RATIONAL ERGODICITY OF STEP FUNCTION SKEW PRODUCTS.

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ABSTRACT. We study `rational step function` skew products over certain rotations of the circle proving ergodicity and bounded rational ergodicity when rotation number is a quadratic irrational. The latter arises from a consideration of the asymptotic temporal statistics of an orbit as modelled by an `associated affine random walk`.

INTRODUCTION

A *rational step function* is a right continuous, step function on $\mathbb{T}$ taking values in $\mathbb{R}^d$, whose discontinuity points are rational.

Let $\varphi : \mathbb{T} \to \mathbb{R}^d$ be a rational step function.

The *skew products* $T_{\alpha,\varphi} = T_\alpha : \mathbb{T} \times \mathbb{R}^d \to \mathbb{T} \times \mathbb{R}^d \quad (\alpha \in \mathbb{T})$ defined by

$$T_{\alpha,\varphi}(x,y) := (x+\alpha, y+\varphi(x))$$

are conservative iff

$$\int_{\mathbb{T}} \varphi(t)dt = 0. \tag{⚖}$$

Necessity follows from the ergodic theorem and sufficiency follows from the Denjoy-Koksma inequality (see below).

Consider the collections of `badly approximable` irrationals

$$\mathtt{BAD} := \{\alpha \in \mathbb{R} \setminus \mathbb{Q} : \ \exists\ \theta > 0,\ |\alpha - \tfrac{p}{q}| \geq \tfrac{\theta}{q^2}\}$$

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and

$$\texttt{QUAD} := \{\alpha \in \mathbb{R} \setminus \mathbb{Q} : \ \alpha \ \ \text{quadratic}\}.$$

It is known that $\texttt{QUAD} \subset \texttt{BAD}$ and that $\texttt{BAD}$ has Lebesgue measure zero (see e.g. [6]).

We prove:

Theorem 1': Ergodicity

There is a collection $\texttt{SBAD} \subset \mathbb{R} \setminus \mathbb{Q}$ *of full Lebesgue measure so that* $\texttt{SBAD} \supset \texttt{BAD}$ *and so that if* $\alpha \in \texttt{SBAD}_Q$ *then* $(\mathbb{T}\times\Gamma, \mathcal{B}(\mathbb{T}\times\Gamma), m_{\mathbb{T}}\times m_\Gamma, T_{\alpha,\varphi})$ *is a* $\texttt{CEMPT}$ *where* $\Gamma := \overline{\langle\varphi(\mathbb{T})\rangle}$ *is the closed subgroup of* $\mathbb{R}^d$ *generated by* $\Phi(\mathbb{Z}_Q)$.

Here and throughout, $m_{\mathbb{G}}$ denotes Haar measure on the locally compact, Polish, Abelian group $\mathbb{G}$, normalized if $\mathbb{G}$ is compact.

Theorem 1' will follow from the stronger theorem 1 (see below).

The technique of the proof of theorem 1 is not new. For older, related results, see [5], [14] and references therein.

Theorem 2: Temporal CLT

If $\alpha \in \texttt{QUAD}$ *and* $\dim \texttt{span}_{\mathbb{R}} \varphi(\mathbb{T}) = d$*, then* $\exists\ \ell_k \in \mathbb{N},\ \ \ell_k \uparrow\ \ \&\ \ell_k \propto \lambda^k$ *for some* $\lambda > 1$ *and* $\mu \in \mathbb{R}^d$ *so that for any box* $I \subset \mathbb{R}^d$,

$$\frac{1}{\ell_k}\#\{1 \le n \le \ell_k : \ \frac{\varphi_n(0) - k\mu}{\sqrt{k}} \in I\} \xrightarrow[k\to\infty]{} \int_I f_Z(t)dt$$

where Z *is a globally supported, centered, normal random variable on* $\mathbb{R}^d$ *and* f_Z *is its probability density function.*

This is a generalization of a subsequence version of theorem 1.1 in [3].

Theorem 3: Rational ergodicity

Suppose that $\alpha \in \texttt{QUAD}$ *and that* $\langle\varphi(\mathbb{T})\rangle = \mathbb{Z}^d$*, then*

$$(\mathbb{T}\times\mathbb{Z}^d, \mathcal{B}(\mathbb{T}\times\mathbb{Z}^d), m_{\mathbb{T}}\times\#, T_{\alpha,\varphi})$$

is boundedly rationally ergodic and $a_n(T_{\alpha,\varphi}) \asymp \frac{n}{(\log n)^{\frac{d}{2}}}$.

See [2] for a definition of `bounded rational ergodicity`. Bounded rational ergodicity of $T_{\alpha,\varphi}$ for $\varphi = 1_{[0,\frac{1}{2})} - 1_{[\frac{1}{2},1)}$ was established in [2] for $\alpha \in \texttt{QUAD}$ and in [1] for $\alpha \in \texttt{BAD}$.

Full statement of theorem 1

Rational step functions.

Fix $d,\ Q \in \mathbb{N},\ Q \geq 2$ and $\Phi : \mathbb{Z}_Q \to \mathbb{R}^d$.

The *rational step function with denominator* Q *and values* Φ is the step function $\varphi = \varphi^{(\Phi)} : \mathbb{T} \to \mathbb{R}^d$ defined by

$$\varphi(x) = \Phi(K(x)),$$

where $K : [0,1) \to \mathbb{Z}_Q$ is defined by $K(x) := \lfloor Qx \rfloor$. Every rational step function is of this form.

If $\varphi : \mathbb{T} \to \mathbb{R}^d$ is a rational step function with denominator Q, then

$$\int_{\mathbb{T}} \varphi(x)dx = \frac{1}{Q}\sum_{k=0}^{Q-1} \Phi(k).$$

Regular continued fractions.

Recall that the *regular continued fraction expansion* of $\alpha \in (0,1) \setminus \mathbb{Q}$ is

$$\alpha = \cfrac{1}{a_1 + \frac{1}{a_2 + \ddots + \frac{1}{a_n + \ddots}}}$$

$$=: \frac{1|}{|a_1} + \frac{1|}{|a_2} + \dots + \frac{1|}{|a_n} \quad + \dots$$
$$= (a_1, a_2, \dots)$$

where $a_n := a(G^{n-1}\alpha) \in \mathbb{N}$ with $a(\alpha) := \lfloor \frac{1}{\alpha} \rfloor$ & $G(\alpha) := \{\frac{1}{\alpha}\}$ for $\alpha \in \mathbb{T} \setminus \mathbb{Q}$.

Note that $G((0,1) \setminus \mathbb{Q}) \subset (0,1) \setminus \mathbb{Q}$ and so every irrational in $(0,1)$ indeed has an infinite regular continued fraction expansion. On the other hand, if $\alpha \in (0,1) \cap \mathbb{Q}$ then $\exists\ n \geq 1,\ G^n(\alpha) = 0$ and α has only a finite regular continued fraction expansion.

Fix $\alpha = (a_1, a_2, \dots) \in (0,1) \setminus \mathbb{Q}$ and define the *principal convergents* $\frac{p_n}{q_n},\ p_n,\ q_n \in \mathbb{Z}_+,\ \gcd(p_n, q_n) = 1$ by

$$\frac{p_n}{q_n} := \frac{1|}{|a(\alpha)} + \frac{1|}{|a(G\alpha)} + \dots + \frac{1|}{|a(G^{n-1}\alpha)}.$$

The *principal denominators* q_n of α are given by

$$q_0 = 1,\ q_1 = a_1,\ q_{n+1} = a_{n+1}q_n + q_{n-1};$$

the numerators p_n are given by

$$p_0 = 0,\ p_1 = 1,\ p_{n+1} = a_{n+1}p_n + p_{n-1}$$

and the principal convergents $\frac{p_n}{q_n}$ satisfy

$$|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}.$$

The following is well known (see e.g. [6], [11]):

Proposition *Let $\alpha = (a_1, a_2, \dots) \in (0,1) \setminus \mathbb{Q}$, then*
(i) $\alpha \in \mathtt{QUAD}$ *iff* $\exists\ K,\ L \geq 1$ *so that* $a_{k+L} = a_k\ \forall\ k \geq K$;
(ii) $\alpha \in \mathtt{BAD}$ *iff* $\sup_{k\geq 1} a_k < \infty$.

For $Q \geq 2$, we'll also need the collection

$$\mathtt{SBAD}_Q := \{\alpha : \overline{\lim_{n\to\infty}} \tfrac{q_{n'}}{q_{n+1}} > 0 \ \text{ where } n' := \max\{1 \leq m < n : \tfrac{q_m}{q_n} < \tfrac{1}{Q}\}\}.$$

Evidently, $\mathtt{BAD} \subset \mathtt{SBAD} := \bigcap_{Q\geq 2} \mathtt{SBAD}_Q$ and it is not hard to show that $\mathtt{SBAD}$ has full Lebesgue measure.

In the following, $\varphi = \varphi^{(\Phi)} : \mathbb{T} \to \mathbb{R}^d$ is a rational step function with denominator $Q \geq 2$.

Theorem 1
Let $\Gamma := \overline{\langle \Phi(\mathbb{Z}_Q)\rangle}$ be the closed subgroup of $\mathbb{R}^d$ generated by $\Phi(\mathbb{Z}_Q)$.
Suppose that either (i) $\alpha \in \mathtt{SBAD}_Q$, *or* (ii) $\alpha \notin \mathbb{Q}$ & Q *is prime, then* $(\mathbb{T} \times \Gamma, \mathcal{B}(\mathbb{T} \times \Gamma), m_{\mathbb{T}} \times m_\Gamma, T_{\alpha,\varphi})$ *is a* $\mathtt{CEMPT}$.

Proof of theorem 1

Essential values and Periods. Let $(X, \mathcal{B}, m)$ be a standard probability space, and let $T : X \to X$ be an invertible, ergodic, $\mathtt{PPT}$.

Suppose that $\mathbb{G}$ is a locally compact, Polish, abelian group equipped with the translation invariant metric ρ (e.g. $\rho(x,y) = \|x - y\|$ if $\mathbb{G} \leq \mathbb{R}^d$).

Let $\phi : X \to \mathbb{G}$ be measurable.

The collection of *essential values* of ϕ (as in [15]) is

$$E(\phi) := \{a \in \mathbb{G} : \forall\ A \in \mathcal{B}_+,\ \epsilon > 0,\ \exists\ n \in \mathbb{Z},\ m(A \cap T^{-n} A \cap [\rho(\phi_n, a) < \epsilon]) > 0\}.$$

The *skew product* $T_\phi : X \times \mathbb{G} \to X \times \mathbb{G}$ is defined by

$$T_\phi(x,y) := (Tx, y + \phi(x))$$

and preserves the measure $m \times m_{\mathbb{G}}$.

Define the collection of *periods* for T_ϕ-invariant functions:

$$\mathtt{Per}\,(\phi) = \{a \in \mathbb{G} : \tau_a A = A \mod m\ \forall\ A \in \mathcal{B}(X \times \mathbb{G}),\ T_\phi(A) = A\}$$

where $\tau_a(x,y) = (x, y+a)$.

It is not hard to see that T_ϕ is ergodic iff T is ergodic & $\mathtt{Per}\,(\phi) = \mathbb{G}$.

Schmidt's Theorem [15]

$E(\phi)$ is a closed subgroup of $\mathbb{G}$ and $E(\phi) = \mathtt{Per}\,(\phi)$.

In view of this,the conclusion of theorem 1 is equivalent to

$$\Gamma := \overline{\Phi(\mathbb{Z}_Q)} = E(\varphi). \tag{♿}$$

We prove this first in the case that $\overline{\Phi(\mathbb{Z}_Q)}$ is countable and then deduce the uncountable case.

Let

$$D(\alpha) := \{q \in \mathbb{N} : \ \exists\ p \in \mathbb{N},\ |\alpha - \tfrac{p}{q}| < \tfrac{1}{q^2}\}$$

and define $\varphi_q : \mathbb{T} \longrightarrow \mathbb{R}$ by

$$\varphi_q := \sum_{k=0}^{q-1} \varphi \circ r_\alpha^k.$$

We'll need

Denjoy-Koksma inequality ([9], [8])

$$\|\varphi_q\|_\infty \le \bigvee_{\mathbb{T}} \varphi\ \ \forall\ q \in\ D(\alpha).$$

Remark. Consequently, when $\Gamma = \overline{\Phi(\mathbb{Z}_Q)}$ is countable, there is a finite set $F \subset \mathcal{F}$ such that $\varphi_q(x) \in F$ for every $x \in \mathbb{T}$ & $q \in D(\alpha)$.

Proof of theorem 1 in the countable case

Sublemma 1 *For theorem 1 in the countable case, it suffices that*

$$\Phi(t+1) - \Phi(t) \in \mathtt{Per}\,(\varphi)\ \ \forall\ t \in \mathbb{Z}_Q. \tag{🚹}$$

Proof

Let $\Gamma_0 \subset \Gamma$ be the group generated by

$$\Gamma_0 := \langle\{\Phi(t+1) - \Phi(t)\ :\ t \in \mathbb{Z}_Q\}\rangle\ \le\ \Gamma.$$

Evidently, $\Phi(t) + \Gamma_0 = \Phi(0) + \Gamma_0\ \forall\ t \in \mathbb{Z}_Q$ whence $\varphi + \Gamma_0 \equiv \Phi(0) + \Gamma_0$ and Γ/Γ_0 is cyclic.

We claim moreover that $\#\Gamma/\Gamma_0 \le Q$. To see this, using $\sum_{t\in\mathbb{Z}_Q} \Phi(t) = 0$, we have

$$\Gamma_0 \ni \sum_{t\in\mathbb{Z}_Q} (\Phi(0) - \Phi(t)) = Q\Phi(0)$$

whence indeed $\#\Gamma/\Gamma_0 \le Q$.

By assumption, $\Gamma_0 \subset \texttt{Per}\,(\varphi)$.

By Schmidt's theorem, if $f \in L^\infty(\mathbb{T}\times\Gamma)$ and $f\circ T_{\alpha,\varphi} = f$, then $f\circ\tau_a = f$ a.e. $\forall\ a \in \Gamma_0$ and $\exists\ F \in L^\infty(X\times\Gamma/\Gamma_0)$ so that

$$f(x,\gamma) = F(x,\gamma+\Gamma_0) \text{ for a.e. } (x,\gamma) \in X\times\Gamma.$$

Evidently

$$F\circ T_{\alpha,\psi} = F \quad \text{a.e.}$$

where $\psi : X \to \Gamma/\Gamma_0,\ \psi := \varphi + \Gamma_0 \equiv \Phi(0) + \Gamma_0$ (as above).

Defining $T_{\alpha,\psi} : X\times\Gamma/\Gamma_0 \to X\times\Gamma/\Gamma_0$ as usual, we have

$$T_{\alpha,\psi} \cong r_\alpha \times r_{\Phi(0)+\Gamma_0} : X\times\Gamma/\Gamma_0 \to X\times\Gamma/\Gamma_0.$$

which is ergodic, being a product of two ergodic group rotations with disjoint spectra.

Thus F is constant a.e., whence also f, and $T_{\alpha,\varphi}$ is ergodic. ☑

Sublemma 2

(🚹) $$\Phi(t+1) - \Phi(t) \in \texttt{Per}\,(\varphi)\ \ \forall\ t \in \mathbb{Z}_Q.$$

Proof

We'll prove the sublemma using

Oren's Lemma [14] *If there exists $n_k \in \mathbb{N}$ and $A_k \subset \mathbb{T}$ such that φ_{n_k} is constant on A_k and $\varphi_{n_k}|_{A_k} \longrightarrow a$, $\inf \mu(A_k) > 0$ and $\lim_{k\longrightarrow\infty} |||n_k\alpha||| = 0$ then $a \in \texttt{Per}\,(\varphi)$.*

Here and and throughout, $|||x||| := \min_{k\in\mathbb{Z}} |x-k|$.

Note that a version of Oren's lemma is implicit in [4].

Next, we claim that for (🚹), it suffices to show

☺ There are sequences of measurable sets $A_k, B_k \subset \mathbb{T}$ such that $\varphi_{q_{n_k}}$ is constant on A_k and B_k,

$$\varphi_{q_{n_k}}|_{B_k} - \varphi_{q_{n_k}}|_{A_k} = \Phi(t+1) - \Phi(t)$$

and $m(A_k), m(B_k) > c > 0$ where c does not depend on k.

Indeed, by the remark after Denjoy Koksma inequality, there is a finite set F so that $\varphi_{q_{n_k}}(x) \in F\ \forall\ k \ge 1,\ x \in \mathbb{T}$.

Thus $\exists\ f \in F$ & $k_\ell \to \infty$ such that $\varphi_{q_{n_{k_\ell}}}|_{A_k} = f\ \ \forall\ \ell \ge 1$ whence $\varphi_{q_{n_{k_\ell}}}|_{B_k} = f + \Phi(t+1) - \Phi(t)\ \forall\ \ell \ge 1$.

By Oren's lemma, $f, f + \Phi(t+1) - \Phi(t) \in \texttt{Per}\,(\varphi)$, whence, since $\texttt{Per}\,(\varphi)$ is a group, $\Phi(t+1) - \Phi(t) \in \texttt{Per}\,(\varphi)$ and sufficiency of ☺ is established.

Finally, we construct the sequences of measurable sets $A_k, B_k \subset \mathbb{T}$ as in ☺.

To this end, we prove first that the discontinuities of φ are "dynamically separated".

Let $q \in \mathbb{N}$. Since φ is a step function with the set of discontinuities contained in $\{\frac{\ell}{Q} \ : \ 0 \le \ell \le Q-1\}$ the set of discontinuities of φ_q is contained in the set

$$\left\{\frac{\ell}{Q} - t\alpha \ : \ 0 \le \ell \le Q-1 \text{ and } 0 \le t \le q-1\right\}.$$

Hence the distance between the discontinuities is bounded below by

$$\mathtt{disc}(q) = \min_{|\ell| \le Q-1, |t| \le q-1, \ (\ell,t) \ne \underline{0}} |||\tfrac{\ell}{Q} - t\alpha|||.$$

Claim

(🚹) $$\exists \ n_k \uparrow \infty \ \& \ \theta > 0 \ s.t. \ \mathtt{disc}(q_{n_k}) \ge \frac{\theta}{q_{n_k}} \quad \forall \ k \ge 1.$$

Proof of (🚹) when $\alpha \in \mathtt{SBAD}$

By definition of $\mathtt{SBAD}$, there exists a sequence $m_k \in \mathbb{N}$, $\nu \in \mathbb{N}$ and $\epsilon > 0$ such that

$$\epsilon \ < \ \frac{q_{m_k-\nu}}{q_{m_k+1}} \ < \ \frac{q_{m_k-\nu}}{q_{m_k}} < \frac{1}{Q}$$

whence

$|\frac{r}{Q} - l\alpha| = \frac{1}{Q}|r - Ql\alpha| > \frac{1}{Q}|p_{m_k} - q_{m_k}\alpha| > \frac{1}{Q(q_{m_k} + q_{m_k+1})} > \frac{1}{2Qq_{m_k+1}} > \frac{\epsilon}{2Qq_{m_k-\nu}}$

for all $r \in \mathbb{Z}$ and $|l| \le q_{m_k-\nu} < \frac{q_{m_k}}{Q}$ and (🚹). ☑

Proof of (🚹) when Q is prime This further splits into two separate cases.

(i) *There are only finitely many n's such that $q_n = 0$ mod Q*: Choose N large enough such that $q_n \ne 0$ mod Q for $n \ge N$. For $n > N$,

$$\mathtt{disc}(q_n) \ge \frac{1}{Q} \min_{0<t<q_n} \{|||tQ\alpha|||, |||t\alpha|||\}.$$

As before, we have

$$\min_{0<t<q_n} |||t\alpha||| > \frac{1}{2q_n}.$$

Since q_i is prime to Q for all $i \geq n$, for $0 < t < q_n$, tQ is not a multiple of q_{n+r} for $r \geq 0$. Thus by [11, Theorem 19] if

$$|||Qt\alpha||| \leq \frac{1}{2Qt}$$

then Qt is a multiple of q_r for some $r < n$; in this case

$$|||Qt\alpha||| \geq \frac{1}{2q_{r+1}} \geq \frac{1}{2q_n}.$$

Therefore

$$\min_{0<t<q_n} |||Qt\alpha||| \geq \min\left(|||q_{n-1}\alpha|||, \frac{1}{2Qq_n}\right) = \frac{1}{2Qq_n} \text{ implying } \mathtt{disc}(q_n) \geq \frac{1}{2Q^2 q_n}.$$

(ii) *There are infinitely many n's such that $q_n = 0$ mod Q:* Let the i-th term of the continued fraction expansion of α be given by a_i; we know

$$(\star) \qquad \prod_{i=1}^{n} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix}.$$

Since $\det \left(\begin{smallmatrix} a_i & 1 \\ 1 & 0 \end{smallmatrix}\right) = -1$ for all i, we have that $\left(\begin{smallmatrix} a_i & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are invertible matrices in $\mathbb{Z}_Q$. Consequently there does not exist n for which $q_{n-1}, q_n = 0 \bmod Q$. Thus from the assumption for this case, we have a subsequence $\{n_k\}$ such that $q_{n_k} = 0 \bmod Q$ while $q_{n_k+1} \neq 0 \bmod Q$. Now by the recursion as in ($\star$) we have that $q_{n_k+r+1} = M(r)q_{n_k} + S(r)q_{n_k+1}$ for some $M(r), S(r) \in \mathbb{N}$. Again,

$$\min_{0<t<q_{n_k+1}} |||t\alpha||| > \frac{1}{2q_{n_k+1}}.$$

If

$$|||tQ\alpha||| < \frac{1}{2tQ} \text{ for some } 0 < t < q_{n_k+1}$$

then by Theorem 19 in [11], tQ is a multiple of q_i for some $i < n_k + 1$: Since $q_{n_k+1} \neq 0 \bmod Q$ we have that tQ is not a multiple of q_{n_k+1}. The number tQ cannot be a multiple of q_{n_k+r+1} for any $r \in \mathbb{N}$: Since $q_{n_k} \neq 0 \bmod Q$ if

$$tQ = sq_{n_k+r+1} = sM(r)q_{n_k} + sS(r)q_{n_k+1} = 0 \bmod Q \text{ for some } s$$

then $sS(r)$ is multiple of Q implying $t \geq q_{n_k+1}$. Hence we have

$$\min_{0<t<q_{n_k+1}} |||Qt\alpha||| \geq \min\left(|||q_{n_k}\alpha|||, \frac{1}{2Qq_{n_k}}\right) = \frac{1}{2Qq_{n_k}}$$

$$\text{implying } \mathtt{disc}(q_n) \geq \frac{1}{2Q^2 q_{n_k}}.$$

This proves (🚹). ☑

Construction of measurable sets as in ☺

By (🚹) there exists a subsequence $n_k \uparrow \infty$ and $\theta > 0$ such that $\texttt{disc}(q_{n_k}) > \frac{\theta}{q_{n_k}}$ and such that $q_{n_k} \gg |F|^2$, where F is the finite set of values taken by $\phi_{q_{n_k}}$ as in the remark after the Denjoy-Koksma inequality.

Fix $0 \le t \le Q-1$. To obtain the periodicity $\Phi(t+1) - \Phi(t)$, we build a sequence of measurable sets $A_k, B_k \subset \mathbb{T}$ such that

- $\phi_{q_{n_k}}$ is constant on A_k and B_k,
- $\phi_{q_{n_k}}|_{B_k} - \phi_{q_{n_k}}|_{A_k} = \Phi(t+1) - \Phi(t)$ and
- $m(A_k), m(B_k) > c > 0$.

Fix k and let ∂ be the partition of $\mathbb{T}$ by the discontinuities $\{\frac{l}{Q} - i\alpha : 0 \le i \le q_{n_k} - 1\}$ of the step function $\phi_{q_{n_k}}$.

For $0 \le i < q_{n_k}$, let $I_i^- \in \partial$ be the interval with right endpoint $\frac{t}{Q} - i\alpha$ and $I_i^+ \in \partial$ be the interval with left endpoint $\frac{t}{Q} - i\alpha$.

We can choose $0 < i_1, i_2, \ldots, i_{\lfloor \frac{q_{n_k}}{|F|^2} \rfloor} < q_{n_k}$ such that $\phi_{q_{n_k}}$ is constant on

$$A_k := \bigcup_{t=1}^{\lfloor \frac{q_{n_k}}{|F|^2} \rfloor} I_{i_t}^- \text{ and } B_k = \bigcup_{t=1}^{\lfloor \frac{q_{n_k}}{|F|^2} \rfloor} I_{i_t}^+.$$

Evidently,

$$\phi_{q_{n_k}}|_{B_k} - \phi_{q_{n_k}}|_{A_k} = \Phi(t+1) - \Phi(t)$$

and by (🚹)

$$m(A_k), m(B_k) \ge \texttt{disc}(q_{n_k}) \left\lfloor \frac{q_{n_k}}{|F|^2} \right\rfloor \ge \frac{\theta}{2|F|^2}.$$

These sets are as in ☺ and the proof of theorem 1 in the countable case is now complete.

Proof of theorem 1 in the uncountable case

Let

$$V := \texttt{Span}_{\mathbb{Q}} \Phi(\mathbb{Z}_Q) \subset \mathbb{R}^d,$$

let $K := \dim V$ and let $\{e_k : 1 \le k \le K\}$ be a basis for V so that each

$$\Phi(t) = \sum_{k=1}^{K} \phi_k(t) e_k \quad \text{with } \phi_k(t) \in \mathbb{Z} \quad (1 \le k \le K,\ t \in \mathbb{Z}_Q).$$

Consider the cocycle $\Psi : \mathbb{T} \to \mathbb{Z}^K$ defined by

$$\Psi(x) := \phi(\lfloor Qx \rfloor) \text{ where } \phi(t) := (\phi_1(t), \dots, \phi_K(t)) \quad (t \in \mathbb{Z}_Q).$$

It follows that $\langle \Psi(\mathbb{T}) \rangle = \mathbb{Z}^K$. We claim that

$$\int_{\mathbb{T}} \Psi(x)dx = \tfrac{1}{Q} \sum_{t \in \mathbb{Z}_Q} \phi(t) = 0.$$

To see this,

$$\begin{aligned} 0 &= Q \int_{\mathbb{T}} \varphi(x)dx \\ &= \sum_{t \in \mathbb{Z}_Q} \Phi(t) \\ &= \sum_{k=1}^{K} e_k \left(\sum_{t \in \mathbb{Z}_Q} \phi_k(t) \right). \end{aligned}$$

By linear independence of $\{e_k : \ 1 \le k \le K\}$, for each $1 \le k \le K$ $\sum_{t \in \mathbb{Z}_Q} \phi_k(t) = 0$ showing that indeed $\int_{\mathbb{T}} \Psi(x)dx = 0$.

Thus, by (♣) in the countable case, and Schmidt's theorem,

$$\langle \phi(\mathbb{Z}_Q) \rangle \subset \texttt{Per}\,(\Psi) = E(\Psi).$$

It follows that $\Phi = L \circ \phi$ (and $\varphi = L \circ \Psi$) where $L : \mathbb{Z}^K \to V \subset \mathbb{R}^d$ is given by

$$L(z_1, \dots, z_K) := \sum_{k=1}^{K} z_k e_k.$$

By linearity of L,

$$L(E(\Psi)) \subset E(L \circ \Psi) = E(\varphi)$$

and

$$\Phi(\mathbb{Z}_Q) = L(\phi(\mathbb{Z}_Q)) \subset \ E(\varphi). \quad ☑$$

The affine random walk

Theorems 2 and 3 both depend on the modeling of the *orbit sequence*

$$(\varphi_n(0) : \ n \ge 1)$$

by an associated `affine random walk`. To extract this affine random walk we first obtain a sequential substitution construction of the *jump sequence*

$$(\varphi(n\alpha) : \ n \ge 1) \text{ for } \ \alpha \in (0,1) \setminus \mathbb{Q}.$$

To this end, let $\beta = Q\alpha \mod 1$ & $P := \lfloor Q\alpha \rfloor$ so that $\alpha = \frac{P+\beta}{Q}$.

Define the map $\pi : [0,1) \to [0,1) \times \mathbb{Z}_Q$ by

$$\pi(x) := (Qx \mod 1, \lfloor Qx \rfloor),$$

the transformation $\tau : [0,1) \times \mathbb{Z}_Q \to [0,1) \times \mathbb{Z}_Q$ by $\tau := \pi \circ r_\alpha \circ \pi^{-1}$ and

$$\kappa(x,k) := K \circ \pi^{-1}(x,k) = k;$$

then

$$\begin{aligned}\tau(y,k) &= \pi\left(\tfrac{y+k}{Q} + \alpha\right)\\ &= \pi\left(\tfrac{y+k+P+\beta}{Q}\right)\\ &= (r_\beta(y), \lfloor k+P+y+\beta \rfloor \mod Q)\\ &= (r_\beta(y), k+P+1_{[1-\beta,1)}(y) \mod Q).\end{aligned}$$

Thus

(✉)
$$\begin{aligned}K(n\alpha) &= K \circ r_\alpha^n(0) = \kappa \circ \pi \circ r_\alpha^n(0)\\ &= \kappa \circ \tau^n \pi(0)\\ &= nP + \sum_{k=0}^{n-1} 1_{[1-\beta,1)}(k\beta)\\ &= nP + \sum_{k=1}^{n} \psi_k \mod Q\end{aligned}$$

where $\psi_k := 1_{[1-\beta,1)}((k-1)\beta)$. The sequence $(\psi_k : \ k \ge 1)$ is generated by

Modified continued fractions.

The *modified continued fraction expansion* of $\beta \in (0,1)$ is

$$\begin{aligned}\beta &= \cfrac{1}{n_1 - \cfrac{1}{n_2 - \ddots - \cfrac{1}{n_k - \ddots}}}\\ &=: \frac{1|}{|n_1} - \frac{1|}{|n_2} - \dots - \frac{1|}{|n_k} \ - \dots\\ &= [n_1, n_2, \dots]\end{aligned}$$

where $n_k(\beta) := n(s^{k-1}(\beta))$ with $n(\beta) := \lceil \frac{1}{\beta} \rceil$ and $s(\beta) := 1 - \{\frac{1}{\beta}\} = n(\beta) - \frac{1}{\beta}$.

See [10] & [12].

The quadratic case. If $\alpha \in \mathtt{QUAD}$, then so is $\beta = \{Q\alpha\}$ and

$$\beta = [n_1, n_2, \dots, n_K, \overline{m_1, \dots, m_L}]$$

where

$$(n_1, n_2, \dots, n_K) \in \mathbb{N}_2^K \ \& \ (m_1, \dots, m_L) \in \mathbb{N}_2^L \setminus 2\mathbb{1}.$$

Theorem 2.1 in [2] *For $\beta = [n_1, n_2, \dots]$, let $b_0(0) = 0,\ b_0(1) = 1$ &*

$$b_{k+1}(0) = b_k(0)^{\odot(n_{k+1}-1)} \odot b_k(1) \quad \& \quad b_{k+1}(1) = b_k(0)^{\odot(n_{k+1}-2)} \odot b_k(1),$$

then

$$(\psi_1, \dots, \psi_{\ell_k(i)}) = b_k(i) \qquad (k \geq 1)$$

if the last symbol in $b_k(1)$ is changed from a one to a zero.

Here $\ell_k(i) = |b_k(i)|$ $(i = 0, 1)$ are the

Block lengths.

Let $\underline{\ell}_k := \begin{pmatrix} \ell_k(0) \\ \ell_k(1) \end{pmatrix}$, then

$$\underline{\ell}_0 := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \& \ \underline{\ell}_{k+1} := \begin{pmatrix} n_{k+1} - 1 & 1 \\ n_{k+1} - 2 & 1 \end{pmatrix} \underline{\ell}_k.$$

Parities, Jumps, Orbits & RATs

Next, we compute the $\mathtt{jump}$ blocks.

We call $K(x) = \lfloor Qx \rfloor \in \mathbb{Z}_Q$ the *parity* of x and we begin by calculating the *parity blocks* with a generalization of Theorem 2.2 in AK.

For $\beta = \{Q\alpha\} = [n_1, n_2, \dots]$, $\epsilon \in \mathbb{Z}_Q$, $i = 0, 1$, define

$$B_k(i, \epsilon) := (\epsilon + K((n-1)\alpha) : 1 \leq n \leq \ell_k(i)),$$

then by (✉) and theorem 2.1 respectively,

$$(🦇) \quad B_k(i, \epsilon) = (\epsilon + (n-1)P + \sum_{t=1}^{n-1} 1_{[1-\beta,1)}((t-1)\beta) : \ 1 \leq n \leq \ell_k(i)))$$
$$= (\epsilon + (n-1)P + \sum_{t=1}^{n-1} b_k(i)_t : \ 1 \leq n \leq \ell_k(i))$$

where the addition is mod Q and $\sum_{t \in \varnothing} := 0$.

Theorem 2.2 (Parity recursions)

$$(☕) \qquad B_{k+1}(i, \epsilon) =$$
$$\bigodot_{j=1}^{n_{k+1}-1-i} B_k(0, \epsilon + (j-1)\epsilon_k) \odot B_k(1, \epsilon + (n_{k+1} - 1 - i)\epsilon_k).$$

Here (as before) the addition is $\mod Q$, $\epsilon_k := \sum_{j=1}^{\ell_k(0)} (b_k(0))_j + \ell_k(0)P \mod Q$ and $\bigodot_{j\in\varnothing} H_j \odot K := K$.

Proof Fix $i = 0, 1,\ \epsilon \in \mathbb{Z}_Q, k \geq 1$ and $1 \leq n \leq \ell_{k+1}(i)$, then $n = r\ell_k(0) + s$ where $0 \leq r \leq n_{k+1} - i - 1$ and $1 \leq s \leq \ell_k(j_r)$ with $j_{n_{k+1}-i-1} = 1$ & $j_r = 0$ for $r < n_{k+1} - i - 1$.

Set

$$\widetilde{B}_{k+1}(i, \epsilon) := \bigodot_{j=1}^{n_{k+1}-1-i} B_k(0, \epsilon + (j-1)\epsilon_k) \odot B_k(1, \epsilon + (n_{k+1} - 1 - i)\epsilon_k).$$

Using theorem 2.1 and (🦇), we have $\mod Q$,

$$\begin{aligned} B_{k+1}(i, \epsilon)_n &= \epsilon + (n-1)P + \sum_{t=1}^{n-1} b_{k+1}(i)_t \\ &= \epsilon + (r\ell_k(0) + s - 1)P + \sum_{t=1}^{r\ell_k(0)+s-1} b_{k+1}(i)_t \\ &= \epsilon + r\epsilon_k + (s-1)P + \sum_{t=1}^{s-1} b_k(j_r)_t \\ &= \widetilde{B}_{k+1}(i, \epsilon)_n. \quad ☑ \end{aligned}$$

Parity states and transition algorithm.

The k^{th} *parity states* are $\epsilon_k(i)\ \ (i = 0, 1)$ where

$$\epsilon_k(i) := \sum_{j=1}^{\ell_k(i)} (b_k(i))_j + \ell_k(i)P \mod Q.$$

In (☕), $\epsilon_k = \epsilon_k(0) \mod Q$.

The parity states are given by

$$\epsilon_0(i) = P + i \ \&\ \epsilon_{k+1}(i) = (n_{k+1} - i - 1)\epsilon_k(0) + \epsilon_k(1) \quad (i = 0, 1,\ k \geq 1).$$

Parity proposition *For every $k \geq 1$, $\langle\{\epsilon_k, \epsilon_{k+1}\}\rangle = \mathbb{Z}_Q$.*

Proof Define $\zeta_k = (\zeta_k(0), \zeta_k(1))$ by

$$\zeta_0(i) = P + i \ \&\ \zeta_{k+1}(i) = (n_{k+1} - i)\zeta_k(0) + \zeta_k(1).$$

It follows that

- $\epsilon_k(i) = \zeta_k(i) \mod Q$;
- $\mathtt{gcd}\,\{\zeta_k(0), \zeta_k(1)\} = 1\ \forall\ k \geq 1$;
- $\mathtt{gcd}\,\{\zeta_k(0), \zeta_{k+1}(0)\} = 1\ \forall\ k \geq 1$;
- $\langle\{\zeta_k(0), \zeta_{k+1}(0)\}\rangle = \mathbb{Z}\ \forall\ k \geq 1$;

- $\langle\{\epsilon_k, \epsilon_{k+1}\}\rangle = \mathbb{Z}_Q$. ☑

To obtain the transition algorithm based on theorem 2.2, set $B_0(i,\epsilon) = \epsilon$, then

$$B_1(i,\epsilon) = (\epsilon, \epsilon + P, \dots, \epsilon + (n_1 - 1 - i)P)$$

and the $B_k(i,\epsilon)$ $(i = 0, 1,\ \epsilon \in \mathbb{Z}_Q,\ k \geq 1)$ are given by (☕).

Jump blocks.

Next, for $k \geq 1,\ \epsilon \in \mathbb{Z}_Q$ & $i = 0, 1$, define the *auxiliary jump blocks*

$$J_k(i,\epsilon) := \Phi(B_k(i,\epsilon))$$

where

$$\Phi((a_1, \dots, a_n)) := (\Phi(a_1), \dots, \Phi(a_n)).$$

It follows from (☕) that for $i = 0, 1$:

$$(*) \quad J_{k+1}(i,\epsilon) = \bigodot_{j=1}^{n_{k+1}-1-i} J_k(0, \epsilon + (j-1)\epsilon_k) \odot J_k(1, \epsilon + (n_{k+1} - 1 - i)\epsilon_k);$$

and that the *jump block*

$$(☼) \qquad (\varphi(\{j\alpha\}))_{j=0}^{\ell_k(0)-1} = J_k(0,0).$$

Orbit blocks.

Define the *auxiliary orbit blocks*

$$\Sigma_k(i,\epsilon) := (\sum_{t=1}^{j} J_k(i,\epsilon)_t :\ 1 \leq j \leq \ell_k(i)).$$

In particular by (☼)

$$\Sigma_k(0,0) := (\varphi_1(0), \varphi_2(0), \dots, \varphi_{\ell_k(0)}(0)).$$

Our goal here is to obtain the auxiliary

Orbit block transitions.

The *simple displacement* over the aux. jump block $J_k(i,\epsilon)$ is

$$\sigma_k(i,\epsilon) := \Sigma_k(i,\epsilon)_{\ell_k(i)} = \sum_{t=1}^{\ell_k(i)} J_k(i,\epsilon)_t.$$

The *cumulative displacements* over the concatenation jump blocks $\bigodot_{j=1}^{K}(J_k(0, \epsilon + (j-1)\epsilon_k)$ $(K \geq 0)$ are

$$s_k(K,\epsilon) := \sum_{t=1}^{K} \sigma_k(0, \epsilon + (t-1)\epsilon_k).$$

By (*), for $k \ge 1,\ \epsilon \in \mathbb{Z}_Q,\ i = 0,1,$

$$\Sigma_{k+1}(i,\epsilon) = \bigodot_{j=1}^{n_{k+1}-1-i} (\Sigma_k(0,\epsilon+(j-1)\epsilon_k) + s_k(j-1,\epsilon)\mathbb{1})\odot$$
$$\odot\,(\Sigma_k(1,\epsilon+(n_{k+1}-i-1)\epsilon_k) + s_k(n_{k+1}-i-1,\epsilon)\mathbb{1}).$$

Generating functions of orbit blocks.

Define the functions $x_k(i,\epsilon) : \Omega_k(i) = [1,\ell_k(i)] \to \mathbb{R}^d$ by

$$x_k(i,\epsilon)(\omega) := \Sigma_{k+1}(i,\epsilon)_\omega \quad (\omega \in \Omega_k(i))$$

and their *generating functions*

$$U_k(i,\epsilon,\theta) := \sum_{\omega\in\Omega_k(i)} e^{i\theta\cdot x_k(i,\epsilon)(\omega)} \qquad (\theta \in \mathbb{T}^d).$$

Transition matrices. Noting that

$$\Omega_{k+1}(i) = \biguplus_{j=1}^{n_{k+1}-i-1} (\Omega_k(0)+(j-1)\ell_k(0)) \uplus (\Omega_k(1)+(n_{k+1}-i-1)\ell_k(0)),$$

we have

$$\begin{aligned}
U_{k+1}(i,\epsilon,\theta) &:= \sum_{\omega\in\Omega_{k+1}(i)} e^{i\theta\cdot x_{k+1}(i,\epsilon)(\omega)}\\
&= \Big(\sum_{j=1}^{n_{k+1}-i-1}\ \sum_{\omega\in\Omega_k(0)+(j-1)\ell_k(0)} + \sum_{\omega\in\Omega_k(1)+(n_{k+1}-i-1)\ell_k(0)}\Big) e^{i\theta\cdot x_{k+1}(i,\epsilon)(\omega)}\\
&= \sum_{j=1}^{n_{k+1}-i-1}\ \sum_{\omega\in\Omega_k(0)} e^{i\theta\cdot(x_k(0,\epsilon+(j-1)\epsilon_k)(\omega)+s_k(j-1,\epsilon))} + \sum_{\omega\in\Omega_k(1)} e^{i\theta\cdot(x_k(1,\epsilon+(n_{k+1}-i-1)\epsilon_k)(\omega)+s_k(n_{k+1}-i-1,\epsilon))}\\
&= \sum_{j=1}^{n_{k+1}-i-1} e^{i\theta\cdot s_k(j-1,\epsilon)} U_k(0,\epsilon+(j-1)\epsilon_k,\theta) + e^{i\theta\cdot s_k(n_{k+1}-i-1,\epsilon)} U_k(1,\epsilon+(n_{k+1}-i-1)\epsilon_k,\theta)\\
&= \sum_{\Delta\in\mathbb{Z}_Q}\ \sum_{j\in\mathfrak{m}(\epsilon_k,\Delta)\cap[1,n_{k+1}-i-1]} e^{i\theta\cdot s_k(j-1,\epsilon)} U_k(0,\epsilon+\Delta,\theta) + e^{i\theta\cdot s_k(n_{k+1}-i-1,\epsilon)} U_k(1,\epsilon+(n_{k+1}-i-1)\epsilon_k,\theta)
\end{aligned}$$

where for $\epsilon,\ \Delta \in \mathbb{Z}_Q$,

$$\mathfrak{m}(\epsilon,\Delta) := \{j \in \mathbb{N} :\ (j-1)\epsilon = \Delta \mod Q\}$$

(with $\sum_{\omega\in\varnothing} := 0$ as before).

Equivalently,

$$U_{k+1}(\theta) = A^{(k+1)}(\theta) U_k(\theta)$$

where

$$U_k := (U_k(i,\epsilon)\quad (i,\epsilon) \in S := \{0,1\}\times\mathbb{Z}_Q)\quad \&$$
$$A^{(k+1)} : \mathbb{T}^d \to M_{S\times S}(\mathbb{C}) := \{a : S\times S \to \mathbb{C}\}$$

is given by:

$$A^{(k+1)}_{(i,\epsilon),(0,\epsilon+\Delta)}(\theta) = \sum_{j\in\mathfrak{m}(\epsilon_k,\Delta)\cap[1,n_{k+1}-i-1]} e^{i\theta\cdot s_k(j-1,\epsilon)} \quad \text{if} \quad (n_{k+1},i) \neq (2,1),$$
$$A^{(k+1)}_{(i,\epsilon),(0,\epsilon+\Delta)}(\theta) = 0 \quad \text{if} \quad (n_{k+1},i) = (2,1),$$
$$A^{(k+1)}_{(i,\epsilon),(1,\epsilon+\Delta)}(\theta) = e^{i\theta\cdot s_k(n_{k+1}-i-1,\epsilon)} 1_{\{\Delta\}}((n_{k+1}-i-1)\epsilon_k).$$

It follows that

$$A^{(k+1)}_{(i,\epsilon),(0,\epsilon+\Delta)}(0) = N_{k+1}(i,\Delta) \ \ \& \ \ A^{(k+1)}_{(i,\epsilon),(1,\epsilon+\Delta)}(0) = 1_{\{\Delta\}}((n_{k+1}-i-1)\epsilon_k)$$

where $N_{k+1}(i,\Delta) := \#\,\mathfrak{m}(\epsilon_k,\Delta)\cap[1,n_{k+1}-i-1]$.

The random affine model

Probabilities. Here, we consider the probabilities

$$P^{(i)}_k := \frac{\#}{\ell_k(i)} \in \mathcal{P}(\Omega_k(i))$$

and each $x_k(i,\epsilon) \in \mathbb{R}^d$ as a random variable with sample space $(\Omega_k(i), P^{(i)}_k)$, and show that the resulting stochastic processes

$$(x_k(i,\epsilon): \ k \geq 1) \quad ((i,\epsilon) \in S)$$

have the same marginal distributions as the coordinate processes of an $(\mathbb{R}^d)^S$-valued `affine random walk`.

Construction of associated affine random walk.

Let

$$\Xi_k(i,\epsilon,\theta) := E(e^{i\theta\cdot x_k(i,\epsilon)}) = \frac{1}{\ell_k(i)} U_k(i,\epsilon,\theta),$$

then

$$\Xi_{k+1}(\theta) = \Pi^{(k+1)}(\theta)\Xi_k(\theta) \tag{⚖}$$

where $\Xi_k := (\Xi_k(i,\epsilon): \ (i,\epsilon) \in S)$ & $\Pi^{(k+1)}(\theta): S\times S \to \mathbb{C}$ is given by

$$\Pi^{(k+1)}_{(i,\epsilon),(j,\Delta)}(\theta) = \frac{\ell_k(j)}{\ell_{k+1}(i)} A^{(k+1)}_{(i,\epsilon),(j,\Delta)}(\theta).$$

The random variables.

Consider any sequence of independent random vectors

$$(\mathcal{L}_s^{(k+1)},\ W_{s,t}^{(k+1)}:\quad s,t\in S)\quad (k\ge 0) \tag{✉}$$

$$(\mathcal{L}_s^{(k+1)},\ W_{s,t}^{(k+1)}:\quad s,t\in S)\quad (k\ge 0)$$

on $S^S\times(\mathbb{R}^d)^{S\times S}$ whose marginals satisfy

$$P(\mathcal{L}_{(i,\epsilon)}^{(k+1)}=(0,\epsilon+\Delta))=\frac{\ell_k(0)N_{k+1}(i,\Delta)}{\ell_{k+1}(i)},$$
$$P(\mathcal{L}_{(i,\epsilon)}^{(k+1)}=(1,\epsilon+\Delta)=\frac{\ell_k(1)}{\ell_{k+1}(i)}1_{\{\Delta\}}((n_{k+1}-i-1)\epsilon_k);$$
$$P([W_{(i,\epsilon),(0,\epsilon+\Delta)}^{(k+1)}=s_k(J-1,\epsilon)]\|[\mathcal{L}_{(i,\epsilon)}^{(k+1)}=(0,\epsilon+\Delta)])=$$
$$=\frac{\#\{j\in\mathfrak{m}(\epsilon_k,\Delta)\cap[1,n_{k+1}-i-1]):\ s_k(j-1,\epsilon)=s_k(J-1,\epsilon)\}}{N_{k+1}(i,\Delta)};$$
$$\texttt{for}\quad J\in\mathfrak{m}(\epsilon_k,\Delta)\cap[1,n_{k+1}-i-1]\ \&$$
$$P([W_{(i,\epsilon),(1,\epsilon+\Delta)}^{(k+1)}=s_k(n_{k+1}-i-1,\epsilon)]\|[\mathcal{L}_{(i,\epsilon)}^{(k+1)}=(1,\epsilon+\Delta)])=1.$$

Note that when $n_{k+1}=2$, then $\mathcal{L}_{(1,\epsilon)}^{(k+1)}=(1,\epsilon+(n_{k+1}-i-1)\epsilon_k)$ a.s. $\forall\ \epsilon\in\mathbb{Z}_Q$ and that $W_{s,t}^{(k+1)}$ is defined when and only when $P(\mathcal{L}_s^{(k+1)}=t)>0$

Random affine transformations.

Given a finite set $S,\ d\ge 1$ and a random variable

$$(\mathcal{L},W)\in\texttt{RV}(S^S\times(R^d)^{S\times S}),$$

the associated *random affine transformation* (RAT) $F\in\texttt{RV}\,(M_{S\times S}(\mathbb{Z})\times(R^d)^S)$ defined by

$$F(x)_s:=x_{\mathcal{L}_s}+W_{s,\mathcal{L}_s}=:(a(F)x)_s+b(F)_s. \tag{🧺}$$

This RAT is of *flip-type* in the sense of [1].

Throughout this paper we'll often denote a flip-type RAT

$$F=(a(F),b(F))\in\texttt{RV}\,(M_{S\times S}(\{0,1\})\times(\mathbb{R}^d)^S)$$

by

$$F=(\mathcal{L},W)=(\mathcal{L}(F),W(F))\in\texttt{RV}\,(S^S\times(\mathbb{R}^d)^{S\times S}).$$

Here

$$a_{s,t}=\delta_{t,\mathcal{L}_s}\ \&\ b_s=W_{s,\mathcal{L}_s}.$$

Given a sequence $(\mathcal{L}_s^{(k+1)},\ W_{s,t}^{(k+1)}:\ \ s,t \in S)\ \ (k \geq 0)$ of independent random vectors as above, consider the associated RAT sequence

$$(F_k:\ k \geq 1) \in \mathtt{RV}\,(M_{S\times S}(\{0,1\}) \times (\mathbb{R}^d)^S)^{\mathbb{N}}$$

of independent RATS defined by (🧺).

RAT characteristic function.

The *characteristic function* of the RAT $F = (\mathcal{L}, W) \in \mathtt{RV}\,(S^S \times (\mathbb{R}^d)^{S\times S})$ (RAT-CF) is $\Pi_F : \mathbb{R}^d \to M_{S\times S}(\mathbb{C})$ defined by

(📖) $$\Pi_F(\theta)_{s,t} = P(\mathcal{L}_s = t)E(e^{i\langle\theta, W_{s,t}\rangle}) \quad (s,t \in S).$$

Note that $\Pi^{(k+1)}$ in (⚖) on page 16 is the RAT-CF of the RAT $(\mathcal{L}^{(k+1)}, W^{(k+1)})$ where $\mathcal{L}^{(k+1)}$ & $W^{(k+1)}$ are as in (✉) on page 17.

RAT lemma

For each $k \geq 1,\ s \in S$:

$$\mathtt{dist}\, F_1^k(0)_s \ = \ \mathtt{dist}\, x_k(s).$$

Here and throughout for $K \leq L$ & RATs $(F_j:\ K \leq j \leq L)$

$$F_K^L := F_L \circ F_{L-1} \circ \cdots \circ F_{K+1} \circ F_K.$$

Proof For $k \geq 1$, define

$$X^{(k)} := F_k \circ F_{k-1} \circ \cdots \circ F_1(0)$$

and

$$\widehat{\Xi}_k := (E(e^{i\theta\cdot X^{(k)}(i,\epsilon)} \quad (i,\epsilon) \in S).$$

By construction,

$$\widehat{\Xi}_{k+1}(\theta) = \Pi^{(k+1)}(\theta)\widehat{\Xi}_k(\theta).$$

By (⚖),

$$\Xi_{k+1}(\theta) = \Pi^{(k+1)}(\theta)\Xi_k(\theta).$$

The result follows by induction since $\Xi_0 = \widehat{\Xi}_0 \equiv \mathbb{1}$. ☑

Associated affine random walks.

We associate to a sequence

$$(F_k:\ k \geq 1) \in \mathtt{RV}\,(M_{S\times S}(\{0,1\}) \times (\mathbb{R}^d)^S)^{\mathbb{N}}$$

of independent RATS an *affine random walk* (ARW).

This is the $(\mathbb{R}^d)^S$-valued stochastic process

$$(X^{(k)} = (X^{(k)}s:\ s \in S):\ k \geq 1)$$

defined by

$$X^{(k)} := F_1^k(0).$$

Elementary presentation.

We now split the random vectors $(\mathcal{L}_s^{(k+1)},\ W_{s,t}^{(k+1)}:\ \ s,t\in S)\ \ (k\ge 0)$ into more elementary components.

Write

$$\mathcal{L}_{(i,\epsilon)}^{(k+1)} = (\mathfrak{r}_{(i,\epsilon)}^{(k+1)}, \mathfrak{s}_{(i,\epsilon)}^{(k+1)}),$$

then $\mathfrak{r}_{(i,\epsilon)}^{(k+1)} = \mathfrak{r}_i^{(k+1)} \mod Q$ where

$$P(\mathfrak{r}_i^{(k+1)} = 0) = \frac{\ell_k(0)(n_{k+1}-i-1)}{\ell_{k+1}(i)}\ \&\ P(\mathfrak{r}_i^{(k+1)} = 1) = \frac{\ell_k(1)}{\ell_{k+1}(i)}$$

and

$$\mathfrak{s}_{(i,\epsilon)}^{(k+1)} = \epsilon + \mathfrak{e}_i^{(k+1)}$$

where

$$P(\mathfrak{e}_i^{(k+1)} = \Delta \| \mathfrak{r}_i^{(k+1)} = 0) = \frac{N_{k+1}(i,\Delta)}{n_{k+1}-i-1} \quad (\Delta\in\mathbb{Z}_Q)$$

and

$$P(\mathfrak{e}_i^{(k+1)} = (n_{k+1}-i-1)\epsilon_k \| \mathfrak{r}_i^{(k+1)} = 1) = 1.$$

Next define random variables $\mathfrak{u}^{(k+1)}(i)\ \ (k\ge 1,\ i=0,1)$ by

$$\mathfrak{u}^{(k+1)}(i)\ \ \begin{cases} \text{uniform on } \mathfrak{m}(\epsilon_k, \mathfrak{e}_i^{(k+1)}) \cap [1, n_{k+1}-i-1] \text{ if } \mathfrak{r}_i^{(k+1)} = 0 \\ \equiv\ n_{k+1}-i \quad \text{if} \quad \mathfrak{r}_i^{(k+1)} = 1. \end{cases}$$

Now we define random variables $W_s^{(k)}\ \ (k\ge 1,\ s\in S)$ by

$$W_{(i,\epsilon)}^{(k+1)} := s_k(\mathfrak{u}^{(k+1)}(i)-1,\epsilon).$$

It is not hard to see that

$$P([W_{(i,\epsilon),(j,\epsilon+\Delta)}^{(k+1)} = J] \| [\mathcal{L}_{(i,\epsilon)}^{(k+1} = (j,\epsilon+\Delta)]) = P(W_{(i,\epsilon)}^{(k+1)} = J \| \mathfrak{r}_i^{(k+1)} = j,\ \mathfrak{e}_i^{(k+1)} = \Delta).$$

In the sequel, we'll have recourse to the *elementary random vector sequence* $(\mathfrak{X}^{(k)}:\ k\ge 1)\in(\{0,1\}\times\mathbb{Z}_Q\times\mathbb{N}_0)^{\mathbb{N}}$ where

$$\mathfrak{X}^{(k)} := (\mathfrak{r}_i^{(k)},\ \mathfrak{e}_i^{(k)},\ \mathfrak{u}^{(k)}(i):\ i,j=0,1).$$

The `RAT` F_k is constructed from $\mathfrak{X}^{(k)}$ & the (deterministic) cumulative displacements s_{k-1}.

Visits to zero and RATs

Recall that we assume $Q \ge 1,\ \alpha \in \mathbb{T} \setminus \mathbb{Q}$, with $\{Q\alpha\} = [n_1, n_2, \dots]$.

Let $\Phi : \mathbb{Z}_Q \to \mathbb{Z}^d$ satisfy $\sum_{k \in \mathbb{Z}_Q} \Phi(k) = 0$ & $\langle \Phi(\mathbb{Z}_Q) \rangle = \mathbb{Z}^d$ and define $\varphi : \mathbb{T} \to \mathbb{Z}^d$ by

$$\varphi(x) := \Phi(\lfloor Qx \rfloor)$$

and $T = T_{\alpha,\Phi} : \mathbb{T} \times \mathbb{Z}^d \to \mathbb{T} \times \mathbb{Z}^d$ by

$$T(x, z) := (x + \alpha, z + \varphi(x)).$$

Visit lemma *Let $(X^{(k)} :\ k \ge 1)$ be the associated* `ARW`*, then*

$$\text{(4.1')} \qquad \int_0^1 \Psi_{\ell_k(0)}(x)dx \ \ge\ \frac{\ell_k(1)^2}{3\ell_k(0)} \min_{\epsilon \in \mathbb{Z}_Q} \int_{\mathbb{T}^d} |E(e^{i\theta \cdot X^{(k)}(1,\epsilon)})|^2 d\theta - \frac{1}{2}$$

$$\text{(4.2')} \qquad \|\Psi_{\ell_k(1)}\|_\infty \le 2\ell_k(0) \max_{\epsilon \in \mathbb{Z}_Q} \int_{\mathbb{T}^d} |E(e^{i\theta \cdot X^{(k)}(0,\epsilon)})| d\theta.$$

Where here and throughout,

$$\int_{\mathbb{T}^d} f(\theta)d\theta := \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} f(\theta)d\theta \ \&$$

$$\Psi_N(x) := \#\{1 \le n \le N :\ \varphi_n(x) = 0\} = S_N(1_{\mathbb{T}^d \times \{0\}})(x, 0).$$

Visit sets.

The *visit set to $\nu \in \mathbb{Z}^d$* is

$$K_\nu := \{n \ge 1 :\ \varphi_n(0) = \nu\}$$

and the *visit distributions* are the discrete measures $U_k^{(i)}$ on $\mathbb{Z}^d$ defined by

$$U_k^{(i)}(\nu) := \#(K_\nu \cap [1, \ell_k(i)]) \quad (k \ge 1,\ i = 0, 1).$$

The *auxiliary visit sets to $\nu \in \mathbb{Z}^d$* are

$$K_k(i, \epsilon, \nu) := \{1 \le j \le \ell_k(i) :\ \Sigma_k(i, \epsilon)_j = \nu\}.$$

and the *auxiliary visit distributions* are the discrete measures $V_k(i, \epsilon)$ on $\mathbb{Z}^d$ defined by

$$V_k(i, \epsilon)(\nu) := \#(K_k(i, \epsilon, \nu)) \quad (k \ge 1,\ i = 0, 1).$$

As above,

$$K_k(0, 0, \nu) = K_\nu \cap [1, \ell_k(0)] \ \&\ \ U_k^{(0)} = V_k(0, 0).$$

Visit sublemma

$$(4.1)\qquad \int_0^1 \Psi_{\ell_k(0)}(x)dx \ \geq\ \frac{1}{3\ell_k(0)} \min_{\epsilon\in\mathbb{Z}_Q} \sum_{\nu\in\mathbb{Z}^d} V_k(1,\epsilon)(\nu)^2 - \frac{1}{2};$$

$$(4.2)\qquad \int_0^1 \Psi_{\ell_k(1)}(x)^N dx \ \leq\ \frac{2^N}{\ell_k(1)} \max_{\epsilon\in\mathbb{Z}_Q} \sum_{\nu\in\mathbb{Z}^d} V_k(0,\epsilon)(\nu)^{N+1} \quad \forall\ N \geq 1.$$

Proof Fix $N,\ k \geq 1$ & $i = 0, 1,\ \Psi^N_{\ell_k(i)}$ is a step function on $\mathbb{T}$, whence Riemann integrable. Using the unique ergodicity of $x \mapsto x + \alpha$,

☯
$$\begin{aligned}
\ell_{k+r}(0) \int_0^1 & \Psi_{\ell_k(i)}(x)^N dx \\
&\underset{r\to\infty}{\sim} \sum_{j=1}^{\ell_{k+r}(0)} \Psi_{\ell_k(i)}(j\alpha)^N \\
&= \sum_{j=1}^{\ell_{k+r}(0)} \#(\{1 \leq m \leq \ell_k(i) :\ \varphi_m(j\alpha) = 0\})^N \\
&= \sum_{j=1}^{\ell_{k+r}(0)} \#(\{1 \leq m \leq \ell_k(i) :\ \varphi_{m+j}(0) = \varphi_j(0)\})^N \\
&= \sum_{j=1}^{\ell_{k+r}(0)} \#(\{j+1 \leq m \leq \ell_k(i) + j :\ \varphi_m(0) = \varphi_j(0)\})^N \\
&= \sum_{\nu\in\mathbb{Z}^d} \sum_{j\in[1,\ell_{k+r}(0)]\cap K_\nu} \#\left(K_\nu \cap [j+1, j+\ell_k(i)]\right)^N.
\end{aligned}$$

By Theorems 2.1 & the orbit block transitions, for $r \geq 1,\ \exists\ J = J_{r,k} \geq 1,\ 0 = m_1 < \cdots < m_J = \ell_{k+r}(0)$ and $\eta_1, \dots \eta_{J-1} \in \mathbb{Z}_Q,\ i_1, \dots, i_{J-1} = 0, 1$ so that

$$m_{j+1} - m_j = \ell_k(i_j)\ \forall\ j, \quad [1, \ell_{k+r}(0)] = \bigcup_{j=1}^{J-1}{}^{\!\!\!\!\!\!\cdot}\ \ (m_j, m_{j+1}]$$

and

$$(⚔)\qquad (\varphi_{m_j+1}(0), \varphi_{m_j+2}(0), \dots, \varphi_{m_{j+1}}(0)) = \varphi_{m_j}(0) + \Sigma_k(i_j, \eta_j).$$

Since $m_{j+1} - m_j = \ell_k(i_j)$, it follows that

$$(\text{🧺})\qquad \ell_{k+r}(0) \in [J\ell_k(1), J\ell_k(0)].$$

Also by (⚔), for fixed $\nu \in \mathbb{Z}^d$,

$$K_\nu \cap (m_j, m_{j+1}] = m_j + K_k(i_j, \eta_j, \nu - \varphi_{m_j}(0)).$$

Proof of (4.1) We have for fixed $\nu \in \mathbb{Z}^d$

$$\begin{aligned}
&\textstyle\sum_{i\in[1,\ell_{k+r}(0)]\cap K_\nu} \#\left(K_\nu \cap [i+1, i+\ell_k(0)]\right)\\
&\quad = \sum_{j=1}^{J-1} \sum_{i\in(m_j,m_{j+1}]\cap K_\nu} \#\left(K_\nu \cap [i+1, i+\ell_k(0)]\right)\\
&\quad \ge \sum_{j=1}^{J-1} \sum_{i\in(m_j,m_{j+1}]\cap K_\nu} \#\left(K_\nu \cap [i+1, m_{j+1}]\right) \quad \because\ \forall\ i,j:\ i+\ell_k(0) \ge i+\ell_k(i_j) \ge m_{j+1}\\
&\quad \ge \sum_{j=1}^{J-1} \sum_{k=1}^{L_j-1} k \quad \text{where} \quad L_j := \#((m_j, m_{j+1}] \cap K_\nu) = V_k(i_j,\eta_j)(\nu - \varphi_{m_j}(0))\\
&\quad = \sum_{j=1}^{J-1} \mathfrak{s}(L_j) \text{ where } \mathfrak{s}(x) := \frac{x(x-1)}{2}\\
&\quad = \sum_{j=1}^{J-1} \mathfrak{s}(V_k(i_j,\eta_j)(\nu - \varphi_{m_j}(0))).
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{\nu\in\mathbb{Z}^d} \sum_{j\in[1,\ell_{k+r}(0)]\cap K_\nu} \#\left(K_\nu \cap [j+1, j+\ell_k(0)]\right)\\
&\qquad \ge \sum_{\nu\in\mathbb{Z}^d} \sum_{j=1}^{J-1} \mathfrak{s}(V_k(i_j,\eta_j)(\nu - \varphi_{m_j}(0)))\\
&\qquad \ge (J-1) \min_{\epsilon\in\mathbb{Z}_Q} \sum_{\nu\in\mathbb{Z}^d} \mathfrak{s}(V_k(1,\epsilon)(\nu)) \quad \text{since } \mathfrak{s} \uparrow \text{ on } \mathbb{N}_0\\
&\qquad = \frac{J-1}{2} \min_{\epsilon\in\mathbb{Z}_Q} \sum_{\nu\in\mathbb{Z}^d} V_k(1,\epsilon)(\nu)^2 - \frac{(J-1)\ell_k(1)}{2}
\end{aligned}$$

and using ☯ with $N = 1$ & $i = 0$

$$\begin{aligned}
\int_0^1 \Psi_{\ell_k(0)}(x)dx &\xleftarrow[r\to\infty]{} \frac{1}{\ell_{k+r}(0)} \sum_{j=1}^{\ell_{k+r}(0)} \Psi_{\ell_k(0)}(j\alpha)\\
&\ge \frac{J-1}{2\ell_{k+r}(0)} \min_{\epsilon\in\mathbb{Z}_Q} \sum_{\nu\in\mathbb{Z}^d} V_k(1,\epsilon)(\nu)^2 - \frac{J\ell_k(1)}{2\ell_{k+r}(0)}\\
&\ge \frac{1}{3\ell_k(0)} \min_{\epsilon\in\mathbb{Z}_Q} \sum_{\nu\in\mathbb{Z}^d} V_k(1,\epsilon)(\nu)^2 - \frac{1}{2} \quad \text{by .} \quad ☑ \text{ (4.1).}
\end{aligned}$$

Proof of (4.2)

Using ☯ with $k,\ N \ge 1$ arbitrary and fixed & $i = 1$ we have

$$\ell_{k+r}(0)\int_0^1 \Psi_{\ell_k(1)}(x)^N dx \underset{r\to\infty}{\sim} \sum_{\nu\in\mathbb{Z}^d}\sum_{j\in[1,\ell_{k+r}(0)]\cap K_\nu} \#\,(K_\nu\cap[j+1,j+\ell_k(1)])^N.$$

Similar to (※),

$$(\varphi_{m_j+1}(0),\varphi_{m_j+2}(0),\dots,\varphi_{m_{j+1}+\ell_k(1)}(0)) = [\varphi_{m_j}(0)\mathbb{1}+\Sigma_k(i_j,\eta_j)]\odot[\varphi_{m_{j+1}}(0)\mathbb{1}+\Sigma_k(1,\Delta_j)].$$

for some $\Delta_j\in\mathbb{Z}_Q$.

We have as before, for fixed $\nu\in\mathbb{Z}^d$,

$$\begin{aligned}\sum_{i\in[1,\ell_{k+r}(0)]\cap K_\nu}\#\,(K_\nu\cap[i+1,i+\ell_k(1)])^N &= \sum_{j=1}^{J-1}\sum_{i\in(m_j,m_{j+1}]\cap K_\nu}\#\,(K_\nu\cap[i+1,i+\ell_k(1)])^N\\ &\le \sum_{j=1}^{J-1}\sum_{i\in(m_j,m_{j+1}]\cap K_\nu}\#\,(K_\nu\cap[i+1,m_{j+1}+\ell_k(1)])^N.\end{aligned}$$

Fix j. For fixed $i\in(m_j,m_{j+1}]$,

$$\begin{aligned}\#\,(K_\nu\cap[i+1,m_{j+1}+\ell_k(1))]) &= \#\,(K_\nu\cap[i+1,m_{j+1}]) + \#\,(K_\nu\cap[m_{j+1}+1,m_{j+1}+\ell_k(1))])\\ &= \#\,(K_\nu\cap[i+1,m_{j+1}]) + V_k(1,\Delta_j)(\nu-\varphi_{m_{j+1}}(0)).\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{i\in(m_j,m_{j+1}]\cap K_\nu}\#\,(K_\nu\cap[i+1,m_{j+1}+\ell_k(1)])^N =\\ &= \sum_{i\in(m_j,m_{j+1}]\cap K_\nu}\#\,(K_\nu\cap[i+1,m_{j+1}]) + V_k(1,\Delta_j)(\nu-\varphi_{m_{j+1}}(0))^N\\ &= \sum_{s=0}^N\binom{N}{s}\Bigg(\sum_{i\in(m_j,m_{j+1}]\cap K_\nu}\#\,(K_\nu\cap[i+1,m_{j+1}])^s\Bigg)V_k(1,\Delta_j)(\nu-\varphi_{m_{j+1}}(0))^{N-s}\\ &\le \sum_{s=0}^N\binom{N}{s}V_k(i_j,\eta_j)(\nu-\varphi_{m_j}(0))^{s+1}V_k(1,\Delta_j)(\nu-\varphi_{m_{j+1}}(0))^{N-s}\\ &\le \sum_{s=0}^N\binom{N}{s}V_k(0,\eta_j)(\nu-\varphi_{m_j}(0))^{s+1}V_k(0,\Delta_j)(\nu-\varphi_{m_{j+1}}(0))^{N-s}.\end{aligned}$$

Using this and Hölder's inequality,

$$\sum_{\nu\in\mathbb{Z}^d}\sum_{j\in[1,\ell_{k+r}(0)]\cap K_\nu}\#\left(K_\nu\cap[j+1,j+\ell_k(1)]\right)^N\le$$
$$\le\sum_{j=1}^{J}\sum_{s=0}^{N}\binom{N}{s}\sum_{\nu\in\mathbb{Z}^d}V_k(0,\eta_j)(\nu-\varphi_{m_j}(0))^{s+1}V_k(0,\Delta_j)(\nu-\varphi_{m_{j+1}}(0))^{N-s}$$
$$\le\sum_{j=1}^{J}\sum_{s=0}^{N}\binom{N}{s}\left(\sum_{\nu\in\mathbb{Z}^d}V_k(0,\eta_j)(\nu-\varphi_{m_j}(0))^{N+1}\right)^{\frac{s+1}{N+1}}\left(\sum_{\nu\in\mathbb{Z}^d}V_k(0,\Delta_j)(\nu-\varphi_{m_{j+1}}(0))^{N+1}\right)^{\frac{N-s}{N+1}}$$
$$=2^N J\max_{\epsilon\in\mathbb{Z}_Q}\sum_{\nu\in\mathbb{Z}^d}V_k(0,\epsilon)(\nu)^{N+1}$$

whence

$$\begin{aligned}\int_0^1\Psi_{\ell_k(1)}(x)^N dx &\underset{r\to\infty}{\longleftarrow}\frac{1}{\ell_{k+r}(0)}\sum_{\nu\in\mathbb{Z}^d}\sum_{j\in[1,\ell_{k+r}(0)]\cap K_\nu}\#\left(K_\nu\cap[j+1,j+\ell_k(1)]\right)^N\\ &\le\frac{2^N J}{\ell_{k+r}(0)}\max_{\epsilon\in\mathbb{Z}_Q}\sum_{\nu\in\mathbb{Z}^d}V_k(0,\epsilon)(\nu)^{N+1}\\ &\le\frac{2^N}{\ell_k(1)}\max_{\epsilon\in\mathbb{Z}_Q}\sum_{\nu\in\mathbb{Z}^d}V_k(0,\epsilon)(\nu)^{N+1}.\quad ☑(4.2)\end{aligned}$$

Proof of the Visit Lemma

Let

$$\widehat{V}_k(i,\epsilon)(\theta):=\sum_{\nu\in\mathbb{Z}^d}V_k(i,\epsilon)(\nu)e^{i\theta\cdot\nu}\qquad(\epsilon\in\mathbb{Z}_Q,\ i=0,1,\ \theta\in\mathbb{T}^d),$$

then

$$\widehat{V}_k(i,\epsilon)(\theta)=\ell_k(i)E(e^{i\theta\cdot X^{(k)}(i,\epsilon)}).$$

Using (4.1) in the sublemma and the Riesz-Fischer theorem, we see that

$$\begin{aligned}\int_0^1\Psi_{\ell_k(0)}(x)dx\ &\ge\ \frac{1}{4\ell_k(0)}\min_{\epsilon\in\mathbb{Z}_Q}\int_{\mathbb{T}^d}|\widehat{V}_k(1,\epsilon)(\theta)|^2d\theta-\frac{1}{2}\\ &=\frac{\ell_k(1)^2}{4\ell_k(0)}\min_{\epsilon\in\mathbb{Z}_Q}\int_{\mathbb{T}^d}|E(e^{i\theta\cdot X^{(k)}(1,\epsilon)})|^2d\theta-\frac{1}{2}.\end{aligned}$$

This is (4.1'). To see (4.2'),

$$\begin{aligned}
\|\Psi_{\ell_k(1)}\|_\infty &\xleftarrow[N\to\infty]{} \left(\int_0^1 \Psi_{\ell_k(1)}(x)^N dx\right)^{\frac{1}{N}} \\
&\le \left(\frac{2^N}{\ell_k(1)} \max_{\epsilon\in\mathbb{Z}_Q} \sum_{\nu\in\mathbb{Z}^d} V_k(0,\epsilon)(\nu)^{N+1}\right)^{\frac{1}{N}} \quad \text{by (4.2)} \\
&= \frac{2}{\ell_k(1)^{\frac{1}{N}}} \max_{\epsilon\in\mathbb{Z}_Q}\left(\sum_{\nu\in\mathbb{Z}^d} V_k(0,\epsilon)(\nu)^{N+1}\right)^{\frac{1}{N}} \\
&\le \frac{2}{\ell_k(1)^{\frac{1}{N}}} \max_{\epsilon\in\mathbb{Z}_Q} \int_{\mathbb{T}^d} |\widehat{V}_k(0,\epsilon)(\theta)|^{1+\frac{1}{N}} d\theta \quad \text{by the Hausdorff-Young theorem} \\
&\xrightarrow[N\to\infty]{} 2\max_{\epsilon\in\mathbb{Z}_Q} \int_{\mathbb{T}^d} |\widehat{V}_k(0,\epsilon)(\theta)| d\theta \\
&= 2\ell_k(0)\max_{\epsilon\in\mathbb{Z}_Q} \int_{\mathbb{T}^d} |E(e^{i\theta\cdot X^{(k)}(0,\epsilon)})| d\theta.
\end{aligned}$$

This is (4.2'). ☑

In order to use the visit lemma, we show next that if $\alpha \in$ `QUAD`, then the associated `RAT` sequence is "asymptotically eventually periodic".

Displacement asymptotics

The modified continued fraction expansion of $\beta \in (0,1) \setminus \mathbb{Q}$ is eventually periodic iff $\beta \in$ `QUAD`. Let

⚽ $$[n_1, n_2, \dots] = [n_1, \dots, n_K, \overline{m_1, \dots, m_L}].$$

We next establish that the centered `RAT` sequence (as in [1]) corresponding to a quadratic irrational and a rational step function is "`asymptotically eventually periodic`".

The proofs of theorems 2 & 3 rely on this fact.

This asymptotic eventual periodicity is obtained via centering. We'll see that elementary random vector sequence is always asymptotically eventually periodic, however, the cumulative displacements may have linear growth. The centering is needed to offset this possibility.

In this section, we'll often "`possibly extend the period in` ⚽" to demonstrate eventual periodicity of related sequences.

This means that for some $M \in \mathbb{N}$, we'll modify ⚽ to

$$[n_1, n_2, \dots] = [n_1, \dots, n_K, \overline{\underbrace{m_1, \dots, m_L, \dots, m_1, \dots, m_L}_{M\text{-times}}}].$$

Recall that the parity state transitions are given by

$$\epsilon_{k+1}(i) = (n_{k+1} - i - 1)\epsilon_k(0) + \epsilon_k(1) \mod Q.$$

In the quadratic case, these transitions form an eventually periodic sequence, whence

$$(\underline{\epsilon}_k = (\epsilon_k(0), \epsilon_k(1)) : \quad k \ge 1)$$

is also eventually periodic.

These parity transitions only depend on $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ & $Q \ge 2$.

If $\alpha \in \mathtt{QUAD}$, then by possibly extending the period in ⚽, we may assume that $\underline{\epsilon}_{k+L} = \underline{\epsilon}_k \ \forall \ k > K$.

Simple displacement transitions.

Consider the *simple displacement vectors*

$$\sigma_k := (\sigma_k(i,\epsilon) : \ (i,\epsilon) \in S) \in (\mathbb{R}^d)^S.$$

By theorem 2.2, for $i = 0, 1$ and $(n_{k+1}, i) \ne (2, 1)$:

$$\sigma_{k+1}(i,\epsilon) = \sum_{j=1}^{n_{k+1}-1-i} \sigma_k(0, \epsilon + (j-1)\epsilon_k) + \sigma_k(1, \epsilon + (n_{k+1} - 1 - i)\epsilon_k);$$

where $\epsilon_k = \epsilon_k(0)$ as above.

In matrix form,

$$\sigma_{k+1} = M^{(k+1)}\sigma_k$$

where each $M^{(k+1)} : S \times S \to \mathbb{Z}$.

The simple displacement transformations also only depend on $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ & $Q \ge 2$.

Seeing $\sigma_k = (\sigma_k(s) : \ s \in S) \in (\mathbb{R}^d)^S$ as

$$\sigma_k = ((\sigma_k^{(j)}(s) : \ s \in S) : \ 1 \le j \le d) \in (\mathbb{R}^S)^d,$$

we note that each $\sigma_k^{(j)} \in \mathbb{R}^S$ is a linear image of $\Phi^{(j)} \in \mathbb{R}^Q$ (the j^{th} coordinate of Φ) and $\sigma_{k+1}^{(j)} = M^{(k+1)}\sigma_k^{(j)}$ for each j.

Displacement lemma

Suppose that $\alpha \in \mathtt{QUAD}$, then $\exists \ K, \ L \in \mathbb{N}$ and $\mathfrak{c}, \ \mathfrak{d} \in (\mathbb{R}^d)^S$, so that

$$\sigma_{K+Ln} = \mathfrak{c} + n\mathfrak{d}.$$

For $\alpha \in \texttt{QUAD}$, the simple displacement transitions are eventually periodic and the proof of the displacement lemma rests on the Denjoy-Koksma inequality and a spectral analysis of the simple displacement transformations on $\mathbb{C}^S$ over a period (as in the "eigenvalue lemma" below).

Subspace decomposition & eigenvalues.

For $\alpha \in \texttt{QUAD}$, the parity sequence $(\epsilon_k : \ k \geq 1)$ is eventually periodic, whence the above sequence of matrices $(M^{(k)} : \ k \geq 1)$ giving the displacement transitions is also eventually periodic.

Suppose that

$$\begin{aligned}
&[n_1, n_2, \dots] = [n_1, \dots, n_K, \overline{m_1, \dots, m_L}];\\
&(\epsilon_k : \ k \geq 1) = (\epsilon_1, \dots, \epsilon_K, \overline{\eta_1, \dots, \eta_L});\\
&(M^{(k)} : \ k \geq 1) = (M^{(1)}, \dots, M^{(K)}, \overline{E_1, \dots, E_L}).
\end{aligned}$$

Thus

$$\sigma_{K+Ln} = B^n \sigma_K \text{ where } B = E_L \dots E_1 \ \& \ \sigma_k := (\sigma_k(i,\epsilon) : \ (i,\epsilon) \in S).$$

Next, write $\mathbb{C}^S = (\mathbb{C}^Q)^{\{0,1\}}$ and $z \in C^S$ as $z = (z^{(0)}, z^{(1)}) \in (\mathbb{C}^Q)^{\{0,1\}}$.

The parity state transitions are now given by

$$\sigma_{k+1} = M^{(k+1)} \begin{pmatrix} \sigma(0)_k \\ \sigma(1)_k \end{pmatrix}$$

where

$$\sigma(i)_k(\epsilon) = \sigma_k(i,\epsilon)$$

and

$$(\text{☞}) \qquad \begin{aligned}
M^{(k+1)} &:= \mathfrak{P}^{(k+1)}(\rho_{\epsilon_k})\\
&= \begin{pmatrix} \mathfrak{P}^{(k+1)}_{(0,0)}(\rho_{\epsilon_k}) & \mathfrak{P}^{(k+1)}_{(0,1)}(\rho_{\epsilon_k}) \\ \mathfrak{P}^{(k+1)}_{(1,0)}(\rho_{\epsilon_k}) & \mathfrak{P}^{(k+1)}_{(1,1)}(\rho_{\epsilon_k}) \end{pmatrix}\\
&= \begin{pmatrix} p_{n_{k+1}}(\rho_{\epsilon_k}) & q_{n_{k+1}}(\rho_{\epsilon_k}) \\ p_{n_{k+1}-1}(\rho_{\epsilon_k}) & q_{n_{k+1}-1}(\rho_{\epsilon_k}) \end{pmatrix}
\end{aligned}$$

with $\rho_\epsilon \in \texttt{Hom}(\mathbb{C}^{\mathbb{Z}_Q}, \mathbb{C}^{\mathbb{Z}_Q})$ defined by

$$\rho_\epsilon z(\delta) := z(\delta + \epsilon).$$

and

$$p_\nu(x) := \sum_{j=1}^{\nu-1} x^{j-1}$$
$$q_\nu(x) := x^{\nu-1}.$$

Set $\gamma_r = e^{\frac{2\pi i r}{Q}}$ and let $\vec{e}_r \in \mathbb{C}^Q$ be given by

$$(\vec{e}_r)_s := \gamma_{rs}$$

for $0 \le r \le Q-1$ and $1 \le s \le Q$.

Since $\vec{e}_s \perp \vec{e}_t \ \forall \ s,\ t \in \mathbb{Z}_Q,\ s \neq t$, we have that $(\vec{e}_r : \ 0 \le r \le Q-1)$ form an orthogonal basis for $\mathbb{C}^Q$ and

$$\texttt{Span}\{\vec{e}_r \ : \ 1 \le r \le Q-1\} = \mathbb{1}^\perp =: \{\vec{v} = (v_i) \in \mathbb{C}^Q \ : \ \sum_{i=0}^{Q-1} v_i = 0\}.$$

Moreover,

$$(\circledast) \qquad \rho_\epsilon \vec{e}_r = \gamma_{r\epsilon}\vec{e}_r.$$

Next, define the bracket $[\cdot,\cdot] : \mathbb{C}^{\{0,1\}} \times \mathbb{C}^Q \to C^S = (C^Q)^{\{0,1\}}$ by

$$[\vec{c},\vec{z}] := \begin{pmatrix} c_0\vec{z} \\ c_1\vec{z} \end{pmatrix}$$

where $\vec{c} = (c_0, c_1)$.

It follows from $(\circledast)$ that

$$M^{(k+1)}[\vec{c},\vec{e}_r] = \mathfrak{P}^{(k+1)}(\rho_{\epsilon_k})[\vec{c},\vec{e}_r] = [\mathfrak{P}^{(k+1)}(\gamma_{\epsilon_k r})\vec{c},\vec{e}_r].$$

To summarize, letting for $0 \le r \le Q-1$,

$$V_r := \{[\vec{c},\vec{e}_r] : \ \vec{c} \in \mathbb{C}^{\{0,1\}}\},$$

then

$\bigoplus_{r=0}^{Q-1} V_r = (\mathbb{C}^Q)^{\{0,1\}}$ and $BV_r = V_r \ \ (0 \le r \le Q-1)$.

Eigenvalue lemma

For $1 \le r \le Q-1$, all the eigenvalues of $B|_{V_r}$ are roots of unity.

Proof

We have that $B|_{V_0}$ is a product of integer matrices of form $\begin{pmatrix} N & 1 \\ N-1 & 1 \end{pmatrix}$ with $N \in \mathbb{N}$ including matrices with $N \ge 2$ and is therefore a positive matrix with integer coefficients with unit determinant. It follows that the characteristic polynomial of $B|_{V_0}$ is an integer polynomial of form $z^2 - Jz + 1$ for some $J \in \mathbb{N}$ (and that $B|_{V_0}$ is hyperbolic).

For each $1 \le r \le Q-1$,

$$|\det B|_{V_r}| = |\det \mathfrak{P}^{(k+1)}(\gamma_{\epsilon_k r})| = 1.$$

We claim first that no $B|_{V_r}$ $(1 \le r \le Q-1)$ is hyperbolic. If this were not the case for $1 \le r \le Q-1$, there would be $\lambda > 1$ and a rational cocycle $0 \ne \Phi \perp \mathbb{1}$ with $\langle \Phi, \vec{e}_r \rangle \ne 0$ giving rise to either

- $\|\sigma_{K+Lt}\| \gg \lambda^t$ which is impossible by the Denjoy-Koksma estimate; or
- $\|\sigma_{K+Lt}\| \ll \frac{1}{\lambda^t}$ which is impossible by theorem 1.

To continue, since B is an integer matrix, $\det(B - z\mathtt{Id})$ is a polynomial with integer coefficients.

It follows that

$$\det(B - z\mathtt{Id})|_{V_0^\perp} = \frac{\det(B - z\mathtt{Id})}{\det(B - z\mathtt{Id})|_{V_0}}$$

is also a polynomial with integer coefficients. As shown above, all its roots are of unit modulus. By Kronecker's theorem ([13]), all these roots are roots of unity. ☑

Proof of the displacement lemma

Let $\{\gamma_{\mathfrak{j}} : \mathfrak{j} \in \mathcal{J}\}$ be the collection of eigenvalues of B counting multiplicity which are all roots of unity. Let $V_{\mathfrak{j}}$ be the corresponding Jordan subspace, then by the above,

$$\dim V_{\mathfrak{j}} = 2.$$

We may extend the period in ⚽ so that $\gamma_{\mathfrak{j}} = 1 \ \forall \ \mathfrak{j} \in \mathcal{J}$.

For each $\mathfrak{j} \in \mathcal{J}$ let $(e_j(\mathfrak{j}) : \ j = 1, 2)$ be the Jordan basis of $V_{\mathfrak{j}}$.

For $\mathfrak{j} \in \mathcal{J}$, $x = x_1 e_1(\mathfrak{j}) + x_2 e_2(\mathfrak{j})$ and $N \ge 1$, we have that

$$B^N x = N x_1 e_1(\mathfrak{j}) + x_2 e_2(\mathfrak{j})$$

Thus for $\Phi : \mathbb{Z}_Q \to \mathbb{R}^d$ & $1 \le k \le d$,

$$\begin{aligned}\sigma_{K+Ln}^{(k)} &= B^N \sigma_K^{(k)} \\ &= \sum_{\mathfrak{j} \in \mathcal{J}} N \langle \sigma_K^{(k)}, e_1(\mathfrak{j}) \rangle e_1(\mathfrak{j}) + \langle \sigma_K^{(k)}, e_2(\mathfrak{j}) \rangle e_2(\mathfrak{j}) \\ &=: \mathfrak{c}^{(k)} + N \mathfrak{d}^{(k)}.\end{aligned}$$

This proves the displacement lemma. ☑

In the sequel, we'll also need the

Positivity proposition *By possibly extending the period in ⚽, we may ensure that* $B_{s,t} > 0 \ \forall \ s, t \in S$.

Remark. Evidently $E(a(F_{K+1}^{K+L})_{s,t}) = \Pi_{F_{K+1}^{K+L}}(0) > 0$ iff $B_{s,t} > 0$.

Proof

It follows from (☞) that

$$E_{t+1} = \begin{pmatrix} \sum_{j=1}^{m_{t+1}-1} \rho_{\eta_t}^{j-1} & \rho_{\eta_t}^{m_{t+1}-1} \\ \sum_{j=1}^{m_{t+1}-2} \rho_{\eta_t}^{j-1} & \rho_{\eta_t}^{m_{t+1}-2} \end{pmatrix}.$$

Choose $1 \le r \le L$ such that $m_r \ne 2$. A direct calculation shows

$$B^2 > E_{r+1}E_r \ge \begin{pmatrix} \rho_0 + \rho_{\eta_r} + \rho_{\eta_{r-1}} & D_1 \\ D_2 & D_3 \end{pmatrix}$$

where $D_1, D_2, D_3 \in M_{Q\times Q}(\mathbb{N} \cup \{0\})$ are matrices where each row and column has at least one non-zero entry.

By the parity proposition, η_t and η_{t+1} generate the group $\mathbb{Z}_Q$.

Applying this to $t = r-1$, we get that there exists an N such that for all $n \ge N$, $(\rho_0 + \rho_{\eta_r} + \rho_{\eta_{r-1}})^n > 0$, meaning all of its entries are positive. Thus $B^{N+2} > 0$. This proves that B is aperiodic and irreducible. ☑

Asymptotic eventual periodicity & centering

Let $\alpha \in \mathtt{QUAD}$ and φ be a step function with rational discontinuities with associated $\mathtt{RAT}$ sequence $(F_k: \ k \ge 1)$ and $\mathtt{ARW}$ $(X^{(t)}: \ t \ge 1)$.

By the displacement lemma, we may suppose that

$$[n_1, n_2, \dots] = [n_1, \dots, n_K, \overline{m_1, \dots, m_L}];$$

$$(\epsilon_k: \ k \ge 1) = (\epsilon_1, \dots, \epsilon_K, \overline{\eta_1, \dots, \eta_L}), \ \sigma_{k+1} = M^{(k+1)}\sigma_k;$$

$$(M^{(k)}: \ k \ge 1) = (M^{(1)}, \dots, M^{(K)}, \overline{E_1, \dots, E_L}) \ \& \ \sigma_{K+Ln} = \mathfrak{e} + n\mathfrak{d}.$$

Next, we examine the asymptotic, distributional periodicity of the $\mathtt{RAT}$ sequence.

Elementary periodic approximation lemma

There are constants $\lambda, \ M > 1$ *and ,for* $1 \le r \le L$ *there are random vectors*

$$\mathfrak{x}^{(r)} := (\mathfrak{R}_i^{(r)}, \ \mathfrak{E}_i^{(r)}, \ \mathfrak{U}^{(r)}(i): \ i = 0, 1, \ \epsilon \in \mathbb{Z}_Q) \in \{0, 1\} \times \mathbb{Z}_Q \times \mathbb{N}_0$$

so that

$$\mathtt{dist}\,(\mathfrak{e}_i^{(K+Ln+r)}, \ \mathfrak{u}^{(K+Ln+r)}(i): \ i = 0, 1 \| \mathfrak{r}_0^{(K+Ln+r)}, \mathfrak{r}_1^{(K+Ln+r)}) = \mathtt{dist}\,(\mathfrak{E}_i^{(r)}, \ \mathfrak{U}^{(r)}(i): \ i, j = 0, 1 \| \mathfrak{R}_0^{(r)}, \mathfrak{R}_1^{(r)})$$

$$|P(\mathfrak{X}^{(K+Ln+r)} = Z) - P(\mathfrak{x}^{(r)} = Z)| \le \frac{M}{\lambda^n} \quad \forall \ n \ge 1, \ Z \in \{0, 1\} \times \mathbb{Z}_Q \times \mathbb{N}_0.$$

Proof We have that

$$\underline{\ell}_{K+Ln+r} = B(m_r)B(m_{r-1})\dots B(m_1)C^n\underline{\ell}_K$$

where

$$B(m) := \begin{pmatrix} m-1 & 1 \\ m-2 & 1 \end{pmatrix} \ \&\ C := B(m_L)B(m_{L-1})\dots B(m_1).$$

Now $\det C = \prod_{r=1}^L \det B(m_r) = 1$ and each $C_{i,j} \in \mathbb{N}$, so C is hyperbolic, having eigenvalues of are λ & $\frac{1}{\lambda}$ for some $\lambda = \lambda_C > 1$.

Moreover, $\exists\ c_r(i) \quad (i = 0, 1\ \&\ 0 \le r \le L)$ with $c_L = \lambda c_0$ so that

$$\ell_{K+Ln+r}(i) = c_r(i)\lambda^n + O(\tfrac{1}{\lambda^n}),$$

whence

$$\begin{aligned} c_{r+1}(i) &= \frac{\ell_{K+Ln+r+1}(i)}{\lambda^n} + O(\tfrac{1}{\lambda^n}) \\ &= \frac{1}{\lambda^n}[(m_{r+1} - i - 1)\ell_{K+Ln+r}(0) + \ell_{K+Ln+r}(1)] + O(\tfrac{1}{\lambda^n}) \\ &= (m_{r+1} - i - 1)c_r(0) + c_r(1) + O(\tfrac{1}{\lambda^n}). \end{aligned}$$

Define random variables $\mathfrak{R}_i^{(r)} \quad (i = 0, 1,\ 1 \le r \le L)$ by

$P(\mathfrak{R}_i^{(r+1)} = 0) = \frac{(m_{r+1}-i-1)c_r(0)}{c_{r+1}(i)}$ & $P(\mathfrak{R}_i^{(r+1)} = 1) = \frac{c_r(1)}{c_{r+1}(i)} = 1 - P(\mathfrak{R}_i^{(r+1)} = 0)$.

It follows that for $i, j = 0, 1$ & $1 \le r \le L$,

$$P(\mathfrak{r}_i^{(K+Ln+r)} = j) = P(\mathfrak{R}_i^{(r)} = j) + O(\tfrac{1}{\lambda^n}).$$

Next, we observe that for $n \ge 1,\ 1 \le r \le L,\ j = 0, 1$, the distribution of $\mathfrak{e}_i^{(K+Ln+r)}$ given $\mathfrak{r}_i^{(K+Ln+r)}$ does not depend on $n \ge 1$ and define:

$$P([\mathfrak{E}_i^{(r+1)} = \Delta] \| [\mathfrak{R}_i^{(r+1)} = 0]) = \frac{\#\mathfrak{m}(\eta_r, \Delta) \cap [1, m_{r+1} - i - 1]}{m_{r+1} - i - 1} \quad (\Delta \in \mathbb{Z}_Q)$$

and

$$P([\mathfrak{E}_i^{(r+1)} = (m_{r+1} - i - 1)\eta_r] \| [\mathfrak{R}_i^{(r+1)} = 1]) = 1.$$

Analogously, $\mathfrak{u}^{(L+Ln+r+1)}(i)$ has a conditional distribution independent of n and we define

$$\mathfrak{U}^{(r+1)}(i) \quad \begin{cases} \text{uniform on } \mathfrak{m}(\eta_r, \mathfrak{E}_i^{(r+1)}) \cap [1, m_{r+1} - i - 1] \text{ if } \mathfrak{R}_i^{(r+1)} = 0 \\ m_{r+1} - i \quad \text{if} \quad \mathfrak{R}_i^{(r+1)} = 1. \end{cases}$$

The random vectors $\mathfrak{x}^{(r)} \in \mathtt{RV}(\{0,1\} \times \mathbb{Z}_Q \times \mathbb{N}_0) \quad 1 \le r \le L$ where

$$(\text{✐}) \qquad \mathfrak{x}^{(r)} := (\mathfrak{R}_i^{(r)},\ \mathfrak{E}_i^{(r)},\ \mathfrak{U}^{(r)}(i) :\ i = 0, 1)$$

are as advertised by construction. ☑

RAT periodic approximation lemma

There are random variables $a \in \mathtt{RV}\,(M_{S\times S}(\mathbb{Z}))$, $\mathfrak{v}$, $\mathfrak{w} \in \mathtt{RV}\,((\mathbb{R}^d)^S)$ *so that if*

$$H^{(n)}(x) = ax + \mathfrak{v} + n\mathfrak{w},$$

then $\exists\ M > 0$ *so that* $\forall\ n \ge 1$

$$|P(\widetilde{F}_n = f) - P(H^{(n)} = f)| \le \frac{M}{\lambda^n} \tag{🚲}$$

where

$$\widetilde{F}_n := F_{K+Ln+1}^{K+Ln+L}.$$

Proof

Let $\mathfrak{x}^{(r)}$ $(1 \le r \le L)$ be independent, each distributed as in (✒).

Define

$$\mathfrak{l}_{(i,\epsilon)}^{(r+1)} := (\mathfrak{R}_i^{(r+1)}, \epsilon + \mathfrak{E}_i^{(r+1)}),$$

then, since

$$\mathcal{L}_{(i,\epsilon)}^{(K+Ln+r+1)} = (\mathfrak{r}_i^{(K+Ln+r+1)}, \epsilon + \mathfrak{e}_i^{(K+Ln+r+1)}),$$

we have by the elementary periodic approximation lemma,

$$\sup_{s,t\in S} |P(\mathfrak{l}_s^{(r+1)} = t) - P(\mathcal{L}_s^{(K+Ln+r+1)} = t)| = O(\tfrac{1}{\lambda^n}).$$

To study the random variables $W_s^{(K+Ln+r)}$, we'll need formulae for the cumulative displacements.

Using the displacement lemma, for $1 \le r \le L$,

$$\begin{aligned} s_{K+Ln+r}(K,\epsilon) &= \sum_{t=1}^{K} \sigma_{K+Ln+r}(0, \epsilon + (t-1)\epsilon_{K+Ln+r}) \\ &= \sum_{t=1}^{K} (\mathfrak{c}_r + n\mathfrak{d}_r)(0, \epsilon + (t-1)\eta_r) \\ &=: \mathfrak{C}_r(K,\epsilon) + n\mathfrak{D}_r(K,\epsilon). \end{aligned}$$

It follows that

$$\begin{aligned} &\mathtt{dist}\,(W_{(i,\epsilon)}{}^{(K+Ln+r+1)} | \mathfrak{r}_i^{(K+Ln+r+1)}) \\ &\qquad = \mathtt{dist}\,(s_{K+Ln+r+1}(\mathfrak{u}^{(K+Ln+r+1)}(i) - 1, \epsilon) | \mathfrak{r}_i^{(K+Ln+r+1)}) \\ &\qquad = \mathtt{dist}\,(s_{K+Ln+r+1}(\mathfrak{U}^{(r+1)}(i) - 1, \epsilon) | \mathfrak{R}_i^{(r+1)}) \\ &\qquad = \mathtt{dist}\,(\mathfrak{C}_{r+1}(\mathfrak{U}^{(r+1)}(i) - 1, \epsilon) + n\mathfrak{D}_{r+1}(\mathfrak{U}^{(r+1)}(i) - 1, \epsilon) | \mathfrak{R}_i^{(r+1)}). \end{aligned}$$

Now let $G_r^{(n)}$ $(1 \le r \le L,\ n \ge 1)$ be the RATs defined by

$$G_r^{(n)}(x)_{(i,\epsilon)} := x_{\mathfrak{l}^{(r)}_{(i,\epsilon)}} + \mathfrak{C}_r(\mathfrak{U}^{(r)}(i) - 1, \epsilon) + n\mathfrak{D}_r(\mathfrak{U}^{(r)}(i) - 1, \epsilon),$$

then there is a constant $M > 0$ so that $\forall\ f \in M_{S\times S}(\mathbb{Z}) \times (\mathbb{R}^d)^S)$,

$$(\ddagger) \qquad |P(F_{K+Ln+r} = f) - P(G_r^{(n)} = f)| \le \frac{M}{\lambda^n}.$$

Finally, let

$$H^{(n)} := G_L^{(n)} \circ G_{L-1}^{(n)} \cdots \circ G_2^{(n)} \circ G_1^{(n)}.$$

This has the form

$$H^{(n)}(x) = ax + \mathfrak{v} + n\mathfrak{w}$$

where $a \in \mathtt{RV}(M_{S\times S}(\mathbb{Z}))$, $\mathfrak{v},\ \mathfrak{w} \in \mathtt{RV}((\mathbb{R}^d)^S)$.

It follows from (‡) that $H^{(n)}$ satisfies (🚲). ☑

Coupling.

It follows that we can define a probability space $(\Omega, \mathcal{A}, P)$ and independent random vectors $(H^{(n)}, \widetilde{F}_n)$ ($n \ge 1$) on $(\Omega, \mathcal{A}, P)$ so that

$$P(H^{(n)} \ne \widetilde{F}_n) \le \frac{M}{\lambda^n}.$$

Consider the ARW

$$Y_J^{(n)} := H_{J+1}^n(X^{(K+LJ)}) \qquad (n > J).$$

ARW periodic approximation lemma

There is a constant $M > 1$ so that $\forall\ J,\ n \ge 1$,

$$(⤫) \qquad |P(Y_J^{(n)} \ne X^{(K+Ln)})| \le \frac{M}{\lambda^J};$$

and

$$(⧄) \qquad \sup_{n>J} |E(Y_J^{(n)\nu}) - E(X^{(K+Ln)\nu})| \xrightarrow[J\to\infty]{} 0\ \forall\ \nu \ge 1.$$

Proof of (⤫)

$$\begin{aligned} P(Y_J^{(n)} \ne X^{(K+Ln)}) &\le P(\widetilde{F}_J^n \ne H_J^n) \\ &\le \sum_{t=J}^n P(H^{(t)} \ne \widetilde{F}_t) \\ &\le \sum_{t=J}^n \frac{M}{\lambda^t} \\ &= O(\frac{1}{\lambda^J}). \quad ☑ \end{aligned}$$

Proof of (⊘) For fixed $\nu \geq 1$ & $f : \Omega \to \mathbb{R}^S$, let

$$\|f\|_\nu := E(\|f\|^\nu)^{\frac{1}{\nu}},$$

then $\|\cdot\|_\nu$ is a norm and:

$$\begin{aligned}\|Y_J^{(n+1)} - X^{(K+L(n+1))}\|_\nu &= \|H^{(n+1)}(Y_J^{(n)}) - \widetilde{F}_{n+1}(X^{(K+Ln)})\|_\nu \\ &\leq \|b(H^{(n+1)}) - b(\widetilde{F}_{n+1})\|_\nu + \|(a(H^{(n+1)}) - a(\widetilde{F}_{n+1}))X^{(K+Ln)}\|_\nu + \\ &\qquad + \|a(H^{(n+1)})(Y_J^{(n)} - X^{(K+Ln)})\|_\nu \\ &\leq \|Y_J^{(n)} - X^{(K+Ln)}\|_\nu + \frac{M'n}{\lambda^{\frac{n}{\nu}}}.\end{aligned}$$

Thus possibly increasing M,

$$\|Y_J^{(n)} - X^{(K+L(n))}\|_\nu \leq \sum_{j \geq J} \frac{Mj}{\lambda^{\frac{j}{\nu}}} \xrightarrow[J\to\infty]{} 0$$

and (⊘) follows. ☑

Corollary *There are constants $\mu,\ \xi,\ \xi_J \in (\mathbb{R}^d)^S\ \ (J \geq 1)$ and $0 < \rho < 1$ so that*

$$\begin{aligned}&E(Y_J^{(t)}) = \mu t + \xi_J + O(\rho^t)\ \forall\ J \geq 1,\ \xi_J \xrightarrow[J\to\infty]{} \xi, \\ \& \quad &E(X^{(K+Lt)}) = \mu t + \xi + O(\rho^t).\end{aligned}$$

Note that $\rho \in (0,1)$ is the second largest eigenvalue of $E(a(\mathcal{H})) = \Pi_{\mathcal{H}}(0)$.

Proof We have

$$Y_J^{(t+1)} = H^{(t+1)}(Y_J^{(t)})$$

where

$$H^{(n)}(x) = a^{(n)}x + \mathfrak{v}^{(n)} + n\mathfrak{w}^{(n)}$$

and $(a^{(n)}, \mathfrak{v}^{(n)}, \mathfrak{w}^{(n)} :\ \ n \geq 1)$ are independent and identically distributed.

It follows as in [1] that

$$(⚗) \qquad E(Y_J^{(t)}) = E(a)^t E(X^{(K+LJ)}) + \sum_{s=1}^{t} E(a)^{t-s} E(\mathfrak{v} + s\mathfrak{w})$$

By the positivity proposition, by possibly extending the period in ⚽, we may ensure that $E(a(H^{(n)})) = \Pi_{H^{(n)}}(0)$ is an aperiodic stochastic matrix whence 1 is a simple, dominant eigenvalue.

Suppose that $\pi^* \in \mathbb{R}_+^S$ satisfies $\langle \pi^*, \mathbb{1} \rangle = 1$ & $\pi^* E(a) = \pi^*$.

Let $N : \mathbb{C}^S \to \mathbb{C}\cdot\mathbb{1},\ N(x) := \langle \pi^*, x\rangle \mathbb{1}$, then by (⛱), $\exists\ \xi_J \in \mathbb{R}^S$ & $0 < \rho < 1$ so that

$$\begin{aligned} E(Y_J^{(t)}) &= \xi_J + \sum_{s=1}^{t} E(a)^{t-s} N(E(\mathfrak{v} + s\mathfrak{w})) + O(\rho^t) \\ &= \xi_J + tN(E(\mathfrak{v})) + \frac{t(t+1)}{2} N(E(\mathfrak{w})) + O(\rho^t). \end{aligned}$$

Finally, we claim that $N(E(\mathfrak{w})) = 0$.
This will follow from the Denjoy-Koksma estimate.
By (⛉), we have

$$|E(X^{(K+Lt)}) - E(Y_J^{(t)})| = O(1).$$

Thus, if $N(E(\mathfrak{w})) \neq 0$, then by (⛱), $|E(X^{(K+Lt)})| \asymp t^2$ contradicting the Denjoy-Koksma estimate that $|E(X^{(K+Lt)})| = O(t)$. The lemma follows from this. ☑

Centering.
As in [1], set $(\widehat{X}^{(n)} :\ n \geq 1)$ be the centered `ARW` defined by

$$\widehat{X}^{(n)} := X^{(n)} - E(X^{(n)})$$

and let $(\mathcal{F}_n :\ n \geq 1)$ be the independent `RAT` sequence so that

$$\widehat{X}^{(K+Ln)} = \mathcal{F}_1^n(\widehat{X}^{(K)}).$$

`ARW` centering lemma
There is a centered, iid `RAT` *sequence* $(\mathcal{H}_n :\ n \geq 1)$ *and* $0 < \rho < r < 1$ *so that if for* $J \geq 1$, $(Z_J^{(t)} :\ t \geq J)$ *is defined by*

$$Z_J^{(t)} := \mathcal{H}_{J+1}^t(\widehat{X}^{(K+LJ)}),$$

then

(⟁) $$\sup_{n>J} |E(Z_J^{(n)\nu}) - E(\widehat{X}^{(K+Ln)\nu})| \xrightarrow[J\to\infty]{} 0\ \forall\ \nu \geq 1;$$

(ii) $$P(\exists\ t \geq J,\ |Z_J^{(t)} - \widehat{X}^{(K+Lt)}| \geq r^t) = O(\rho^J)\ \textit{as}\ J \to \infty.$$

Proof
Define

$$\widehat{Y}_J^{(t)} := Y_J^{(t)} - E(Y_J^{(t)}) = Y_J^{(t)} - c_t.$$

As in [1], $(\widehat{Y}_J^{(t)}:\ t \geq 1)$ is given by the centered `RAT` sequence $(\mathcal{G}_n:\ n \geq 1)$ where $a(\mathcal{G}_n) = a^{(n)}$ and

$$\begin{aligned} b(\mathcal{G}_{n+1}) &= a^{(n+1)}c_n - c_{n+1} + \mathfrak{v}^{(n+1)} + n\mathfrak{w}^{(n+1)} \\ &= (a^{(n+1)} - I)\xi_J + \mathfrak{v}^{(n+1)} - \mu + n\big[(a^{(n+1)} - I)\mu + \mathfrak{w}^{(n+1)}\big] + O(\rho^n) \\ &=: \mathfrak{v}'^{(n+1)} + n\mathfrak{w}'^{(n+1)} + O(\rho^n) \end{aligned}$$

where $(a^{(n)}, \mathfrak{v}^{(n)}, \mathfrak{w}^{(n)})$ are independent, identically distributed random variables.

By the remark after the positivity proposition, $E(a)$ is irreducible and aperiodic.

Thus, by the variance lemma in [1], for each $s \in S$,

$$\begin{aligned} E((\widehat{Y}_J^{(t)2})_s) &\asymp \sum_{k=1}^{t} E(b(\mathcal{G}_k)_s^2) \\ &= \sum_{n=1}^{t} E((\mathfrak{v}'_s + n\mathfrak{w}'_s)^2 + O(\rho^n)) \\ &\sim tE(\mathfrak{v}'^2_s) + t^2E(\mathfrak{w}'_s\mathfrak{v}'_s) + \frac{t^3}{3}E(\mathfrak{w}'^2_s). \end{aligned}$$

By ($\varnothing$),

$$|E((\widehat{Y}_J^{(t)2})_s) - E((\widehat{X}^{(K+Lt)2})_s)| = O(1)$$

whence, by the Denjoy-Koksma estimate, $E((\widehat{Y}_J^{(t)2})_s) \ll t^2$ and $\mathfrak{w}'^{(n+1)} \equiv 0$.

Thus

$$b(\mathcal{G}_{n+1}) = (a^{(n+1)} - I)\xi_J + \mathfrak{v}^{(n+1)} - \mu + O(\rho^n).$$

Accordingly, define $\mathcal{H}_n$ by

$$a(\mathcal{H}_n) := a^{(n)} \ \&\ b(\mathcal{H}_n) := (a^{(n)} - I)\xi_J + \mathfrak{v}^{(n)} - \mu.$$

The lemma follows. ☑

Spectral theory and theorem 2

By the Perron-Frobenius theorem, 1 is a simple, dominant eigenvalue of $\Pi_{\mathcal{H}}(0)$ (see (𝕸) on page 18) with right eigenvector $\mathbb{1} \in \mathbb{R}_+^S$ and left eigenvector $\pi^* \in \mathbb{R}_+^S$ satisfying $\langle \pi^*, \mathbb{1} \rangle = 1$.

By the implicit function theorem $\exists\ r = r_{\mathcal{H}} > 0$ and smooth functions

$$\lambda : (-r, r)^d \to \mathbb{C},\ v : (-r, r)^d \to \mathbb{C}^S,\ \pi : (-r, r)^d \to \mathbb{C}^S$$

so that

- $\langle \pi(0), v(\theta)\rangle = \langle \pi(\theta), v(\theta)\rangle = 1$;
- $\lambda(0) = 1,\ v(0) = \mathbb{1}\ \&\ \pi(0) = \pi^*$;
- for each $1 \le k \le \infty$, $\theta \in (-r, r)^d$, $\lambda(\theta)$ is a simple, dominant eigenvalue of $\Pi_{\mathcal{H}}(\theta)$ with eigenvector $v(\theta)$ and left eigenvector $\pi(\theta)$.

As in [7], consider the *principal projections* $N(\theta) : \mathbb{C}^S \to \mathbb{C}^S$ defined by

$$N(\theta)x := \langle \pi(\theta), x\rangle v(\theta)$$

then possibly reducing $r_{\mathcal{H}} > 0$, we ensure $\exists\ 0 < \rho < 1$ so that

$$\Pi(\theta)^n - \lambda(\theta)^n N(\theta) = Q(\theta)^n = O(\rho^n) \text{ uniformly in } |\theta| \le r_{\mathcal{H}}$$

where $Q(\theta) := \Pi(\theta)(I - \lambda(\theta))N(\theta)$.

Lemma: `Taylor expansion of the eigenvalue`

(🚲) $$\lambda(\theta) = 1 - \langle D\theta, \theta\rangle + o(\|\theta\|^2)$$

as $\theta \to 0$ where $D \in M_{d\times d}(\mathbb{C})$ is positive definite.

Proof We have

$$\lambda(\theta) = 1 + \langle \nabla\lambda(0), \theta\rangle + \langle d^2\lambda(0)\theta, \theta\rangle + o(\|\theta\|^2)$$

where $d^2\lambda(\theta)$ is the matrix of second partial derivatives:

$$d^2\lambda(\theta)_{i,j} := \frac{\partial^2\lambda}{\partial\theta_i\partial\theta_j}(\theta).$$

and we must show that $\nabla\lambda(0) = 0$ and that $D := -d^2\lambda(0)$ is positive definite.

Fix $\sigma \in \mathbb{R}^d$, $\|\sigma\| = 1$ and write, for differentiable $f : \mathbb{R}^d \to \mathbb{C}$,

$$D^k_\sigma f(\theta) := \tfrac{d^k}{dt^k} f(\theta + t\sigma)|_{t=0},$$

then

$$D_\sigma f(\theta) = \langle \sigma, \nabla f(\theta)\rangle\ \&\ D^2_\sigma f(\theta) = \langle d^2 f(\theta)\sigma, \sigma\rangle.$$

Accordingly, it suffices to show that for each $\sigma \in \mathbb{R}^d$, $\|\sigma\| = 1$,
(i) $D_\sigma\lambda(0) = \frac{d}{dt}\lambda(t\sigma)|_{t=0} = 0$ and (ii) $D^2_\sigma\lambda(0) = \frac{d^2}{dt^2}\lambda(t\sigma)|_{t=0} < 0$.
¶1 $D_\sigma(\Pi_{\mathcal{H}})(0)\mathbb{1} = 0$.

Proof of ¶1 For fixed $s \in S$:

$$\begin{aligned}(D_\sigma(\Pi_{\mathcal{H}})(0)\mathbb{1})_s &= i\sum_{t\in S} P(\mathcal{L}_s(\mathcal{H}) = t)E(\langle \sigma, b_s(\mathcal{H})\rangle | \mathcal{L}_s(\mathcal{H}) = t)\\ &= iE(\langle \sigma, b_s(\mathcal{H})\rangle)\\ &= 0 \quad \because\ \mathcal{H} \text{ is centered.} \quad ☑\ ¶1\end{aligned}$$

Proof of (i)

Since $\langle \pi(0), v(\theta)\rangle \equiv 1$, we have that $D_\sigma v(\theta) \perp \pi(0)$. Also

$$D_\sigma(\Pi_{\mathcal{H}} v)(\theta) = D_\sigma(\Pi_{\mathcal{H}})(\theta)v(\theta) + \Pi_{\mathcal{H}}(\theta) D_\sigma v(\theta),$$
$$D_\sigma(\lambda v)(\theta) = D_\sigma \lambda(\theta) v(\theta) + \lambda(\theta) D_\sigma v(\theta)$$

whence

$$\begin{aligned} 0 &= D_\sigma(\Pi_{\mathcal{H}} v - \lambda v)(\theta) \\ &= D_\sigma(\Pi_{\mathcal{H}} - \lambda)(\theta) v(\theta) + (\Pi_{\mathcal{H}}(\theta) - \lambda) D_\sigma v(\theta) \end{aligned}$$

and in particular

$$\begin{aligned} 0 &= \langle \pi(0), D_\sigma(\Pi_{\mathcal{H}} v - \lambda v)(0) \\ &= \langle \pi(0), D_\sigma(\Pi_{\mathcal{H}} - \lambda)(0)\mathbb{1}\rangle + \langle \pi(0), (\Pi_{\mathcal{H}}(0) - 1) D_\sigma v(0)\rangle \\ &= \langle \pi(0), D_\sigma(\Pi_{\mathcal{H}} - \lambda)(0)\mathbb{1}\rangle \ \because\ \Pi_{\mathcal{H}}(0)^* \pi(0) = \pi(0). \end{aligned}$$

Thus

$$D_\sigma \lambda(0) = \langle \pi(0), D_\sigma(\Pi_{\mathcal{H}})(0)\mathbb{1}\rangle = 0 \quad \text{by} \quad ¶1. \quad ☑ \quad \text{(i)}$$

¶2 $D_\sigma v(0) = 0$.

Proof Differentiating $\Pi_{\mathcal{H}}(\theta) v(\theta) = \lambda(\theta) v(\theta)$ at 0:

$$D_\sigma(\Pi_{\mathcal{H}})\mathbb{1} + \Pi_{\mathcal{H}}(0) D_\sigma v(0) = D_\sigma \lambda(0)\mathbb{1} + D_\sigma v(0).$$

By ¶1 & (i),

$$\Pi_{\mathcal{H}}(0) D_\sigma v(0) = D_\sigma v(0).$$

Thus $D_\sigma v(0) \propto \mathbb{1}$. But $D_\sigma v(0) \perp \mathbb{1}$. ☑ ¶2

¶3 $D_\sigma^2(\lambda)(0) = \langle \pi(0), D_\sigma^2(\Pi_{\mathcal{H}})(0)\mathbb{1}\rangle$.

Proof Differentiating $\Pi_{\mathcal{H}}(\theta) v(\theta) = \lambda(\theta) v(\theta)$ twice at 0 in direction σ:

$$\begin{aligned} &D_\sigma^2(\Pi_{\mathcal{H}})(0)\mathbb{1} + 2D_\sigma(\Pi_{\mathcal{H}})(0) D_\sigma v(0) + \Pi_{\mathcal{H}}(0) D_\sigma^2 v(0) \\ &\qquad = D_\sigma^2(\lambda)(0)\mathbb{1} + 2D_\sigma(\lambda)(0) D_\sigma v(0) + D_\sigma^2 v(0). \end{aligned}$$

By (i) & ¶2

$$D_\sigma^2(\Pi_{\mathcal{H}})(0)\mathbb{1} + \Pi_{\mathcal{H}}(0) D_\sigma^2 v(0) = D_\sigma^2(\lambda)(0)\mathbb{1} + D_\sigma^2 v(0)$$

and in particular

$$\begin{aligned} D_\sigma^2(\lambda)(0) &= \langle \pi(0), D_\sigma^2(\lambda)(0)\mathbb{1}\rangle \\ &= \langle \pi(0), D_\sigma^2(\Pi_{\mathcal{H}})(0)\mathbb{1}\rangle + \langle \pi(0), \Pi_{\mathcal{H}}(0) D_\sigma^2 v(0)\rangle - \langle \pi(0), D_\sigma^2 v(0)\rangle \\ &= \langle \pi(0), D_\sigma^2(\Pi_{\mathcal{H}})(0)\mathbb{1}\rangle \quad \because\ \Pi_{\mathcal{H}}(0)^* \pi(0) = \pi(0). \quad ☑ \quad ¶3 \end{aligned}$$

Proof of (ii) For fixed $s \in S$:

$$\begin{aligned}(D_\sigma^2(\Pi_{\mathcal{H}})(0)\mathbb{1})_s &= -\sum_{t\in S} P(\mathcal{L}_s(\mathcal{H}) = t) E(\langle \sigma, b_s(\mathcal{H})\rangle^2 | \mathcal{L}_s(\mathcal{H}) = t) \\ &= -E(\langle \sigma, b_s(\mathcal{H})\rangle^2)\end{aligned}$$

whence by ¶3,

$$\begin{aligned}D_\sigma^2(\lambda)(0) &= \langle \pi(0), D_\sigma^2(\Pi_{\mathcal{H}})(0)\mathbb{1}\rangle \\ &= -\sum_{s\in S} \pi_s(0) E(\langle \sigma, b_s(\mathcal{H})\rangle^2) \ \le 0\end{aligned}$$

with equality iff $\langle \sigma, b_s(\mathcal{H})\rangle = 0 \ \forall \ s \in S$.

Next recall that $Z^{(n)} := \mathcal{H}_1^n(0) = X^{(K+Ln)} - \mu n + O(1)$.

If $\langle \sigma, b_s(\mathcal{H})\rangle = 0 \ \forall \ s \in S$,then, taking $s = (0,0)$ we have

$$\sup_n |\langle \sigma, Z_s^{(n)}\rangle| < \infty \implies | \ \sup_n |\langle \sigma, X_s^{(K+Ln)} - \mu_s n\rangle| =: W < \infty$$

whence

$$|\langle \sigma, \varphi_j(0) - \mu_s n\rangle| \le W \ \forall \ n \ge 1, \ 1 \le j \le \ell_{K+Ln}(0).$$

It follows from this that $\mu_s = 0$ and that $\langle \sigma, \varphi\rangle$ is a coboundary. This contradicts theorem 1. ☑ (ii) $\square$

Proof of theorem 2 It suffices to prove that for fixed $s \in S, \ \theta \in \mathbb{R}^d$

$$(🚂) \qquad E(\exp[i\langle \theta, \tfrac{X_s^{(K+Ln)} - n\mu_s}{\sqrt{n}}]) \xrightarrow[n\to\infty]{} \exp[-\tfrac{i\langle \theta, D\theta\rangle}{2}].$$

By asymptotic, eventual periodicity $\exists \ \rho \in (0,1)$ so that for any fixed $J, \ s \in S, \ \theta \in \mathbb{R}^d, \ \exists \ M = M_J > 1$ so that $\forall \ n \ge 1$,

$$\begin{aligned}E(\exp[i\langle \theta, \tfrac{X_s^{(K+L(J+n))} - (J+n)\mu_s}{\sqrt{n}}\rangle]) &= E(\exp[i\langle \theta, \tfrac{(Z_J{}^{(n)})_s}{\sqrt{n}}\rangle]) \pm M\rho^J \\ &= \lambda(\tfrac{\theta}{\sqrt{n}})^n E(\exp[i\langle \theta, \tfrac{(X_s^{(K+LJ)} - J\mu_s}{\sqrt{n}}\rangle]) \pm M\rho^J.\end{aligned}$$

Now

$$\lambda(\tfrac{\theta}{\sqrt{n}})^n E(\exp[i\langle \theta, \tfrac{(X_s^{(K+LJ)} - J\mu_s}{\sqrt{n}}\rangle]) \xrightarrow[n\to\infty]{} \ \exp[-\tfrac{i\langle \theta, D\theta\rangle}{2}] \ \forall \ J \ge 1.$$

To deduce (🚂) from this, let $\epsilon > 0$ and choose $J = J_\epsilon \ge 1$ so that

$$|E(\exp[i\langle \theta, \tfrac{X_s^{(K+L(J+n))} - (J+n)\mu_s}{\sqrt{n}}\rangle]) - \lambda(\tfrac{\theta}{\sqrt{n}})^n E(\exp[i\langle \theta, \tfrac{(X_s^{(K+LJ)} - J\mu_s}{\sqrt{n}}\rangle])| < \frac{\epsilon}{2} \ \forall \ n \ge J$$

and then choose $N = N_{J,\epsilon} > J$ so that

$$|\lambda(\tfrac{\theta}{\sqrt{n}})^n E(\exp[i\langle \theta, \tfrac{(X_s^{(K+LJ)} - J\mu_s}{\sqrt{n}}\rangle]) - \exp[-\tfrac{i\langle \theta, D\theta\rangle}{2}]| < \frac{\epsilon}{2} \ \forall \ n > N.$$

This implies (🚂). $\square$

As in [1], the norm of a matrix $A \in M_{S\times S}$ is given by

$$\|A\| := \sup\{\|Ax\|_\infty :\ x \in \mathbb{R}^S,\ \|x\|_\infty = 1\}$$

where $\|(x_s :\ s \in S)\|_\infty := \sup_{s\in S} |x_s|$.

Adaptedness.

We'll call the $\mathtt{RAT}$ $F \in \mathtt{RV}(M_{S\times S}(\mathbb{R}) \times (\mathbb{R}^d)^S)$ *adapted* if $\exists$ a discrete subgroup $\Gamma = \Gamma_F \le \mathbb{R}^d$ (called the *adaptivity group*) so that

$$\theta \in \mathbb{R}^d \ \&\ \|\Gamma_F(\theta)\| = 1 \implies \theta \in \Gamma.$$

Equivalently, for some $r > 0$,

$$\|\Gamma_F(\theta)\| < 1\ \forall\ \theta \in B(0,r) \smallsetminus \{0\}.$$

Now, writing $F = (\mathcal{L}, W) \in \mathtt{RV}\,(S^S, (\mathbb{R}^d)^{S\times S})$, we have as $\theta \to 0$

$$\begin{aligned}\|\Gamma_F(\theta)\| &= \max_{s\in S} \sum_{t\in S} P(\mathcal{L}_s = t)|E(e^{i\theta\cdot W_{s,t}})| \\ &= 1 - \min_{s\in S} \sum_{t\in S} P(\mathcal{L}_s = t)\langle \mathtt{Cov}\,(W_{s,t})\theta, \theta\rangle + o(\|\theta\|)\end{aligned}$$

where for $V = (V_1, V_2, \ldots, V_d)$ a $\mathbb{R}^d$-valued L^2 random variable, the *covariance matrix* $\mathtt{Cov}\,(V) \in M_{d\times d}(\mathbb{R})$ is defined by

$$\mathtt{Cov}\,(V)_{k,\ell} := E(V_k - E(V_k))(V_\ell - E(V_\ell)).$$

A covariance matrix is *non-negative definite* in the sense that

$$\langle \mathtt{Cov}\,(V)\theta, \theta\rangle = E\left((\sum_{k=1}^d \theta_k (V_k - E(V_k)))^2 \right) \ge 0\ \forall\ \theta \in \mathbb{R}^d$$

and is called *positive definite* if it is invertible. Equivalently, for some $\epsilon > 0$ (the minimum eigenvalue modulus)

$$\langle \mathtt{Cov}\,(V)\theta, \theta\rangle \ge \epsilon\|\theta\|^2\ \ \forall\ \theta \in \mathbb{R}^d.$$

Thus, F is adapted iff

(📐) $\forall\ s \in S$, $\exists\ t \in S$ such that $P(\mathcal{L}_s = t) > 0$ &
$\mathtt{Cov}\,(W_{s,t})$ is strictly positive definite.

It follows as in [1] that if F is adapted, then $\forall\ \epsilon > 0$ & $M > 0$, $\exists\ \delta > 0$ so that

$$\|\Gamma_F(\theta)\| \le 1 - \delta\ \forall\quad n \ge 1 \ \text{ and } \theta \in B(0, M) \smallsetminus B(\Gamma, \epsilon).$$

The following lemma gives a sequence version of adaptedness similar to that in [1].

Adaptedness lemma *For $N,\ J,\ M \geq 1$ large, $\exists$ a discrete subgroup $\Gamma \leq \mathbb{R}^d$ and $\epsilon,\ b,\ c > 0,\ r \in (0,1)$ so that*

(i) $\|\Pi_{\mathcal{E}_n}(\theta+\gamma)\| \leq 1 - c\|\theta\|^2\ \forall\ \gamma \in \Gamma \cap B(0,M),\ \theta \in B(0,r);$

(ii) $\|\Pi_{\mathcal{E}_n}(\theta)\| \leq 1 - \epsilon\ \forall\ \theta \in B(0,M) \smallsetminus\ B(\Gamma,r);$

(iii) $\langle \mathtt{Cov}\,(W_{s,t}(\mathcal{E}_n))\theta,\theta\rangle \geq \epsilon\|\theta\|^2\ \forall\ \theta \in \mathbb{R}^d;$

where $\mathcal{E}_n := \mathcal{F}_{K+L(J+n)+1}^{K+L(J+n+1)}$.

The proof is in a series of steps, the first two of which are as in [16].

¶1 If $\theta \in \mathbb{R}^d,\ \xi \in \mathbb{C}\ \&\ v \in \mathbb{C}^S$ satisfy $\Pi_{\mathcal{H}}(\theta)v = \xi v$, then $|\xi| \leq 1$ with equality iff

(a) $v \in (\mathbb{S}^1)^S\ \&\quad (\Pi_{\mathcal{H}}(\theta))_{s,t} = \xi v_s \overline{v}_t (\Pi_{\mathcal{H}}(0))_{s,t}\ \ \forall\ s,t \in S.$

Proof Write $\Pi := \Pi_{\mathcal{H}}(\theta)\ \&\ P := \Pi_{\mathcal{H}}(0)$. Evidently (a) $\Longrightarrow\ \Pi v = \xi v$.

Now suppose that $\Pi v = \xi v$ with $J \in S,\ |v_J| = \|v\|_\infty = 1$, then

$$\begin{aligned}|\xi| = |\xi v_J| &= \left|\sum_{t\in S}\Pi_{J,t}v_t\right| \leq \sum_{t\in S}|\Pi_{J,t}|\,|v_t| \\ &\leq \sum_{t\in S}P_{J,t}\,|v_t| \leq 1.\end{aligned}$$

If $|\xi| = 1$, then $v \in (\mathbb{S}^1)^S$ and

$$\left|\sum_{t\in S}\Pi_{s,t}v_t\right| = 1\ \forall\ s \in S$$

and $\exists\ z \in (\mathbb{S}^1)^S$ so that

$$\Pi_{s,t}v_t = z_s P_{s,t}\ \forall\ s,t \in S.$$

Thus, for $s \in S$,

$$\xi v_s = \sum_{t\in S}\Pi_{s,t}v_t = \sum_{t\in S} z_s P_{s,t} = z_s$$

which is (a). ☑

Next, for $\theta \in \mathbb{R}^d$, let

$$\mu(\theta) := \max\{|\xi| :\ \xi \in \mathbb{C}\ \&\ \exists\ v \in \mathbb{C}^S,\ \Pi_{\mathcal{H}}(\theta) = \xi v\}.$$

By ¶1, $\mu(\theta) \leq 1$. Set $\Gamma := \{\gamma \in \mathbb{R}^d :\ \mu(\gamma) = 1\}$.

¶2 Γ is a discrete subgroup of $\mathbb{R}^d$ and

(b) $\mu(\gamma+\theta) = |\lambda(\theta)|\ \forall\ \gamma \in \Gamma,\ \|\theta\|_\infty \leq r_{\mathcal{H}}.$

Proof Since $P(\mathcal{L}_s = t) > 0\ \forall\ s,t \in S$,

$$\Gamma = \{\gamma \in \mathbb{R}^d :\ \exists\ x \in \mathbb{R},\ \gamma \cdot W_{s,t} + x \in 2\pi\mathbb{Z}\text{ a.s. }\forall\ s,t \in S\}$$

which is evidently a subgroup of $\mathbb{R}^d$.

Now suppose that $\gamma \in \Gamma$ with $\Pi_{\mathcal{H}}(\gamma)v = \xi v$ where $|v_s| = |\xi| = 1 \ \forall \ s \in S$. By ¶1,

$$(\Pi_{\mathcal{H}}(\theta))_{s,t} = \xi v_s \overline{v}_t (\Pi_{\mathcal{H}}(0))_{s,t} \quad \forall \ s,t \in S.$$

Equivalently $\forall \ s,t \in S$,

$$E(e^{i\gamma \cdot W_{s,t}}) = \xi v_s \overline{v}_t \implies E(e^{i(\gamma+\theta)\cdot W_{s,t}}) = \xi v_s \overline{v}_t E(e^{i\theta \cdot W_{s,t}}) \ \forall \ \theta \in \mathbb{R}^d$$

whence

$$(\Pi_{\mathcal{H}}(\theta + \gamma))_{s,t} = \xi v_s \overline{v}_t (\Pi_{\mathcal{H}}(\theta))_{s,t} \quad \forall \ s,t \in S, \ \theta \in \mathbb{R}^d.$$

Statement (b) follows from this, whence ¶2 via the Taylor expansion of λ. ☑

¶3 For $N \geq 1$ sufficiently large, $\mathcal{H}_1^N$ is an adapted `RAT` with adaptivity group Γ.

Proof Fix $0 < q < 1$ and $N \geq 1$ so that

$$\|Q(\theta)^N\| < q \text{ for } \theta \in B(0, r_{\mathcal{H}}) \text{ \& hence for } \theta \in B(\Gamma, r_{\mathcal{H}}); \ \&$$
$$\mu(\theta)^N < q \ \forall \ \theta \in \mathbb{R}^d \setminus B(\Gamma, r_{\mathcal{H}}).$$

It follows that $\mathcal{H}_1^N$ is adapted with adaptivity group Γ. ☑

To complete the proof of the lemma, fix $J \geq 1$ and let

$$(\mathcal{E}_n := \mathcal{F}_{J+Nn+1}^{J+N(n+1)} : \ n \geq 1).$$

Statements (i) and (ii) follow because for each $M > 0$,

$$\sup_{\|\theta\| \leq M} \|\Pi_{\mathcal{E}_n}(\theta) - \Pi_{\mathcal{H}_1^N}(\theta)\| \xrightarrow[n\to\infty]{} 0$$

and statement (iii) follows from

$$W_{s,t}(\mathcal{E}_n) \xrightarrow[n\to\infty]{\mathtt{RV}(\mathbb{R}^d)} W_{s,t}(\mathcal{H}_1^N) \ \forall \ s,t \in S. \ ☑$$

The `WRLLT` and theorem 3

To establish theorem 3, we use the

Weak, rough local limit theorem

For each $s \in S$ and $1 \leq p \leq 2$,

$$(\mathtt{WRLLT}) \qquad \int_{\mathbb{T}^d} |E(e^{i\langle \theta, X_s^{(J+Ln)}\rangle})|^p d\theta \asymp \frac{1}{n^{\frac{d}{2}}}.$$

The proof is a multidimensional version of the proof of the `WRLLT` in [1].

Proof of $\gg$

By theorem 6.1 in [1], for each $1 \le k \le d$, we have

$$\sum_{\nu=1}^{n} \min_{s \in S} E([b_s(\mathcal{E}_\nu))_k]^2) \le E([(X_s^{(J+Ln)})_k]^2) \le \sum_{\nu=1}^{n} \max_{s \in S} E([b_s(\mathcal{E}_\nu))_k]^2).$$

Thus, by the Adaptedness lemma,$\exists\ G > 0$ so that

$$E(\|X_s^{(J+Ln)}\|^2) \le Gn.$$

Next, fix $M = 2\sqrt{G}$, then by Chebyshev's inequality,

$$P([\|X_s^{(J+Ln)}\| \le M\sqrt{n}]) \ge \frac{3}{4}.$$

Now fix $\Delta > 0$ so that

$$|1 - e^{ix}| < \frac{1}{4}\ \forall\ |x| < \Delta.$$

We have

$$\begin{aligned} n^{\frac{d}{2}} \int_{[-\pi,\pi]^d} |E(e^{i\langle \theta, X_s^{(J+Ln)}\rangle})|^2 d\theta &= \int_{[-\pi\sqrt{n},\pi\sqrt{n}]^d} |E(\exp[i\langle t, \frac{X_s^{(J+Ln)}}{\sqrt{n}}\rangle])|^2 dt \\ &\ge \int_{[-\frac{\Delta}{M},\frac{\Delta}{M}]^d} |E(\exp[i\langle t, \frac{X_s^{(J+Ln)}}{\sqrt{n}}\rangle])|^2 dt. \end{aligned}$$

For $\|t\| < \frac{\Delta}{M}$, we have

$$\begin{aligned} &|E(\exp[i\langle t, \frac{X_s^{(J+Ln)}}{\sqrt{n}}\rangle])| \\ &\qquad \ge |E(\exp[i\langle t, \frac{X_s^{(J+Ln)}}{\sqrt{n}}\rangle]) 1_{[\|X_s^{(J+Ln)}\| < M\sqrt{n}]})| - P([\|X_s^{(J+Ln)}\| \ge M\sqrt{n}] \\ &\qquad \ge \frac{3}{4}\cdot\frac{3}{4} - \frac{1}{4} \\ &\qquad = \frac{5}{16} \end{aligned}$$

whence

$$\begin{aligned} n^{\frac{d}{2}} \int_{[-\pi,\pi]^d} |E(e^{i\langle \theta, X_s^{(J+Ln)}\rangle})|^2 d\theta &\ge \int_{[-\frac{\Delta}{M},\frac{\Delta}{M}]^d} |E(\exp[i\langle t, \frac{X_s^{(J+Ln)}}{\sqrt{n}}\rangle])|^2 dt \\ &\ge \left(\frac{2\Delta}{M}\right)^d \cdot \frac{25}{256}. \quad ☑ \quad \gg \end{aligned}$$

Proof of $\ll$

We have

$$\begin{aligned}|E(e^{i\langle\theta,X_s^{(J+Ln)}\rangle})| &= |(\Pi_{\mathcal{E}_n}(\theta)\Pi_{\mathcal{E}_{n-1}}(\theta)\cdots\Pi_{\mathcal{E}_1}(\theta)\widehat{\Xi}_J(\theta))_s| \\ &\le \prod_{k=1}^n \|\Pi_{\mathcal{E}_k}(\theta)\|.\end{aligned}$$

Fix $M > 0$ so that $[-\pi,\pi]^d \subset B(0,M)$. By the Adaptedness lemma, for $n \ge 1$, $\gamma \in \Gamma$ we have

$$\|\Pi_{\mathcal{E}_n}(\gamma+\theta)\| \le 1 - c\|\theta\|^2 \quad \forall\ |\theta| < r$$

and

$$\|\Pi_{\mathcal{E}_n}(\theta)\| \le 1 - \epsilon \quad \forall\ \theta \in B(0,M) \setminus B(\Gamma, r).$$

$$\begin{aligned}&\int_{[-\pi,\pi]^d} |E(e^{i\langle\theta,X_s^{(J+Ln)}\rangle})|d\theta \\ &\le \Big(\int_{B(0,M)\cap B(\Gamma,r)} + \int_{B(0,M)\setminus B(\Gamma,r)}\Big) \prod_{k=1}^n \|\Pi_{\mathcal{E}_k}(\theta)\| d\theta \\ &\le \sum_{\gamma\in B(0,M)\cap\Gamma} \int_{B(0,r)} \prod_{k=1}^n \|\Pi_{\mathcal{E}_k}(\gamma+\theta)\| d\theta + \int_{B(0,M)\setminus B(\Gamma,r)} \prod_{k=1}^n \|\Pi_{\mathcal{E}_k}(\theta)\| d\theta \\ &\le \# B(0,M)\cap\Gamma \int_{B(0,r)} (1-c\|\theta\|^2)^n d\theta + O((1-\epsilon)^n) \\ &\ll \frac{1}{n^{\frac{d}{2}}}. \quad ☑ \ \ll \ \& \ \texttt{WRLLT}\end{aligned}$$

Proof of theorem 3

Set $\nu_k(i) := \ell_{K+Lk}(i) \quad (i = 1,2)$. The Visit lemma and the WRLLT show that

$$\begin{aligned}S_{\nu_k(i)}(1_{\mathbb{T}^d\times\{0\}}) &\ll \nu_k(0)\int_{\mathbb{T}^d} |E(e^{i\langle\theta,X_{(0,i)}^{(J+Lk)}\rangle})|d\theta \\ &\asymp \frac{\nu_k(0)}{k^{\frac{d}{2}}} \ll \nu_k(0)\int_{\mathbb{T}^d} |E(e^{i\langle\theta,X_s^{(J+Lk)}\rangle})|^2 d\theta \\ &\asymp \int_{\mathbb{T}^d\times\{0\}} S_{\nu_k(i)}(1_{\mathbb{T}^d\times\{0\}})dm \\ &\ll \frac{\nu_k(0)}{k^{\frac{d}{2}}}\end{aligned}$$

Next, $\exists\ \Lambda > 1$ so that $\nu_k(i) \propto \Lambda^k (i = 0,1)$ whence for

$\nu_k(0) \le n \le \nu_{k+1}(0)$,

$$\begin{aligned}\int_{\mathbb{T}^d} S_n(1_{\mathbb{T}^d\times\{0\}})(x,0)dx &\ge \int_{\mathbb{T}^d} S_{\nu_k(0)}(1_{\mathbb{T}^d\times\{0\}})(x,0)dx \\ &\gg \frac{\nu_k(1)}{k^{\frac{d}{2}}} \gg \frac{\nu_{k+1}(0)}{k^{\frac{d}{2}}} \\ &\gg \frac{n}{(\log n)^{\frac{d}{2}}}\end{aligned}$$

and for $\nu_k(1) \le n \le \nu_{k+1}(1)$,

$$\|S_n(1_{\mathbb{T}^d\times\{0\}})\|_{L^\infty(\mathbb{T}^d\times\{0\})} \ll \frac{\nu_{k+1}(1)}{k^{\frac{d}{2}}} \ll \frac{\nu_k(0)}{k^{\frac{d}{2}}} \ll \frac{n}{(\log n)^{\frac{d}{2}}}. \quad ☑$$

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