

DELOCALIZATION OF UNIFORM GRAPH HOMOMORPHISMS FROM \mathbb{Z}^2 TO \mathbb{Z}

NISHANT CHANDGOTIA, RON PELED, SCOTT SHEFFIELD AND MARTIN TASSY

ABSTRACT. Graph homomorphisms from the \mathbb{Z}^d lattice to \mathbb{Z} are functions on \mathbb{Z}^d whose gradients equal one in absolute value. These functions are the height functions corresponding to proper 3-colorings of \mathbb{Z}^d and, in two dimensions, corresponding to the 6-vertex model (square ice). We consider the uniform model, obtained by sampling uniformly such a graph homomorphism subject to boundary conditions. Our main result is that the model delocalizes in two dimensions, having no translation-invariant Gibbs measures. Additional results are obtained in higher dimensions including the fact that every Gibbs measure which is ergodic under even translations is extremal and that these Gibbs measures are stochastically ordered.

1. INTRODUCTION

In this paper we study *homomorphism height functions* on the \mathbb{Z}^d lattice: functions $f : \mathbb{Z}^d \rightarrow \mathbb{Z}$ restricted to satisfy $|f(u) - f(v)| = 1$ when u is adjacent to v and taking even values on the even sublattice. Such functions have received special attention in the literature. In one dimension they are the possible trajectories of simple random walk. In two dimensions they are the height functions of the 6-vertex model (square ice), while in all dimensions they are in bijection with proper 3-colorings of the lattice by taking the modulo 3 operation (after fixing the value at one vertex). There is no single name associated to these height functions in the literature and they have at times been called random graph homomorphisms into \mathbb{Z} [3] (or \mathbb{Z} -homomorphisms [34]), \mathbb{Z}^d -indexed random walk [26], Body-Centered Solid-On-Solid (BCSOS) [40] or the height functions of the 6-vertex model [40].

Our main concern is with the fluctuations exhibited by a homomorphism height function which is uniformly sampled in a domain with zero boundary conditions. The one-dimensional function is a random walk bridge, which fluctuates as the square-root of the length of the domain. In high dimensions the surface localizes, having bounded variance at each vertex, and this is conjectured to occur for all $d \geq 3$ [26, 16, 17, 33]. The two-dimensional case is especially interesting, as integer-valued height functions in two dimensions generally undergo a *roughening transition* from a rough (delocalized) phase to a smooth (localized) phase as the parameters of the model are changed [44].

The uniform model on homomorphism height functions has no parameters and hence its behavior is a priori unclear in two dimensions. However, noting that the model is a special case of the 6-vertex model (when the 6 weights are equal) suggests that it delocalizes, as it falls in the disordered regime of the predicted phase diagram [2]. Indeed, it is particularly natural to consider the *F-model*, a one-parameter subfamily of the 6-vertex model, indexed by $c > 0$, whose description in terms of height functions is the following: A homomorphism height function f in a domain (with boundary conditions) is sampled with probability proportional to $c^{-N(f)}$ with $N(f)$ being the number of diagonally adjacent vertices u, v on which $f(u) \neq f(v)$. The uniform model that we study here is then obtained for $c = 1$, whereas it is predicted that the two-dimensional *F-model* delocalizes for all $c \leq 2$ and localizes when $c > 2$. While the full prediction has not been mathematically justified yet, it has recently been shown that it is correct for $c \geq 2$ [10, 21]. Delocalization at $c = 1$ is further suggested by rapid mixing results which have been proved for the uniform proper 3-coloring model [31, 22]. In this work we prove that the two-dimensional uniform homomorphism height function delocalizes. Additional results are obtained in higher dimensions.

We note that the exact entropy constant for the two-dimensional uniform model has been found by Lieb [30], and the result has been extended to the 6-vertex parameter space; see [11] for a recent survey. However, these results do not seem to suffice to describe the fluctuations of the height function.

It is worth mentioning that there are only a handful of results in the mathematical literature where an integer-valued height function has been shown to delocalize in two dimensions. The breakthrough work of Fröhlich–Spencer [15] established this fact for the high-temperature integer-valued Gaussian free field

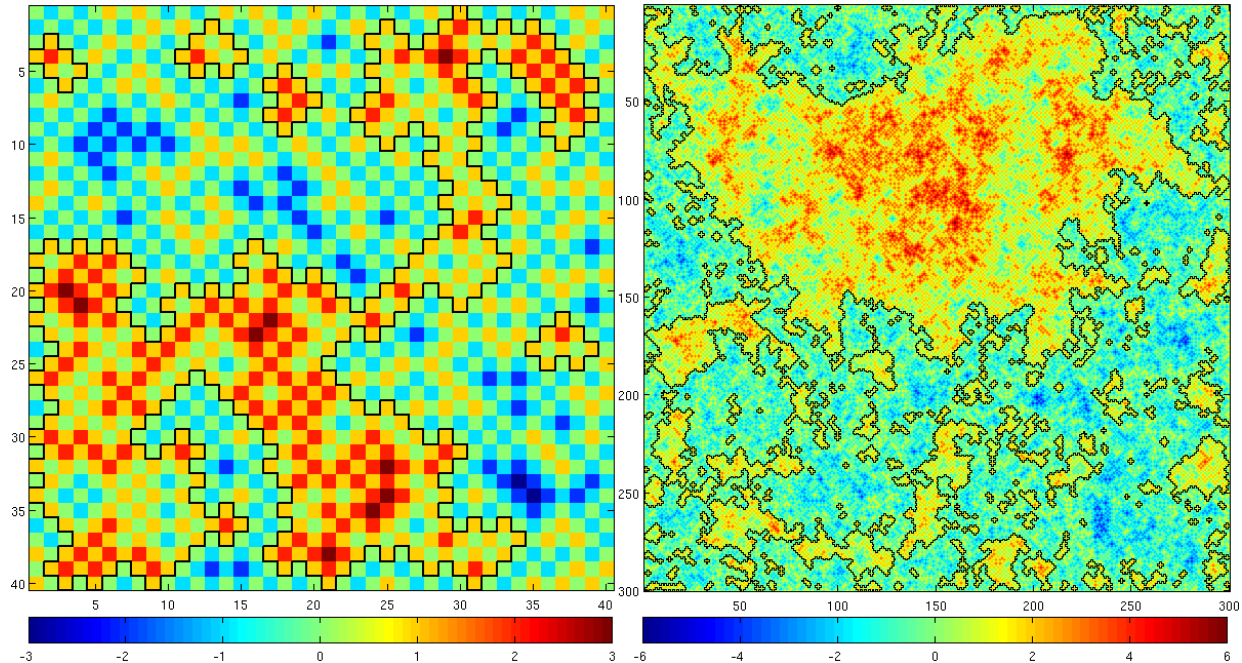


FIGURE 1. the outermost level sets separating zeros and ones of a uniform homomorphism height function with zero boundary values (on every second vertex) on a 40×40 and 300×300 squares, sampled using coupling from the past [35]. Theorem 1.2 shows that the height at the center of the squares diverges as the square size increases.

and Solid-On-Solid models, en route to proving the existence of the Kosterlitz-Thouless transition in the plane-rotator (XY) model (see also [29]). The only other examples concern integrable models either in the sense that the free energy is exactly calculable (as done by Lieb [30] for the model studied here) or having an exactly predicted phase diagram, though, as mentioned above, these facts by themselves do not yield a description of the fluctuations. Here is a list of such examples known to us.

- The dimer model [27].
- The uniform spanning tree (for certain domains in \mathbb{Z}^2 [36, 37], the honeycomb lattice [28] and for planar graphs with some natural restrictions [4]).
- The F -model at the parameter $c = 2$ [10, 21], and around the free fermion point $c = \sqrt{2}$ [9, 19].
- The Lipschitz height function model on the triangular lattice (the loop $O(2)$ model) at the parameter $x = 1/\sqrt{2}$ [12] and $x = 1$ [20].

As a final perspective on our results, we note that the set of homomorphism height functions form a dynamical system under translations by the even sublattice and has the structure of a countable-state shift of finite type. Our results imply that this dynamical system in two dimensions does not have a measure of maximal entropy. A characterisation of one-dimensional countable-state shifts of finite type (also known as topological Markov shifts) which have a measure of maximal entropy can be found in [38]; however such a characterisation is not known in higher dimensions.

1.1. Notation.

1.1.1. \mathbb{Z}^d Lattice. We work throughout on the integer lattice \mathbb{Z}^d , $d \geq 1$. Denote $\mathbf{0} := (0, 0, \dots, 0) \in \mathbb{Z}^d$. For $v = (v_1, v_2, \dots, v_d) \in \mathbb{Z}^d$ let $\|v\|_1 := |v_1| + \dots + |v_d|$. Vertices $v, w \in \mathbb{Z}^d$ are adjacent, denoted $v \sim w$, if $\|v - w\|_1 = 1$. We give names to the two sublattices in \mathbb{Z}^d defined by the parity of the sum of coordinates,

$$\begin{aligned}\mathbb{Z}_{\text{even}}^d &:= \{v \in \mathbb{Z}^d : \|v\|_1 \text{ is even}\}, \\ \mathbb{Z}_{\text{odd}}^d &:= \{v \in \mathbb{Z}^d : \|v\|_1 \text{ is odd}\}.\end{aligned}$$

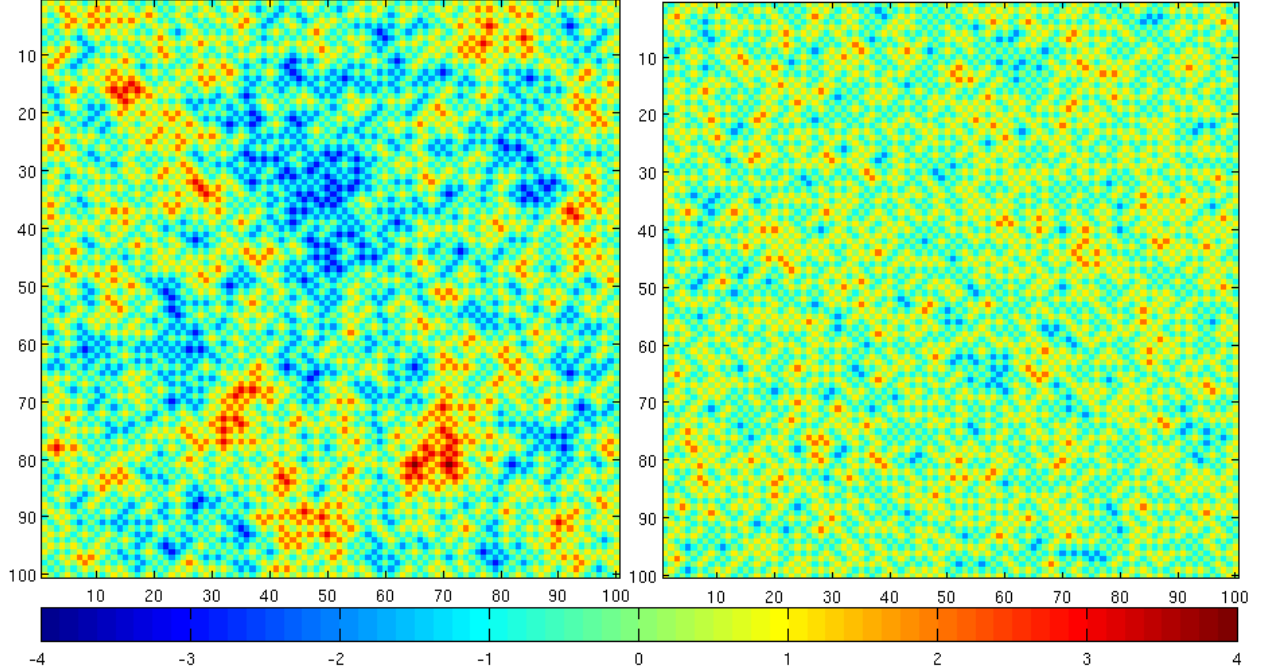


FIGURE 2. Simulation of uniform homomorphism height functions with zero boundary conditions (on every second vertex) on a 100×100 square (left) and on the middle slice of a $100 \times 100 \times 100$ cube. Sampled using coupling from the past [35]. Unlike the two-dimensional case, it is conjectured that homomorphism height functions in three and higher dimensions are localized. This is known in sufficiently high dimensions [17, 33].

The (graph) ball of radius L is

$$\Lambda(L) := \{v \in \mathbb{Z}^d : \|v\|_1 \leq L\}, \quad L \geq 0 \text{ integer.}$$

The external vertex boundary of a set $\Lambda \subset \mathbb{Z}^d$ is denoted

$$\partial_o \Lambda := \{v \in \mathbb{Z}^d : v \notin \Lambda, \exists w \sim v, w \in \Lambda\}$$

and the 1-extension of the set is given by

$$\Lambda^+ := \Lambda \cup \partial_o \Lambda.$$

1.1.2. Homomorphism height functions. An integer-valued function f defined on a subset of \mathbb{Z}^d is called a *homomorphism height function* (or \mathbb{Z} -homomorphism) if it satisfies

$$\text{Lipschitz condition:} \quad |f(u) - f(v)| \leq \|u - v\|_1, \quad (1)$$

$$\text{Parity condition:} \quad f(v) \equiv \|v\|_1 \pmod{2} \quad (2)$$

for u, v in the domain of f . Given (2), on connected domains condition (1) is equivalent to

$$|f(u) - f(v)| = 1 \quad \text{for all adjacent } u, v \text{ in the domain of } f. \quad (3)$$

The name homomorphism height function is motivated by the fact that a function satisfying (3) is a graph homomorphism from its domain to \mathbb{Z} . One may check that a homomorphism height function on any domain may be extended to a homomorphism height function on the whole of \mathbb{Z}^d , see, e.g., [7, Kirszbraun property] or [32].

The restriction of a function f to a subset Λ of its domain is denoted f_Λ .

1.1.3. *Finite-volume Gibbs measures.* Given a non-empty, finite $\Lambda \subset \mathbb{Z}^d$ and homomorphism height function $\tau : \partial_o \Lambda \rightarrow \mathbb{Z}$, the probability measure $\mu^{\Lambda, \tau}$, termed the Gibbs measure in Λ with boundary condition τ , is the uniform measure on homomorphism height functions $f : \Lambda^+ \rightarrow \mathbb{Z}$ which satisfy that $f \equiv \tau$ on $\partial_o \Lambda$.

We write $\mu^{\Lambda, 0}$ to indicate that τ is the identically zero function.

1.1.4. *Infinite-volume Gibbs measures.* A probability measure μ on homomorphism height functions on \mathbb{Z}^d is a *Gibbs measure* if it satisfies the following condition: Sampling f from μ , for any non-empty, finite $\Lambda \subset \mathbb{Z}^d$ the conditional distribution of the restriction f_{Λ^+} , conditioned on f_{Λ^c} , equals $\mu^{\Lambda, \tau}$ with $\tau := f_{\partial_o \Lambda}$, almost surely. One checks easily that if the condition in the definition holds for a set Λ then it holds with Λ replaced by a subset of it. Hence the definition will be unaltered if we require both Λ and Λ^c to be connected subsets.

A sequence of finite-volume Gibbs measures μ^{Λ_n, τ_n} , with Λ_n increasing to \mathbb{Z}^d , is said to *converge in distribution* (i.e., in the thermodynamic limit) if their marginal distribution on any finite subset of \mathbb{Z}^d (defined for large n) converges in distribution. It is standard that the limiting measure is then a Gibbs measure.

A sublattice \mathcal{L} of \mathbb{Z}^d is the set of linear combinations with integer coefficients of a collection of linearly independent (over the rationals) vectors $v_1, \dots, v_k \in \mathbb{Z}^d$. The sublattice has *full rank* if $k = d$.

As we have defined homomorphism height functions to take even values on $\mathbb{Z}_{\text{even}}^d$, it is not possible for a Gibbs measure to be invariant in distribution to all translations of \mathbb{Z}^d . It is thus useful to write that a Gibbs measure μ is *\mathcal{L} -translation-invariant*, for a sublattice $\mathcal{L} \subset \mathbb{Z}^d$, if it is invariant in distribution to all translations in \mathcal{L} . Automatically, we have that $\mathcal{L} \subset \mathbb{Z}_{\text{even}}^d$ and we typically take $\mathcal{L} = \mathbb{Z}_{\text{even}}^d$. Correspondingly, μ is called *\mathcal{L} -ergodic* if it is \mathcal{L} -translation-invariant and every event A which is invariant to translations in \mathcal{L} satisfies $\mu(A) \in \{0, 1\}$.

A Gibbs measure μ is called *extremal* if it satisfies that any event A in the tail-sigma-algebra has $\mu(A) \in \{0, 1\}$. Here, the event A is in the tail-sigma-algebra if its occurrence for a homomorphism height function f may be determined solely from the restriction f_{Λ^c} , for any finite subset Λ . It is standard that an \mathcal{L} -translation-invariant extremal Gibbs measure is also \mathcal{L} -ergodic.

1.2. **Results.** Our first result is a simple dichotomy holding in every dimension between delocalization and localization of homomorphism height functions with zero boundary conditions.

Theorem 1.1. *In each dimension $d \geq 1$ exactly one of the following alternatives hold:*

- (i) *For odd integers $L \geq 1$, let f_L be sampled from $\mu^{\Lambda(L), 0}$. Then the set of random variables $(f_L(\mathbf{0}))$ is not tight. Precisely,*

$$\lim_{M \rightarrow \infty} \sup_L \mathbb{P}(|f_L(\mathbf{0})| > M) > 0. \quad (4)$$

- (ii) *The sequence of finite-volume Gibbs measures $(\mu^{\Lambda(L), 0})$, $L \geq 1$ odd, converges in distribution to a $\mathbb{Z}_{\text{even}}^d$ -translation-invariant Gibbs measure.*

The next theorem is the main result of this paper.

Theorem 1.2. *In dimension $d = 2$, there are no $\mathbb{Z}_{\text{even}}^2$ -translation-invariant Gibbs measures. In particular, two-dimensional homomorphism height functions delocalize in the sense that alternative (i) of Theorem 1.1 holds when $d = 2$.*

The techniques used in the proof of Theorem 1.2 also provide information on the set of Gibbs measures in higher dimensions.

Theorem 1.3. *In all dimensions $d \geq 3$: Suppose μ is an \mathcal{L} -ergodic Gibbs measure for a full-rank sublattice \mathcal{L} . Then μ is $\mathbb{Z}_{\text{even}}^d$ -ergodic and extremal. In addition, if μ_1, μ_2 are $\mathbb{Z}_{\text{even}}^d$ -ergodic Gibbs measures then either μ_1 stochastically dominates μ_2 or μ_2 stochastically dominates μ_1 (with respect to the pointwise partial order on \mathbb{Z} -valued functions on \mathbb{Z}^d).*

We make several remarks regarding the theorems.

The proof of the dichotomy in Theorem 1.1 makes use of a positive association (FKG) property for the *absolute value* of homomorphism height functions shown by Benjamini, Häggström and Mossel [3]. The proof implies that the alternative satisfied in each dimension, as well as the limiting measure in the second alternative, remain unchanged if one replaces the domains $(\Lambda(L))$, $L \geq 1$ odd, by another sequence of subsets

increasing to \mathbb{Z}^d and having their external vertex boundary on the even sublattice. The argument further shows that the limiting measure is also invariant under interchange of coordinates and reflection in coordinate hyperplanes.

It is known that the single-site marginals of homomorphism height functions have a log-concave distribution on the integers ([39, Lemma 8.2.4] and [26, Proposition 2.1]). With this fact several a priori different notions of delocalization coincide. Specifically, for f_L sampled from $\mu^{\Lambda(L),0}$ as in Theorem 1.1 the non-tightness condition (4) is equivalent to

$$\sup_L \mathbb{E}(\exp(\alpha \cdot |f_L(\mathbf{0})|)) = \infty \quad \text{for all } \alpha > 0 \quad (5)$$

and also to

$$\sup_L \mathbb{P}(|f_L(\mathbf{0})| > M) = 1 \quad \text{for all } M. \quad (6)$$

Noting that $\mathbb{E}(f_L(\mathbf{0})) = 0$ as $f_L \stackrel{d}{=} -f_L$, we see that the above conditions are also equivalent to having unbounded variance, $\sup_L \text{Var}(f_L(\mathbf{0})) = \infty$. These ideas are explained in more detail in Section 7.

Delocalization also holds for one-dimensional homomorphism height functions in the sense that there are no \mathbb{Z}_{even} -translation-invariant Gibbs measures and alternative (i) of Theorem 1.1 holds when $d = 1$. This is straightforward to prove using the ergodic theorem and the properties of simple random walk bridge. We thus focus throughout on dimensions $d \geq 2$. As mentioned, it is conjectured that alternative (ii) of Theorem 1.1 holds for all $d \geq 3$ and this is known in sufficiently high dimensions [33, 17].

In his Ph.D. thesis [39], the third author considered a very broad class of random surface models, allowing nearest-neighbor interactions with arbitrary convex potentials whose heights take values either in the reals or in the integers, and obtained very strong results including a variational principle and large deviation results for these models and a detailed discussion of the set of gradient Gibbs measures of the model with reference to their roughness or smoothness properties. The results obtained there imply, and are indeed stronger than, the characterization of Gibbs measures given in Theorem 1.3 and Theorem 3.1 below. The main contribution of this paper is to present in a concise manner the road to obtaining the described results in the context of graph homomorphisms to \mathbb{Z} . The focus on the specific setup and set of results allows for various simplifications to be made resulting in a relatively short presentation, which we hope will be of value to the community both for the actual results proven and as a gateway to the more involved discussion presented in [39].

1.3. Acknowledgements. The work of NC was supported in part by the European Research Council starting grant 678520 (LocalOrder) and ISF grant no. 1289/17. The work of RP was supported in part by Israel Science Foundation grant 861/15 and the European Research Council starting grant 678520 (LocalOrder). The work of SS was supported in part by NSF grants DMS 1209044 and DMS 1712862.

We thank Yinon Spinka for reading our draft and suggesting various improvements. Steven M. Heilman helped prepare Figure 1.

2. LOCALIZATION DICHOTOMY

In this section we will prove Theorem 1.1.

Let $d \geq 1$. For odd integers $L \geq 1$, let f_L be sampled from $\mu^{\Lambda(L),0}$. It is clear that if alternative (ii) of the theorem holds, i.e., the finite-volume Gibbs measures $(\mu^{\Lambda(L),0})$, $L \geq 1$ odd, converge in distribution, then alternative (i) of the theorem does not hold, i.e.,

$$\text{the sequence of random variables } (f_L(\mathbf{0})), L \geq 1 \text{ odd, is tight.} \quad (7)$$

Thus to prove Theorem 1.1 we need only show that (7) implies alternative (ii).

We rely on the following positive association (FKG) property for the *absolute value* of random homomorphism height functions, which follows immediately from a result proven by Benjamini, Häggström and Mossel [3]. Positive association of the values themselves also holds (Lemma 4.1), and will be used later.

Proposition 2.1. *Let $\Lambda \subset \mathbb{Z}^d$ be non-empty and finite, with $\partial_o \Lambda \subset \mathbb{Z}_{\text{even}}^d$. If f is randomly sampled from $\mu^{\Lambda,0}$ then $|f|$ has positive association, in the sense that if $\varphi, \psi : \{0, 1, 2, \dots\}^{\Lambda^+} \rightarrow [0, \infty)$ are non-decreasing*

(in the pointwise partial order on integer-valued functions) one has

$$\mathbb{E}(\varphi(|f|)\psi(|f|)) \geq \mathbb{E}(\varphi(|f|))\mathbb{E}(\psi(|f|)).$$

In [3, Proposition 2.3] the result stated above has been proved for homomorphism height functions on general finite bipartite graphs, normalized to be 0 at a fixed vertex. Proposition 2.1 follows by identifying the vertices of $\partial_o \Lambda$ to a single vertex.

Our use of the proposition is through the following consequence: If $\Lambda_1 \subset \Lambda_2$ are non-empty, finite subsets of \mathbb{Z}^d such that $\partial_o \Lambda_1, \partial_o \Lambda_2 \subset \mathbb{Z}_{\text{even}}^d$ then, when g_1, g_2 are randomly sampled from $\mu^{\Lambda_1, 0}, \mu^{\Lambda_2, 0}$ respectively, one has that

$$|g_2|_{\Lambda_1^+} \text{ stochastically dominates } |g_1|. \quad (8)$$

This is implied by Proposition 2.1 applied with $f = g_2$, $\varphi(f) := 1 - 1_{\{f(v)=0 \text{ for all } v \in \partial_o \Lambda_1\}}$ (with 1_A denoting the indicator function of A) and ψ which depends only on the restriction of f to Λ_1^+ . We may now derive the following lemma.

Lemma 2.2. *Let (Λ_n) be a sequence of non-empty, finite subsets of \mathbb{Z}^d which increase to \mathbb{Z}^d and satisfy $\partial_o \Lambda_n \subset \mathbb{Z}_{\text{even}}^d$ for all n . If (7) holds then the measures $\mu^{\Lambda_n, 0}$ converge in distribution.*

Proof. Let g_n be sampled from $\mu^{\Lambda_n, 0}$. Property (8) implies that the distributions of $|g_n|$ form a stochastically increasing sequence of measures. Convergence in distribution of $|g_n|$ will then follow by verifying that for each $v \in \mathbb{Z}^d$, the set of random variables $g_n(v)$ is tight (for $n \geq n_0(v)$ with $n_0(v)$ the smallest value for which $v \in \Lambda_{n_0(v)}$). This last property is in turn a consequence of the tightness assumption (7) and of (8): For $v \in \mathbb{Z}_{\text{even}}^d$, by (8), the distribution of $|g_n(v)|$ is stochastically bounded by the distribution of $|f_L(\mathbf{0})|$ where L is chosen so that $\Lambda_n \subset \Lambda(L) + v$. For $v \notin \mathbb{Z}_{\text{even}}^d$ the distribution of $|g_n(v)|$ is stochastically bounded by the distribution of $|g_n(w)| + 1$ for a neighbor w of v , by the definition of homomorphism height function.

We have shown that the random functions $|g_n|$ converge in distribution. Let us deduce the same for the g_n themselves. This is done by exhibiting a coupling of the g_n in which $g_n(v)$ converges almost surely for each v .

First, as the distributions of $|g_n|$ stochastically increase to a limit, there exists a coupling of the different $|g_n|$ so that $|g_n(v)|$ is non-decreasing for each v and converges almost surely. We assume that the $|g_n|$ are coupled in this way. Second, to extend this to a coupling of the g_n , introduce a collection $(\varepsilon_v)_{v \in \mathbb{Z}^d}$ of uniform and independent signs ± 1 and fix a natural-numbers ordering of \mathbb{Z}^d . Note that for each n the distribution of g_n conditioned on $|g_n|$ has the following structure: In each connected component of the set $\{v : |g_n(v)| > 0\}$ the values of g_n equal the values of $|g_n|$ times a random sign, with the signs being uniform and independent between different components. We couple the conditional distributions of g_n given $|g_n|$ by deciding that the sign of each component equals ε_v for the vertex v in the component which is lowest in the fixed order. As $|g_n|$ are non-decreasing, this choice ensures that for each v the sign of $g_n(v)$ will eventually stabilize as n increases, and thus that $g_n(v)$ converges almost surely as we wanted to show. \square

Now assume the tightness condition (7). It follows that for any sequence of subsets (Λ_n) as in the lemma, the sequence of finite-volume Gibbs measures $(\mu^{\Lambda_n, 0})$ converges in distribution. As is standard, the limiting measure μ is a Gibbs measure. To see that it is the same for all choices of (Λ_n) one may take two such sequences of subsets and interlace them (omitting elements as necessary to obtain an increasing sequence). This implies that μ is $\mathbb{Z}_{\text{even}}^d$ -translation-invariant, so that alternative (ii) of the theorem holds.

3. NO TRANSLATION-INVARIANT GIBBS MEASURES

Our proof of Theorem 1.2 is based on the following uniqueness statement for ergodic Gibbs measures, which is adapted from [39, Section 9]. The uniqueness statement is proved in the following sections, where Theorem 1.3 is further deduced.

Theorem 3.1. *In dimension $d = 2$: Let μ, μ' be $\mathbb{Z}_{\text{even}}^2$ -ergodic Gibbs measures. Then there is an integer k and a coupling of μ, μ' such that if (\tilde{f}, \tilde{g}) are sampled from the coupling then, almost surely,*

$$\tilde{f} = \tilde{g} + 2k. \quad (9)$$

Theorem 1.2 is derived from this statement with the following additional argument. Suppose, in order to obtain a contradiction, that μ is a $\mathbb{Z}_{\text{even}}^2$ -translation-invariant Gibbs measure. By replacing with an element of its ergodic decomposition, if necessary, we assume that μ is $\mathbb{Z}_{\text{even}}^2$ -ergodic (recalling that the $\mathbb{Z}_{\text{even}}^2$ -ergodic decomposition of a $\mathbb{Z}_{\text{even}}^2$ -translation-invariant Gibbs measure consists of $\mathbb{Z}_{\text{even}}^2$ -ergodic Gibbs measures [18, Theorem 14.15]). Let f be randomly sampled from μ . Define a homomorphism height function g on \mathbb{Z}^2 by

$$g(v) := f(-v + (1, 0)) - 1. \quad (10)$$

One checks in a straightforward way that the distribution of g is also a $\mathbb{Z}_{\text{even}}^2$ -ergodic Gibbs measure, which we denote by μ' . Thus, by Theorem 3.1, there exists an integer k and a coupling of μ, μ' such that, when sampling (\tilde{f}, \tilde{g}) from this coupling, the equality (9) holds almost surely. Continuing, we observe that (10) implies that

$$f((0, 0)) + f((1, 0)) = g((1, 0)) + g((0, 0)) + 2.$$

Thus, the equality in distribution

$$\tilde{f}((0, 0)) + \tilde{f}((1, 0)) \stackrel{d}{=} \tilde{g}((0, 0)) + \tilde{g}((1, 0)) + 2$$

also holds. However, by (9),

$$\tilde{f}((0, 0)) + \tilde{f}((1, 0)) = \tilde{g}((0, 0)) + \tilde{g}((1, 0)) + 4k.$$

This implies that $4k = 2$, which contradicts the fact that k is an integer. The contradiction establishes Theorem 1.2.

4. TOOLS

In this section we detail some well-known tools which will be used in the proof of Theorem 3.1. All of the described results apply in general dimension d except the coexistence results of Section 4.5 which rely on planarity.

4.1. Positive association for pointwise partial order. Since the maximum and minimum of two homomorphism height functions are still homomorphism height functions the following property is an immediate consequence of the standard FKG inequality [14] (see [3, Proposition 2.2]), with its second part derived from the reverse martingale convergence theorem.

Lemma 4.1. *Let $\Lambda \subset \mathbb{Z}^d$ be a non-empty finite subset and $\tau : \partial_o \Lambda \rightarrow \mathbb{Z}$ be a homomorphism height function. If f is randomly sampled from $\mu^{\Lambda, \tau}$ then f has positive association, in the sense that if $\varphi, \psi : (\mathbb{Z})^{\Lambda^+} \rightarrow [0, \infty)$ are non-decreasing (in the pointwise partial order on integer-valued functions) one has*

$$\mathbb{E}(\varphi(f)\psi(f)) \geq \mathbb{E}(\varphi(f))\mathbb{E}(\psi(f)). \quad (11)$$

Consequently, if μ is an extremal Gibbs measure and f is sampled from μ then (11) continues to hold for all measurable, non-decreasing $\varphi, \psi : (\mathbb{Z})^{\mathbb{Z}^d} \rightarrow [0, \infty)$.

4.2. Gibbs measures for homomorphism pairs and infinite cluster swaps. A key idea in the proof is to perform operations on a pair of homomorphism height functions (f, g) (especially involving the difference $f - g$). This idea will be pursued here and in Section 4.3 below.

Say that a measure ρ on the space $(\mathbb{Z}^2)^{\mathbb{Z}^d}$ is a *Gibbs measure for homomorphism pairs* if, when (f, g) is sampled from ρ , then for any non-empty, finite $\Lambda \subset \mathbb{Z}^d$, the conditional distribution of $(f, g)_{\Lambda^+}$, conditioned on $(f, g)_{\Lambda^c}$, equals the product measure $\mu^{\Lambda, \tau_f} \otimes \mu^{\Lambda, \tau_g}$ with $\tau_f := f_{\partial_o \Lambda}$ and $\tau_g := g_{\partial_o \Lambda}$, almost surely (that is, ρ conforms to the product specification). We note two simple properties which will be used repeatedly:

- If μ_1, μ_2 are Gibbs measures (for homomorphism height functions on \mathbb{Z}^d) then the product measure $\mu_1 \otimes \mu_2$ is a Gibbs measure for homomorphism pairs.
- If ρ is a Gibbs measure for homomorphism pairs and (f, g) is sampled from ρ , then each of the marginal distributions of f and of g are Gibbs measures (for homomorphism height functions on \mathbb{Z}^d).

In the following lemma and later in the paper we shorthand expressions such as $\{v \in \mathbb{Z}^d : f_v > g_v\}$ to $\{f > g\}$.

Lemma 4.2. *Let (f, g) be sampled from a Gibbs measure for homomorphism pairs. Denote*

$$C = \{v: v \text{ belongs to an infinite connected component of } \{f > g\}\}. \quad (12)$$

Define a new pair (\tilde{f}, \tilde{g}) of homomorphism height functions on \mathbb{Z}^d by

$$(\tilde{f}(v), \tilde{g}(v)) := \begin{cases} (f(v), g(v)) & v \notin C, \\ (g(v), f(v)) & v \in C. \end{cases} \quad (13)$$

Then the distribution of (\tilde{f}, \tilde{g}) is a Gibbs measure for homomorphism pairs.

The idea of swapping (f, g) on connected components where $f > g$ is called *cluster swapping* [39, Chapter 8] and is a main tool in the analysis there. Related ideas have been used by van den Berg [41] in developing *disagreement percolation* and were further advanced by van den Berg-Steif [43] and van den Berg-Maes [42]. A short review of similar techniques is in [8, Section 2].

Proof of Lemma 4.2. Fix a non-empty, finite $\Lambda \subset \mathbb{Z}^d$ throughout the proof. Our goal is to show that

$$\begin{aligned} &\text{conditioned on } (\tilde{f}, \tilde{g})_{\Lambda^c}, \text{ the distribution of } (\tilde{f}, \tilde{g}) \text{ is uniform} \\ &\text{on the set of homomorphism pairs extending } (\tilde{f}, \tilde{g})_{\Lambda^c}. \end{aligned} \quad (14)$$

Let

$$E := \{v \in \Lambda^c : v \text{ belongs to a finite connected component of } (\{\tilde{f} \neq \tilde{g}\} \setminus \Lambda) \text{ which intersects } \partial_o \Lambda\}. \quad (15)$$

Note that E is necessarily finite, is measurable with respect to $(\tilde{f}, \tilde{g})_{\Lambda^c}$ and that it remains unchanged if one replaces (\tilde{f}, \tilde{g}) by (f, g) in its definition since $\{\tilde{f} \neq \tilde{g}\} = \{f \neq g\}$. We will prove that

$$\begin{aligned} &\text{conditioned on } (\tilde{f}, \tilde{g})_{\Lambda^c} \text{ and } (f, g)_{(\Lambda \cup E)^c}, \text{ the distribution of } (\tilde{f}, \tilde{g}) \text{ is uniform} \\ &\text{on the set of homomorphism pairs extending } (\tilde{f}, \tilde{g})_{\Lambda^c}. \end{aligned} \quad (16)$$

This implies (14) by averaging over $(f, g)_{(\Lambda \cup E)^c}$.

As a first step we prove that

$$\mathcal{L} := \sigma((\tilde{f}, \tilde{g})_{\Lambda^c}, (f, g)_{(\Lambda \cup E)^c}) = \sigma(E, (\tilde{f}, \tilde{g})_E, (f, g)_{(\Lambda \cup E)^c}) =: \mathcal{R}, \quad (17)$$

where we write $\sigma(X, Y, \dots)$ for the sigma algebra generated by X, Y, \dots . As it is clear that $\mathcal{L} \supset \mathcal{R}$, it suffices to prove that $(\tilde{f}, \tilde{g})_{(\Lambda \cup E)^c}$ is measurable with respect to \mathcal{R} . Taking into account the definition (13), we need only show that

$$C \setminus (\Lambda \cup E) \text{ is measurable with respect to } \mathcal{R}. \quad (18)$$

Let $v \notin \Lambda \cup E$ and let D_v be the connected component of $(\{f \neq g\} \setminus \Lambda)$ containing v . The definition of E and the fact that $v \notin \Lambda \cup E$ imply that $D_v \subset (\Lambda \cup E)^c$, as follows by considering separately the cases that D_v is finite or infinite, and also that if D_v is finite then it equals the connected component of v in $\{f \neq g\}$. Thus the information in $(f, g)_{(\Lambda \cup E)^c}$ suffices to determine if $v \in C$, implying (18) and thus also (17).

As a second step we prove that

$$\text{the conditional distribution of } (f, g) \text{ given } \mathcal{R} \text{ is uniform on its support.} \quad (19)$$

Indeed, we may condition on the random variables generating \mathcal{R} sequentially. For each $E_0 \subset \mathbb{Z}^d$ with $\mathbb{P}(E = E_0) > 0$ we may first condition on $(f, g)_{(\Lambda \cup E_0)^c}$ and then, on the event that $\mathbb{P}(E = E_0 | (f, g)_{(\Lambda \cup E_0)^c}) > 0$ we may condition further on $\{E = E_0\}$ and on $(\tilde{f}, \tilde{g})_{E_0}$. After the first conditioning the distribution of (f, g) is uniform on its support due to the fact that (f, g) is sampled from a Gibbs measure for homomorphism pairs, and thus this property remains true also after the subsequent conditioning.

To finish the proof of the lemma we proceed to show (16), noting that we may replace the conditioning on \mathcal{L} in this statement by a conditioning on \mathcal{R} due to (17). Denote by ϕ the map taking (f, g) to (\tilde{f}, \tilde{g}) given by (13). Condition on \mathcal{R} and proceed with $E, (\tilde{f}, \tilde{g})_E$ and $(f, g)_{(\Lambda \cup E)^c}$ fixed. By (17) and (18), $(\tilde{f}, \tilde{g})_{\Lambda^c}$ and $C \setminus (\Lambda \cup E)$ are also fixed. Due to the uniformity (19) we may deduce (16) by showing that ϕ is a bijection between the sets A and B with

$$A := \{(f_0, g_0) \text{ homomorphism height functions: } (f_0, g_0)_{(\Lambda \cup E)^c} = (f, g)_{(\Lambda \cup E)^c}, \phi(f_0, g_0)_E = (\tilde{f}, \tilde{g})_E\},$$

$$B := \{(\tilde{f}_0, \tilde{g}_0) \text{ homomorphism height functions: } (\tilde{f}_0, \tilde{g}_0)_{\Lambda^c} = (\tilde{f}, \tilde{g})_{\Lambda^c}\}$$

(noting that the set E defined with (f_0, g_0) replacing (\tilde{f}, \tilde{g}) in (15) is the same for all $(f_0, g_0) \in A$). Note that $\phi(A) \subset B$ as $C \setminus (\Lambda \cup E)$ is fixed. We prove that ϕ is bijective by constructing a mapping $\psi : B \rightarrow A$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity maps. For $(f_0, g_0) \in A$ define $C(f_0, g_0)$ by (12) with (f_0, g_0) replacing (\tilde{f}, \tilde{g}) . For $(\tilde{f}_0, \tilde{g}_0) \in B$ define

$$\tilde{C}(\tilde{f}_0, \tilde{g}_0) = \{v : v \text{ belongs to the infinite connected component of } \{\tilde{f}_0 < \tilde{g}_0\} \text{ that intersects } C \setminus (\Lambda \cup E)\}$$

and

$$\psi(\tilde{f}_0, \tilde{g}_0) := \begin{cases} (\tilde{f}_0, \tilde{g}_0) & \text{on } \tilde{C}(\tilde{f}_0, \tilde{g}_0)^c, \\ (\tilde{g}_0, \tilde{f}_0) & \text{on } \tilde{C}(\tilde{f}_0, \tilde{g}_0). \end{cases}$$

It is now not difficult to verify that $\psi(B) \subset A$, $C(f_0, g_0) = \tilde{C}(\psi(f_0, g_0))$ for $(f_0, g_0) \in A$ and $C(\psi(\tilde{f}_0, \tilde{g}_0)) = \tilde{C}(\tilde{f}_0, \tilde{g}_0)$ for $(\tilde{f}_0, \tilde{g}_0) \in B$. This implies that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity mappings, as required. \square

4.3. Disagreement percolation. In this section we consider the operation of swapping finite clusters of disagreement of a pair of homomorphism height functions.

Write \mathcal{C}_{fin} for the (countable) collection of all finite, connected (non-empty) subsets of \mathbb{Z}^d .

Lemma 4.3. *Let (f, g) be sampled from a Gibbs measure for homomorphism pairs. Let $\varepsilon : \mathcal{C}_{\text{fin}} \rightarrow \{0, 1\}$. Define a new pair of homomorphism height functions (\tilde{f}, \tilde{g}) as follows:*

$$(\tilde{f}, \tilde{g})(v) = \begin{cases} (g, f)(v) & \text{there is a finite connected component } C \text{ of } \{f \neq g\} \text{ containing } v \text{ and } \varepsilon_C = 1, \\ (f, g)(v) & \text{otherwise.} \end{cases} \quad (20)$$

Then (\tilde{f}, \tilde{g}) has the same distribution as (f, g) .

Corollary 4.4. *Let (f, g) be sampled from a Gibbs measure for homomorphism pairs. Let $\varepsilon : \mathcal{C}_{\text{fin}} \rightarrow \{0, 1\}$. Define a new pair of homomorphism height functions (\bar{f}, \bar{g}) as follows:*

$$(\bar{f}, \bar{g})(v) = \begin{cases} (f, f)(v) & \text{there is a finite connected component } C \text{ of } \{f \neq g\} \text{ containing } v, \text{ and } \varepsilon_C = 0, \\ (g, g)(v) & \text{there is a finite connected component } C \text{ of } \{f \neq g\} \text{ containing } v, \text{ and } \varepsilon_C = 1, \\ (f, g)(v) & \text{otherwise.} \end{cases} \quad (21)$$

Then \bar{f} has the same distribution as f and \bar{g} has the same distribution as g .

Proof of Lemma 4.3. One checks that if $\varepsilon_n : \mathcal{C}_{\text{fin}} \rightarrow \{0, 1\}$ is a sequence converging to ε (i.e., for each $C \in \mathcal{C}_{\text{fin}}$ there exists n_C such that $(\varepsilon_n)_C = \varepsilon_C$ for all $n \geq n_C$) then $(\tilde{f}_n, \tilde{g}_n)$, defined via the recipe (20) with ε_n replacing ε , converges (pointwise) to (\tilde{f}, \tilde{g}) , almost surely. In particular, $(\tilde{f}_n, \tilde{g}_n)$ converges to (\tilde{f}, \tilde{g}) in distribution. Thus it suffices to prove the lemma under the assumption that $\varepsilon_C = 0$ for all but finitely many C ; an assumption which we now impose.

Fix a non-empty, finite $\Lambda \subset \mathbb{Z}^d$, large enough to satisfy that $C \subset \Lambda$ whenever $\varepsilon_C = 1$. By definition, $(\tilde{f}, \tilde{g})_{\Lambda^c} = (f, g)_{\Lambda^c}$. In addition, conditioned on $(f, g)_{\Lambda^c}$, the distribution of $(f, g)_\Lambda$ is uniform over all pairs of homomorphism height functions extending the boundary conditions, as (f, g) is sampled from a Gibbs measure for homomorphism pairs. It thus suffices to prove that this latter uniformity statement holds also for (\tilde{f}, \tilde{g}) . This now follows from the straightforward fact that, given ε and conditioned on $(f, g)_{\Lambda^c}$ (which equals $(\tilde{f}, \tilde{g})_{\Lambda^c}$), the definition (20) yields a bijection (in fact, an involution) between the homomorphism pairs $(f, g)_\Lambda$ and the homomorphism pairs $(\tilde{f}, \tilde{g})_\Lambda$ which extend the boundary conditions. \square

Proof of Corollary 4.4. Lemma 4.3 stipulates that the distribution of (\tilde{f}, \tilde{g}) (defined by (20)) equals the distribution of (f, g) , for any $\varepsilon : \mathcal{C}_{\text{fin}} \rightarrow \{0, 1\}$. However, comparing the definitions (20) and (21) makes it clear that $\bar{f} = \tilde{f}$ when the same sequence ε is used in both definitions. Thus $\bar{f} \stackrel{d}{=} f$. Similarly, $\bar{g} = \tilde{g}$ when the sequence ε is used in (21) and the sequence $1 - \varepsilon$ is used in (20). Thus $\bar{g} \stackrel{d}{=} g$. \square

Corollary 4.5. *Let (f, g) be sampled from a Gibbs measure for homomorphism pairs. If*

$$\mathbb{P}(\text{there is an infinite connected component of } \{f < g\}) = 0 \quad (22)$$

then the distribution of f stochastically dominates the distribution of g . Similarly, if

$$\mathbb{P}(\text{there is an infinite connected component of } \{f \neq g\}) = 0 \quad (23)$$

then the distribution of f equals the distribution of g .

Proof. Define (\bar{f}, \bar{g}) according to the recipe (21) with $\varepsilon \equiv 0$. Assuming (22) we obtain that $\bar{f} \geq \bar{g}$, which yields the required stochastic domination as $\bar{f} \stackrel{d}{=} f$ and $\bar{g} \stackrel{d}{=} g$ by Corollary 4.4. Similarly, assuming (23) it holds that $\bar{f} = \bar{g}$ implying that $f \stackrel{d}{=} g$. \square

Remark 4.6. The corollary generalizes, for homomorphism height functions, the disagreement percolation criterion of van den Berg [41] who proved that the assumption (23) implies equality of the marginal distributions when the Gibbs measure from which (f, g) are sampled is a product measure (f is independent from g). Van den Berg–Maes [42] also discuss generalizations but we have not seen Corollary 4.5 explicitly in the literature. We note that it is important that (f, g) are sampled from a Gibbs measure for homomorphism pairs rather than just from a coupling of Gibbs measures with the same specification, as [23] gives an example of such a coupling of distinct Gibbs measures for which (23) holds.

Remark 4.7. Cluster swapping was applied in [39, Section 8.2] to give short proofs of log-concavity of the distribution of single-site marginals (See Section 7) and monotonicity of the Gibbs measures with respect to boundary conditions, for general class of random surfaces with convex interactions (see also [8, Section 2]).

Corollary 4.8. Let μ_1, μ_2 be Gibbs measures and f, g be independently sampled from μ_1, μ_2 , respectively. If (23) holds then the two measures are equal and extremal.

Proof. The equality $\mu_1 = \mu_2$ follows from Corollary 4.5. We are left to prove that $\mu = \mu_1 = \mu_2$ is extremal. Otherwise $\mu = \frac{1}{2}(\nu_1 + \nu_2)$ for distinct Gibbs measures ν_1 and ν_2 ([18, Theorem 14.15]). Thereby the independent coupling of μ with itself would decompose as the average of the independent couplings of ν_1 with itself, ν_2 with itself, ν_1 with ν_2 and ν_2 with ν_1 . As (23) holds for each of these couplings it follows from Corollary 4.5 that $\nu_1 = \nu_2$, a contradiction. \square

4.4. Uniqueness of infinite clusters (following Burton–Keane). It is well known in percolation theory on \mathbb{Z}^d (and other amenable groups) that, almost surely, there is at most one infinite connected component of open sites. This fact was first proven by Aizenman–Kesten–Newman [1], with a subsequent shorter proof provided by Burton–Keane [5]. We show here that the ideas of Burton and Keane admit extension to the random set $\{f > g\}$ for suitable distributions of (f, g) . Similar arguments are used in [39, Lemma 8.5.6].

Theorem 4.9. Let (f, g) be sampled from an \mathcal{L} -translation-invariant Gibbs measure for homomorphism pairs, for a full-rank sublattice \mathcal{L} . Then

$$\mathbb{P}(\text{the set } \{f > g\} \text{ has two or more infinite connected components}) = 0.$$

Fix an integer $M > 0$. For a configuration $\omega \in \{0, 1\}^{\mathbb{Z}^d}$, regarded as a subset of \mathbb{Z}^d , say that the graph ball $v + \Lambda(M)$ is a *trifurcation ball* in ω if

- there is an infinite connected component \mathcal{C} of ω intersecting $v + \Lambda(M)$, and
- there is a connected set $D \subset \mathcal{C} \cap (v + \Lambda(M))$ such that $\mathcal{C} \setminus D$ has at least three infinite connected components.

The argument of Burton and Keane [5, Theorem 2] adapts to show that, for any $\omega \in \{0, 1\}^{\mathbb{Z}^d}$, the density of trifurcation balls is zero. That is,

$$\lim_{L \rightarrow \infty} \frac{1}{L^2} |\{v \in \Lambda(L) : v + \Lambda(M) \text{ is a trifurcation ball}\}| = 0.$$

The following statement is an immediate consequence, making use of the ergodic theorem.

Lemma 4.10. Let \mathbb{P} be a measure on $\{0, 1\}^{\mathbb{Z}^d}$ which is \mathcal{L} -translation-invariant for some full-rank sublattice \mathcal{L} . Then

$$\mathbb{P}(\Lambda(M) \text{ is a trifurcation ball}) = 0.$$

The proof of Theorem 4.9 relies on Lemma 4.10 and Lemma 4.11 given below.

Flatness of \mathcal{L} -translation-invariant measures: Fix a full rank sublattice $\mathcal{L} \subset \mathbb{Z}^d$. Let us recall some basic results about \mathcal{L} -translation-invariant probability measures on homomorphism height functions.

Suppose f is a given homomorphism height function and $\Delta \subset \mathbb{Z}^d$. Consider the homomorphism height functions given by

$$\begin{aligned} f_{\max, \Delta}(v) &= \min\{f(w) + \|v - w\|_1 : w \in \Delta\}, \\ f_{\min, \Delta}(v) &= \max\{f(w) - \|v - w\|_1 : w \in \Delta\}. \end{aligned}$$

It follows that

$$(f_{\max, \Delta})_{\Delta} = (f_{\min, \Delta})_{\Delta} = f_{\Delta}$$

and if g is a homomorphism height function such that $f_{\Delta} = g_{\Delta}$ then pointwise $f_{\min, \Delta} \leq g \leq f_{\max, \Delta}$.

Lemma 4.11. *Let f be sampled from an \mathcal{L} -translation-invariant probability measure on homomorphism height functions. For all positive integers M there exists m such that*

$$\begin{aligned} \mathbb{P}((f_{\max, \partial_o \Lambda(m)})_{\Lambda(M)} > \frac{m}{2}) &> \frac{3}{4} \\ \mathbb{P}((f_{\min, \partial_o \Lambda(m)})_{\Lambda(M)} < -\frac{m}{2}) &> \frac{3}{4} \end{aligned} \quad (24)$$

Proof. The following limit exists almost surely, for each $v \in \mathbb{Z}^d$, by the ergodic theorem

$$\lim_{n \rightarrow \infty} \frac{f(nv) - f(\mathbf{0})}{n} = \frac{1}{n} \sum_{i=0}^{n-1} (f((i+1)v) - f(iv)). \quad (25)$$

The limit is necessarily 0 as the Poincaré recurrence theorem implies that $f(nv) = f(\mathbf{0})$ for infinitely many n , almost surely. These statements can be used to prove that, almost surely,

$$\lim_{n \rightarrow \infty} \sup_{|v|_1 = n} \frac{|f(v)|}{n} = 0. \quad (26)$$

A detailed proof can be found in [6, Lemma 6.4]. Its idea is as follows: the previous statements on the limit (25) extend to

$$\lim_{n \rightarrow \infty} \frac{f(\lfloor nv \rfloor)}{n} = 0, \quad v \in \mathbb{R}^d, \quad (27)$$

where $\lfloor \cdot \rfloor$ indicates rounding down each coordinate to the next integer. For $v \in \mathbb{Z}^d$ this follows from the previous statements, and this case implies also the case that v has rational coordinates, making use of the homomorphism property (the fact that f changes by 1 between adjacent vertices). A density argument and another use of the homomorphism property then yields the full statement (27). Now apply (27) for a finite ε -dense set in the sphere $\{v \in \mathbb{R}^d : \|v\|_1 = 1\}$ and use the homomorphism property to conclude that, almost surely, $\sup_{|v|_1 = n} \frac{|f(v)|}{n} \leq 2\varepsilon$ for all n larger than some random threshold. This implies (26) as ε is arbitrary.

By (26), for all $\epsilon, \delta > 0$, there exists a positive integer M such that for all $m > M$,

$$\mathbb{P}(f_{\partial_o \Lambda(m)} \in (-m\epsilon, m\epsilon)) > 1 - \delta.$$

It follows that for all $\Lambda \subset \mathbb{Z}^d$ there exists M such that $\Lambda \subset \Lambda(M)$ and for all $m > M$

$$\begin{aligned} \mathbb{P}((f_{\max, \partial_o \Lambda(m)})_{\Lambda} > \frac{m}{2}) &> \frac{3}{4} \\ \mathbb{P}((f_{\min, \partial_o \Lambda(m)})_{\Lambda} < -\frac{m}{2}) &> \frac{3}{4} \end{aligned} \quad (28)$$

□

Proof of Theorem 4.9. By [18, Theorem 14.15], we have that the ergodic components of an \mathcal{L} -translation-invariant Gibbs measure for homomorphism pairs are Gibbs almost surely. Thus it is sufficient to prove the result for \mathcal{L} -ergodic Gibbs measure.

Suppose that there are infinitely many infinite connected components of $\{f > g\}$. Choose M large enough such that

$$\mathbb{P}(\text{at least three infinite connected components of } f > g \text{ intersect } \Lambda(M)) > \frac{3}{4}.$$

By (28) choose $m > M$ such that

$$\begin{aligned}\mathbb{P}((f_{\max, \partial_o \Lambda(m)})_{\Lambda(M)} > \frac{m}{2}) &> \frac{3}{4} \\ \mathbb{P}((g_{\min, \partial_o \Lambda(m)})_{\Lambda(M)} < -\frac{m}{2}) &> \frac{3}{4}.\end{aligned}$$

Putting these together we have that

$$\begin{aligned}\mathbb{P}(\text{at least three infinite connected components of } f > g \text{ intersect } \Lambda(M), \\ (f_{\max, \partial_o \Lambda(m)})_{\Lambda(M)} > \frac{m}{2} \text{ and } (g_{\min, \partial_o \Lambda(m)})_{\Lambda(M)} < -\frac{m}{2}) &> \frac{1}{4}.\end{aligned}$$

Since (f, g) are sampled from a Gibbs measure for homomorphism pairs it follows that

$$\mathbb{P}(\Lambda(m) \text{ is a trifurcation ball}) > 0.$$

By Lemma 4.10, we have a contradiction.

Now suppose that $\{f > g\}$ has more than one (but finitely many) connected components. Choose M such that

$$\mathbb{P}(\text{all the infinite connected components of } \{f > g\} \text{ intersect } \Lambda(M)) > \frac{3}{4}.$$

By the argument above we get that

$$\mathbb{P}(\{f > g\} \text{ has a unique connected component}) > 0.$$

But the number of connected components is constant almost surely. Hence there is a unique infinite connected component almost surely. \square

4.5. No coexistence of primal and dual percolation with unique infinite cluster. In the classical theory of percolation, it was shown by Harris [25] that in bond percolation on \mathbb{Z}^2 there is no infinite cluster at $p = 1/2$, almost surely. Indeed, if the infinite cluster had positive probability to appear, then it would appear almost surely, and by symmetry, also the dual percolation would have an infinite cluster almost surely. However, in a sense, planar geometry places too many constraints on how the two clusters should co-exist for this to be possible. This argument has been generalized considerably in order to weaken the required assumptions. The version required here is based on an idea of Zhang.

Theorem 4.12. [24, Theorem 14.3] *For a full-rank sublattice \mathcal{L} , if μ is a probability measure on $\{0, 1\}^{\mathbb{Z}^2}$ satisfying*

- (1) *(Ergodicity) μ is \mathcal{L} -ergodic.*
- (2) *(Positive association) For any two increasing events A, B , $\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$.*
- (3) *(Symmetry) Invariance under interchange of coordinates and reflection in coordinate hyperplanes.*
- (4) *(Unique infinite cluster) There is at most one infinite connected component of 0's and at most one infinite connected component of 1's.*

Then the probability that there is simultaneous existence of an infinite connected component of 0's and an infinite connected component of 1's is zero.

Strictly speaking, [24, Theorem 14.3] is for measures satisfying assumption (1) with $\mathcal{L} = \mathbb{Z}^d$, (2), (3) and an additional “finite energy” assumption which means roughly that every pattern appears with positive probability. However the proof given there carries forward to \mathcal{L} -ergodic probability measures as well and the only place where the “finite energy” assumption is used is to prove assumption (4) above.

A further generalization which removes the symmetry assumption is proved in ([39, Theorem 9.3.1 and Corollary 9.4.6]) and also in [13, Theorem 1.5].

5. ERGODIC GIBBS MEASURES ARE EXTREMAL

The main goal of this section is to prove Theorem 1.3 and its analogue for $d = 2$.

Fix $d \geq 2$. Suppose μ is an \mathcal{L} -ergodic Gibbs measure for a full-rank sublattice \mathcal{L} . We will first show that μ is extremal. Let f, g be independently sampled from two \mathcal{L} -ergodic Gibbs measures μ_1 and μ_2 . Define the events

$$\begin{aligned} E^+ &:= \{\text{the set } \{f > g\} \text{ contains an infinite connected component}\}, \\ E^- &:= \{\text{the set } \{f < g\} \text{ contains an infinite connected component}\}. \end{aligned}$$

For $\mu_1 = \mu_2 = \mu$ we will show that

$$\mathbb{P}(E^+) = \mathbb{P}(E^-) = 0. \quad (29)$$

By Corollary 4.8 we get that μ is extremal. In the following lemma we show that E^+ and E^- cannot occur simultaneously.

Lemma 5.1.

$$\mathbb{P}(E^+ \cap E^-) = 0. \quad (30)$$

Proof. Define a new pair of functions \tilde{f}, \tilde{g} according to the recipe (12), (13) given in Lemma 4.2. The construction makes it clear that the occurrence of $E^+ \cap E^-$ implies the occurrence of the event

$$E = \{\text{the set } \{\tilde{f} < \tilde{g}\} \text{ has two or more infinite connected components}\}$$

Thus it suffices to show that $\mathbb{P}(E) = 0$. To this end, apply Lemma 4.2 to conclude that the distribution of (\tilde{f}, \tilde{g}) is a Gibbs measure for the homomorphism pairs. In addition, the construction implies that this Gibbs measure retains the \mathcal{L} -translation-invariance property of (f, g) . Thus we may apply Theorem 4.9 (to the pair (\tilde{g}, \tilde{f})) to conclude that $\mathbb{P}(E) = 0$, as required. \square

Our use of the fact that μ is \mathcal{L} -ergodic for a full-rank sublattice \mathcal{L} is in the following property: there exists a sequence $(p_k)_{k \in \mathbb{Z}}$ so that

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda(L)|} |\{v \in \Lambda(L) : f(v) = k\}| = p_k \quad \text{for each } k \in \mathbb{Z}, \text{ almost surely.} \quad (31)$$

Let us also define a function $I(f, g) : \mathbb{Z}^d \rightarrow \{0, 1\}$ by

$$I(f, g)(v) = 1 \text{ if and only if } v \text{ belongs to an infinite connected component of } \{f > g\}.$$

Setting $\mu_1 = \mu_2 = \mu$, the \mathcal{L} -translation-invariance of the product measure of μ with itself implies that

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda(L)|} \sum_{v \in \Lambda(L)} I(f, g)(v) > 0, \quad \text{almost surely on the event } E^+. \quad (32)$$

Now define a pair of homomorphism height functions (\bar{f}, \bar{g}) via the recipe (21) applied with $\varepsilon \equiv 0$. Then

$$\begin{aligned} \bar{f} &\geq \bar{g} \quad \text{on the event } (E^-)^c, \\ I(f, g) &= I(\bar{f}, \bar{g}). \end{aligned} \quad (33)$$

In addition, Corollary 4.4 shows that $\bar{f} \stackrel{d}{=} f$ and $\bar{g} \stackrel{d}{=} g$. In particular, referring to (31),

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda(L)|} |\{v \in \Lambda(L) : \bar{f}(v) = k\}| = \lim_{L \rightarrow \infty} \frac{1}{|\Lambda(L)|} |\{v \in \Lambda(L) : \bar{g}(v) = k\}| \quad \text{for each } k \in \mathbb{Z}, \text{ almost surely.} \quad (34)$$

Assume now, in order to obtain a contradiction, that (by symmetry) $\mathbb{P}(E^+) = \mathbb{P}(E^-) > 0$. Lemma 5.1 then implies that $\mathbb{P}(E^+ \setminus E^-) > 0$. Thus, combining (32) with (33) we conclude that

$$\bar{f} \geq \bar{g} \text{ and } \lim_{L \rightarrow \infty} \frac{1}{|\Lambda(L)|} |\{v : \bar{f}(v) > \bar{g}(v)\}| > 0, \quad \text{almost surely on the event } E^+ \setminus E^-,$$

in contradiction with (34). We conclude that (29) holds proving that μ is extremal.

Corollary 5.2. *Let μ_1 and μ_2 be two distinct \mathcal{L} -ergodic Gibbs measures for a full-rank sublattice \mathcal{L} . Either μ_1 stochastically dominates μ_2 or vice versa.*

Proof. Let (f, g) be sampled from the independent product of μ_1 and μ_2 . By extremality of μ_1 and μ_2 we have that the probability of $\{f > g\}$ having an infinite connected component is either 0 or 1. If it is 1 then by Theorem 5.1 we have that all components of $\{f < g\}$ are finite with probability 1 as well. By Corollary 4.5 we have that μ_1 stochastically dominates μ_2 . Similarly if $\{f < g\}$ has an infinite connected component then μ_2 stochastically dominates μ_1 . \square

We are left to prove that if μ is an \mathcal{L} -ergodic Gibbs measures then it is in fact $\mathbb{Z}_{\text{even}}^d$ -invariant.

Corollary 5.3. *For full rank-sublattices \mathcal{L} , the \mathcal{L} -ergodic Gibbs measures are invariant under translations by $\mathbb{Z}_{\text{even}}^d$, interchange of coordinates and reflections in coordinate hyperplanes.*

This will enable us to rely on Zhang's theorem (Theorem 4.12) in the later parts of the paper.

Proof. Let S be one of the following maps on the space of probability measures on \mathbb{Z}^d : translation by a vector in $\mathbb{Z}_{\text{even}}^d$, reflection along coordinate hyperplanes and permutation of coordinates. Given an ergodic Gibbs measures μ , $S(\mu)$ is also an ergodic Gibbs measure. If μ and $S(\mu)$ are distinct, by Corollary 5.2 we have that either μ strictly stochastically dominates $S(\mu)$ or vice versa. Without loss of generality assume that μ strictly stochastically dominates $S(\mu)$. But then μ strictly stochastically dominates $S^n(\mu)$ for all positive integers n . Since there exists a positive integer n such that $S^n(\mu) = \mu$ this leads to a contradiction and completes the proof. \square

6. UNIQUENESS UP TO ADDITIVE CONSTANT OF ERGODIC GIBBS MEASURES

In this section we prove Theorem 3.1.

Fix the dimension $d = 2$ throughout the section. Let f, g be independent samples from $\mathbb{Z}_{\text{even}}^2$ -ergodic Gibbs measures μ, μ' . Our goal is to show that there is an integer k_0 such that the distribution of f equals the distribution of $g + 2k_0$.

Applying Theorem 1.3 we have that μ and μ' are extremal Gibbs measures. A main consequence of extremality that we shall use is that

both f and g have positive association (for the pointwise partial order),

according to Lemma 4.1. Since $(f, -g)$ have been sampled independently and the random fields of the form $1_{f > g + 2k}$ are increasing functions of $(f, -g)$, they have positive association as well.

For each integer k , define the events

$$E_k^+ := \{\text{the set } \{f \geq g + 2k\} \text{ contains an infinite connected component}\},$$

$$E_k^- := \{\text{the set } \{f < g + 2k\} \text{ contains an infinite connected component}\}.$$

As another consequence of extremality, we note that the (joint) distribution of (f, g) is an extremal $\mathbb{Z}_{\text{even}}^2$ -translation-invariant Gibbs measure for homomorphism pairs. Thus, for each integer k ,

$$\mathbb{P}(E_k^+), \mathbb{P}(E_k^-) \in \{0, 1\}. \quad (35)$$

The following statement is the main claim that we use to prove Theorem 3.1.

Lemma 6.1. *For each integer k , $\mathbb{P}(E_k^+ \cap E_k^-) = 0$.*

Proof. Suppose for contradiction that $\mathbb{P}(E_k^+ \cap E_k^-) = 1$. By Theorem 4.9, $\{f \geq g + 2k\}$ and $\{f < g + 2k\}$ each have a unique infinite connected component. In addition, we have by Corollary 5.3 and the discussion above that

- (1) $1_{f > g + 2k}$ has positive association.
- (2) $1_{f > g + 2k}$ is invariant under interchange of coordinates and reflection in coordinate hyperplanes.

This contradicts Theorem 4.12. \square

We continue to study the events E_k^+ and E_k^- . It is clear that

$$\mathbb{P}(E_{k+1}^+) \leq \mathbb{P}(E_k^+), \quad \mathbb{P}(E_{k+1}^-) \geq \mathbb{P}(E_k^-). \quad (36)$$

Combining the relations (35) and (36) with Lemma 6.1, and exchanging the roles of f and g if necessary, we see that one of the following cases must occur:

- (1) There exists an integer k_0 such that $\mathbb{P}(E_{k_0+1}^+) = 0$ and $\mathbb{P}(E_{k_0}^-) = 0$.
- (2) For all integer k , $\mathbb{P}(E_k^-) = 0$.

Note, however, that if $\mathbb{P}(E_k^-) = 0$ for some k then the distribution of f stochastically dominates the distribution of $g + 2k$ by Corollary 4.5. Thus the second case implies that f dominates $g + 2k$ for all integer k . This cannot occur. Suppose then that the first case occurs for some integer k_0 . Applying Corollary 4.5 again shows that the distribution of f equals the distribution of $g + 2k_0$, completing the proof of Theorem 3.1.

7. LOG-CONCAVITY AND SOME CONSEQUENCES

We further derive the log-concavity of the distribution of the height at a vertex.

Proposition 7.1. *Let $\Lambda \subset \mathbb{Z}^d$ be finite and $\tau : \partial_o \Lambda \rightarrow \mathbb{Z}$ be a homomorphism height function. Let f be sampled from $\mu^{\Lambda, \tau}$. Then for any $v \in \Lambda$, the distribution of $f(v)$ is log-concave in the sense that*

$$\mathbb{P}(f(v) = 2m)^2 \geq \mathbb{P}(f(v) = 2m + 2k) \cdot \mathbb{P}(f(v) = 2m - 2k), \quad k, m \in \mathbb{Z}. \quad (37)$$

Consequently, if f is sampled from an extremal Gibbs measure then (37) remains true at any $v \in \mathbb{Z}^d$ and, in particular, $f(v)$ has finite moments of all orders.

Log-concavity is known more generally for nearest-neighbor convex potentials from [39, Lemma 8.2.4] and for more general graphs (but prescribing boundary values at a single vertex rather than a whole set) by Kahn [26, Proposition 2.1]. While the proofs are similar, we provide a proof for completeness.

Proof. Let $n \in 2\mathbb{Z}$ and $m \in \{2, 4, \dots\}$. We want to prove that

$$\mathbb{P}(f(v) = n - m) \cdot \mathbb{P}(f(v) = n + m) \leq \mathbb{P}(f(v) = n)^2.$$

For this, it is sufficient to build an injection from

$$\{\tilde{f} : \tilde{f}_{\partial_o \Lambda} = f_{\partial_o \Lambda}, \tilde{f}(v) = n - m\} \times \{\tilde{f} : \tilde{f}_{\partial_o \Lambda} = f_{\partial_o \Lambda}, \tilde{f}(v) = n + m\} \rightarrow \{\tilde{f} : \tilde{f}_{\partial_o \Lambda} = f_{\partial_o \Lambda}, \tilde{f}(v) = n\}^2$$

Let \tilde{f}, \tilde{g} be homomorphism height functions on Λ^+ such that $\tilde{f}_{\partial_o \Lambda} = \tilde{g}_{\partial_o \Lambda} = f_{\partial_o \Lambda}$, $\tilde{f}(v) = n + m$ and $\tilde{g}(v) = n - m$. Let Λ' be the smallest connected set containing v on which $\tilde{f} > \tilde{g} + m$. Since $\tilde{f}_{\partial_o \Lambda} = \tilde{g}_{\partial_o \Lambda}$ we must have that $\Lambda' \subsetneq \Lambda$. We can use this to define homomorphism height functions h, h' on Λ^+ given by

$$h_{\Lambda'} := (\tilde{f} - m)_{\Lambda'}, h_{\Lambda \setminus \Lambda'} := \tilde{g}_{\Lambda \setminus \Lambda'} \text{ and } h'_{\Lambda'} := (\tilde{g} + m)_{\Lambda'}, h'_{\Lambda \setminus \Lambda'} := \tilde{f}_{\Lambda \setminus \Lambda'}.$$

Further Λ' can be recovered from the pair (h, h') as the smallest connected set containing v on which $h > h' - m$. Thus the map $(\tilde{f}, \tilde{g}) \rightarrow (h, h')$ is injective. Finally $h(v) = h'(v) = n$. \square

Here is an immediate corollary of Theorem 1.3 and Proposition 7.1 proving that $\mathbb{Z}_{\text{even}}^d$ -ergodic Gibbs measures can be parametrized by the mean of the height at the origin.

Corollary 7.2. *Let f and g be sampled from $\mathbb{Z}_{\text{even}}^d$ -ergodic Gibbs measures. The mean of $f(\mathbf{0})$ is strictly greater than the mean of $g(\mathbf{0})$ if and only if f strictly stochastically dominates g .*

Proof. Let f, g be sampled independently from their respective distributions. By Corollary 5.2, it is sufficient to prove that if f strictly stochastically dominates g then the mean of $f(\mathbf{0})$ is strictly greater than the mean of $g(\mathbf{0})$. Consider (\bar{f}, \bar{g}) as in Corollary 4.4 with $\varepsilon \equiv 0$. Then \bar{f} has the same distribution as f , \bar{g} has the same distribution as g and by Corollary 4.5 we have that $\bar{f} \geq \bar{g}$ pointwise and that $\{\bar{f} > \bar{g}\}$ has an infinite connected component. In particular it follows that there exists $v \in \mathbb{Z}_{\text{even}}^d$ such that $\bar{f}(v)$ strictly stochastically dominates $\bar{g}(v)$. Since the measures are $\mathbb{Z}_{\text{even}}^d$ -translation invariant it follows that $f(\mathbf{0})$ strictly stochastically dominates $g(\mathbf{0})$. By the log concavity of the marginals (Proposition 7.1) it follows that the mean of $f(\mathbf{0})$ exists and is greater than the mean of $g(\mathbf{0})$. \square

Lastly, we deduce that various notions of delocalization are equivalent in our case.

Proposition 7.3. *For odd integers $L \geq 1$, let f_L be sampled from $\mu^{\Lambda(L), 0}$. The following are equivalent:*

- (1) $\inf_L \mathbb{P}(|f_L(\mathbf{0})| = 0) = 0$.
- (2) $\lim_{M \rightarrow \infty} \sup_L \mathbb{P}(|f_L(\mathbf{0})| > M) > 0$.
- (3) $\sup_L \mathbb{P}(|f_L(\mathbf{0})| > M) = 1$ for all $M > 0$.
- (4) $\sup_L \mathbb{E}(\exp(\alpha \cdot |f_L(\mathbf{0})|)) = \infty$ for all $\alpha > 0$.

Proof. Since the random variable $f_L(\mathbf{0})$ is symmetric and log-concave, $\mathbb{P}(f_L(\mathbf{0}) = 2n)$ is non-increasing as a function of integer $n \geq 0$. Thus (1) implies (3). It is obvious that (3) implies (2) and (4). Let us see why each of (2) and (4) imply (1). Let $\theta_L = \mathbb{P}(f_L(\mathbf{0}) = 0)$. If

$$2m_L = \min \left\{ n \in \{2, 4, \dots\} : \mathbb{P}(f_L(\mathbf{0}) = n) < \frac{\theta_L}{2} \right\}$$

then $2m_L < \frac{2}{\theta_L}$. Thus $\mathbb{P}(f_L(\mathbf{0}) = 2km_L) < \frac{\theta_L}{2^k}$ for all k . From here it is easy to see that both (2) and (4) individually imply that $\sup_L m_L = \infty$ whence $\inf_L \theta_L = 0$. This completes the proof. \square

8. DISCUSSIONS AND OPEN QUESTIONS

- (1) In Theorem 1.2 we prove that two-dimensional homomorphism height functions do not admit $\mathbb{Z}_{\text{even}}^2$ -translation-invariant Gibbs measures. It is however easy to construct Gibbs measures which are not translation-invariant. A probability measure on homomorphism height functions is called *almost frozen* if its sample f satisfies

$$\mathbb{P} \left(\lim_{n \rightarrow \infty} \frac{f(u + ne) - f(u)}{n} = 1 \right) = 1$$

for some fixed unit vector e and all $u \in \mathbb{Z}^2$. There are uncountably many extremal Gibbs measures which are almost frozen. The following question naturally arises: Are all extremal Gibbs measures on two-dimensional homomorphism height functions almost frozen?

- (2) Let $d \geq 3$. Let f be sampled from a $\mathbb{Z}_{\text{even}}^d$ -ergodic Gibbs measure μ and fix a unit vector e . For all positive integers n another $\mathbb{Z}_{\text{even}}^d$ -ergodic Gibbs measure ν can be constructed from it such that its sample g satisfies $g(v) = f(v + ne) + n$ in distribution. Are these all the $\mathbb{Z}_{\text{even}}^d$ -ergodic Gibbs measures?

By Corollary 7.2 we know that the $\mathbb{Z}_{\text{even}}^d$ -ergodic Gibbs measures can be parametrized by the mean of the height function at the origin. Thus the question can be rephrased as the following: For any two distinct $\mathbb{Z}_{\text{even}}^d$ -ergodic Gibbs measures do the means of the corresponding height functions at the origin differ by an integer?

- (3) Given graphs G and H , we write $G \times H$ to denote the tensor product of G with H (in the tensor product (g, h) is adjacent to (g', h') if g is adjacent to g' and h is adjacent to h'). It is easy to see that \mathbb{Z}^2 is isomorphic to the both the connected components of $\mathbb{Z} \times \mathbb{Z}$ (by the mapping $(i, j) \mapsto (i+j, i-j)$). It follows that our results for homomorphism height functions also hold for graph homomorphisms from \mathbb{Z}^d to \mathbb{Z}^2 . The following question naturally arises: Let $f_{L,m}$ be a graph homomorphism sampled uniformly from $\Lambda(L)^+ \subset \mathbb{Z}^2$ to \mathbb{Z}^m , normalized by having $f_{L,m}$ map $\partial_o \Lambda(L)$ to the origin of \mathbb{Z}^m . For which m does

$$\sup_L \text{Var}(|f_{L,m}(\mathbf{0})|) = \infty? \tag{38}$$

- (4) As described in the introduction, the model of uniformly sampled homomorphism height functions on \mathbb{Z}^2 is the $c = 1$ case of the *F-model*, which is a one-parameter sub-family of the 6-vertex model indexed by the parameter $c > 0$. In the F-model one weighs a homomorphism height function f by $c^{-N(f)}$ where $N(f)$ is the number of diagonally-adjacent vertex pairs on which f is unequal. It is conjectured that the F-model is delocalized if and only if $c \leq 2$ (as a subset of the disordered regime of the 6-vertex model). Presently, this conjecture has been verified only when $c = 1$, in this work, and when $c \geq 2$ by a coupling with the random-cluster model [10, 21]. It is of interest to verify the conjecture for additional values of c .

REFERENCES

- [1] M. Aizenman, H. Kesten, and C. M. Newman. Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Communications in Mathematical Physics*, 111(4):505–531, 1987.
- [2] R. J. Baxter. *Exactly solved models in statistical mechanics*. Courier Corporation, 2007.
- [3] I. Benjamini, O. Häggström, and E. Mossel. On random graph homomorphisms into \mathbb{Z} . *Journal of Combinatorial Theory, Series B*, 78(1):86–114, 2000.
- [4] N. Berestycki, B. Laslier, and G. Ray. Universality of fluctuations in the dimer model. *arXiv preprint arXiv:1603.09740*, 2016.

- [5] R. M. Burton and M. Keane. Density and uniqueness in percolation. *Comm. Math. Phys.*, 121(3):501–505, 1989.
- [6] N. Chandgotia and T. Meyerovitch. Markov random fields, Markov cocycles and the 3-colored chessboard. *Israel J. Math.*, 215(2):909–964, 2016.
- [7] N. Chandgotia, I. Pak, and M. Tassy. Kirszbraun-type theorems for graphs. *arXiv preprint arXiv:1710.11007*, 2017.
- [8] O. Cohen-Alloro and R. Peled. Rarity of extremal edges in random surfaces and other theoretical applications of cluster algorithms. *arXiv preprint arXiv:1711.00259*, 2017.
- [9] J. Dubédat. Exact Bosonization of the Ising model. *arXiv preprint arXiv:1112.4399*, 2011.
- [10] H. Duminil-Copin, M. Gagnebin, M. Harel, I. Manolescu, and V. Tassion. Discontinuity of the phase transition for the planar random-cluster and Potts models with $q > 4$. *arXiv preprint arXiv:1611.09877*, 2016.
- [11] H. Duminil-Copin, M. Gagnebin, M. Harel, I. Manolescu, and V. Tassion. The Bethe ansatz for the six-vertex and xxz models: An exposition. *Probability Surveys*, 15:102–130, 2018.
- [12] H. Duminil-Copin, A. Glazman, R. Peled, and Y. Spinka. Macroscopic loops in the loop $o(n)$ model at Nienhuis’ critical point. *arXiv preprint arXiv:1707.09335*, 2017.
- [13] H. Duminil-Copin, A. Raoufi, and V. Tassion. Sharp phase transition for the random-cluster and Potts models via decision trees. *arXiv:1705.03104*.
- [14] C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre. Correlation inequalities on some partially ordered sets. *Communications in Mathematical Physics*, 22(2):89–103, 1971.
- [15] J. Fröhlich and T. Spencer. The Kosterlitz-Thouless transition in two-dimensional abelian spin systems and the Coulomb gas. *Comm. Math. Phys.*, 81(4):527–602, 1981.
- [16] D. Galvin. On homomorphisms from the Hamming cube to \mathbb{Z} . *Israel Journal of Mathematics*, 138(1):189–213, 2003.
- [17] D. Galvin, J. Kahn, D. Randall, and G. B. Sorkin. Phase coexistence and torpid mixing in the 3-coloring model on \mathbb{Z}^d . *SIAM Journal on Discrete Mathematics*, 29(3):1223–1244, 2015.
- [18] H.-O. Georgii. *Gibbs measures and phase transitions*, volume 9 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2011.
- [19] A. Giuliani, V. Mastropietro, and F. L. Toninelli. Haldane relation for interacting dimers. *J. Stat. Mech. Theory Exp.*, (3):034002, 47, 2017.
- [20] A. Glazman and I. Manolescu. *In preparation*, 2018.
- [21] A. Glazman and R. Peled. *In preparation*, 2018.
- [22] L. A. Goldberg, R. Martin, and M. Paterson. Random sampling of 3-colorings in \mathbb{Z}^{2*} . *Random Structures Algorithms*, 24(3):279–302, 2004.
- [23] O. Häggström. A note on disagreement percolation. *Random Structures & Algorithms*, 18(3):267–278, 2001.
- [24] O. Häggström and J. Jonasson. Uniqueness and non-uniqueness in percolation theory. *Probab. Surv.*, 3:289–344, 2006.
- [25] T. E. Harris. A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.*, 56:13–20, 1960.
- [26] J. Kahn. Range of cube-indexed random walk. *Israel J. Math.*, 124:189–201, 2001.
- [27] R. Kenyon. Dominos and the Gaussian free field. *Ann. Probab.*, 29(3):1128–1137, 2001.
- [28] R. Kenyon. Height fluctuations in the honeycomb dimer model. *Comm. Math. Phys.*, 281(3):675–709, 2008.
- [29] V. Kharash and R. Peled. The Fröhlich-Spencer proof of the Berezinskii-Kosterlitz-Thouless transition. *arXiv preprint arXiv:1711.04720*, 2017.
- [30] E. H. Lieb. Residual entropy of square ice. *Physical Review*, 162:162–172, 1967.
- [31] M. Luby, D. Randall, and A. Sinclair. Markov chain algorithms for planar lattice structures. *SIAM J. Comput.*, 31(1):167–192, 2001.
- [32] G. Menz and T. Martin. A variational principle for a non-integrable model. *arXiv:1610.08103*.
- [33] R. Peled. High-dimensional Lipschitz functions are typically flat. *The Annals of Probability*, 45(3):1351–1447, 2017.
- [34] R. Peled, W. Samotij, and A. Yehudayoff. Lipschitz functions on expanders are typically flat. *Combinatorics, Probability and Computing*, 22(4):566–591, 2013.
- [35] J. G. Propp and D. B. Wilson. Exact sampling with coupled Markov chains and applications to statistical mechanics. *Random Structures & Algorithms*, 9(1-2):223–252, 1996.
- [36] M. Russkikh. Dimers in piecewise Temperleyan domains. *Comm. Math. Phys.*, 359(1):189–222, 2018.
- [37] M. Russkikh. Dominos in hedgehog domains. *arXiv preprint arXiv:1803.10012*, 2018.
- [38] O. Sarig. Existence of Gibbs measures for countable Markov shifts. *Proc. Amer. Math. Soc.*, 131(6):1751–1758, 2003.
- [39] S. Sheffield. Random surfaces. *Astérisque*, (304):vi+175, 2005.
- [40] H. van Beijeren. Exactly solvable model for the roughening transition of a crystal surface. *Physical Review Letters*, 38(18):993, 1977.
- [41] J. van den Berg. A uniqueness condition for Gibbs measures, with application to the 2-dimensional Ising antiferromagnet. *Comm. Math. Phys.*, 152(1):161–166, 1993.
- [42] J. van den Berg and C. Maes. Disagreement percolation in the study of Markov fields. *The Annals of Probability*, pages 749–763, 1994.
- [43] J. van den Berg and J. Steif. Percolation and the hard-core lattice gas model. *Stochastic Processes and their Applications*, 49(2):179–197, 1994.
- [44] Y. Velenik. Localization and delocalization of random interfaces. *Probability Surveys*, 3:112–169, 2006.

SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY
E-mail address: `nishant.chandgotia@gmail.com`

SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY
E-mail address: `peledron@post.tau.ac.il`

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY
E-mail address: `sheffield@math.mit.edu`

DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE
E-mail address: `mtassy@math.dartmouth.edu`