Four-Cycle Free Graphs and Entropy Minimality

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September, 2015

Outline

- Shifts of Finite Type
- Entropy
- Entropy Minimality
- The 3-coloured Chessboard.
- Universal Covers

Full Shift

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- The space $\mathfrak{A}^{\mathbb{Z}^d}$ is the set of all configurations on $\mathbb{Z}^d$. It is called the full shift.
- It is a compact space under the product topology with a natural $\mathbb{Z}^d$-action given by translations (also called shifts) of configurations.

Shift Spaces

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- A shift space is a subset $X \subset \mathfrak{A}^{\mathbb{Z}^d}$ such that there exists a forbidden list $\mathcal{F}$ satisfying $X = X_{\mathcal{F}}$.

Examples: Full Shift

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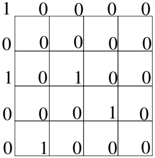
- $\mathfrak{A} = \{0, 1\}$, $\mathcal{F} = \emptyset$. $X_{\mathcal{F}} = \{0, 1\}^{\mathbb{Z}^d}$.

Examples: Hard Square Model

1	0	0	0	0
0	0	0	0	0
1	0	1	0	0
0	0	0	1	0
0	1	0	0	0

A Pattern

Examples: Hard Square Model



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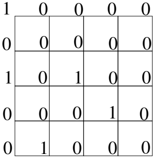

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- $\mathcal{F} = \{11, \begin{smallmatrix}1\\1\end{smallmatrix}\}$

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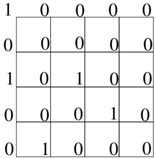

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- By $\mathbb{Z}^d$, we will refer to the Cayley graph of $\mathbb{Z}^d$ with standard generators. Thus $\mathbb{Z}^2$ is the grid.
- $X_{\mathcal{F}} =$ $\{$configurations in 0 and 1 where two $1's$ cannot be adjacent$\}$.

Examples: Even Shift

00001000010011000000100000000100000

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  $\{0, 1 \text{ sequences such that the gap between any two } 1's \text{ is even}\}$.

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- Note that $\mathcal{F}$ is infinite.

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- $X_{\mathcal{F}} =$ $\{0, 1 \text{ sequences such that the gap between any two } 1's \text{ is even}\}$.
- Note that $\mathcal{F}$ is infinite.
- It can be proved that $\mathcal{F}$ cannot be chosen to finite!

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- Both the hard square model and the full shift are shifts of finite type.
- The even shift is not a shift of finite type.

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  - $\mathcal{F}$ consists of patterns on edges of $\mathbb{Z}^d$.
- The hard square model is a nearest neighbour shift of finite type.

Examples: Non-Attacking Kings

1	0	1	0	0	0
0	0	0	0	1	0
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0	0	1	0	0	1
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- $X_{\mathcal{F}}$ is a shift of finite type but not at a nearest neighbour shift of finite type.
- Any shift of finite type can be recoded into a nearest neighbour shift of finite type (for a different alphabet).

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- It is decidable.

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- So we deal with a more restricted class of shift spaces.

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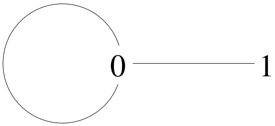
Examples: (Hard Square model)



Graph $\mathcal{H}$

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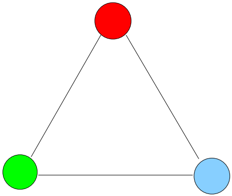
Examples: (3-colourings)



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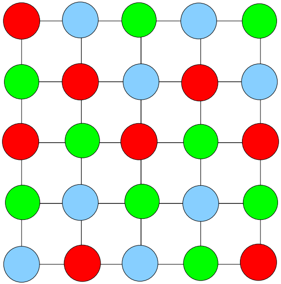

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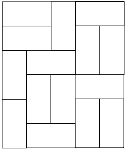
- Then $X_{\mathcal{H}} = X_{\mathcal{F}}$
- This is a nearest neighbour shift of finite type for which every direction has the same constraint.
- $X_{\mathcal{H}}$ is non-empty if and only if $\mathcal{H}$ has an edge.

Examples: Domino Tilings



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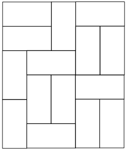

- What is the alphabet?

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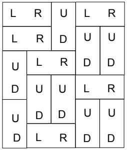

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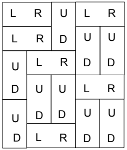

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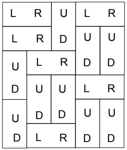

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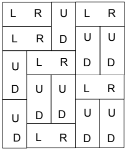

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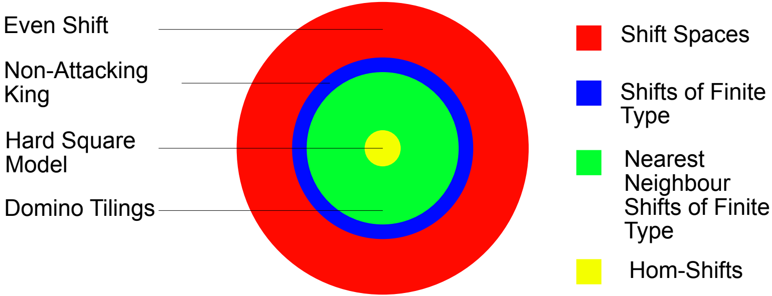
- The constraints in different directions are different. It is not a hom-shift.

Shift Spaces Schematic



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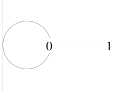

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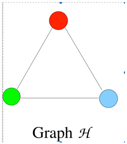
- The entropy of the hard square model is $\log\left(\frac{\sqrt{5}+1}{2}\right)$. Hence it is also called the golden mean shift.
- The entropy of the space of 3-colourings is $\log(2)$.



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- Closed forms are known for very few examples.

Entropy for Hom-Shifts

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- *(Friedlander, 1997)* There are approximating upper and lower bounds for the entropy of hom-shifts.

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- We just proved that if a shift space X which is not irreducible, then it is not entropy minimal.
- In fact it can be proved that a shift space is irreducible if and only if it is entropy minimal.

Entropy Minimality in Higher Dimensions

It is undecidable whether or not a nearest neighbour shift of finite type is entropy minimal.

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Theorem (Chandgotia '14)

If $\mathcal{H}$ is a four-cycle free graph then $X_{\mathcal{H}}$ is entropy minimal.

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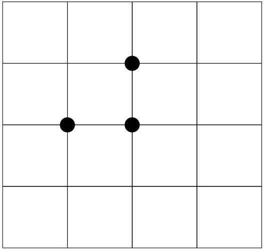

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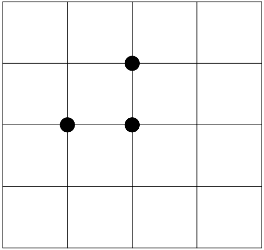

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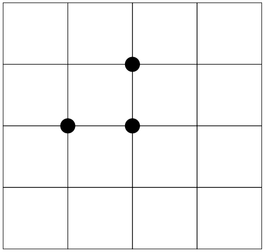

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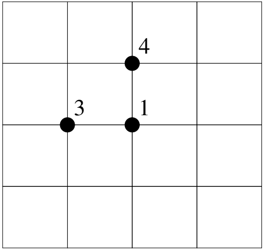
$$[a]_A \quad := \quad \{x \in X \mid x|_A = a\} \text{ (Cylinder set).}$$



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Cylinder set $[4,3,1]_A$

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Theorem

A shift space X is entropy minimal if and only if for every measure of maximal entropy μ, $supp(\mu) = X$.

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Entropy Minimality of X_{C_3}

The Cayley graph of $\mathbb{Z}/3\mathbb{Z}$ is C_3.

Let us see why X_{C_3} is entropy minimal.

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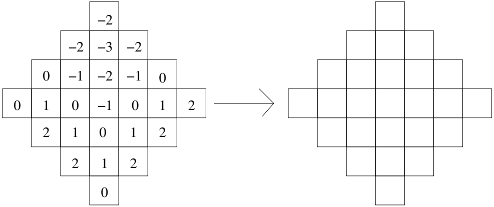

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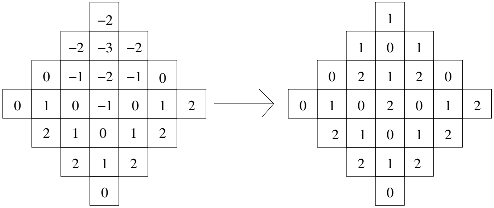

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Pattern in X_{C_3}

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- The slope may be different in different directions.

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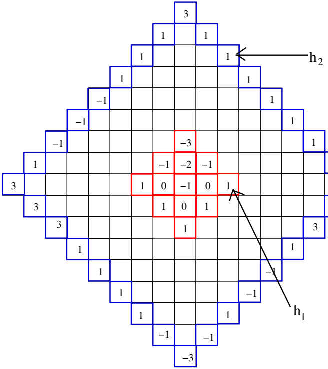

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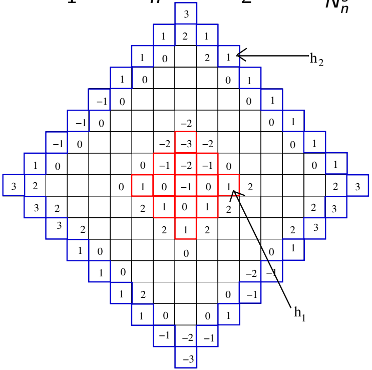

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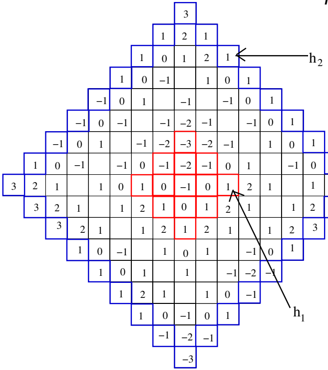

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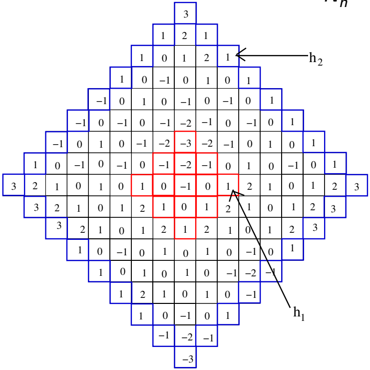

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Thus if μ is a uniform Gibbs measure with slope between -1 and 1 then $supp(\mu) = X_{C_3}$.

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If C_3 is replaced by a connected four-cycle free graph $\mathcal{H}$ then $\mathbb{Z}$ is replaced by the universal cover of $\mathcal{H}$.

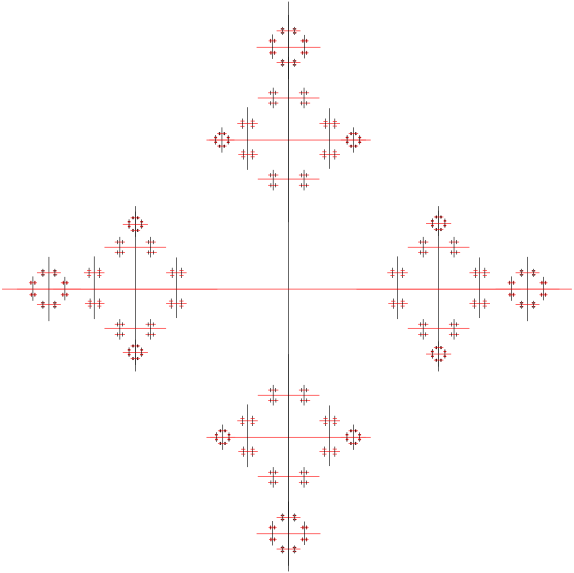
Four-Cycle Free Graphs

Notice that we were using the following commutative diagram:

$$\begin{tikzcd} & \mathbb{Z} \arrow[d, "\text{mod } 3"] \\ \mathbb{Z}^d \arrow[ur, "\textit{height function}"] \arrow[r] & C_3. \end{tikzcd}$$

If C_3 is replaced by a connected four-cycle free graph $\mathcal{H}$ then $\mathbb{Z}$ is replaced by the universal cover of $\mathcal{H}$.

Conjecture: For $d = 2$, $X_{\mathcal{H}}$ is entropy minimal for all connected graphs $\mathcal{H}$.



Thank You!