Four-Cycle Free Graphs and Entropy Minimality

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Outline

- Entropy Minimality and Hom Shifts
- Mixing Conditions and Entropy Minimality
- Measures of Maximal Entropy
- Rigidity and Flexibility in the Space of 3-Colourings.

Nearest Neighbour Shifts of Finite Type

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$\mathbb{Z}^d$ acts by translations(shifts) on the shift spaces.

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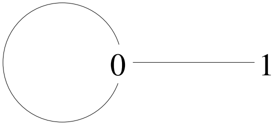
Examples:(Hard Square model)



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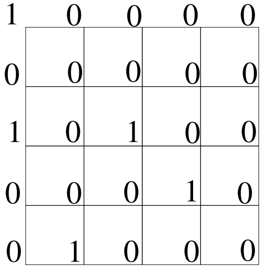

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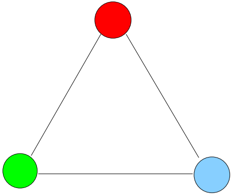
Examples:(3-colourings)



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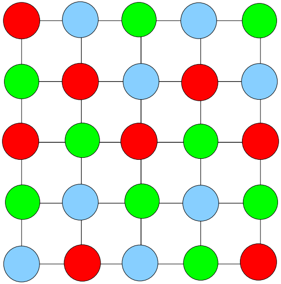

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$$h_{top}(X) := \lim_{n \longrightarrow \infty} \frac{\log |\mathcal{B}(X) \cap \mathcal{A}^{\{1,2,\dots,n\}^d}|}{n^d}.$$

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(Hochman and Meyerovitch, '07) The set of entropies of shifts of finite type for $d > 1$ is the set of non-negative right recursively ennumerable numbers.

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(Quas and Trow '00) Every shift space X contains an entropy minimal shift space $Y \subset X$ such that $h_{top}(X) = h_{top}(Y)$.

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Remark: We will concentrate on X_{C_3}, the space of all 3-colourings.

Mixing Conditions and Entropy Minimality

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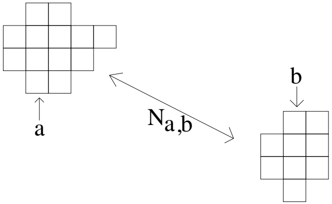

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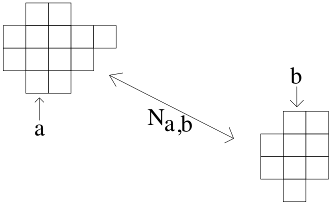

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(Coven and Smítal '93) If a shift space is entropy minimal then it is topologically transitive.

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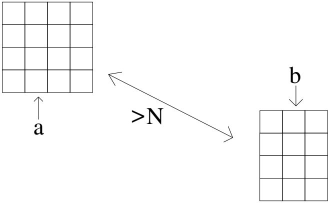

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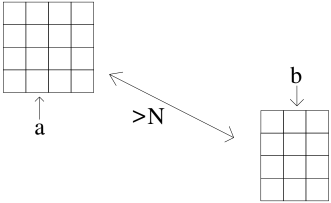

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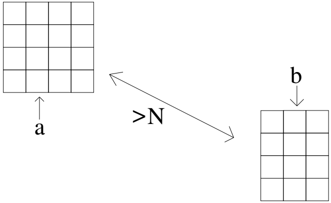

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(Boyle, Pavlov and Schraudner '09) There exists a block-gluing shift space which is not entropy minimal.

A Block-Gluing Shift Which Is Not Entropy Minimal

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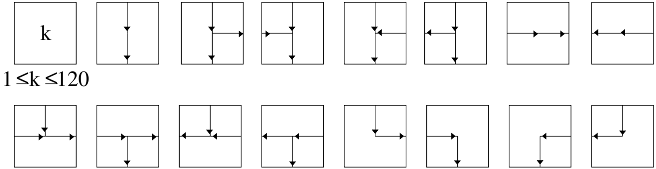

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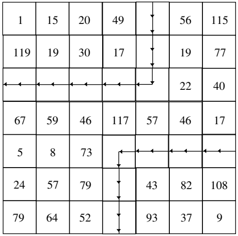

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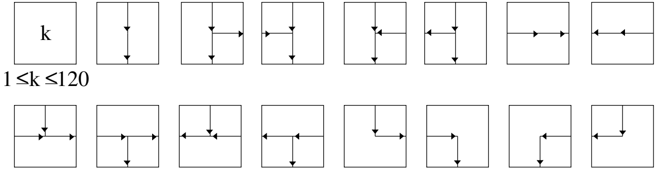

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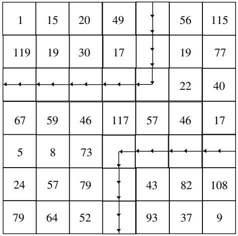

The symbols with arrows do not contribute any entropy.

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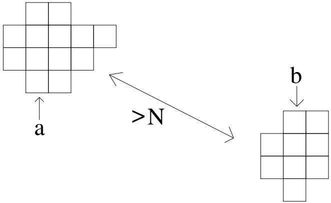

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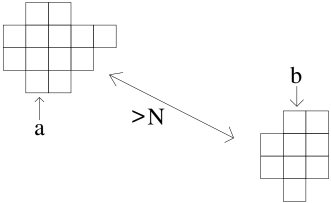

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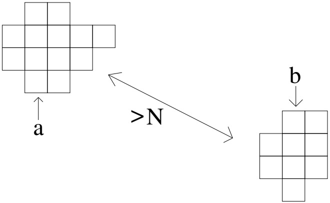

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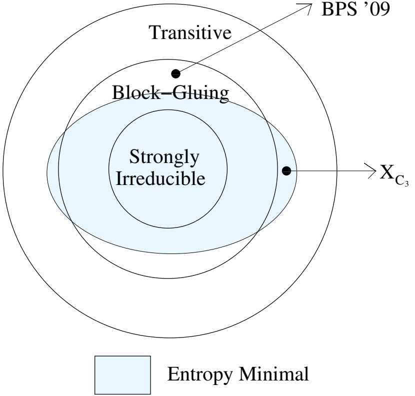

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If μ is a measure of maximal entropy for X then $h_\mu = h_{Top}(X)$. Therefore, a shift space X is entropy minimal if and only if for every measure of maximal entropy μ, $supp(\mu) = X$.

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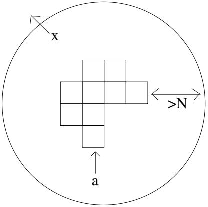

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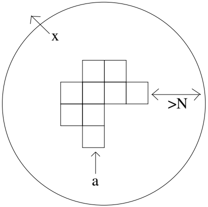

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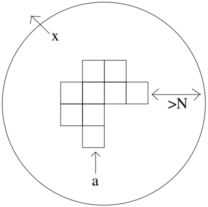

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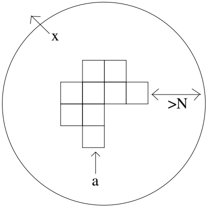

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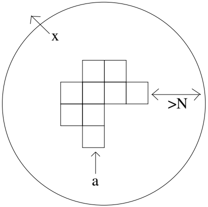

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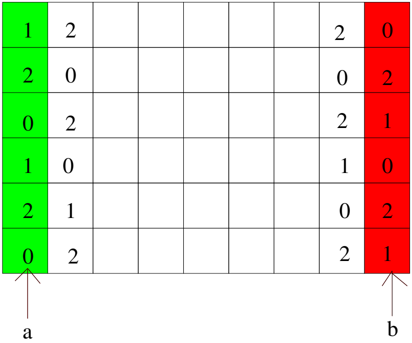

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2	0	2				2	0	2
0	2	0				0	2	1
1	0	2				2	1	0
2	1	0				1	0	2
0	2	1				0	2	1
a								b

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a

Distance depends on the size of a and b

b

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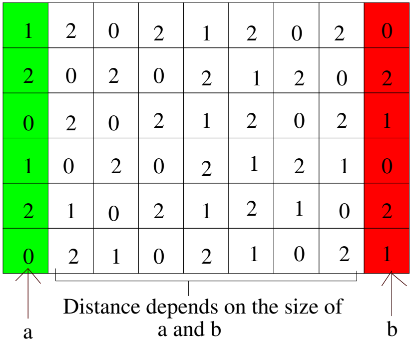

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$$|h(v) - h(w)| = 1$$

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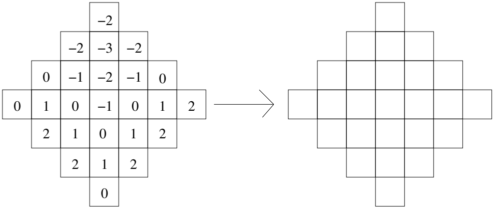

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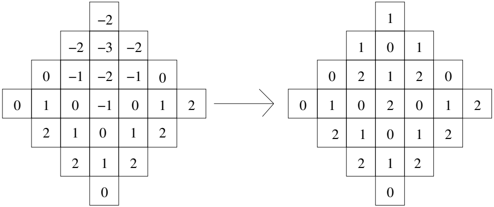

Height Function

Pattern in X_{C_3}

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X_{C_3} satisfies the pivot property, that is, given distinct configurations $x, y \in X_{C_3}$ which differ only at finitely many sites there is a chain

$$((x_1 = x), x_2, \ldots, (x_n = y)) \in X_{C_3}$$

where (x_i, x_{i+1}) differ only at a single site.

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2	0	1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0	1	2
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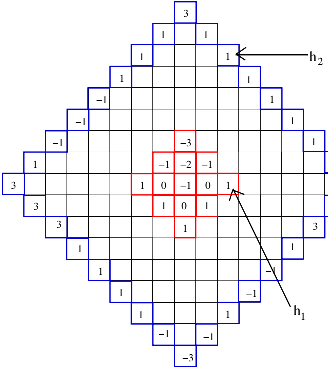

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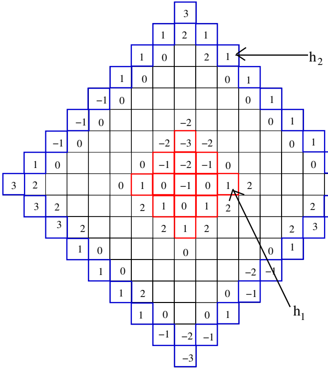

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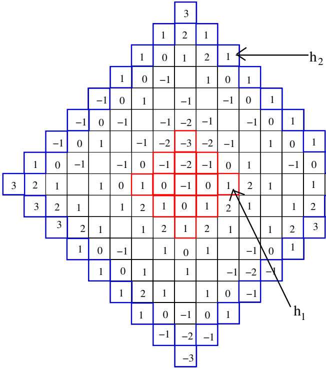

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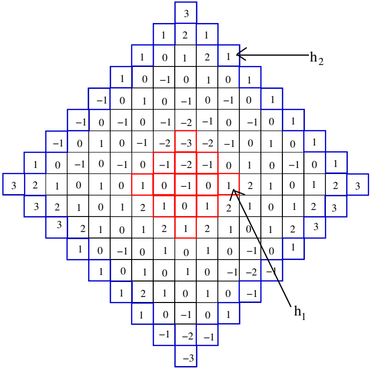

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$\mathbb{Z}$ is replaced by the universal cover of $\mathcal{H}$.

Conjecture: If $\mathcal{H}$ is any connected graph then $X_{\mathcal{H}}$ is entropy minimal.(d=2)

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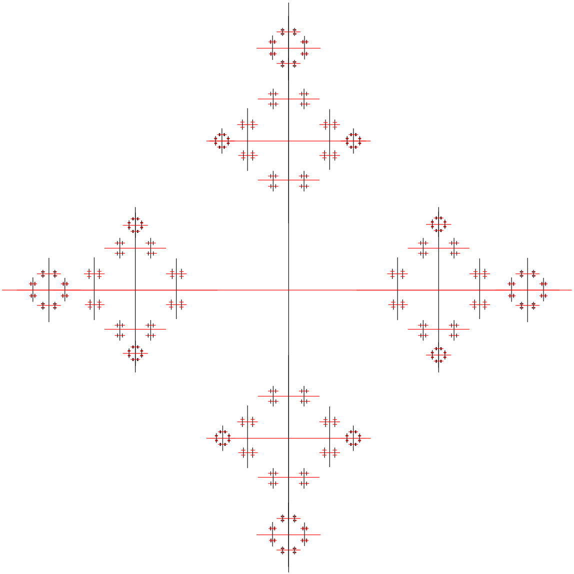
Question: What shift spaces are conjugate to $X_{\mathcal{H}}$ for some graph $\mathcal{H}$?



Thank You!