Many questions and a few answers about hom-shifts and rectangular tiling shifts in higher dimensions

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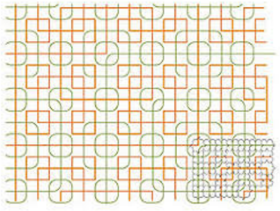
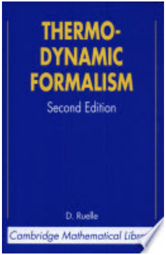
May, CMO

Motivation



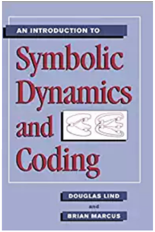


TILINGS, SUBSTITUTION SYSTEMS AND
DYNAMICAL SYSTEMS GENERATED BY THEM

By
SHAHAR MOZES

A characterization of the entropies of multidimensional shifts of finite type

By MICHAEL HOCHMAN and TOM MEYEROVITCH



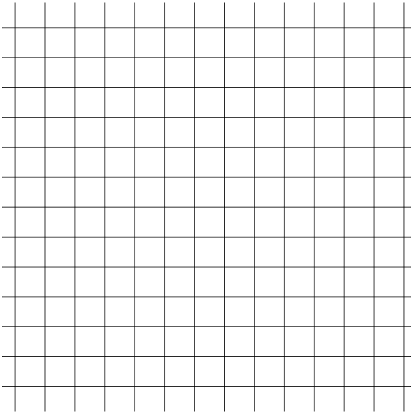
Basic Outline of the talk

Models in statistical physics are known for the simplicity of their description and intricacies of their questions. Inspired from such models we will discuss some very simple (to describe) shifts of finite type in higher dimensions.

1. Graph homomorphisms
    1. Mixing properties
    2. Universality
    3. Reflection positivity
    4. Measures of maximal entropy
2. Tilings
    1. Mixing properties
    2. The special case of domino tilings

The Cayley graph structure

We will fix the Cayley structure on $\mathbb{Z}^d$ given by the standard generators. All graphs in this talk will be undirected.



Hom-Shifts

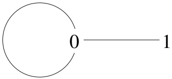
Graph homomorphisms and hom-shifts

A graph homomorphism is a map between graphs which preserves adjacencies. Given a graph H, a hom-shift associated with H is the space

$$X_H^d = \{\text{graph homomorphisms from } \mathbb{Z}^d \text{ to } H\}.$$

Some examples: The hard-core shift

The symbols are 0 and 1. Adjacent symbols in this shift cannot be both 1. This is the space X_H^d where H is given by the graph below.



Graph $\mathcal{H}$

1	0	0	0	0
0	0	0	0	0
1	0	1	0	0
0	0	0	1	0
0	1	0	0	0

Some examples: Proper q-colorings

The symbols are $1, 2, 3, \ldots, q$. Adjacent symbols in this shift are distinct. This is the space $X^d_{K_q}$ where K_q is the complete graph on q vertices.

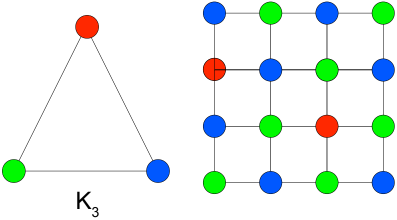
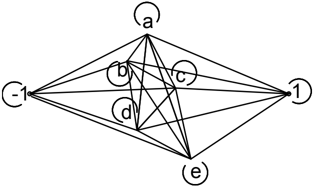


Figure : Proper 3-colorings

Some examples: Iceberg Model

The symbols are $-M, -M+1, \ldots, -1, 1, \ldots, M$ and a, b, c, d, e. The only restriction is that the positives cannot sit next to the negatives.

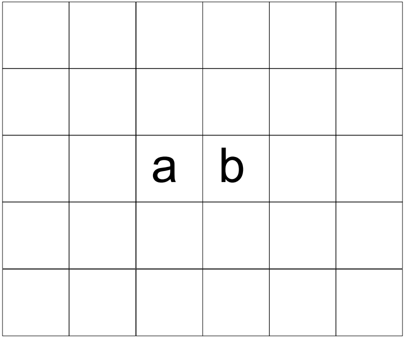


-1	-1	-1	a	b	c	a	e
b	c	b	e	a	d	a	b
b	a	e	a	1	b	a	e
a	b	1	1	1	d	d	e
b	a	c	c	c	e	a	d
e	e	b	c	-1	e	1	d
d	c	a	e	d	b	c	d

Figure : Iceberg model for $M = 1$

Some features of hom-shifts

There are numerous features of hom-shifts which make them more tractable. If symbol a can sit next to b in some direction then it can sit next to a in all directions.



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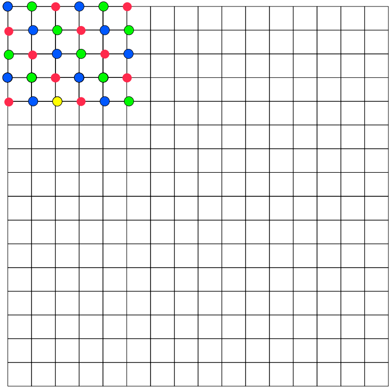
a	b	a	b	a	b
b	a	b	a	b	a
a	b	a	b	a	b
b	a	b	a	b	a
a	b	a	b	a	b

Some features of hom-shifts

Thus given any pattern from a hom-shift, its reflection is also a valid pattern. By multiple reflections of patterns on a rectangular box we get periodic configurations. Thus periodic configurations are dense.



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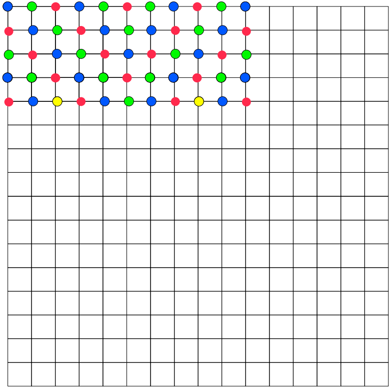

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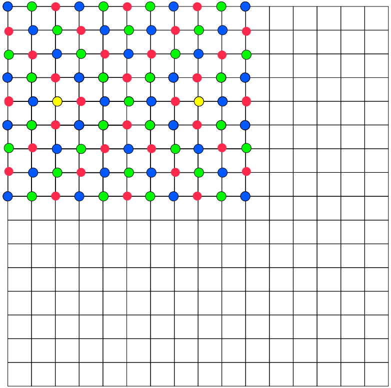

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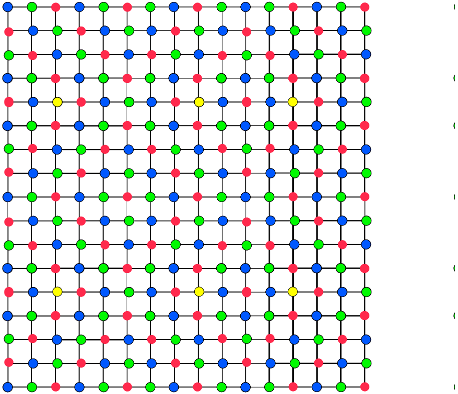


Let us look at some more subtle properties.

Topological mixing and strong irreducibility

Recall that a shift space is topologically mixing if for any two patterns a and b in the shift space there exists an integer $N_{a,b}$ such that copies of a and b which are separated by distance greater than $N_{a,b}$ can be extended to a configuration in the shift space.

A shift space is strongly irreducible if there is an integer N such that any two patterns a and b in the shift space separated by distance greater than N can be extended to a configuration in the shift space.

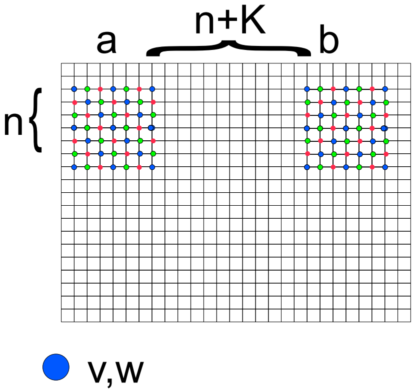
Topological mixing for hom-shifts

If H is a connected graph which is not bipartite then X_H^d is topologically mixing. Let us see why this is true.

There exists $k \in \mathbb{N}$ such that for any two vertices v and w and $K \geq k$, there is a path from v to w of exactly length K.

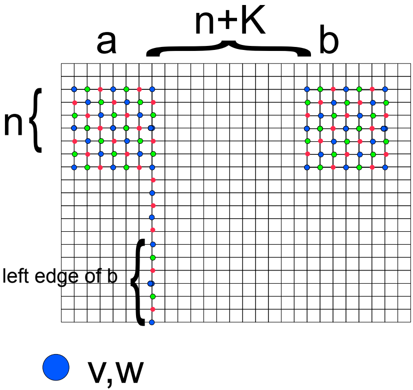
Now take any two patterns a and b from the hom-shift of size n which are separated by distance exactly $n + K$ where $K \geq k$.

Topological mixing for hom-shifts



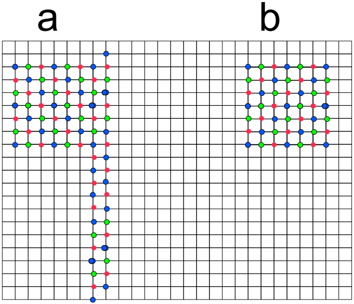
Mark the bottom right symbol of a as v and the top left symbol of b as w.

Topological mixing for hom-shifts

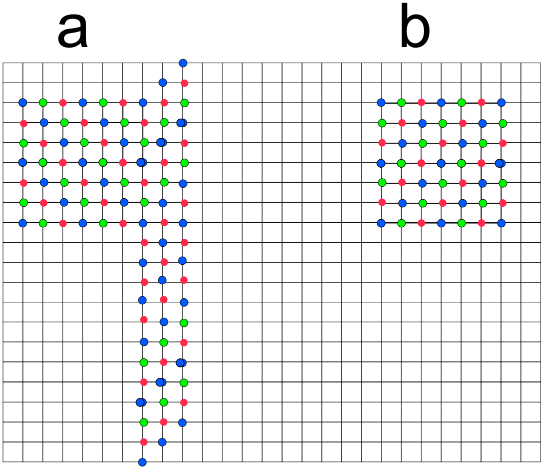


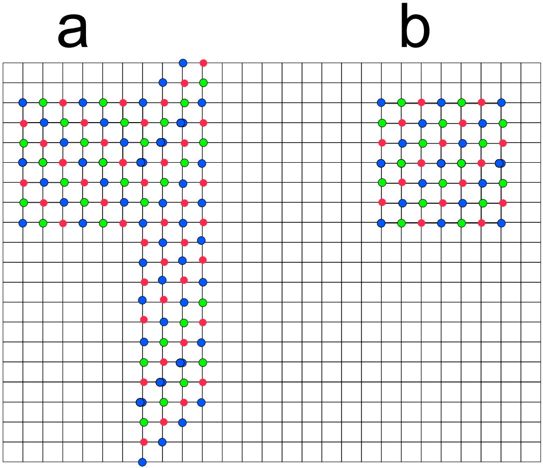
There is a path from v to w of length K. Extend the right edge of a by this path and follow it by the pattern on the left edge of b.

Topological mixing for hom-shifts



Now keep moving the extended right edge of a up till we reach b.


a
b


a
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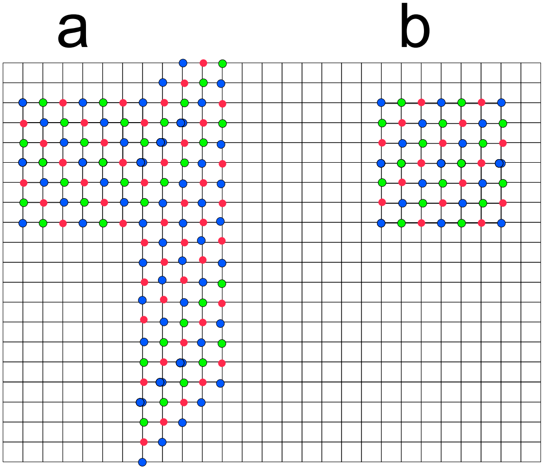
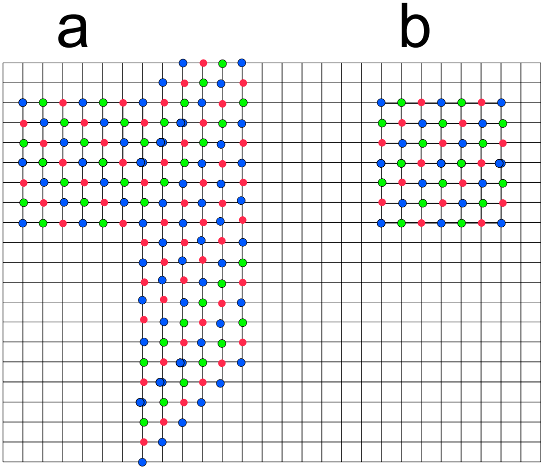

a
b


a
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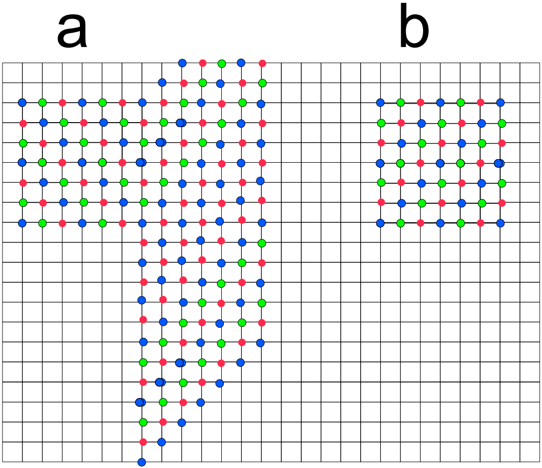
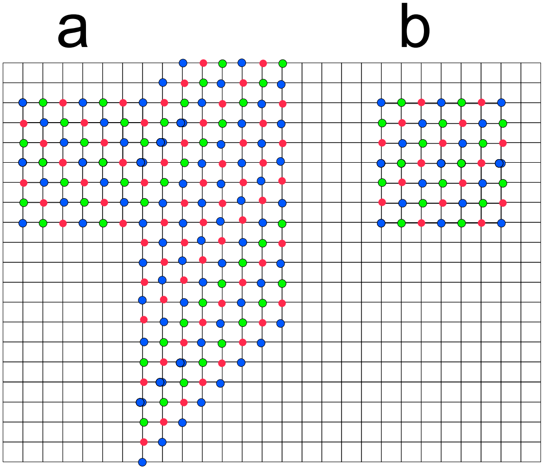
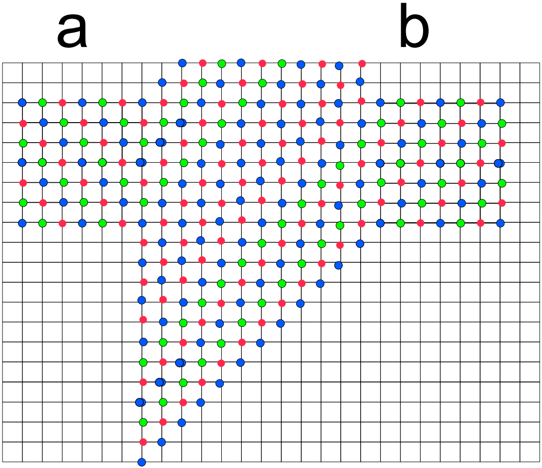
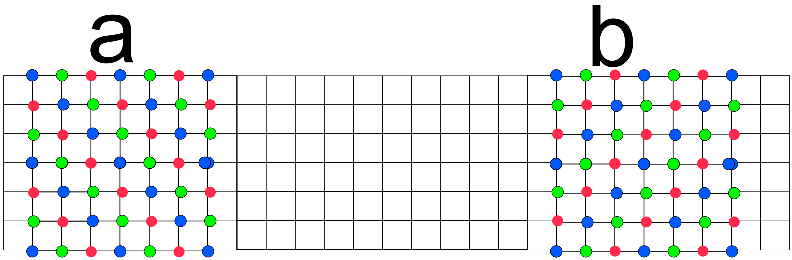

a
b


a
b


a
b

Topological mixing for hom-shifts

Using this one can prove that if a and b are sufficiently far apart (distance depending on a and b) they can be extended to a graph homomorphism on $\mathbb{Z}^d$.



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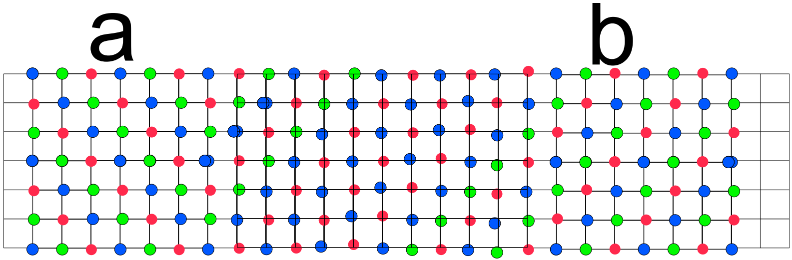


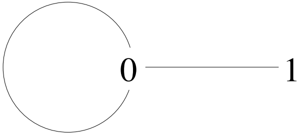
Thus if H is connected and not bipartite then the hom-shift X_H^d is topologically mixing. (Chandgotia and Marcus, 2018)

Strong irreducibility for hom-shifts

However strong irreducibility is a much more subtle property as compared to topological mixing.

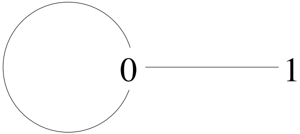
The hard-core shift is clearly strongly irreducible: Two patterns separated by distance 2 can be extended to a configuration in the hard-core shift by placing 0 on the complement of the pattern.

Hard-core shift is strongly irreducible



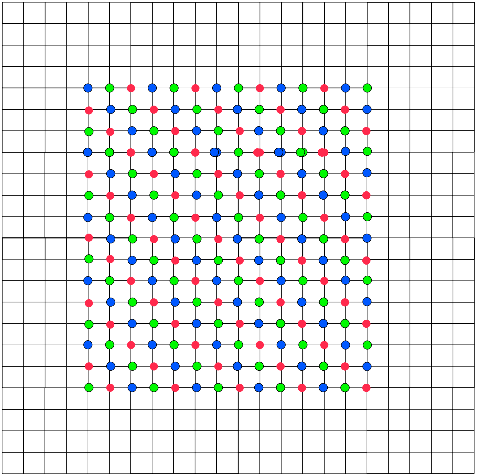
0	0	1	0	1	0	1	0	1	0	a
1									1	
0		0	1	0	1	0	1		0	
1		1	0	1	0	1	0		1	
0		0	1	0	1	0	1		0	
1		1	0	1	0	1	0		1	b
0		0	1	0	1	0	1		0	
0		1	0	1	0	1	0		1	
1									0	
0	1	0	1	0	1	0	1	0	1	

Hard-core shift is strongly irreducible



0	0	1	0	1	0	1	0	1	0	a
1	0	0	0	0	0	0	0	0	1	
0	0	0	1	0	1	0	1	0	0	
1	0	1	0	1	0	1	0	0	1	
0	0	0	1	0	1	0	1	0	0	
1	0	1	0	1	0	1	0	0	1	b
0	0	0	1	0	1	0	1	0	0	
0	0	0	0	0	0	0	0	0	1	
1	0	0	0	0	0	0	0	0	0	
0	1	0	1	0	1	0	1	0	1	

While proper 3-colourings are not strongly irreducible



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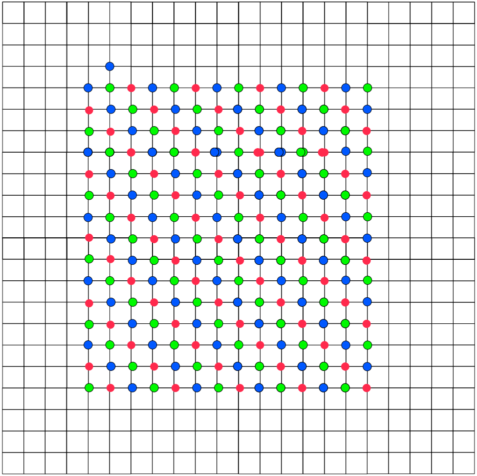

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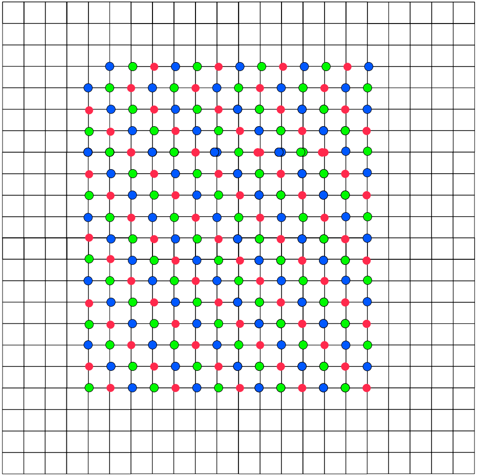

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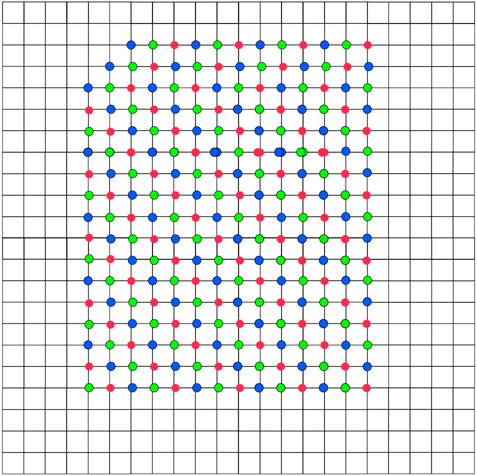

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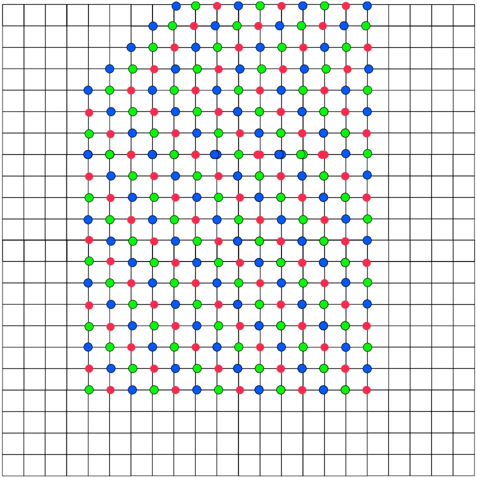

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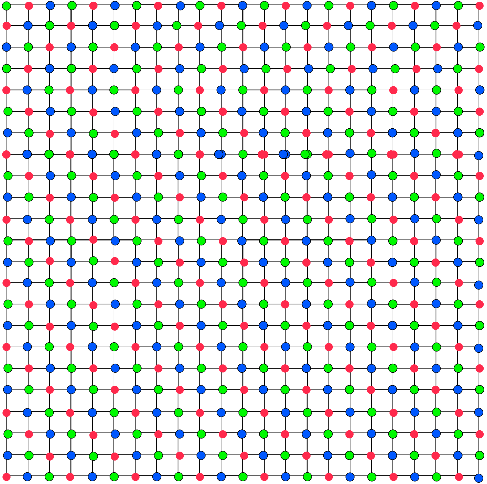


Is strongly irreduciblity decidable for hom-shifts?

Question

Is strong irreducibility decidable for hom-shifts?

Assume that H has no loops.

A graph homomorphism can be thought of as a cube complex map from $\mathbb{Z}^d$ to H. The vertices of the graphs can be thought of as the 0-cubes, the edges as 1-cubes and each simple 4-cycle as a 2-cubes.

The space of graph homomorphisms to H lifts to graph homomorphisms to a graph H' if and only if H lifts to H' as a cube complex.

Is strongly irreduciblity decidable for hom-shifts?

If homomorphisms to H lift to an infinite graph, then the hom-shift X_H^d is not strongly irreducible.

Thus H lifts to infinite graph if and only if the first homotopy group of the cube complex obtained from H is infinite.

But it is undecidable whether a given cube complex has an infinite first homotopy group.

Theorem

It is undecidable whether the space of graph homomorphisms from $\mathbb{Z}^d$ to a graph H lift to homomorphisms to an infinite graph H'.

However in specific cases a lot more can be said.

Theorem (Alon, Briceño, C., Magazinov, Spinka 2019)

The space of proper q colourings of $\mathbb{Z}^d$ is strongly irreducible if and only if $q \geq d + 2$.

There is also more work related to other mixing properties like block-gluing and topological strong spatial mixing (Briceño, Pavlov 2016; C. and Marcus 2018; Alon, Briceño, C., Magazinov, Spinka 2019).

Why do we care about mixing properties?

Let σ denote the shift action on shift spaces X. A shift space X is called universal if for any free $\mathbb{Z}^d$ action (Y, S) (with smaller entropy) there exists an equivariant injection from (Y, S) to (X, σ) defined on a universally full set.

Theorem (Krieger 1972, Hochman 2015)

Every mixing SFT for $d = 1$ is universal.

Hochman in fact proved that the injection can be defined on the entire set Y.

Theorem (Şahin and Robinson 2002)

Every strongly irreducible shift of finite type with dense periodic points is universal.

Şahin and Robinson proved that for every ergodic measure there exist an equivariant injection defined almost everywhere.

Theorem (C. and Meyerovitch, 2019)

The space of proper 3-colourings does not contain any strongly irreducible subshift.

Universality among hom-shifts

Theorem (C. and Meyerovitch 2019)

If H is an undirected graph which is not bipartite then the hom-shift X_H^d is universal.

While we won't go into details of the proof let me indicate the main combinatorial estimate that we need.

The main combinatorial estimate for graph homomorphisms

To prove the universality of (X^d_H, σ) we need the following:

Let $Free_n$ be the set of graph homomorphisms from the box $\{-n, -n+1, \ldots, n\}^d$ to H. Let $Checker_n$ be the set of graph homomorphisms from the box $\{-n, -n+1, \ldots, n\}^d$ to H such that the boundary only uses two symbols alternatively.

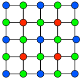


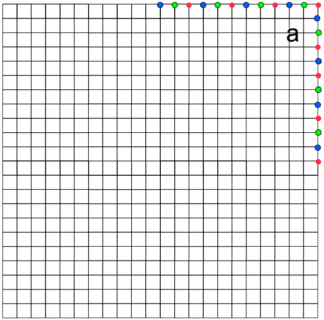
Figure : An element of P_n: Only two specified symbols are used on the boundary.

We need $h_{top}(X^d_H) = \lim_{n\to\infty} \frac{1}{(2n+1)^d} \log |Free_n| = \lim_{n\to\infty} \frac{1}{(2n+1)^d} \log |Checker_n|$.

The main combinatorial estimate for graph homomorphisms

Put the uniform probability measure on $Free_n$ and lets restrict our attention to $d = 2$. There must exist some graph homomorphism a from $\partial\{-n, -n+1, \dots, n\}^2 \cap \mathbb{N}^2$ to H such that

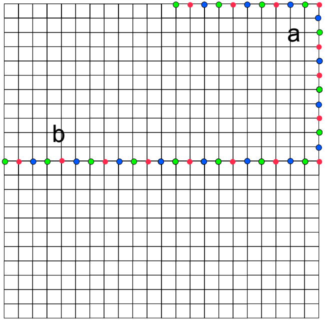
$$\mathbb{P}(a) \geq |H|^{-3n}.$$



The main combinatorial estimate for graph homomorphisms

It follows that for any graph homomorphism b from $\{-n, -n+1, \ldots, n\} \times \{0\}$ to H,

$$\mathbb{P}(a,\ \text{reflection of } a \text{ about } \{-n, -n+1, \ldots, n\} \times \{0\} \mid b) = \mathbb{P}(a \mid b)^2.$$



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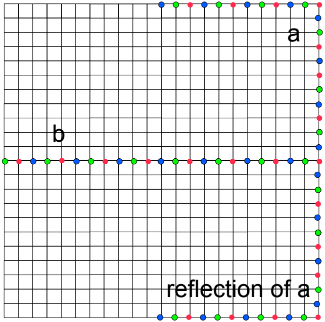
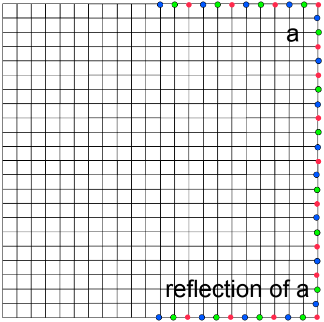


The main combinatorial estimate for graph homomorphisms

By integrating over all possible values of b, we have that

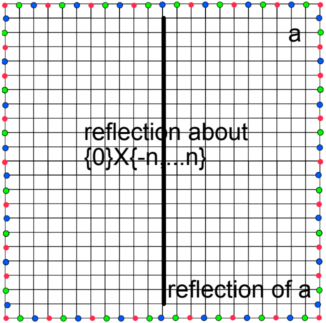
$$\begin{aligned} \mathbb{P}(a,\ \text{reflection of } a \text{ about } \{-n, -n+1, \ldots, n\} \times \{0\}) &\geq \mathbb{P}(a)^2 \\ &\geq |H|^{-6n}. \end{aligned}$$



The main combinatorial estimate for graph homomorphisms

By applying another reflection about $\{0\} \times \{-n, -n+1, \dots, n\}$ we have that

$$\mathbb{P}(\text{periodic boundary conditions}) \geq |H|^{-12n}.$$



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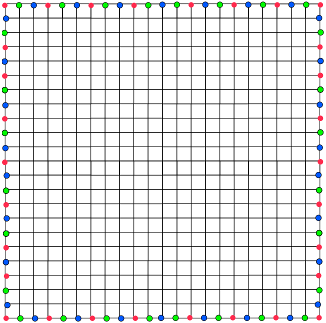


The main combinatorial estimate for graph homomorphisms

This implies that (Friedland, 1997)

$$\begin{aligned} 1 &\leq \lim_{n\to\infty} \frac{\log|\mathit{Free}_n|}{\log|\text{periodic elements of } \mathit{Free}_n|} \\ &\leq \lim_{n\to\infty} \frac{\log|\mathit{Free}_n|}{-12n\log(|H|) + \log|\mathit{Free}_n|} \\ &\leq 1 \end{aligned}$$

which proves that

$$1 = \lim_{n\to\infty} \frac{\log|\mathit{Free}_n|}{\log|\text{periodic elements of } \mathit{Free}_n|}.$$

This is a simple application of a technique called reflection positivity. By applying these ideas iteratively the estimate that we want can be obtained.

Reflection positivity

This technique extends the result of Galvin, Kahn, Randall and Sorkin (2012) and of Benjamini, Häggström and Mossel (2000) that we could have used for proper colourings.

Computability of entropy

Corollary

The entropy of a hom-shift is computable.

Since the growth rate of the number of periodic elements of $Free_n$ gives a lower bound for the topological entropy, the corollary follows easily.

Question

How fast can the computation be done?

What else can we use mixing properties for?

A shift space X is called entropy minimal if it does not contain another subshift with the same entropy.

Equivalently, a shift space X is entropy minimal if and only if every measure of maximal entropy is fully supported (otherwise the support of the measure of maximal entropy has equal entropy to that of X).

This can be used to prove that every strongly irreducible shift of finite type is entropy minimal but further is true.

Theorem (Schraudner)

Every strongly irreducible shift space is entropy minimal.

Theorem

Transitive SFTs are entropy minimal for $d = 1$.

Conjecture

If H is a connected graph then the space X_H^d is entropy minimal. In other words if H is a connected graph then measures of maximal entropy of X_H^d are fully supported.

By extensive use of reflection positivity the following result can be proved.

Theorem (Peled)

If H is a connected graph then there exists a measure of maximal entropy which is fully supported on X_H^d.

What about measures of maximal entropy?

Shifts of finite type for $d = 1$ have finitely many measures of maximal entropy.

For higher dimensions it is not difficult to construct mixing shifts of finite type which have uncountably many ergodic measures of maximal entropy. We believe that such examples can't exist for hom-shifts.

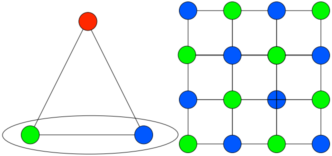
Conjecture

Hom-shifts X_H^d have at most finitely many ergodic measures of maximal entropy.

How to build a graph homomorphism?

Before we discuss more about this question, let us do some elementary exercises. Suppose that we are given a graph H. How can one quickly build a graph homomorphism from $\mathbb{Z}^d$ to H?

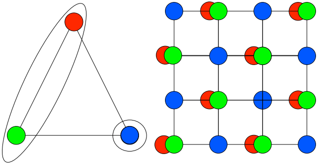
Choose two adjacent vertices of H and alternate between them.



But this has entropy zero.

A little more randomness

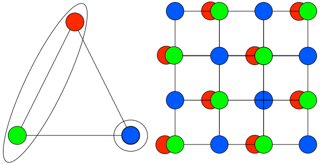
A phase of a graph H is an unordered pair of subsets (A, B) such that every vertex in A is adjacent to every vertex in B. Thus now we can build graph homomorphisms by alternating between elements of A and elements of B.



A phase is maximal if it maximises $|A||B|$. We believe that the number of maximal phases essentially dictates the number of ergodic measures of maximal entropy.

Maximal phases and the number of ergodic measures of maximal entropy

Every phase (A, B) gives rise to an ergodic measure: Place with uniform probability elements of A and B on different partite classes of $\mathbb{Z}^d$. This measure has entropy $\frac{1}{2}\log(|A||B|)$.



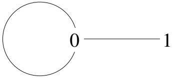
Maximal phases and the number of ergodic measures of maximal entropy

Let $\mathbb{Z}^\infty$ denote the direct limits of $\mathbb{Z}^d$ as $d \to \infty$ and let X_H^∞ denote the corresponding hom-shift.

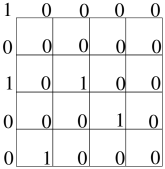
Theorem (Pavlov, Meyerovitch 2014)

The only ergodic measures of maximal entropy invariant to the change of coordinates are those which arise from maximal phases.

Some examples: Hard Core model



Graph $\mathcal{H}$

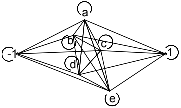


There is a unique maximal phase: $\{0, 1\}, \{0\}$.

For $d = 2$, (van den Berg and Steif 1994) it is known that the hard square model has a unique measure of maximal entropy.

Some examples: Iceberg Model (Burton and Steif, 1994)

The symbols are $-M, -M+1, \ldots, -1, 1, \ldots M$ and a, b, c, d, e. The only restriction is that the positives cannot sit next to the negatives.



-1	-1	-1	a	b	c	a	e
b	c	b	e	a	d	a	b
b	a	e	a	1	b	a	e
a	b	1	1	1	d	d	e
b	a	c	c	c	e	a	d
e	e	b	c	-1	e	1	d
d	c	a	e	d	b	c	d

Figure : Iceberg model for $M = 1$

Some examples: Iceberg Model (Burton and Steif, 1994)

For M large enough the maximal phases are

$$(\{-M,\ldots,-1,a,b,c,d,e\},\{-M,\ldots,-1,a,b,c,d,e\})$$

and

$$(\{1,\ldots,M,a,b,c,d,e\},\{1,\ldots,M,a,b,c,d,e\}).$$

For $d=2$ and M large enough there are exactly two measures of maximal entropy. For $d=2$ and $M=1$ there is a unique measure of maximal entropy.

Some examples: Proper colourings

The symbols are $1, 2, 3, \ldots, q$. Adjacent symbols in this shift are distinct. This is the space $X^d_{K_q}$ where K_q is the complete graph on q vertices.

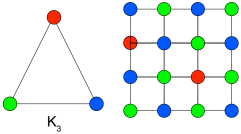


Figure : Proper 3-colorings

Maximal phases are partitions of $1, 2, \ldots q$ into cardinalities $\lfloor q/2 \rfloor$ and $\lceil q/2 \rceil$.

Some examples: Proper colourings

(Sheffield 2006) X_3^2 has a unique measure of maximal entropy.

(Goldberg, Martin and Paterson 2005) X_q^d has a unique measure of maximal entropy for $q \geq 3.6d$.

What about higher dimensions?

Recent results of Peled and Spinka

For all the models stated above (and much much more), Peled and Spinka recently proved that for large enough dimension, the number of ergodic measures of maximal entropy is precisely equal to the number of distinct maximal phases.

Rectangular tiling shifts

Tilings by rectangular tiles

A rectangular tile is a subset of $\mathbb{Z}^d$ of the form $[1, i_1] \times [1, i_2] \times \cdots \times [1, i_d]$ for $i_1, i_2, \ldots, i_d \in \mathbb{N}$.

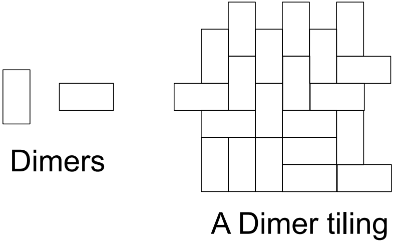
Given a set of rectangular tiles, T, we denote by X_T the set of tilings of $\mathbb{Z}^d$ by elements of T. It comes with the natural $\mathbb{Z}^d$-shift action which makes it a shift of finite type.

Examples: Dimer Tilings

Dimers are the set of rectangular tiles given by

$$T_{dim} = \{[1, i_1] \times [1, i_2] \times \cdots \times [1, i_d] \ : \ \prod_{t=1}^{d} i_t = 2\}.$$

Let X_{dim} be the set of dimer tilings.

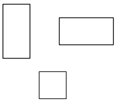


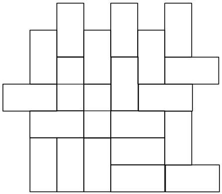
Examples: Monomer k-mer tilings

Let

$$T = \{[1]^d\} \cup \{[1, i_1] \times [1, i_2] \times \cdots \times [1, i_d] \; : \; i_t = 1 \text{ or } k \text{ and } \prod_{t=1}^{d} i_t = k\}$$

The set of tilings by T is called the monomer k-mer shift.

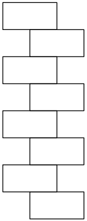




A Monomer 2-mer tiling

Mixing properties of tiling shifts

Dimer tilings are not strongly irreducible.



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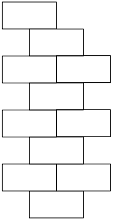

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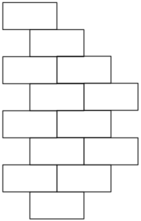

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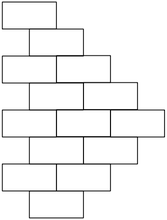
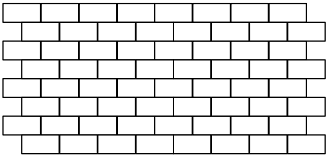


Mixing properties of tiling shifts

However monomer k-mer shifts are clearly strongly irreducible. Given tilings of regions A and B which do not intersect, the complement can be tiled by monomers.



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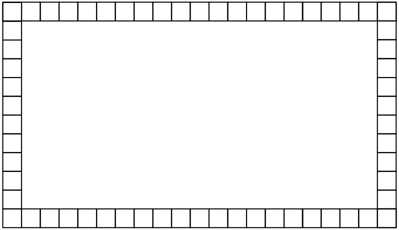

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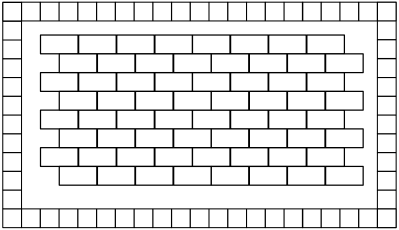

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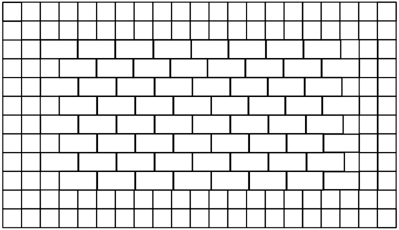


Mixing properties of tiling shifts

Question

Is strong irreducibility of rectangular tiling shifts decidable?

Mixing properties of tiling shifts

A set of tiles T is called prime if the greatest common divisor of side lengths along any given direction is 1.

It is easy to see that if T is not prime then X_T is not topologically mixing.

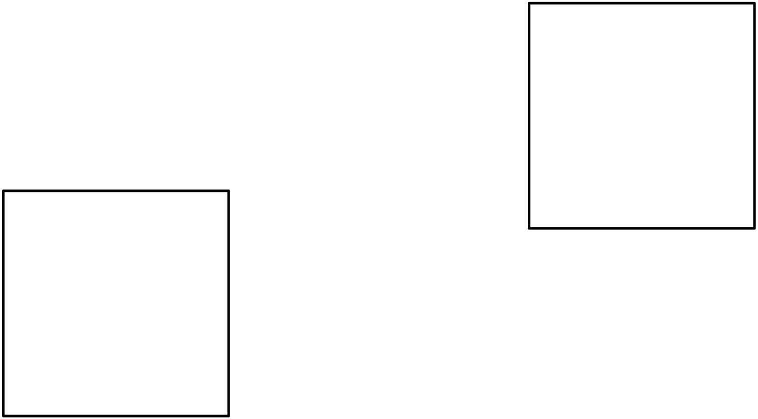
Theorem (Einsedler 2001)

For $\mathbb{Z}^2$ tiling shifts, if $|T| = 2$, then X_T is mixing if and only if T is prime.

This is true in general as well but very messy.

The main idea for proving mixing

If a set of tiles T is prime, then T can tile the complement of any two rectangles provided the rectangles are far enough apart.



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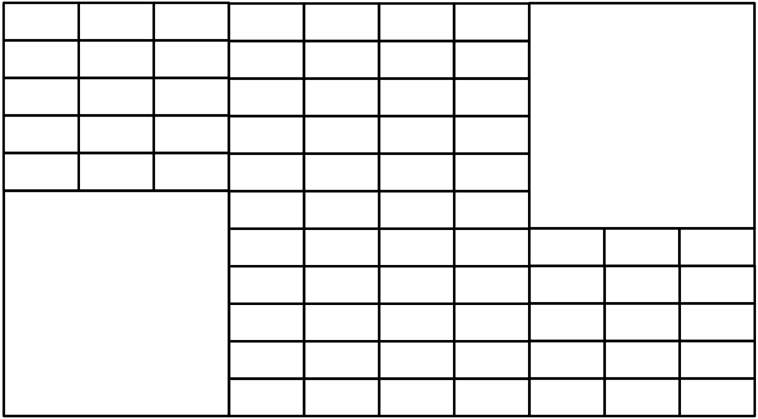
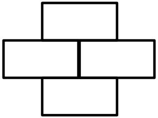


The main idea for proving mixing

So to prove mixing is sufficient to prove that the restriction of any tiling of $\mathbb{Z}^2$ to a finite region can be extended to a tiling of a rectangle.



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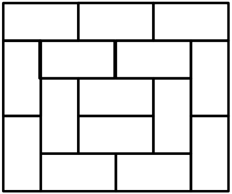


Extending tilings

Question

How often can we extend a tiling of a region to a tiling of a slightly bigger rectangle?

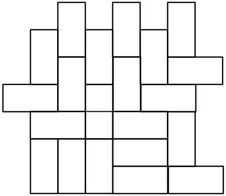
Fix P to be the product of the sides of the tiles in T.

Let $Free_n$ be the set of tilings of $\mathbb{Z}^d$ restricted to $[1, nP]^d$.

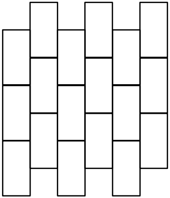
Let $Periodic_n$ be the set of nP-periodic tilings of $\mathbb{Z}^d$ restricted to $[1, nP]^d$.

Let $Perfect_n$ be the set of tilings of $\mathbb{Z}^d$ restricted to $[1, nP]^d$.

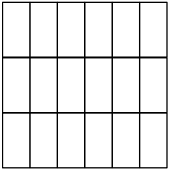
Topological, Periodic and Perfect



Free



Periodic



Perfect

$$h_{top}(X_T) = \lim_{n\to\infty} \frac{1}{(np)^d} \log(Free_n).$$

$$h_{Periodic}(X_T) = \liminf_{n\to\infty} \frac{1}{(np)^d} \log(Periodic_n).$$

$$h_{Perfect}(X_T) = \liminf_{n\to\infty} \frac{1}{(np)^d} \log(Perfect_n).$$

Topological, Periodic and Perfect

We know immediately

$$h_{top}(X_T) \geq h_{Periodic}(X_T) \geq h_{Perfect}(X_T).$$

If T is prime and $h_{top}(X_T) = h_{Perfect}(X_T)$ then X_T is universal.

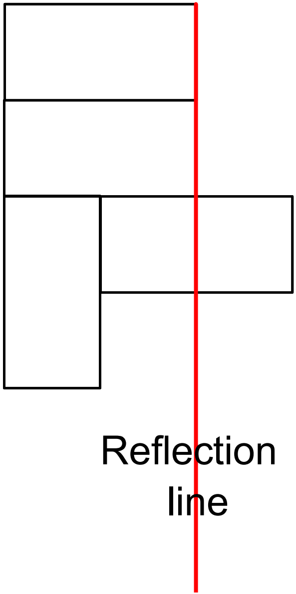
If $h_{top}(X_T) = h_{Periodic}(X_T)$ then $h_{top}(X_T)$ is computable.

For $d = 2$, this follows from Kastelyn's formalism for domino tilings.

Theorem

(C, 2019) For domino tilings $h_{top}(X_T) = h_{Perfect}(X_T)$ for all dimensions d.

The proof uses the fact that dominos can be reflected



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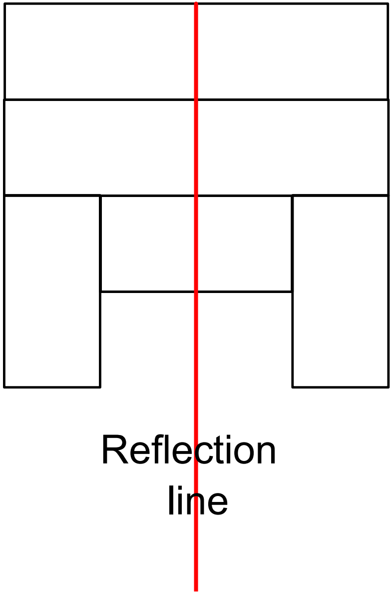

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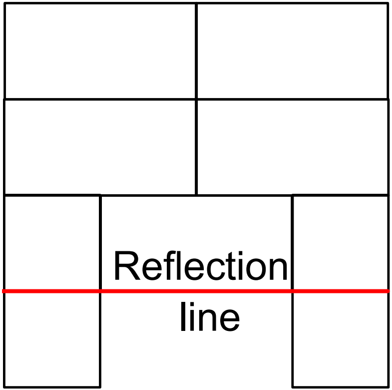

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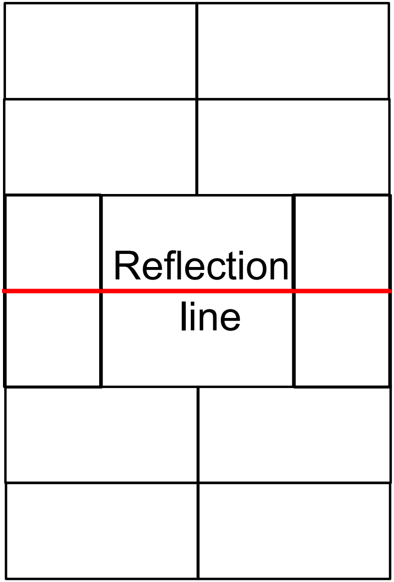

and some $d-1$-cube cohomology.

Conjecture

Conjecture

X_T is universal for all prime tiling sets T. In other words prove that $h_{top}(X_T) = h_{Perfect}(X_T)$.

A few things I would like to know about.

1. Hom-shifts
    1. Algorithm for deciding strong irreducibility
    2. Entropy minimality
    3. Bound on the number of measures of maximal entropy
2. Rectangular tiling shifts
    1. A deeper understanding of various mixing properties.
    2. Computability of entropy: Counting periodic points.
    3. Universality.