The series on the right is not a geometric series because of the presence of the factor k. The key is to realize that k could appear in this way through differentiation; specifically, something like $\frac{d}{dx}(x^{2k}) = 2kx^{2k-1}$. To achieve terms of this form, we write

$$\underbrace{\sum_{k=1}^{\infty}\frac{(-1)^k k}{4^k}x^{2k}}_{\text{original series}} = \sum_{k=1}^{\infty} k\left(-\frac{1}{4}\right)^k x^{2k}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} 2k\left(-\frac{1}{4}\right)^k x^{2k} \qquad \text{Multiply and divide by 2.}$$

$$= \frac{x}{2}\sum_{k=1}^{\infty} 2k\left(-\frac{1}{4}\right)^k x^{2k-1}. \qquad \text{Remove } x \text{ from the series.}$$

Now we identify the last series as the derivative of another series:

$$\underbrace{\sum_{k=1}^{\infty}\frac{(-1)^k k}{4^k}x^{2k}}_{\text{original series}} = \frac{x}{2}\sum_{k=1}^{\infty}\left(-\frac{1}{4}\right)^k 2kx^{2k-1}$$

$$= \frac{x}{2}\sum_{k=1}^{\infty}\left(-\frac{1}{4}\right)^k \frac{d}{dx}(x^{2k}) \qquad \text{Identify a derivative.}$$

$$= \frac{x}{2}\frac{d}{dx}\sum_{k=1}^{\infty}\left(-\frac{x^2}{4}\right)^k \qquad \text{Combine factors; term by term differentiation.}$$

This last series is a geometric series with a ratio $r = -x^2/4$ and first term $-x^2/4$; therefore, its value is $\frac{-x^2/4}{1 + (x^2/4)}$, provided $\left|\frac{x^2}{4}\right| < 1$. We now have

$$\underbrace{\sum_{k=1}^{\infty}\frac{(-1)^k k}{4^k}x^{2k}}_{\text{original series}} = \frac{x}{2}\frac{d}{dx}\sum_{k=1}^{\infty}\left(-\frac{x^2}{4}\right)^k$$

$$= \frac{x}{2}\frac{d}{dx}\left(\frac{-x^2/4}{1 + (x^2/4)}\right) \qquad \text{Sum of geometric series}$$

$$= \frac{x}{2}\frac{d}{dx}\left(\frac{-x^2}{4 + x^2}\right) \qquad \text{Simplify.}$$

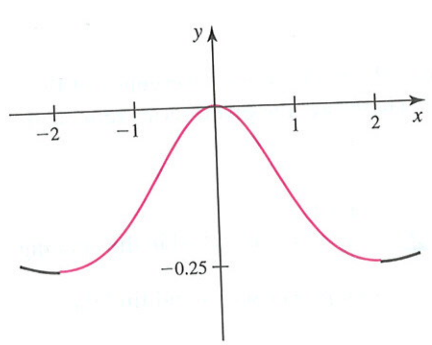
$$= -\frac{4x^2}{(4 + x^2)^2} \qquad \text{Differentiate and simplify.}$$

Therefore, the function represented by the power series on $(-2, 2)$ has been uncovered; it is

$$f(x) = -\frac{4x^2}{(4 + x^2)^2}.$$

Notice that f is defined for $-\infty < x < \infty$ (Figure 9.19), but its power series centered at 0 converges to f only on $(-2, 2)$.

Related Exercises 49–58



$$\boxed{\sum_{k=1}^{\infty}\frac{(-1)^k k}{4^k}x^{2k} = -\frac{4x^2}{(4+x^2)^2} \text{ on } (-2, 2)}$$

FIGURE 9.19

SECTION 9.4 EXERCISES

Review Questions

1. Explain the strategy presented in this section for evaluating a limit of the form $\lim_{x\to a} f(x)/g(x)$, where f and g have Taylor series centered at a.

2. Explain the method presented in this section for evaluating $\int_a^b f(x)\,dx$, where f has a Taylor series with an interval of convergence centered at a that includes b.

3. How would you approximate $e^{-0.6}$ using the Taylor series for e^x?

4. Suggest a Taylor series and a method for approximating π.

5. If $f(x) = \sum_{k=0}^{\infty} c_k x^k$ and the series converges for $|x| < b$, what is the power series for $f'(x)$?

6. What condition must be met by a function f for it to have a Taylor series centered at a?

Basic Skills

7–20. Limits *Evaluate the following limits using Taylor series.*

7. $\lim_{x\to 0}\frac{e^x - e^{-x}}{x}$

8. $\lim_{x\to 0}\frac{1 + x - e^{-x}}{4x^2}$

9. $\lim_{x\to 0}\frac{2\cos 2x - 2 + 4x^2}{2x^4}$

10. $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

11. $\lim_{x\to 0}\frac{3\tan x - 3x - x^3}{x^5}$

12. $\lim_{x\to 4}\frac{x^2 - 16}{\ln(x - 3)}$

13. $\lim_{x\to 0}\frac{3\tan^{-1}x - 3x + x^3}{x^5}$

14. $\lim_{x\to 0}\frac{\sqrt{1 + x} - 1 - (x/2)}{4x^2}$

15. $\lim_{x\to 0}\frac{\sin x - \tan x}{3x^3\cos x}$

16. $\lim_{x\to 1}\frac{x - 1}{\ln x}$

17. $\lim_{x\to 2}\frac{x - 2}{\ln(x - 1)}$

18. $\lim_{x\to\infty}(x^4(e^{1/x} - 1) - x^3)$

19. $\lim_{x\to 0^+}\frac{(1 + x)^{-2} - 4\cos\sqrt{x} + 3}{2x^2}$

20. $\lim_{x\to 0}\frac{(1 - 2x)^{-1/2} - e^x}{8x^2}$

21–26. Power series for derivatives

a. *Differentiate the Taylor series about 0 for the following functions.*
b. *Identify the function represented by the differentiated series.*
c. *Give the interval of convergence of the power series for the derivative.*

21. $f(x) = e^x$

22. $f(x) = \cos x$

23. $f(x) = \ln(1 + x)$

24. $f(x) = \sin(x^2)$

25. $f(x) = e^{-2x}$

26. $f(x) = \sqrt{1 + x}$

27–30. Differential equations

a. *Find a power series for the solution of the following differential equations.*
b. *Identify the function represented by the power series.*

27. $y'(t) - y(t) = 0,\ y(0) = 2$

28. $y'(t) + 4y(t) = 8,\ y(0) = 0$

29. $y'(t) - 3y(t) = 10,\ y(0) = 2$

30. $y'(t) = 6y(t) + 9,\ y(0) = 2$

T **31–38. Approximating definite integrals** *Use a Taylor series to approximate the following definite integrals. Retain as many terms as needed to ensure the error is less than 10^{-4}.*

31. $\int_0^{0.25} e^{-x^2}\,dx$

32. $\int_0^{0.2}\sin x^2\,dx$

33. $\int_{-0.35}^{0.35}\cos 2x^2\,dx$

34. $\int_0^{0.2}\sqrt{1 + x^4}\,dx$

35. $\int_0^{0.15}\frac{\sin x}{x}\,dx$

36. $\int_0^{0.1}\cos\sqrt{x}\,dx$

37. $\int_0^{0.5}\frac{dx}{\sqrt{1 + x^6}}$

38. $\int_0^{0.2}\frac{\ln(1 + t)}{t}\,dt$

39–44. Approximating real numbers *Use an appropriate Taylor series to find the first four nonzero terms of an infinite series that is equal to the following numbers.*

39. e^2

40. $\sqrt{e}$

41. $\cos 2$

42. $\sin 1$

43. $\ln\left(\frac{3}{2}\right)$

44. $\tan^{-1}\left(\frac{1}{2}\right)$

45. **Evaluating an infinite series** Let $f(x) = (e^x - 1)/x$ for $x \neq 0$ and $f(0) = 1$. Use the Taylor series for f about 0 and evaluate $f(1)$ to find the value of $\sum_{k=0}^{\infty}\frac{1}{(k + 1)!}$.

46. **Evaluating an infinite series** Let $f(x) = (e^x - 1)/x$ for $x \neq 0$ and $f(0) = 1$. Use the Taylor series for f and f' about 0 to evaluate $f'(2)$ and to find the value of $\sum_{k=1}^{\infty}\frac{k2^{k-1}}{(k + 1)!}$.

47. **Evaluating an infinite series** Write the Taylor series for $f(x) = \ln(1 + x)$ about 0 and find the interval of convergence. Assume the Taylor series converges to f on the interval of convergence. Evaluate $f(1)$ to find the value of $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ (the alternating harmonic series).

48. **Evaluating an infinite series** Write the Taylor series for $f(x) = \ln(1 + x)$ about 0 and find the interval of convergence. Evaluate $f\left(-\frac{1}{2}\right)$ to find the value of $\sum_{k=1}^{\infty}\frac{1}{k\cdot 2^k}$.

49–58. Representing functions by power series *Identify the functions represented by the following power series.*

49. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{2^k}$

50. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{3^k}$

51. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k}$

52. $\displaystyle\sum_{k=0}^{\infty} 2^k x^{2k+1}$

53. $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k}$

54. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{4^k}$

55. $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{kx^{k+1}}{3^k}$

56. $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k}}{k}$

57. $\displaystyle\sum_{k=2}^{\infty} \frac{k(k-1)x^k}{3^k}$

58. $\displaystyle\sum_{k=2}^{\infty} \frac{x^k}{k(k-1)}$

Further Explorations

59. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. To evaluate $\displaystyle\int_0^2 \frac{dx}{1-x}$, one could expand the integrand in a Taylor series and integrate term by term.

b. To approximate $\pi/3$, one could substitute $x = \sqrt{3}$ into the Taylor series for $\tan^{-1} x$.

c. $\displaystyle\sum_{k=0}^{\infty} \frac{(\ln 2)^k}{k!} = 2.$

60–62. Limits with a parameter *Use Taylor series to evaluate the following limits. Express the result in terms of the parameter(s).*

60. $\displaystyle\lim_{x\to 0} \frac{e^{ax}-1}{x}$

61. $\displaystyle\lim_{x\to 0} \frac{\sin ax}{\sin bx}$

62. $\displaystyle\lim_{x\to 0} \frac{\sin ax - \tan ax}{bx^3}$

63. **A limit by Taylor series** Use Taylor series to evaluate

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}.$$

64. **Inverse hyperbolic sine** A function known as the *inverse of the hyperbolic sine* is defined in several ways; among them are

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) = \int_0^x \frac{dt}{\sqrt{1+t^2}}.$$

Find the first four terms of the Taylor series for $\sinh^{-1} x$ using these two definitions (and be sure they agree).

65–68. Derivative trick *Here is an alternative way to evaluate higher derivatives of a function f that may save time. Suppose you can find the Taylor series for f centered at the point a without evaluating derivatives (for example, from a known series). Explain why $f^{(k)}(a) = k!$ multiplied by the coefficient of $(x-a)^k$.*

Use this idea to evaluate $f^{(3)}(0)$ and $f^{(4)}(0)$ for the following functions. Use known series and do not evaluate derivatives.

65. $f(x) = e^{\cos x}$

66. $f(x) = \dfrac{x^2+1}{\sqrt[3]{1+x}}$

67. $f(x) = \displaystyle\int_0^x \sin(t^2)\, dt$

68. $f(x) = \displaystyle\int_0^x \frac{1}{1+t^4}\, dt$

Applications

69. **Probability: tossing for a head** The expected (average) number of tosses of a fair coin required to obtain the first head is $\displaystyle\sum_{k=1}^{\infty} k\left(\tfrac{1}{2}\right)^k$. Evaluate this series and determine the expected number of tosses. (*Hint:* Differentiate a geometric series.)

70. **Probability: sudden death playoff** Teams A and B go into sudden death overtime after playing to a tie. The teams alternate possession of the ball and the first team to score wins. Each team has a $\frac{1}{6}$ chance of scoring when it has the ball, with Team A having the ball first.

a. The probability that Team A ultimately wins is $\displaystyle\sum_{k=0}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{2k}$. Evaluate this series.

b. The expected number of rounds (possessions by either team) required for the overtime to end is $\displaystyle\frac{1}{6}\sum_{k=1}^{\infty} k\left(\frac{5}{6}\right)^{k-1}$. Evaluate this series.

T 71. **Elliptic integrals** The period of a pendulum is given by

$$T = 4\sqrt{\frac{\ell}{g}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \equiv 4\sqrt{\frac{\ell}{g}}\, F(k),$$

where ℓ is the length of the pendulum, $g \approx 9.8\ \text{m/s}^2$ is the acceleration due to gravity, $k = \sin(\theta_0/2)$, and θ_0 is the initial angular displacement of the pendulum (in radians). The integral in this formula $F(k)$ is called an **elliptic integral** and it cannot be evaluated analytically.

a. Approximate $F(0.1)$ by expanding the integrand in a Taylor (binomial) series and integrating term by term.

b. How many terms of the Taylor series do you suggest using to obtain an approximation to $F(0.1)$ with an error less than 10^{-3}?

c. Would you expect to use fewer or more terms (than in part (b)) to approximate $F(0.2)$ to the same accuracy? Explain.

T 72. **Sine integral function** The function $\mathrm{Si}(x) = \displaystyle\int_0^x \frac{\sin t}{t}\, dt$ is called the **sine integral function**.

a. Expand the integrand in a Taylor series about 0.

b. Integrate the series to find a Taylor series for Si.

c. Approximate Si (0.5) and Si (1). Use enough terms of the series so the error in the approximation does not exceed 10^{-3}.

T 73. **Fresnel integrals** The theory of optics gives rise to the two **Fresnel integrals**

$$S(x) = \int_0^x \sin(t^2)\, dt \quad\text{and}\quad C(x) = \int_0^x \cos(t^2)\, dt.$$

a. Compute $S'(x)$ and $C'(x)$.

b. Expand $\sin(t^2)$ and $\cos(t^2)$ in a Maclaurin series and then integrate to find the first four nonzero terms of the Maclaurin series for S and C.

c. Use the polynomials in part (b) to approximate $S(0.05)$ and $C(-0.25)$.

d. How many terms of the Maclaurin series are required to approximate $S(0.05)$ with an error no greater than 10^{-4}?

e. How many terms of the Maclaurin series are required to approximate $C(-0.25)$ with an error no greater than 10^{-6}?

T 74. **Error function** An essential function in statistics and the study of the normal distribution is the **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

a. Compute the derivative of erf (x).

b. Expand e^{-t^2} in a Maclaurin series, then integrate to find the first four nonzero terms of the Maclaurin series for erf.

c. Use the polynomial in part (b) to approximate erf (0.15) and erf (-0.09).

d. Estimate the error in the approximations of part (c).

T 75. **Bessel functions Bessel functions** arise in the study of wave propagation in circular geometries (for example, waves on a circular drum head). They are conveniently defined as power series. One of an infinite family of Bessel functions is

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k}.$$

a. Write out the first four terms of J_0.

b. Find the radius and interval of convergence of the power series for J_0.

c. Differentiate J_0 twice and show (by keeping terms through x^6) that J_0 satisfies the equation $x^2y''(x) + xy'(x) + x^2y(x) = 0$.

Additional Exercises

76. **Power series for sec x** Use the identity $\sec x = \dfrac{1}{\cos x}$ and long division to find the first three terms of the Maclaurin series for sec x.

77. **Symmetry**

a. Use infinite series to show that cos x is an even function. That is, show $\cos x = \cos(-x)$.

b. Use infinite series to show that sin x is an odd function. That is, show $\sin x = -\sin(-x)$.

78. **Behavior of csc x** We know that $\displaystyle\lim_{x\to 0^+} \csc x = \infty$. Use long division to determine exactly how csc x grows as $x \to 0^+$. Specifically, find a, b, and c (all positive) in the following sentence: As $x \to 0^+$, $\csc x \approx \dfrac{a}{x^b} + cx$.

79. **L'Hôpital's Rule by Taylor series** Suppose f and g have Taylor series about the point a.

a. If $f(a) = g(a) = 0$ and $g'(a) \neq 0$, evaluate $\displaystyle\lim_{x\to a} f(x)/g(x)$ by expanding f and g in their Taylor series. Show that the result is consistent with l'Hôpital's Rule.

b. If $f(a) = g(a) = f'(a) = g'(a) = 0$ and $g''(a) \neq 0$, evaluate $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ by expanding f and g in their Taylor series. Show that the result is consistent with two applications of l'Hôpital's Rule.

T 80. **Newton's derivation of the sine and arcsine series** Newton discovered the binomial series and then used it ingeniously to obtain many more results. Here is a case in point.

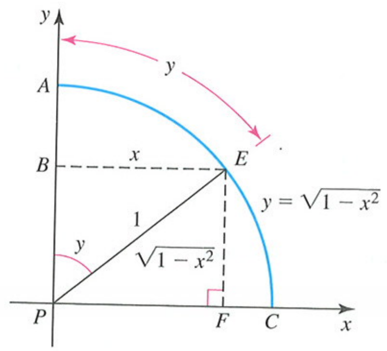
a. Referring to the figure, show that $x = \sin y$ or $y = \sin^{-1} x$.

b. The area of a circular sector of radius r subtended by an angle θ is $\frac{1}{2}r^2\theta$. Show that the area of the circular sector APE is $y/2$, which implies that

$$y = 2\int_0^x \sqrt{1-t^2}\, dt - x\sqrt{1-x^2}.$$

c. Use the binomial series for $f(x) = \sqrt{1-x^2}$ to obtain the first few terms of the Taylor series for $y = \sin^{-1} x$.

d. Newton next inverted the series in part (c) to obtain the Taylor series for $x = \sin y$. He did this by assuming that $\sin y = \sum a_k y^k$ and solving $x = \sin(\sin^{-1} x)$ for the coefficients a_k. Find the first few terms of the Taylor series for sin y using this idea (a computer algebra system might be helpful as well).



QUICK CHECK ANSWERS

1. $\dfrac{\sin x}{x} = \dfrac{x - x^3/3! + \cdots}{x} = 1 - \dfrac{x^2}{3!} + \cdots \to 1$ as $x \to 0$.
2. The result is the power series for $-\sin x$.
3. $x = 1/\sqrt{3}$ (which lies in the interval of convergence)

CHAPTER 9 REVIEW EXERCISES

1. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. Let p_n be the nth-order Taylor polynomial for f centered at 2. The approximation $p_3(2.1) \approx f(2.1)$ is likely to be more accurate than the approximation $p_2(2.2) \approx f(2.2)$.
   b. If the Taylor series for f centered at 3 has a radius of convergence of 6, then the interval of convergence is $[-3, 9]$.
   c. The interval of convergence of the power series $\sum c_k x^k$ could be $\left(-\frac{7}{3}, \frac{7}{3}\right)$.
   d. The Taylor series for $f(x) = (1+x)^{12}$ centered at 0 has a finite number of terms.

2–7. Taylor polynomials *Find the nth-order Taylor polynomial for the following functions with the given center point a.*

2. $f(x) = \sin 2x$, $n = 3, a = 0$
3. $f(x) = \cos x^2$, $n = 2, a = 0$
4. $f(x) = e^{-x}$, $n = 2, a = 0$
5. $f(x) = \ln(1+x)$, $n = 3, a = 0$
6. $f(x) = \cos x$, $n = 2, a = \pi/4$
7. $f(x) = \ln x$, $n = 2, a = 1$

8–11. Approximations
   a. *Find the Taylor polynomials of order n = 0, 1, and 2 for the given functions centered at the given point a.*
   b. *Make a table showing the approximations and the absolute error in these approximations using a calculator for the exact function value.*

8. $f(x) = \cos x$, $a = 0$; approximate $\cos(-0.08)$.
9. $f(x) = e^x$, $a = 0$; approximate $e^{-0.08}$.
10. $f(x) = \sqrt{1+x}$, $a = 0$; approximate $\sqrt{1.08}$.
11. $f(x) = \sin x$, $a = \pi/4$; approximate $\sin(\pi/5)$.

12–14. Estimating remainders *Find the remainder term $R_n(x)$ for the Taylor series centered at 0 for the following functions. Find an upper bound for the magnitude of the remainder on the given interval for the given value of n. (The bound is not unique.)*

12. $f(x) = e^x$; bound $R_3(x)$ for $|x| < 1$.
13. $f(x) = \sin x$; bound $R_3(x)$ for $|x| < \pi$.
14. $f(x) = \ln(1-x)$; bound $R_3(x)$ for $|x| < 1/2$.

15–20. Radius and interval of convergence *Use the Ratio or Root Test to determine the radius of convergence of the following power series. Test the endpoints to determine the interval of convergence, when appropriate.*

15. $\sum \frac{k^2 x^k}{k!}$
16. $\sum \frac{x^{4k}}{k^2}$
17. $\sum (-1)^k \frac{(x+1)^{2k}}{k!}$
18. $\sum \frac{(x-1)^k}{k \cdot 5^k}$
19. $\sum \left(\frac{x}{9}\right)^{3k}$
20. $\sum \frac{(x+2)^k}{\sqrt{k}}$

21–26. Power series from the geometric series *Use the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x| < 1$ to determine the Maclaurin series and the interval of convergence for the following functions.*

21. $f(x) = \frac{1}{1-x^2}$
22. $f(x) = \frac{1}{1+x^3}$
23. $f(x) = \frac{1}{1-3x}$
24. $f(x) = \frac{10x}{1+x}$
25. $f(x) = \frac{1}{(1-x)^2}$
26. $f(x) = \ln(1+x^2)$

27–32. Taylor series *Write out the first three terms of the Taylor series for the following functions centered at the given point a. Then write the series using summation notation.*

27. $f(x) = e^{3x}$, $a = 0$
28. $f(x) = \frac{1}{x}$, $a = 1$
29. $f(x) = \cos x$, $a = \pi/2$
30. $f(x) = -\ln(1-x)$, $a = 0$
31. $f(x) = \tan^{-1} x$, $a = 0$
32. $f(x) = \sin 2x$, $a = -\pi/2$

33–36. Binomial series *Write out the first three terms of the Maclaurin series for the following functions.*

33. $f(x) = (1+x)^{1/3}$
34. $f(x) = (1+x)^{-1/2}$
35. $f(x) = (1+x/2)^{-3}$
36. $f(x) = (1+2x)^{-5}$

37–40. Convergence *Write the remainder term $R_n(x)$ for the Taylor series for the following functions centered at the given point a. Then show that $\lim_{n\to\infty} R_n(x) = 0$ for all x in the given interval.*

37. $f(x) = e^{-x}$, $a = 0, -\infty < x < \infty$
38. $f(x) = \sin x$, $a = 0, -\infty < x < \infty$
39. $f(x) = \ln(1+x)$, $a = 0, -\frac{1}{2} \le x \le \frac{1}{2}$
40. $f(x) = \sqrt{1+x}$, $a = 0, -\frac{1}{2} \le x \le \frac{1}{2}$

41–46. Limits by power series *Use Taylor series to evaluate the following limits.*

41. $\lim_{x\to 0} \frac{x^2/2 - 1 + \cos x}{x^4}$
42. $\lim_{x\to 0} \frac{2\sin x - \tan^{-1} x - x}{2x^5}$
43. $\lim_{x\to 4} \frac{\ln(x-3)}{x^2 - 16}$
44. $\lim_{x\to 0} \frac{\sqrt{1+2x} - 1 - x}{x^2}$
45. $\lim_{x\to 0} \frac{\sec x - \cos x - x^2}{x^4}$
46. $\lim_{x\to 0} \frac{(1+x)^{-2} - \sqrt[3]{1-6x}}{2x^2}$

47. **A differential equation** Find a power series solution of the differential equation $y'(x) - 4y(x) + 12 = 0$, subject to the condition $y(0) = 4$. Identify the solution in terms of known functions.

48. **Rejected quarters** The probability that a random quarter is *not* rejected by a vending machine is given by the integral $11.4\int_0^{0.14} e^{-102x^2}\,dx$ (assuming that the weights of quarters are normally distributed with a mean of 5.670 g and a standard deviation of 0.07 g). Expand the integrand in $n = 2$ and $n = 3$ terms of a Taylor series and integrate to find two estimates of the probability. Check for agreement between the two estimates.

49. **Approximating ln 2** Consider the following three ways to approximate ln 2.
   a. Use the Taylor series for $\ln(1+x)$ centered at 0 and evaluate it at $x = 1$ (convergence was asserted in Table 9.5). Write the resulting infinite series.
   b. Use the Taylor series for $\ln(1-x)$ centered at 0 and the identity $\ln 2 = -\ln\left(\frac{1}{2}\right)$. Write the resulting infinite series.
   c. Use the property $\ln(a/b) = \ln a - \ln b$ and the series of parts (a) and (b) to find the Taylor series for $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ centered at 0.
   d. At what value of x should the series in part (c) be evaluated to approximate ln 2? Write the resulting infinite series for ln 2.
   e. Using four terms of the series, which of the three series derived in parts (a)–(d) gives the best approximation to ln 2? Which series gives the worst approximation? Can you explain why?

50. **Graphing Taylor polynomials** Consider the function $f(x) = (1+x)^{-4}$.
   a. Find the Taylor polynomials p_0, p_1, p_2, and p_3 centered at 0.
   b. Use a graphing utility to plot the Taylor polynomials and f for $-1 < x < 1$.
   c. For each Taylor polynomial, give the interval on which its graph appears indistinguishable from the graph of f.

Chapter 9 Guided Projects

Applications of the material in this chapter and related topics can be found in the following Guided Projects. For additional information, see the Preface.

- Euler's formula (Taylor series with complex numbers)
- Fourier Series
- Series approximations to π
- Stirling's formula and $n!$
- Three-sigma quality control