Four-Cycle Free Graphs and Entropy Minimality

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Outline

- Entropy Minimality and Hom Shifts
- Mixing Conditions and Entropy Minimality
- Measures of Maximal Entropy
- Rigidity and Flexibility in the Space of 3-Colourings.

Nearest Neighbour Shifts of Finite Type

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$\mathbb{Z}^d$ acts by translations(shifts) on the shift spaces.

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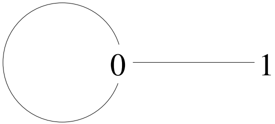
Examples:(Hard Square model)



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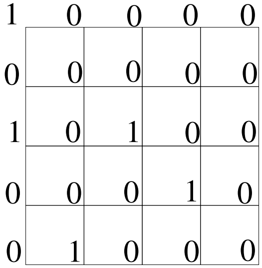

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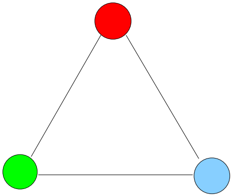

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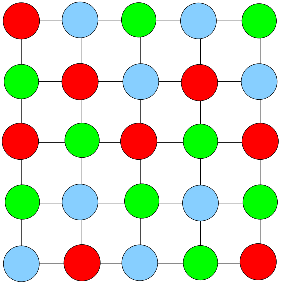

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$$h(X) := \lim_{n \longrightarrow \infty} \frac{\log \left|\mathcal{B}(X) \cap \mathcal{A}^{\{1,2,\ldots,n\}^d}\right|}{n^d}.$$

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(Quas and Trow '00) Every shift space X contains an entropy minimal shift space $Y \subset X$ such that $h_{top}(X) = h_{top}(Y)$.

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Remark: We will concentrate on X_{C_3}, the space of all 3-colourings.

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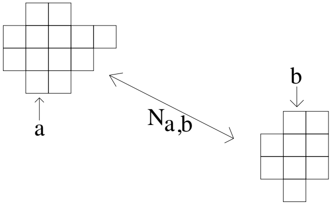

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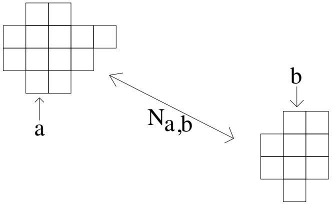

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(Coven and Smítal '93) If a shift space is entropy minimal then it is topologically transitive.

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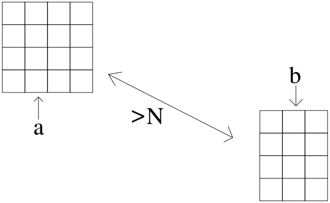

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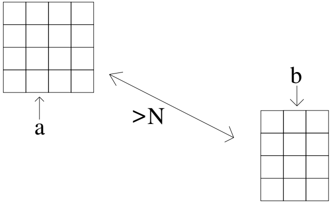

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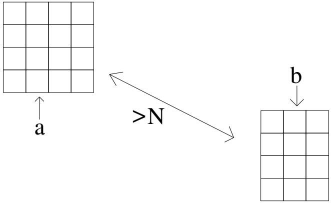

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(Boyle, Pavlov and Schraudner '09) There exists a block-gluing shift space which is not entropy minimal.

A Block-Gluing Shift Which Is Not Entropy Minimal

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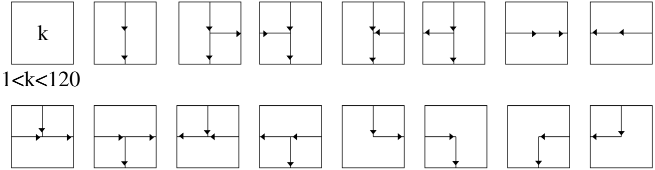

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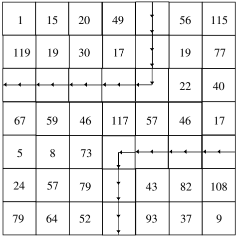

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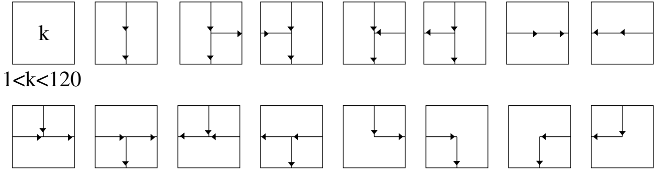

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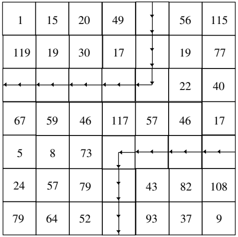

The symbols with arrows do not contribute any entropy.

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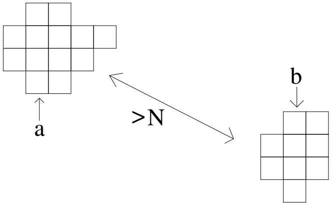

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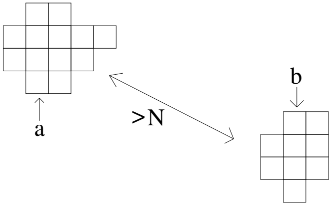

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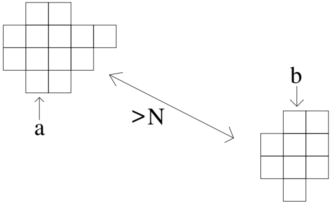

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(Schraudner '09) Every strongly irreducible shift space is entropy minimal.

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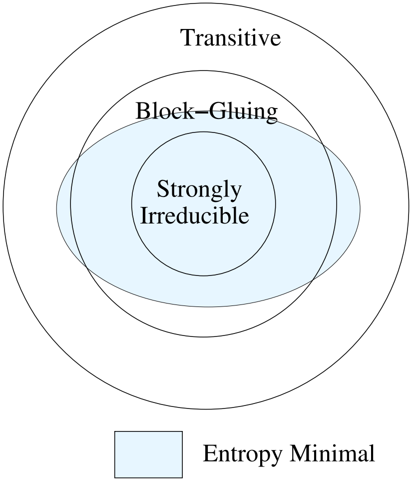

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If μ is a measure of maximal entropy for X then $h_\mu = h_{Top}(X)$. Therefore, a shift space X is entropy minimal if and only if for every measure of maximal entropy μ, $supp(\mu) = X$.

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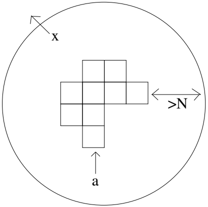

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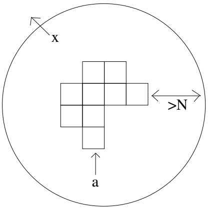

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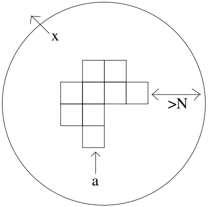

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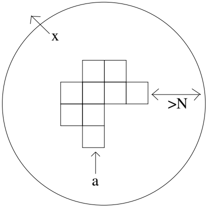

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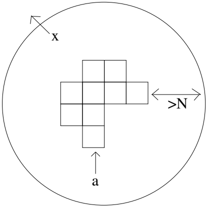
Strongly Irreducible Implies Entropy Minimal

(Meester and Steif '01) Every strongly irreducible shift of finite type is entropy minimal.
Proof: Suppose X is a strongly irreducible shift of finite type with a measure of maximal entropy μ. We want to prove $supp(\mu) = X$. Consider a pattern $a \in \mathcal{B}(X)$ and $x \in supp(\mu)$. Since X is strongly irreducible



is 'allowed' in X. Since μ is a measure of maximal entropy it is uniform Gibbs. Together with $x \in supp(\mu)$ we get $\mu([a]) > 0$. Therefore $supp(\mu) = X$. This completes the proof.

Mixing Properties of the Space of 3-colourings

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X_{C_3} is transitive but not block-gluing.

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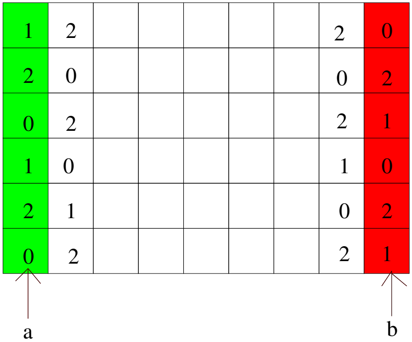

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2	0	2	0		1	2	0	2
0	2	0	2		2	0	2	1
1	0	2	0		1	2	1	0
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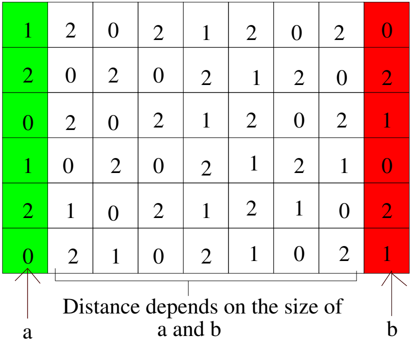

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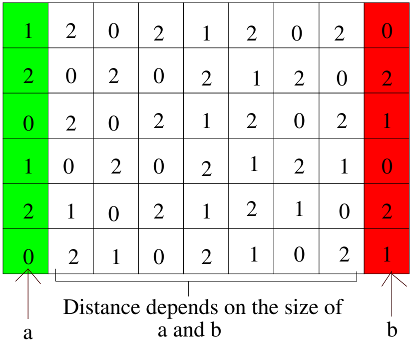

Yet, X_{C_3} is entropy minimal.

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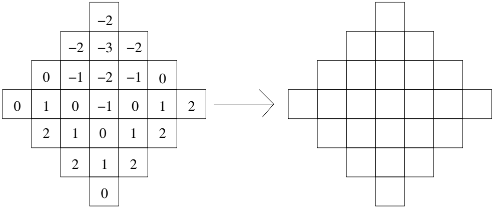

Height Function

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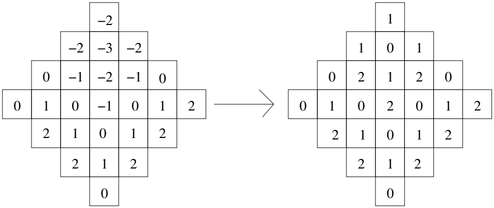

Height Function   Pattern in X_{C_3}

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Given an ergodic measure μ on X_{C_3}, the slope is a constant μ-almost everywhere in every direction.

Note that the slope may be different in different directions.

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2	0	1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0	1	2
2	0	1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0	1	2
2	0	1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0	1	2
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2	0	1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0	1	2
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If X_{frozen} is the space of such elements then

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If X_{frozen} is the space of such elements then $h_{Top}(X_{frozen}) = 0$.

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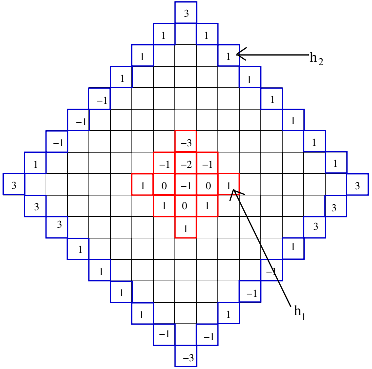

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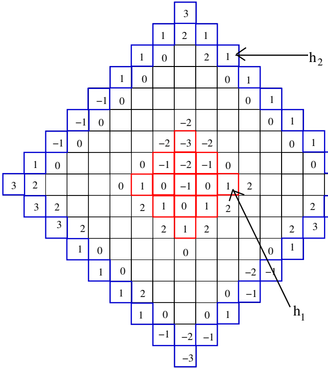

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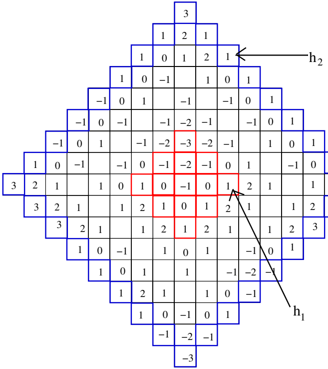

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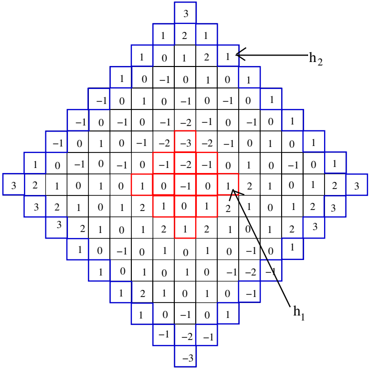

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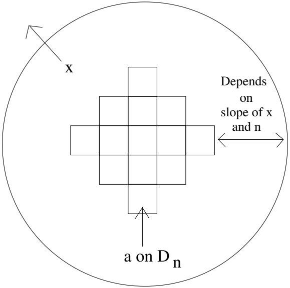

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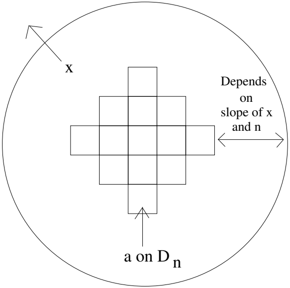

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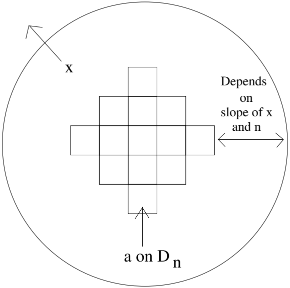

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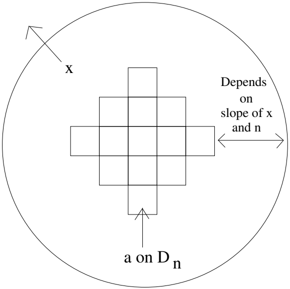

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If C_3 is replaced by a connected four-cycle free graph $\mathcal{H}$ then $\mathbb{Z}$ is replaced by the universal cover of $\mathcal{H}$.

Conjecture: If $\mathcal{H}$ is any connected graph then $X_{\mathcal{H}}$ is entropy minimal.(d=2)

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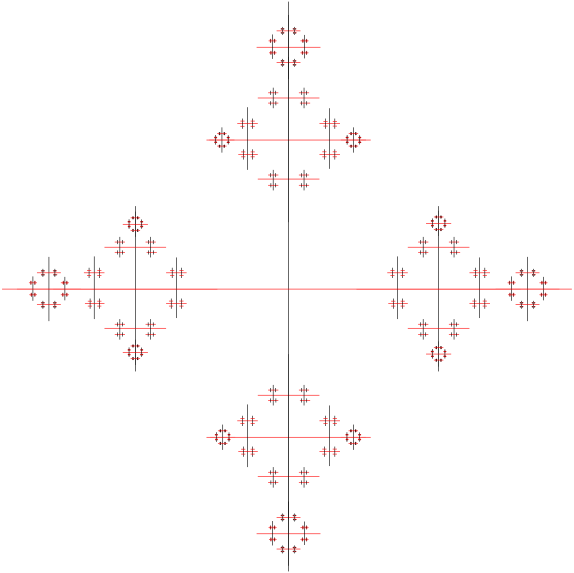
Question: What shift spaces are conjugate to $X_{\mathcal{H}}$ for some graph $\mathcal{H}$?



Thank You!