EXAMPLE 6 Approximating a *p*-series

a. How many terms of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to obtain an approximation that is within 10^{-3} of the exact value of the series?

b. Find an approximation to the series using 50 terms of the series.

SOLUTION The function associated with this series is $f(x) = 1/x^2$.

a. Using the bound on the remainder, we have

$$R_n \leq \int_n^{\infty} f(x)\,dx = \int_n^{\infty} \frac{dx}{x^2} = \frac{1}{n}.$$

To ensure that $R_n \leq 10^{-3}$, we must choose n so that $1/n \leq 10^{-3}$, which implies that $n \geq 1000$. In other words, we must sum at least 1000 terms of the series to be sure that the remainder is less than 10^{-3}.

b. Using the bounds on the series itself, we have $L_n \leq S \leq U_n$, where S is the exact value of the series, and

$$L_n = S_n + \int_{n+1}^{\infty} \frac{dx}{x^2} = S_n + \frac{1}{n+1} \quad \text{and} \quad U_n = S_n + \int_n^{\infty} \frac{dx}{x^2} = S_n + \frac{1}{n}.$$

Therefore, the series is bounded as follows:

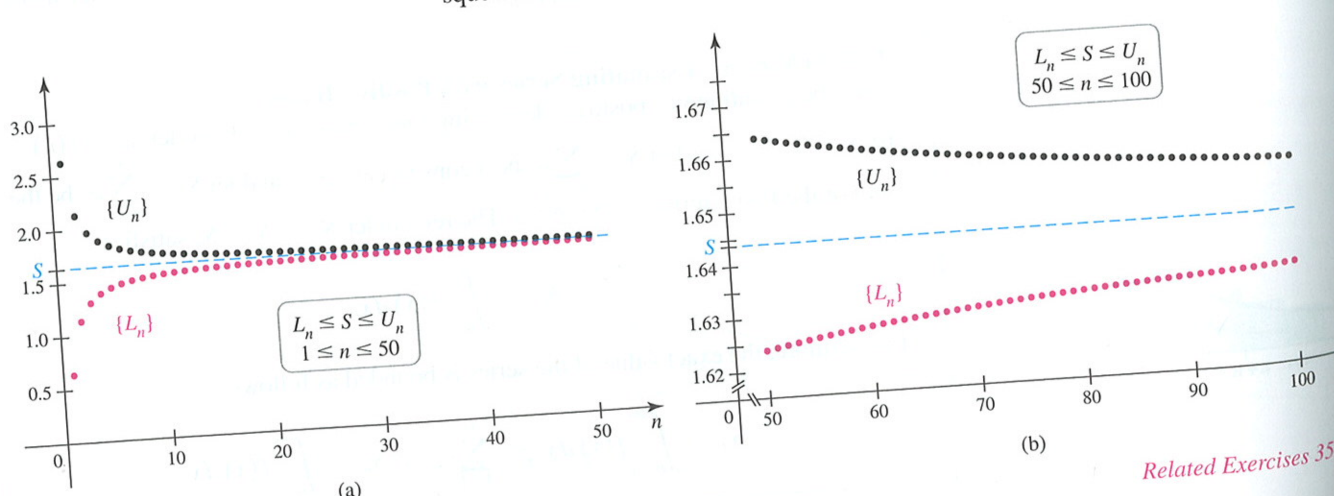
$$S_n + \frac{1}{n+1} \leq S \leq S_n + \frac{1}{n},$$

where S_n is the sum of the first n terms. Using a calculator to sum the first 50 terms of the series, we find that $S_{50} \approx 1.625133$. The exact value of the series is in the interval

$$S_{50} + \frac{1}{50+1} \leq S \leq S_{50} + \frac{1}{50},$$

or $1.644741 < S < 1.645133$. Taking the average of these two bounds as our approximation of S, we find that $S \approx 1.644937$. This estimate is better than simply using S_{50}. Figure 8.31a shows the lower and upper bounds, L_n and U_n, respectively, for $n = 1, 2, \ldots, 50$. Figure 8.31b shows these bounds on an enlarged scale for $n = 50, 51, \ldots, 100$. These figures illustrate how the exact value of the series is squeezed into a narrowing interval as n increases.

➤ The values of p-series with even values of p are generally known. For example, with $p = 2$ the series converges to $\pi^2/6$ (a proof is outlined in Exercise 58); with $p = 4$, the series converges to $\pi^4/90$. The values of p-series with odd values of p are not known.



Related Exercises 35–42 ◀

FIGURE 8.31

SECTION 8.4 EXERCISES

Review Questions

1. Explain why computation alone may not determine whether a series converges.

2. Is it true that if the terms of a series of positive terms decrease to zero, then the series converges? Explain using an example.

3. Can the Integral Test be used to determine whether a series diverges?

4. For what values of p does the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge? For what values of p does it diverge?

5. For what values of p does the series $\sum_{k=10}^{\infty} \frac{1}{k^p}$ converge (initial index is 10)? For what values of p does it diverge?

6. Explain why the sequence of partial sums for a series with positive terms is an increasing sequence.

7. Define the remainder of an infinite series.

8. If a series of positive terms converges, does it follow that the remainder R_n must decrease to zero as $n \to \infty$? Explain.

Basic Skills

9–14. Properties of series *Use the properties of infinite series to evaluate the following series.*

9. $\sum_{k=0}^{\infty} \left[3\left(\frac{2}{5}\right)^k - 2\left(\frac{5}{7}\right)^k\right]$

10. $\sum_{k=1}^{\infty} \left[2\left(\frac{3}{5}\right)^k + 3\left(\frac{4}{9}\right)^k\right]$

11. $\sum_{k=1}^{\infty} \left[\frac{1}{3}\left(\frac{5}{6}\right)^k + \frac{3}{5}\left(\frac{7}{9}\right)^k\right]$

12. $\sum_{k=0}^{\infty} \left[\frac{1}{2}(0.2)^k + \frac{3}{2}(0.8)^k\right]$

13. $\sum_{k=1}^{\infty} \left[\left(\frac{1}{6}\right)^k + \left(\frac{1}{3}\right)^{k-1}\right]$

14. $\sum_{k=0}^{\infty} \frac{2 - 3^k}{6^k}$

15–22. Divergence Test *Use the Divergence Test to determine whether the following series diverge or state that the test is inconclusive.*

15. $\sum_{k=0}^{\infty} \frac{k}{2k+1}$

16. $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

17. $\sum_{k=2}^{\infty} \frac{k}{\ln k}$

18. $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

19. $\sum_{k=0}^{\infty} \frac{1}{1000 + k}$

20. $\sum_{k=1}^{\infty} \frac{k^3}{k^3+1}$

21. $\sum_{k=2}^{\infty} \frac{\sqrt{k}}{\ln^{10} k}$

22. $\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{k}$

23–30. Integral Test *Use the Integral Test to determine the convergence or divergence of the following series. Check that the conditions of the test are satisfied.*

23. $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$

24. $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+4}}$

25. $\sum_{k=1}^{\infty} ke^{-2k^2}$

26. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k+10}}$

27. $\sum_{k=0}^{\infty} \frac{1}{\sqrt{k+8}}$

28. $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

29. $\sum_{k=1}^{\infty} \frac{k}{e^k}$

30. $\sum_{k=2}^{\infty} \frac{1}{k \ln k \ln(\ln k)}$

31–34. *p*-series *Determine the convergence or divergence of the following series.*

31. $\sum_{k=1}^{\infty} \frac{1}{k^{10}}$

32. $\sum_{k=2}^{\infty} \frac{k^e}{k^\pi}$

33. $\sum_{k=3}^{\infty} \frac{1}{(k-2)^4}$

34. $\sum_{k=1}^{\infty} 2k^{-3/2}$

T **35–42. Remainders and estimates** *Consider the following convergent series.*

a. *Find an upper bound for the remainder in terms of n.*
b. *Find how many terms are needed to ensure that the remainder is less than 10^{-3}.*
c. *Find lower and upper bounds (L_n and U_n, respectively) on the exact value of the series.*
d. *Find an interval in which the value of the series must lie if you approximate it using ten terms of the series.*

35. $\sum_{k=1}^{\infty} \frac{1}{k^6}$

36. $\sum_{k=1}^{\infty} \frac{1}{k^8}$

37. $\sum_{k=1}^{\infty} \frac{1}{3^k}$

38. $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

39. $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

40. $\sum_{k=1}^{\infty} e^{-k}$

41. $\sum_{k=1}^{\infty} \frac{1}{k^3}$

42. $\sum_{k=1}^{\infty} ke^{-k^2}$

Further Explorations

43. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. If $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=10}^{\infty} a_k$ converges.
b. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=10}^{\infty} a_k$ diverges.
c. If $\sum a_k$ converges, then $\sum (a_k + 0.0001)$ also converges.
d. If $\sum p^k$ diverges, then $\sum (p + 0.001)^k$ diverges, for a fixed real number p.
e. If $\sum k^{-p}$ converges, then $\sum k^{-p+0.001}$ converges.
f. If $\lim_{k\to\infty} a_k = 0$, then $\sum a_k$ converges.

44–49. Choose your test *Determine whether the following series converge or diverge.*

44. $\sum_{k=1}^{\infty} \sqrt{\frac{k+1}{k}}$

45. $\sum_{k=1}^{\infty} \frac{1}{(3k+1)(3k+4)}$

46. $\sum_{k=0}^{\infty} \frac{10}{k^2+9}$

47. $\sum_{k=0}^{\infty} \frac{k}{\sqrt{k^2+4}}$

48. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + 3^k}{4^k}$ **49.** $\displaystyle\sum_{k=2}^{\infty} \frac{4}{k \ln^2 k}$

50. Log *p*-series Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$, where p is a real number.

a. Use the Integral Test to determine the values of p for which this series converges.

b. Does this series converge faster for $p = 2$ or $p = 3$? Explain.

51. Loglog *p*-series Consider the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k (\ln \ln k)^p}$, where p is a real number.

a. For what values of p does this series converge?

b. Which of the following series converges faster? Explain.

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} \quad \text{or} \quad \sum_{k=2}^{\infty} \frac{1}{k \ln k (\ln \ln k)^2}?$$

52. Find a series Find a series that . . .

a. converges faster than $\sum \frac{1}{k^2}$ but slower than $\sum \frac{1}{k^3}$.

b. diverges faster than $\sum \frac{1}{k}$ but slower than $\sum \frac{1}{\sqrt{k}}$.

c. converges faster than $\sum \frac{1}{k \ln^2 k}$ but slower than $\sum \frac{1}{k^2}$.

Additional Exercises

53. A divergence proof Give an argument, similar to that given in the text for the harmonic series, to show that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges.

54. Properties proof Use the ideas in the proof of Property 1 of Theorem 8.8 to prove Property 2 of Theorem 8.8.

55. Property of divergent series Prove that if $\sum a_k$ diverges, then $\sum ca_k$ also diverges, where $c \neq 0$ is a constant.

56. Prime numbers The prime numbers are those positive integers that are divisible by only 1 and themselves (for example, 2, 3, 5, 7, 11, 13, . . .). A celebrated theorem states that the sequence of prime numbers $\{p_k\}$ satisfies $\displaystyle\lim_{k\to\infty} p_k/(k \ln k) = 1$. Show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges, which implies that the series $\sum_{k=1}^{\infty} 1/p_k$ diverges.

57. The zeta function The Riemann zeta function is the subject of extensive research and is associated with several renowned unsolved problems. It is defined by $\zeta(x) = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^x}$. When x is a real number, the zeta function becomes a p-series. For even positive integers p, the value of $\zeta(p)$ is known exactly. For example,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}, \quad \sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945}, \ldots.$$

Use the estimation techniques described in the text to approximate $\zeta(3)$ and $\zeta(5)$ (whose values are not known exactly) with a remainder less than 10^{-3}.

58. Showing that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ In 1734, Leonhard Euler informally proved that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. An elegant proof is outlined here that uses the inequality

$$\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x \quad (\text{provided that } 0 < x < \pi/2)$$

and the identity

$$\sum_{k=1}^{n} \cot^2 (k\theta) = \frac{n(2n-1)}{3}, \text{ for } n = 1, 2, 3 \ldots, \text{where } \theta = \frac{\pi}{2n+1}.$$

a. Show that $\displaystyle\sum_{k=1}^{n} \cot^2 (k\theta) < \frac{1}{\theta^2} \sum_{k=1}^{n} \frac{1}{k^2} < n + \sum_{k=1}^{n} \cot^2 (k\theta)$.

b. Use the inequality in part (a) to show that

$$\frac{n(2n-1)\pi^2}{3(2n+1)^2} < \sum_{k=1}^{n} \frac{1}{k^2} < \frac{n(2n+2)\pi^2}{3(2n+1)^2}.$$

c. Use the Squeeze Theorem to conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

[*Source: The College Mathematics Journal*, **24**, No. 5 (November, 1993).]

59. Reciprocals of odd squares Given that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ (Exercises 57 and 58) and that the terms of this series may be rearranged without changing the value of the series. Determine the sum of the reciprocals of the squares of the odd positive integers.

T **60. Shifted *p*-series** Consider the sequence $\{F_n\}$ defined by

$$F_n = \sum_{k=1}^{\infty} \frac{1}{k(k+n)},$$

for $n = 0, 1, 2, \ldots$. When $n = 0$, the series is a p-series, and we have $F_0 = \pi^2/6$ (Exercises 57 and 58).

a. Explain why $\{F_n\}$ is a decreasing sequence.

b. Plot approximations to $\{F_n\}$ for $n = 1, 2, \ldots, 20$.

c. Based on your experiments, make a conjecture about $\displaystyle\lim_{n\to\infty} F_n$.

61. A sequence of sums Consider the sequence $\{x_n\}$ defined for $n = 1, 2, 3, \ldots$ by

$$x_n = \sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

a. Write out the terms x_1, x_2, x_3.

b. Show that $\frac{1}{2} \leq x_n < 1$ for $n = 1, 2, 3, \ldots$.

c. Show that x_n is the right Riemann sum for $\displaystyle\int_1^2 \frac{dx}{x}$ using n subintervals.

d. Conclude that $\displaystyle\lim_{n\to\infty} x_n = \ln 2$.

T **62. The harmonic series and Euler's constant**

a. Sketch the function $f(x) = 1/x$ on the interval $[1, n+1]$, where n is a positive integer. Use this graph to verify that

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n.$$

b. Let S_n be the sum of the first n terms of the harmonic series, so part (a) says $\ln(n+1) < S_n < 1 + \ln n$. Define the new sequence $\{E_n\}$, where

$$E_n = S_n - \ln(n+1), \quad \text{for } n = 1, 2, 3, \ldots.$$

Show that $E_n > 0$ for $n = 1, 2, 3, \ldots$.

c. Using a figure similar to that used in part (a), show that

$$\frac{1}{n+1} > \ln(n+2) - \ln(n+1).$$

d. Use parts (a) and (c) to show that $\{E_n\}$ is an increasing sequence ($E_{n+1} > E_n$).

e. Use part (a) to show that $\{E_n\}$ is bounded above by 1.

f. Conclude from parts (d) and (e) that $\{E_n\}$ has a limit less than or equal to 1. This limit is known as **Euler's constant** and is denoted γ (the Greek lowercase letter gamma).

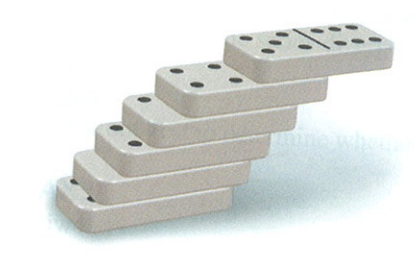
g. By computing terms of $\{E_n\}$, estimate the value of γ and compare it to the value $\gamma \approx 0.5772$. (It has been conjectured, but not proved, that γ is irrational.)

h. The preceding arguments show that the sum of the first n terms of the harmonic series satisfy $S_n \approx 0.5772 + \ln(n+1)$. How many terms must be summed for the sum to exceed 10?

63. Stacking dominoes Consider a set of identical dominoes that are 2 in long. The dominoes are stacked on top of each other with their long edges aligned so that each domino overhangs the one beneath it *as far as possible* (see figure).

a. If there are n dominoes in the stack, what is the *greatest* distance that the top domino can be made to overhang the bottom domino? (*Hint:* Put the nth domino beneath the previous $n - 1$ dominoes.)

b. If we allow for infinitely many dominoes in the stack, what is the greatest distance that the top domino can be made to overhang the bottom domino?



QUICK CHECK ANSWERS

1. Adding a finite number of nonzero terms does not change whether the series converges. It does, however, change the value of the series. 2. Given the nth term of the sequence of partial sums S_n, the next term is obtained by adding a positive number. So $S_{n+1} > S_n$, which means the sequence is increasing. 3. The series diverges for $|r| \geq 1$. 4. a. Divergent p-series b. Convergent geometric series c. Convergent p-series 5. The remainder is $R_n = a_{n+1} + a_{n+2} + \cdots$, which consists of positive numbers.

8.5 The Ratio, Root, and Comparison Tests

We now consider several more convergence tests: the Ratio Test, the Root Test, and two comparison tests. The Ratio Test will be used frequently throughout the next chapter, and comparison tests are valuable when no other test works. Again, these tests determine *whether* an infinite series converges, but they do not establish the value of the series.

The Ratio Test

The Integral Test is powerful, but limited, because it requires evaluating integrals. For example, the series $\sum 1/k!$, with a factorial term, cannot be handled by the Integral Test. The next test significantly enlarges the set of infinite series that we can analyze.

➤ In words, the Ratio Test says the limit of the ratio of successive terms of the series must be less than 1 for convergence of the series.

THEOREM 8.14 The Ratio Test

Let $\sum a_k$ be an infinite series with positive terms and let $r = \displaystyle\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.

1. If $0 \leq r < 1$, the series converges.
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. If $r = 1$, the test is inconclusive.