = 1$. Then A is weakly periodic.

Proof of weak periodicity

$$1_A = 1 - \sum_{j=1}^{m} \phi_j \qquad \star$$

where $\phi_j : \mathbb{Z}^d \to [0,1]$ is $\langle h_j \rangle$ periodic.

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1. e is linearly independent of all h_i.
2. Choose N such that $e \in N\mathbb{Z}^2 \subset \langle h_i, h_j \rangle$ for all i, j.
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Let $\Lambda = \langle e, NMh_1 \rangle$. We will prove $A \cap (x + \Lambda)$ is periodic.

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Let $\Lambda = \langle e, NMh_1 \rangle$. We will prove $A \cap (x + \Lambda)$ is periodic. Rewriting $\star$ we get that

$$1_A(x + se + tNMh_1) = 1 - \sum_{j=1}^{m} \phi_j(x + se + tNMh_1).$$

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But $NMh_1 = a_i h_i + b_i e$. Hence

$$\begin{aligned} \phi_i(x + se + tNMh_1) &= \phi_i(x + (s + b_i t)e + ta_i h_i) \\ &= \phi_i(x + (s + b_i t)e) \\ &= P_{x,i}(s + b_i t). \end{aligned}$$

$$\text{Thus } 1_A(x + se + tNMh_1) = 1 - \sum_{j=1}^{m} P_{x,j}(s + b_j t).$$

Proof of weak periodicity

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where $\phi_j : \mathbb{Z}^d \to [0,1]$ is $\langle qh_j \rangle$ periodic.

$$P_{x,j}(n) := \phi_j(x + ne).$$

Properties of $P_{x,j}$.

1. One has $\sum_{j=1}^m P_{x,j}(n) = 0 \bmod 1$ for all $n \in \mathbb{Z}$.
2. For each $1 \le j \le m$, the map $n \mapsto P_{x,j}(n) \bmod 1$ is a polynomial of degree at most $m-2$.
3. For any $1 \le i < j \le m$, one has $\sup_{n\in\mathbb{Z}} P_{x,i}(n) + \sup_{n\in\mathbb{Z}} P_{x,j}(n) \le 1$.

We had $NMh_1 = a_j h_j + b_j e$ and

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Case 1: $\sup_{n \in \mathbb{N}} P_{x,j}(n) = 1$ for some $1 \leq j \leq m$. By the third property we have that $P_{x,i} = 0$ for all $i \neq j$.

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$$1_A(x + s'e + ta_j h_j) = 1 - P_{x,j}(s').$$

This is true for all s' and t.

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This is true for all s' and t. Thus $A \cap (x + \Lambda)$ is $\langle a_j h_j \rangle$-periodic.

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What is left to do?

1. Understanding the structure of higher level tilings.
    1. Suppose a finite shape can tile at level k. Can it tile periodically at level k.
    2. Given $h \in \mathbb{Z}^2$ we will say that $A \subset \mathbb{Z}^2$ is a Bohr open set in the direction h if for all $x \in \mathbb{Z}^2$, the set

       $$\{n \in \mathbb{Z} \ : \ x + nh \in A\}$$

       is the visit times of a compact group rotation to an open set.

       Does there exist a full-rank sublattice Λ such that for all $A \in X_c$ and $y \in \mathbb{Z}^2$, $A \cap (y + \Lambda)$ is Bohr open in some direction?
2. Higher dimensions.