# How much do we need to know to recognise a process?

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October, Expanding Dynamics

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#### Question

For which sets  $P \subset \mathbb{Z}$ , can we identify the process given its sample on the set P, that is,  $X_i$ ;  $i \in P$ ?

## Some notation: Cylinder Sets

Given a finite set  $B \subset \mathbb{Z}$  and  $w \in A^B$ , we write

$$[w] := \{ x \in A^{\mathbb{Z}} : x|_B = w \}.$$

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We will frequently use the correspondence between processes and infinite measure preserving transformations: The process will be denoted by  $X_i$ ;  $i \in \mathbb{Z}$  while the corresponding measure preserving action will be denoted by  $(A^{\mathbb{Z}}, \mu, \sigma)$  where  $\sigma$  is the shift map.

## What if we know everything?

Hopf's ratio ergodic theorem says that if  $f, g \in L^1(\mu)$ ;  $g \ge 0$  then for almost every  $x \in A^{\mathbb{Z}}$ ,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n f(\sigma^i(x))}{\sum_{i=1}^n g(\sigma^i(x))} = \frac{\int f \ d\mu}{\int g \ d\mu}.$$

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Thus if a sample of a process is given on  $\mathbb N$  then (up to measure zero), we can apply Hopf's ratio ergodic theorem to find the ratio of  $\mu([u])$  and  $\mu([v])$  for all cylinder sets [u], [v] enabling us to recognise the underlying process.

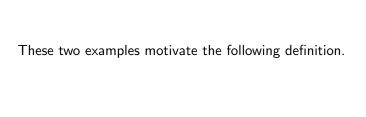
What if we just know the process only on the even integers?

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Then given a sample of either process, we will not be able to identify it with probability 1/2.



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A set  $P \subset \mathbb{Z}$  is called a recognition set if there is a measurable function

$$f: A^P \to \textit{Meas}(A^{\mathbb{Z}})$$

such that for all processes  $X_i$ ;  $i \in \mathbb{Z}$  corresponding to measure  $\mu$ ,

$$f(X_i; i \in P) = k\mu$$

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 $\mathbb{N}$  is a recognition set while  $2\mathbb{N}$  is not.

#### Question

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While we have not found one, we do have a partial understanding.

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Theorem (Chandgotia, Weiss)

P is a recognition set for processes on probability spaces if and only if it is thick, that is, it contains intervals of arbitrary size.

## There are thick sets which are not recognition sets

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In particular, there are thick sets which do not have bounded gaps so there are recognition sets for processes on probability spaces which are not recognition sets (on spaces of infinite measure).

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This is a consequence of more general theorem which we now state. Given a sequence of positive integers  $i_1, i_2, \ldots$ , we write

$$IP(i_1, i_2, \ldots) := \{ \sum_{t \in \mathbb{N}} \epsilon_t i_t : \epsilon_t \text{ is 0 or 1} \}.$$

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#### Maharam sets

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A set  $P \subset \mathbb{N}$  is called a Maharam set if for all conservative measure preserving actions,  $(X, \mu, T)$  and sets  $A \subset X$  of positive measure, there exists  $n \in P$  such that

$$\mu(A\cap T^{-n}A)>0.$$

#### Maharam sets

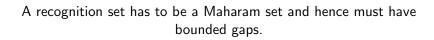
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A set  $P \subset \mathbb{N}$  is a Maharam set if and only if it is  $IP^*$ . In particular it must have bounded gaps.



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$$\lim_{k\to\infty}\frac{f(\mathit{T^{n_1}}x)+f(\mathit{T^{n_2}}x)+\ldots+f(\mathit{T^{n_k}}x)}{f(\mathit{Tx})+f(\mathit{T^2}x)+\ldots+f(\mathit{T^{n_k}}x)}\to 0.$$

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It follows that  $\mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}\$  is a recognition set.

Question

Do all recognition sets have density one?

## Summary

- We are interested in conservative, infinite, ergodic, invariant Radon measures on  $\mathbb{N}^{\mathbb{Z}}$ .
- A set  $P \subset \mathbb{Z}$  is called recognition set if processes can be recognised by their samples restricted to P.
- Recognition sets in the natural numbers have bounded gaps.
- If  $n_k$  forms a sparse sequence then  $\mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}$  is a recognition set.

#### Questions:

- Is there a nice characterisation of recognition sets?
- Do they necessarily have density one?