Some universal Models

orbit equivalence
Dougherty, Jackson, Kechris 1994

I will begin with two questions.

Ques. 1) X – Standard Borel space

$T : \mathbb{Z}^d \times X \to X$ action by Borel automorphisms.

(X, T) — Borel dynamical system.

$\mathcal{E}(X, T)$ – ~~invar~~ ergodic invariant probability measures.

$X_0 \subset X$ is "full" if $\mu(X_0) = 1$ for all $\mu \in \mathcal{E}(X, T)$

Defn: (X, T) & (Y, S) are "almost" B. isomorphic if $\exists$ $X_0 \subset X$, $Y_0 \subset Y$ full and invariant & an isomorphism from (X_0, T) to (Y_0, S).

(X, T), (Y, S) Borel automorphisms

Question: Suppose $\Psi : \mathcal{E}(X, T) \to \mathcal{E}(Y, S)$ ~~are~~ is an isomorphism such that $(X, \mu, T) \cong (Y, \Psi(\mu), S)$ $\forall \mu \in \mathcal{E}(X, T)$

Are (X, T) & (Y, S) almost ~~is~~ B. isomorphic?

— x —

Question 2: Dominos are rectangular parallelopiped with ~~one~~ side length 2 and rest 1.

X_{dom} = space of tilings of $\mathbb{Z}^d$ by dominos.

σ – ~~shi~~ $\mathbb{Z}^d$ actions ~~of~~ on X_{dom} by translations (shifts).

Question: Prove

$$\lim_{n \to \infty} \frac{1}{(2n)^d} \log \left(\# \text{domino tilings of } [1, 2n]^d \right) = h_{top}(X_{dom})$$

For $d = 2$. follows from Kasteleyn (1962).

(X, T) – Borel dy. system

$$h_{gur}(X,T) = \sup_{\mu \in \mathcal{E}(X,T)} h_\mu$$

(X, T) is called "universal" if for all free, ergodic (Y, μ, S)

$$h_\mu < h_{gur}(X,T)$$

$$\exists_{\text{invariant}}\, Y_0 \subset Y, \quad \mu(Y_0) = 1$$

and an embedding of (Y_0, S) into (X, T).

– x –

~~What about non-ergodic measures?~~

(X, T) ------ $\boxed{t}$ – (almost) ~~almost~~ Borel universal – if for all free Borel (Y, S).

$$h_{gur}(Y,S) < h^{\boxed{t}}_{gur}(X,T)$$

(almost) $\exists$ embedding ~~from~~ of (Y, S) into (X, T)

<u>Hochman</u> (2015) All mixing SFTs are Borel universal.

Motivation for question 1)

(What is the real distinction between universal and almost Borel universal.)

~~Topologica~~

Two topological dynamical system

- $\mathbb{Z}^d$ - Cayley graph with standard generators

Proper k- colouring of $\mathbb{Z}^d$

$X_{k, col.} = \{ x \in \{1, \dots k\}^{\mathbb{Z}^d} :$ $\vec{i}$ is adjacent to $\vec{j}$ in $\mathbb{Z}^d$ then $x(\vec{i}) \neq x(\vec{j}) \}$

Tiling Shifts

- ~~Tiles~~

$\mathcal{F} = \{$ ~~set~~ ~~set~~ $\{F_1, \dots F_n\}$ are ~~rectn pa~~ ~~||piped~~s rectangular parallelepipeds in $\mathbb{Z}^d$ s.t for all coordinate directions the g.c.d. of the corr. side lengths $= 1$ (mixing)

e.g. dominos X_{dom}

$X_{\mathcal{F}}$ = set of tilings of $\mathbb{Z}^d$ by ~~$\mathcal{F}$~~ $F_1 \dots F_n$.

(general class of shifts and very little is known).

Prikhodko (2002)
(a) Şahin (2008) proved.
for all (Y, μ, S) free.

$\exists$ equivariant meas. map

$\varphi : Y_0 \to X_{\mathcal{F}}$ defined a.μ.a.e.

(2d - Alpern's Lemma
equivariant tiling of
orbits).

(b)

Şahin & Robinson (2001)
Are $X_{3,col}$ and X_{dom} universal in $d=2$?

(c)

Gao & Jackson (2015) ~~Let~~ (X, T) Borel dynamical -

Is there a Borel equivariant map
into $X_{3,col}$ and X_{dom}?

~~Thm: $X_{\mathcal{F}}$ is t- ~~Borel~~ almost Borel universal for some t~~
~~(~~strengthens (a) & "almost" answers (c)~~)~~

~~$X_{3,col}$, ~~is un alm~~ X_{dom} are ~~universal~~.~~

Thm (C & M) $X_{\mathcal{F}}$ is t- almost Borel universal for some t.
$(d=2)$
- X_{dom} & $X_{3,col}$ are almost Borel universal.

[~~strengthen~~

strengthens (a) (Stronger $\mathbb{Z}^d$ - Alpern's Lemma)

Completely answers (b).

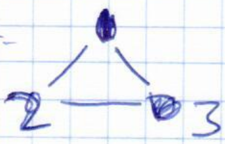
"almost" answers (c) - [We donot know how to deal with part ~~with~~ which does not support probability measures]

("almost" equivariant tiling ~~of an~~ and colouring of any Borel dynamical system.)

— x —

We prove further: H - finite undirected graph

X_H vertex shift for H = maps from $\mathbb{Z}^d$ to H which preserve adjacency.

H =  — X_H ~~is~~ $\cong X_{3,col}$.

<u>Thm:</u> H - not bipartite $\Rightarrow X_H$ is almost Borel universal.

<u>Question (2)</u> $\Longrightarrow X_{dom}$ for $d>2$ is universal.

<u>General Scheme</u> <u>Want:</u> (X,T) universal.

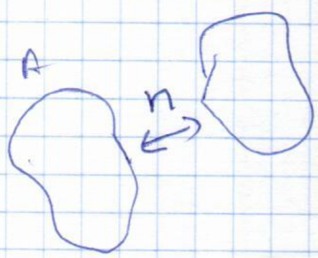
- prove (X,T) has specification-like condition
- Specification like condition $\Longrightarrow$ universality.

(Let us discuss two such conditions)

x – shift space $X \subset A^{\mathbb{Z}^d}$.

- Strong irreducibility

$\exists n$ s.t. $\forall A, B \subset \mathbb{Z}^d$



$\forall x, y \in X$

$\exists z$ s.t. $z|_A = x|_A$

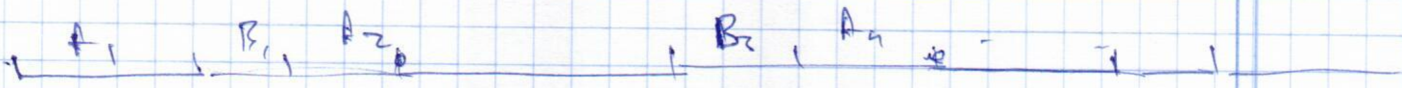
$z|_B = y|_B$.

+ ~~technical~~ condition $\Rightarrow$ universality (Şahin & Robinson '2001)

$\hookrightarrow$ (not required is a result of our theorem.

- $\boxed{d = 1}$ Almost weak specification.

$\exists f : \mathbb{N} \to \mathbb{N}$ $f(n) = o(n)$ s.t.



$|B_i| \geq f(|A_i|)$

then for all $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(n)}, \ldots \in X$

$\exists$ $x \in X$ s.t. $x|_{A_i} = x^{(i)}|_{A_i}$ $\Rightarrow$ universality (Quas & Soo 2012)

Answered Lind & Thouvenot 6

~~Quas & Soo used this to prove~~

to prove quasi-hyperbolic aut. of $\mathbb{T}^d$ are ...

~~Ques & Soo~~ (2012). 7.

First Idea: Why not try to approximate our systems with these.

Thm: $X_{3,col}$ & X_{dom} does not contain any such system!

~~Second Idea.~~ ($X_{3,col}$ ~~too~~ & X_{dom} ~~do not contain any such~~ have a non-trivial cocycle into $\mathbb{Z}$).

Second Idea: Why not use their method of proof?

They use Burton - Rothstein Machinery.

First step: μ on (X,T) ~~ies~~, ~~unig~~ μ is not an mme.

find μ' closest to μ (weak* sense).
st $h_{\mu'} > h_{\mu}$.

~~For X_H It is believe.~~

(Belief) $\exists$ μ on X_H (vertex shifts) st. μ is not an mme. But μ is a local mme.

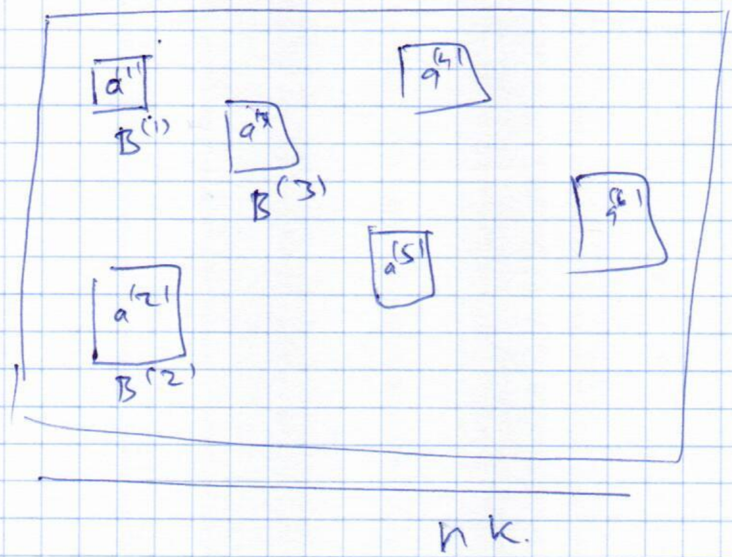
(statistical physics entropic repulsion)

$L(X, B) = \{x|_B : x \in X\}$.

B_n - box of side length n. Fix $k \in \mathbb{N}$.

$C_{nk} \subset \mathcal{L}(X, B_{nk})$ is called t-flexible if

- $\liminf_{n \to \infty} \frac{1}{|B_{nk}|} \log |C_{nk}| = t$.



$$a^{(i)}\text{'s} \in \bigcup_{s \in \mathbb{N}} C_{sk}.$$

$$\Rightarrow \exists\ a \in C_{nk} \text{ st } a|_{B^{(i)}} = a^{(i)}$$

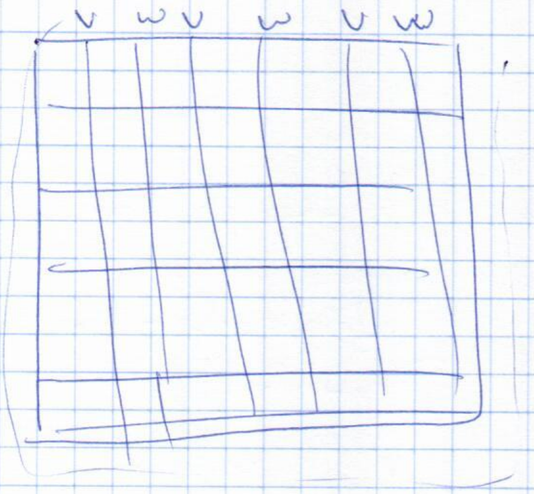
<u>Thms</u> t-flexible $\Rightarrow$ t- ~~Bo~~ almost Borel universal segn.

H – undirected graph (connected).

(v, w) edge

C_n = graph homomorphisms from B_n to H.

such that on $B_n \setminus B_{n-1}$ only v, w appear



F – set of rectangular tiles. K = product of side lengths of tiles in F.

X_F C_{nK}.

C_{nK} = complete tilings of B_{nK}.

Question: Is $h_{top}(X_F)$ always computable?