sequence of partial sums related to the series. If the sequence of partial sums $\{S_n\}$ has a limit, then the infinite series $\sum_{k=1}^{\infty} a_k$ converges to that limit. If the sequence of partial sums does not have a limit, the infinite series diverges.

Table 8.2 shows the correspondences between sequences/series and functions, and between summing and integration. For a sequence, the index n plays the role of the independent variable and takes on integer values; the terms of the sequence $\{a_n\}$ correspond to the dependent variable.

With sequences $\{a_n\}$, the idea of accumulation corresponds to summation, whereas with functions, accumulation corresponds to integration. A finite sum is analogous to integrating a function over a finite interval. An infinite series is analogous to integrating a function over an infinite interval.

Table 8.2

	Sequences/Series	Functions
Independent variable	n	x
Dependent variable	a_n	$f(x)$
Domain	Integers e.g., $n = 0, 1, 2, 3, \ldots$	Real numbers e.g., $\{x: x \geq 0\}$
Accumulation	Sums	Integrals
Accumulation over a finite interval	$\sum_{k=0}^{n} a_k$	$\int_0^n f(x)\,dx$
Accumulation over an infinite interval	$\sum_{k=0}^{\infty} a_k$	$\int_0^{\infty} f(x)\,dx$

SECTION 8.1 EXERCISES

Review Questions

1. Define *sequence* and give an example.
2. Suppose the sequence $\{a_n\}$ is defined by the explicit formula $a_n = 1/n$, for $n = 1, 2, 3, \ldots$. Write out the first five terms of the sequence.
3. Suppose the sequence $\{a_n\}$ is defined by the recurrence relation $a_{n+1} = na_n$, for $n = 1, 2, 3, \ldots$, where $a_1 = 1$. Write out the first five terms of the sequence.
4. Define *finite sum* and give an example.
5. Define *infinite series* and give an example.
6. Given the series $\sum_{k=1}^{\infty} k$, evaluate the first four terms of its sequence of partial sums $S_n = \sum_{k=1}^{n} k$.
7. The terms of a sequence of partial sums are defined by $S_n = \sum_{k=1}^{n} k^2$, for $n = 1, 2, 3, \ldots$. Evaluate the first four terms of the sequence.
8. Consider the infinite series $\sum_{k=1}^{\infty} \frac{1}{k}$. Evaluate the first four terms of the sequence of partial sums.

Basic Skills

9–12. Explicit formulas *Write the first four terms of the sequence* $\{a_n\}_{n=1}^{\infty}$.

9. $a_n = 1/10^n$
10. $a_n = n + 1/n$
11. $a_n = 1 + \sin(\pi n/2)$
12. $a_n = 2n^2 - 3n + 1$

13–16. Recurrence relations *Write the first four terms of the sequence* $\{a_n\}$ *defined by the following recurrence relations.*

13. $a_{n+1} = 3a_n - 12;\ a_1 = 10$
14. $a_{n+1} = a_n^2 - 1;\ a_1 = 1$
15. $a_{n+1} = 3a_n^2 + n + 1;\ a_1 = 0$
16. $a_{n+1} = a_n + a_{n-1};\ a_1 = 1, a_0 = 1$

17–22. Enumerated sequences *Several terms of a sequence* $\{a_n\}_{n=1}^{\infty}$ *are given.*

a. *Find the next two terms of the sequence.*
b. *Find a recurrence relation that generates the sequence (supply the initial value of the index and the first term of the sequence).*
c. *Find an explicit formula for the general nth term of the sequence.*

17. $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$
18. $\{1, -2, 3, -4, 5, \ldots\}$
19. $\{1, 2, 4, 8, 16, \ldots\}$
20. $\{1, 4, 9, 16, 25, \ldots\}$
21. $\{1, 3, 9, 27, 81, \ldots\}$
22. $\{64, 32, 16, 8, 4, \ldots\}$

23–30. Limits of sequences *Write the terms* a_1, a_2, a_3, *and* a_4 *of the following sequences. If the sequence appears to converge, make a conjecture about its limit. If the sequence diverges, explain why.*

23. $a_n = 10^n - 1;\ n = 1, 2, 3, \ldots$
24. $a_n = n^8 + 1;\ n = 1, 2, 3, \ldots$
25. $a_n = \dfrac{(-1)^n}{n};\ n = 1, 2, 3, \ldots$
26. $a_n = 1 - 10^{-n};\ n = 1, 2, 3, \ldots$
27. $a_{n+1} = \dfrac{a_n^2}{10};\ a_0 = 1$
28. $a_{n+1} = 0.5a_n(1 - a_n);\ a_0 = 0.8$
29. $a_{n+1} = 0.5a_n + 50;\ a_0 = 100$
30. $a_{n+1} = 0.9a_n + 100;\ a_0 = 50$

T 31–36. Explicit formulas for sequences *Consider the explicit formulas for the following sequences.*

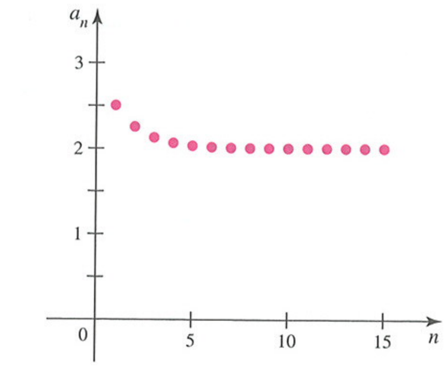
a. *Find the first four terms of the sequence.*
b. *Using a calculator, make a table with at least 10 terms and determine a plausible value for the limit of the sequence or state that it does not exist.*

31. $a_n = n + 1;\ n = 0, 1, 2, \ldots$
32. $a_n = 2\tan^{-1}(1000n);\ n = 1, 2, 3, \ldots$
33. $a_n = n^2 - n;\ n = 1, 2, 3, \ldots$
34. $a_n = \dfrac{2n - 3}{n};\ n = 1, 2, 3, \ldots$
35. $a_n = \dfrac{(n-1)^2}{(n^2 - 1)};\ n = 2, 3, 4, \ldots$
36. $a_n = \sin(n\pi/2);\ n = 0, 1, 2, \ldots$

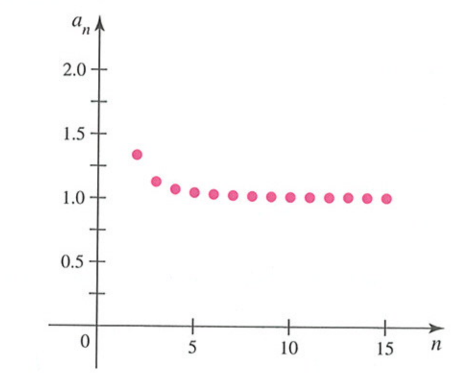
37–38. Limits from graphs *Consider the following sequences.*

a. *Find the first four terms of the sequence.*
b. *Based on part (a) and the figure, determine a plausible limit of the sequence.*

37. $a_n = 2 + 2^{-n};\ n = 1, 2, 3, \ldots$



38. $a_n = \dfrac{n^2}{n^2 - 1};\ n = 2, 3, 4, \ldots$



T 39–44. Recurrence relations to formulas *Consider the following recurrence relations.*

a. *Find the terms* a_0, a_1, a_2, a_3 *of the sequence.*
b. *If possible, find an explicit formula for the nth term of the sequence.*
c. *Using a calculator, make a table with at least 10 terms and determine a plausible value for the limit of the sequence or state that it does not exist.*

39. $a_{n+1} = a_n + 2;\ a_0 = 3$
40. $a_{n+1} = a_n - 4;\ a_0 = 36$
41. $a_{n+1} = 2a_n + 1;\ a_0 = 0$
42. $a_{n+1} = \dfrac{a_n}{2};\ a_0 = 32$
43. $a_{n+1} = \frac{1}{2}a_n + 1;\ a_0 = 1$
44. $a_{n+1} = \sqrt{1 + a_n};\ a_0 = 1$

45–48. Heights of bouncing balls *Suppose a ball is thrown upward to a height of* h_0 *meters. Each time the ball bounces, it rebounds to a fraction r of its previous height. Let* h_n *be the height after the nth bounce. Consider the following values of* h_0 *and r.*

a. *Find the first four terms of the sequence of heights* $\{h_n\}$.
b. *Find a general expression for the nth term of the sequence* $\{h_n\}$.

45. $h_0 = 20,\ r = 0.5$
46. $h_0 = 10,\ r = 0.9$
47. $h_0 = 30,\ r = 0.25$
48. $h_0 = 20,\ r = 0.75$

75. $0.\overline{1} = 0.111\ldots$

76. $0.\overline{5} = 0.555\ldots$

77. $0.\overline{09} = 0.090909\ldots$

78. $0.\overline{27} = 0.272727\ldots$

79. $0.\overline{037} = 0.037037\ldots$

80. $0.\overline{027} = 0.027027\ldots$

QUICK CHECK ANSWERS

1. $a_{10} = 28$ **2.** $a_n = 2^n - 1,\ n = 1, 2, 3, \ldots$ **3.** $0.33333\ldots = \frac{1}{3}$ **4.** Both diverge **5.** $S_1 = -1, S_2 = 1, S_3 = -2, S_4 = 2$; the series diverges.

8.2 Sequences

The overview of the previous section sets the stage for an in-depth investigation of sequences and infinite series. This section is devoted to sequences, and the remainder of the chapter deals with series.

Limit of a Sequence

A fundamental question about sequences concerns the behavior of the terms as we go out farther and farther in the sequence. For example, in the sequence

$$\{a_n\}_{n=0}^{\infty} = \left\{\frac{1}{n^2 + 1}\right\}_{n=0}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \ldots\right\},$$

the terms remain positive and decrease to 0. We say that this sequence **converges** and its **limit** is 0, written $\lim_{n\to\infty} a_n = 0$. Similarly, the terms of the sequence

$$\{b_n\}_{n=1}^{\infty} = \left\{(-1)^n \frac{n(n+1)}{2}\right\}_{n=1}^{\infty} = \{-1, 3, -6, 10, \ldots\}$$

increase in magnitude and do not approach a unique value as n increases. In this case, we say that the sequence **diverges**.

Limits of sequences are really no different from limits at infinity of functions except that the variable n assumes only integer values as $n \to \infty$. This idea works as follows.

Given a sequence $\{a_n\}$, we define a function f such that $f(n) = a_n$ for all indices n. For example, if $\{a_n\} = \{n/(n+1)\}$, then we let $f(x) = x/(x+1)$. By the methods of Section 2.5, we know that $\lim_{x\to\infty} f(x) = 1$; because the terms of the sequence lie on the graph of f, it follows that $\lim_{n\to\infty} a_n = 1$ (Figure 8.11). This reasoning is the basis of the following theorem.

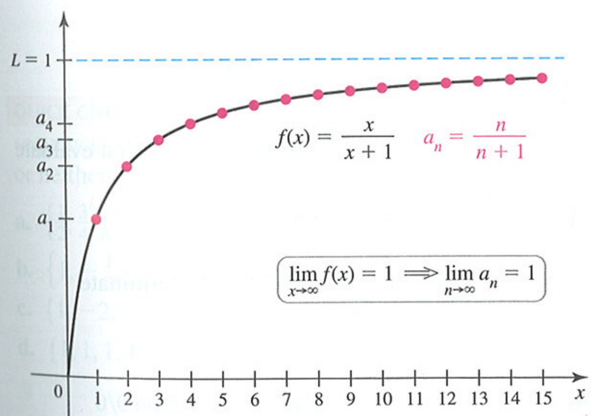


FIGURE 8.11

➤ The converse of Theorem 8.1 is not true. For example, if $a_n = \cos 2\pi n$, then $\lim_{n\to\infty} a_n = 1$, but $\lim_{x\to\infty} \cos 2\pi x$ does not exist.

> **THEOREM 8.1 Limits of Sequences from Limits of Functions**
> Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L.

Because of the correspondence between limits of sequences and limits at infinity of functions, we have the following properties that are analogous to those for functions given in Theorem 2.3.