Modelling processes on the $\mathbb{Z}^d$-lattice

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No! We are constrained by the size of the sample space.

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We can use functions $f : \mathbb{Z} \longrightarrow \{1, 2, 3, 4, 5, 6\}$. $f(i)$ is used to record the result of the i^{th} dice throw.

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Let us first define ‘recording’ rigorously.

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Let $\mathbb{A}$ be a finite set. An element $\omega \in \mathbb{A}^{\mathbb{Z}}$ can be thought both as a function

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Given $j \in \mathbb{N}$, the sequence $(\omega_{i-j})_{i\in\mathbb{Z}}$ represents the function whose values have been shifted j entries to the left.

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$$\Omega_0, \Omega_1, \ldots, \Omega_i$$

has the same distribution as

$$\Omega_j, \Omega_{j+1}, \ldots, \Omega_{j+i}$$

for all $i \in \mathbb{N}$ and $j \in \mathbb{Z}$.

Example: Bernoulli process

Let Ω be a fixed finite-valued random variable. Let $(\Omega_i)_{i \in \mathbb{Z}}$ be a sequence of independent copies of Ω. $(\Omega_i)_{i \in \mathbb{Z}}$ is called a Bernoulli process.

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A sequence of dice throws forms a Bernoulli process where Ω takes values $1, 2, \ldots, 6$ with equal probability.

A slightly more complicated stochastic process

Consider the stochastic process $\overline{\Omega} := (\Omega_i)_{i \in \mathbb{Z}}$ where

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It can be verified that this defines a stochastic process.

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Recording of the stochastic process

In other words, $\overline{\Omega}$ has been recorded by $\{0, 1\}^{\mathbb{Z}}$.

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Embedding captures the idea of recording that we have spoken about until now!

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To understand why, we need to introduce entropy in our setting. This replaces the size of the sample space of random variables that we were using before.

Suppose Ω is a random variable which takes values $1, 2, \ldots, k$ with probabilities $p_1, p_2, \ldots, p_k$. Then the Shannon entropy of Ω is given by

$$H(\Omega) := -\sum_{i=1}^{k} p_i \log(p_i).$$

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In general if Ω is the uniform random variable taking k values then

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Note that log is a strictly concave function. Thus if Ω takes only k values with positive probability then by Jensen's inequality

$$H(\Omega) = \sum_{i=1}^{k} p_i \log \frac{1}{p_i} \leq \log k,$$

where equality is attained if and only if Ω is a uniform random variable ($p_i = \frac{1}{k}$ for all $1 \leq i \leq k$).

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If Ω_1 and Ω_2 are independent copies of Ω then (Ω_1, Ω_2) is a random variable taking k^2 distinct values with probabilities

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If $(\Omega_1, \Omega_2, \ldots, \Omega_n)$ are independent copies of Ω then

$$H(\Omega_1, \Omega_2, \ldots, \Omega_n) = nH(\Omega).$$

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For example, if

$$\Omega := \begin{cases} 1 & \text{with probability } \frac{19}{20} \\ 2, 3 & \text{with probability } \frac{1}{40} \text{ each} \end{cases}$$

then $H(\Omega) = .101 < \log 2$ but takes three different values.

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For stochastic processes, we consider 'entropy-per-site' instead.

Kolmogorov-Sinai entropy (1958-1959)

Given a stationary stochastic process $\overline{\Omega} = (\Omega_i)_{i \in \mathbb{Z}}$ we define its entropy by

$$h(\overline{\Omega}) := \lim_{n \longrightarrow \infty} \frac{1}{n} H(\Omega_1, \Omega_2, \ldots, \Omega_n).$$

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In general, $h(\overline{\Omega}) \leq H(\Omega_1)$.

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Theorem (Kolmogorov and Sinai, 1958-1959)

If $\overline{\Omega}$ can embedded in $\{1, 2, \ldots k\}^{\mathbb{Z}}$ then $h(\overline{\Omega}) \leq \log k$.

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Theorem (Kolmogorov and Sinai, 1958-1959)

If $\overline{\Omega}$ *can embedded in* $\{1, 2, \ldots k\}^{\mathbb{Z}}$ *then* $h(\overline{\Omega}) \leq \log k$.

If $\overline{\Omega}$ is an infinite sequence of dice throws then

$$h(\overline{\Omega}) = H(\Omega_1) = \log 6;$$

thus dice throws cannot be embedded in $\{1, 2\}^{\mathbb{Z}}$.

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If $\overline{\Omega}$ can embedded in $\{1, 2, \ldots k\}^{\mathbb{Z}}$ then $h(\overline{\Omega}) \leq \log k$.

If $\overline{\Omega}$ is an infinite sequence of dice throws then

$$h(\overline{\Omega}) = H(\Omega_1) = \log 6;$$

thus dice throws cannot be embedded in $\{1, 2\}^{\mathbb{Z}}$. Conversely

Theorem (Krieger's generator theorem (1972))

If $h(\overline{\Omega}) < \log k$ then $\overline{\Omega}$ can be embedded in $\{1, 2, \ldots, k\}^{\mathbb{Z}}$.

The results are sharp.

$\mathbb{Z}^d$-stochastic processes

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A shift-invariant $\mathbb{Z}^d$-stochastic processes $\overline{\Omega} = (\Omega_{\vec{i}})_{\vec{i} \in \mathbb{Z}^d}$ is a collection of random variables indexed by $\mathbb{Z}^d$, such that for all $\vec{j} \in \mathbb{Z}^d$,

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For $d = 1$, this is the same as a stationary stochastic processes.

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Let B_n be a cube in $\mathbb{Z}^d$ of side length n. The entropy is defined by

$$h(\overline{\Omega}) := \lim_{n \longrightarrow \infty} \frac{1}{n^d} H(\Omega_{\vec{i}}; \vec{i} \in B_n).$$

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Embedding of $\mathbb{Z}^d$-stochastic processes

Suppose $\overline{\Omega} = (\Omega_{\vec{i}})_{\vec{i} \in \mathbb{Z}^d}$ is a stationary stochastic process where the $\Omega_{\vec{i}}$'s take values in a finite set $\mathbb{A}$.

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$$\Phi : \mathbb{A}^{\mathbb{Z}^d} \to \{1, 2, \ldots, k\}$$

for which the map $\phi : \mathbb{A}^{\mathbb{Z}^d} \to \{1, 2, \ldots, k\}^{\mathbb{Z}^d}$ given by

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We say that $\overline{\Omega}$ can be embedded in a set $X \subset \{1, 2, \ldots, k\}^{\mathbb{Z}^d}$ if in addition there exists ϕ as above for which $\phi(\omega) \in X$ with probability one.

Embedding of $\mathbb{Z}^d$-stochastic processes

Again, we have,

Theorem (Robinson and Ruelle, 1967)

If $\overline{\Omega}$ *can embedded in* $\{1, 2, \ldots k\}^{\mathbb{Z}^d}$ *then* $h(\overline{\Omega}) \leq \log k$.

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Theorem (Rosenthal, 1988 ($d = 2$) and Kammeyer, 1990 ($d > 2$))

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The results are sharp.

But what if we want to embed in some $X \subset \{1, 2, 3, \dots, k\}^{\mathbb{Z}^d}$?

Embedding under constraints

Let $X \subset \{1, 2, \ldots, k\}^{\mathbb{Z}^d}$ be closed and invariant under translations of the $\mathbb{Z}^d$-lattice. We define the topological entropy of X as

$$h_{top}(X) := \lim_{n \to \infty} \frac{1}{n^d} \log(\#\{x|_{B_n} \ : \ x \in X\}).$$

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$$\begin{aligned} & h_{top}(\{1, 2, \ldots, k\}^{\mathbb{Z}^d}) \\ = \quad & \lim_{n\to\infty} \frac{1}{n^d} \log(\#\{x|_{B_n} \ : \ x \in \{1, 2, \ldots, k\}^{\mathbb{Z}^d}\}) \\ = \quad & \lim_{n\to\infty} \frac{1}{n^d} \log |\{1, 2, \ldots, k\}|^{n^d} = \log k. \end{aligned}$$

Universality

X is said to be universal if all stochastic process $\overline{\Omega}$ for which

$$h(\overline{\Omega}) < h_{top}(X)$$

$\overline{\Omega}$ can be embedded in X.

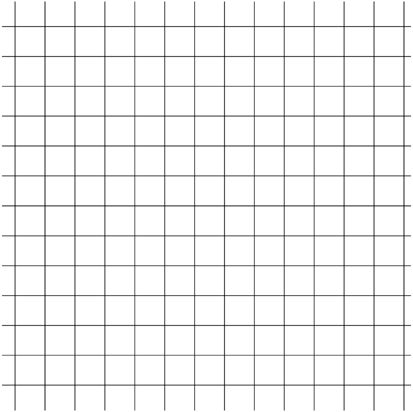
By the aforementioned results of Krieger, Rosenthal and Kammeyer, $\{1, 2, \ldots, k\}^{\mathbb{Z}^d}$ are universal.

Motivating Question

When is X universal?

Example: Hom-shifts

We are going to think of $\mathbb{Z}^d$ as both the group and the Cayley graph with respect to standard generators. For instance, $\mathbb{Z}^2$ is the infinite grid.



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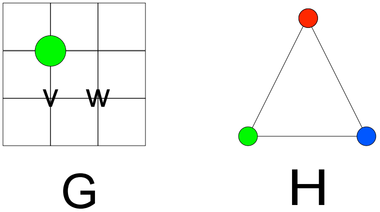

Figure : If $f(v)$ is green

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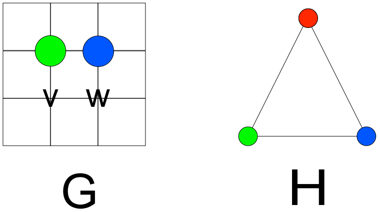

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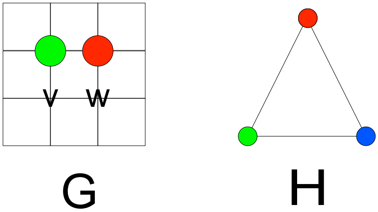

Figure : If $f(v)$ is green then $f(w)$ is either blue or red.

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Hom-shifts $X_{\mathcal{H}}$ are the space of graph homomorphisms from $\mathbb{Z}^d$ to $\mathcal{H}$.

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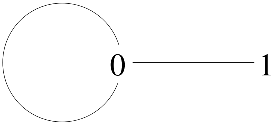
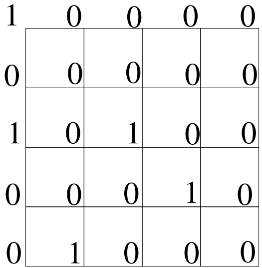
Examples: (Hard core model)



Graph $\mathcal{H}$



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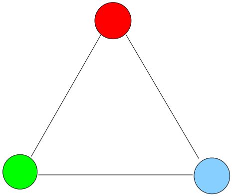
Examples: (Proper 3-colourings)



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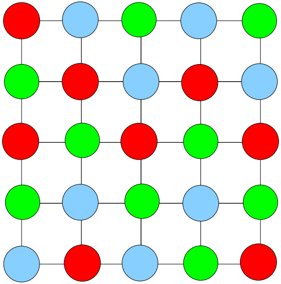

Example: Domino tilings

The space of domino tilings X_{dom} are all possible partitions of $\mathbb{Z}^d$ by rectangular parallelepipeds one of whose side lengths is 2 and rest are 1.

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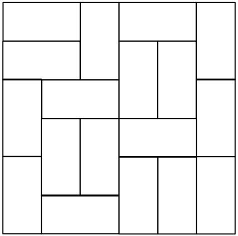

Figure : A domino tiling in $d = 2$.

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In 2015, Jackson and Gao reiterated the question (in a stronger form).

Main result (Contd.)

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Theorem (Chandgotia and Meyerovitch, 2018)

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A similar (but more technical) result holds for graphs $\mathcal{H}$ which are bipartite; we will skip it.

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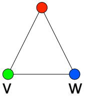

Figure : The graph H

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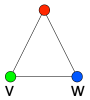

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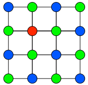

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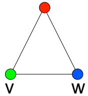

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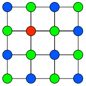

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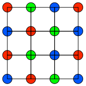

Figure : This is an element of $G_4 \setminus L_4$ (all the three colours appear on the boundary)

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Note that when $X_{\mathcal{H}} := \{1, 2, \ldots, k\}^{\mathbb{Z}^d}$ we have that

$$|L_n| = 2k^{(n-1)^d} \text{ while } |G_n| = k^{(n)^d}$$

So the equation mentioned above follows automatically.

What is at stake? (Hom-shifts)

In fact, we prove that there is a $c_H > 0$ then given the uniform distribution on G_n

$$\texttt{Prob}(L_n) \geq e^{-c_H n^{d-1}}.$$

What is at stake? (Domino tilings)

Question (Open)

Are domino tilings universal in all dimensions d?

Recall that B_n is the box of side length n in $\mathbb{Z}^d$. Let L_{2n} be the set of tilings of B_{2n} by dominos and G_{2n} be the set of tilings of $\mathbb{Z}^d$ by dominos restricted to B_{2n}. It follows that $L_{2n} \subset G_{2n}$.

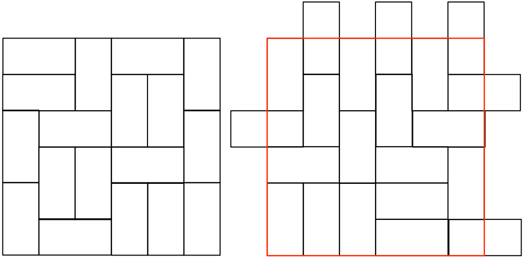

Figure : An element of L_6 (on the left) and of $G_6 \setminus L_6$ (on the right)

What is at stake? (Domino tilings)

If the equation

$$\lim_{n\to\infty} \frac{\log |L_{2n}|}{(2n)^d} = \lim_{n\to\infty} \frac{\log |G_{2n}|}{(2n)^d}$$

holds then domino tilings are universal for all dimensions d.

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For $d = 2$, the equation follows from some deep ideas from Kastelyn (1961) and also from the work of Cohn, Kenyon and Propp (2001). These ideas fail to extend to higher dimensions.

Thank You!