EXAMPLE 4 Using the Limit Comparison Test Determine whether the following series converge.

a. $\displaystyle\sum_{k=1}^{\infty} \frac{k^4 - 2k^2 + 3}{2k^6 - k + 5}$ **b.** $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$

SOLUTION In both cases, we must find a comparison series whose terms behave like the terms of the given series as $k \to \infty$.

a. As $k \to \infty$, a rational function behaves like the ratio of the leading (highest-power) terms. In this case, as $k \to \infty$,

$$\frac{k^4 - 2k^2 + 3}{2k^6 - k + 5} \approx \frac{k^4}{2k^6} = \frac{1}{2k^2}.$$

Therefore, a reasonable comparison series is the convergent p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ (the factor of 2 does not affect whether the given series converges). Having chosen a comparison series, we compute the limit L:

$$\begin{aligned} L &= \lim_{k\to\infty} \frac{(k^4 - 2k^2 + 3)/(2k^6 - k + 5)}{1/k^2} && \text{Ratio of terms of series} \\ &= \lim_{k\to\infty} \frac{k^2(k^4 - 2k^2 + 3)}{2k^6 - k + 5} && \text{Simplify.} \\ &= \lim_{k\to\infty} \frac{k^6 - 2k^4 + 3k^2}{2k^6 - k + 5} = \frac{1}{2} && \text{Simplify and evaluate the limit.} \end{aligned}$$

We see that $0 < L < \infty$; therefore, the given series converges.

b. Why is this series interesting? We know that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges and that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. The given series $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ is "between" these two series. This observation suggests that we use either $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ or $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ as a comparison series. In the first case, letting $a_k = \ln k/k^2$ and $b_k = 1/k^2$, we find that

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k/k^2}{1/k^2} = \lim_{k\to\infty} \ln k = \infty.$$

Case (3) of the Limit Comparison Test does not apply here because the comparison series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. So the test is inconclusive.

If, instead, we use the comparison series $\sum b_k = \sum \frac{1}{k}$, then

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k/k^2}{1/k} = \lim_{k\to\infty} \frac{\ln k}{k} = 0.$$

Case (2) of the Limit Comparison Test does not apply here because the comparison series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Again, the test is inconclusive.

With a bit more cunning, the Limit Comparison Test becomes conclusive. A series that lies "between" $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ is the convergent p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$; we try it as a comparison series. Letting $a_k = \ln k/k^2$ and $b_k = 1/k^{3/2}$, we find that

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k/k^2}{1/k^{3/2}} = \lim_{k\to\infty} \frac{\ln k}{\sqrt{k}} = 0.$$

(This limit is evaluated using l'Hôpital's Rule or by recalling that $\ln k$ grows more slowly than any positive power of k.) Now case (2) of the limit comparison test applies; the comparison series $\sum \frac{1}{k^{3/2}}$ converges, so the given series converges.

Related Exercises 27–38 ◂

Guidelines

We close by outlining a procedure that puts the various convergence tests in perspective. Here is a reasonable course of action when testing a series of positive terms $\sum a_k$ for convergence.

1. Begin with the Divergence Test. If you show that $\lim_{k\to\infty} a_k \neq 0$, then the series diverges and your work is finished. The order of growth rates of sequences given in Section 8.2 is useful for evaluating $\lim_{k\to\infty} a_k$.
2. Is the series a special series? Recall the convergence properties for the following series.
   - Geometric series: $\sum ar^k$ converges if $|r| < 1$ and diverges for $|r| \geq 1$.
   - p-series: $\sum \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.
   - Check also for a telescoping series.
3. If the general kth term of the series looks like a function you can integrate, then try the Integral Test.
4. If the general kth term of the series involves $k!$, k^k, or a^k, where a is a constant, the Ratio Test is advisable. Series with k in an exponent may yield to the Root Test.
5. If the general kth term of the series is a rational function of k (or a root of a rational function), use the Comparison or the Limit Comparison Test. Use the families of series given in Step 2 as comparison series.

These guidelines will help, but in the end, convergence tests are mastered through practice. It's your turn.

SECTION 8.5 EXERCISES

Review Questions

1. Explain how the Ratio Test works.
2. Explain how the Root Test works.
3. Explain how the Limit Comparison Test works.
4. What is the first test you should use in analyzing the convergence of a series?
5. What tests are advisable if the series involves a factorial term?
6. What tests are best for the series $\sum a_k$ when a_k is a rational function of k?
7. Explain why, with a series of positive terms, the sequence of partial sums is an increasing sequence.
8. Do the tests discussed in this section tell you the value of the series? Explain.

Basic Skills

9–18. The Ratio Test *Use the Ratio Test to determine whether the following series converge.*

9. $\sum_{k=1}^{\infty} \frac{1}{k!}$
10. $\sum_{k=1}^{\infty} \frac{2^k}{k!}$
11. $\sum_{k=1}^{\infty} \frac{k^2}{4^k}$
12. $\sum_{k=1}^{\infty} \frac{2^k}{k^k}$
13. $\sum_{k=1}^{\infty} ke^{-k}$
14. $\sum_{k=1}^{\infty} \frac{k!}{k^k}$
15. $\sum_{k=1}^{\infty} \frac{2^k}{k^{99}}$
16. $\sum_{k=1}^{\infty} \frac{k^6}{k!}$
17. $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$
18. $\sum_{k=1}^{\infty} k^4 2^{-k}$

19–26. The Root Test *Use the Root Test to determine whether the following series converge.*

19. $\sum_{k=1}^{\infty} \left(\frac{4k^3+k}{9k^3+k+1}\right)^k$
20. $\sum_{k=1}^{\infty} \left(\frac{k+1}{2k}\right)^k$
21. $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$
22. $\sum_{k=1}^{\infty} \left(1+\frac{3}{k}\right)^{k^2}$
23. $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{2k^2}$
24. $\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)}\right)^k$
25. $1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^3+\left(\frac{1}{4}\right)^4+\cdots$
26. $\left(\frac{1}{2}\right)^2+\left(\frac{2}{3}\right)^3+\left(\frac{3}{4}\right)^4+\cdots$

27–38. Comparison tests *Use the Comparison Test or Limit Comparison Test to determine whether the following series converge.*

27. $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$
28. $\sum_{k=1}^{\infty} \frac{k^2+k-1}{k^4+4k^2-3}$
29. $\sum_{k=1}^{\infty} \frac{k^2-1}{k^3+4}$
30. $\sum_{k=1}^{\infty} \frac{0.0001}{k+4}$
31. $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}+1}$
32. $\sum_{k=1}^{\infty} \sqrt{\frac{k}{k^3+1}}$
33. $\sum_{k=1}^{\infty} \frac{\sin(1/k)}{k^2}$
34. $\sum_{k=1}^{\infty} \frac{1}{3^k-2^k}$
35. $\sum_{k=1}^{\infty} \frac{1}{2k-\sqrt{k}}$
36. $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k+2}}$
37. $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^2+1}}{\sqrt{k^3+2}}$
38. $\sum_{k=2}^{\infty} \frac{1}{(k\ln k)^2}$

Further Explorations

39. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
    a. Suppose that $0 < a_k < b_k$. If $\sum a_k$ converges, then $\sum b_k$ converges.
    b. Suppose that $0 < a_k < b_k$. If $\sum a_k$ diverges, then $\sum b_k$ diverges.
    c. Suppose $0 < b_k < c_k < a_k$. If $\sum a_k$ converges, then $\sum b_k$ and $\sum c_k$ converge.

40–57. Choose your test *Use the test of your choice to determine whether the following series converge.*

40. $\sum_{k=1}^{\infty} \frac{(k!)^3}{(3k)!}$
41. $\sum_{k=1}^{\infty} \left(\frac{1}{k}+2^{-k}\right)$
42. $\sum_{k=2}^{\infty} \frac{5\ln k}{k}$
43. $\sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$
44. $\sum_{k=1}^{\infty} \left(1-\frac{1}{k}\right)^{k^2}$
45. $\sum_{k=1}^{\infty} \frac{k^8}{k^{11}+3}$
46. $\sum_{k=1}^{\infty} \frac{1}{(1+p)^k},\ p>0$
47. $\sum_{k=1}^{\infty} \frac{1}{k^{1+p}},\ p>0$
48. $\sum_{k=2}^{\infty} \frac{1}{k^2\ln k}$
49. $\sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k+1}\right)$
50. $\sum_{k=1}^{\infty} k^{-1/k}$
51. $\sum_{k=2}^{\infty} \frac{1}{k^{\ln k}}$
52. $\sum_{k=1}^{\infty} \sin^2\left(\frac{1}{k}\right)$
53. $\sum_{k=1}^{\infty} \tan\left(\frac{1}{k}\right)$
54. $\sum_{k=2}^{\infty} 100k^{-k}$
55. $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\cdots$
56. $\frac{1}{2^2}+\frac{2}{3^2}+\frac{3}{4^2}+\cdots$
57. $\frac{1}{1!}+\frac{4}{2!}+\frac{9}{3!}+\frac{16}{4!}+\cdots$

58–65. Convergence parameter *Find the values of the parameter p for which the following series converge.*

58. $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^p}$
59. $\sum_{k=2}^{\infty} \frac{\ln k}{k^p}$
60. $\sum_{k=2}^{\infty} \frac{1}{k\ln k(\ln\ln k)^p}$
61. $\sum_{k=2}^{\infty} \left(\frac{\ln k}{k}\right)^p$
62. $\sum_{k=0}^{\infty} \frac{k!p^k}{(k+1)^k}$
63. $\sum_{k=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2k-1)}{kp^{k+1}k!}$
64. $\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)^p$
65. $\sum_{k=1}^{\infty} \left(1-\frac{p}{k}\right)^k$

66. **Series of squares** Prove that if $\sum a_k$ is a convergent series of positive terms, then the series $\sum a_k^2$ also converges.

67. **Geometric series revisited** We know from Section 8.3 that the geometric series $\sum r^k$ converges if $|r| < 1$ and diverges if $|r| > 1$. Prove these facts using the Integral Test, the Ratio Test, and the Root Test. What can be determined about the geometric series using the Divergence Test?

68. **Two sine series** Determine whether the following series converge.
    a. $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$
    b. $\sum_{k=1}^{\infty} \frac{1}{k}\sin\left(\frac{1}{k}\right)$

Additional Exercises

69. **Limit Comparison Test proof** Use the proof of case (1) of the Limit Comparison Test to prove cases (2) and (3).

70–75. A glimpse ahead to power series *Use the Ratio Test to determine the values of $x \ge 0$ for which each series converges.*

70. $\sum_{k=1}^{\infty} \frac{x^k}{k!}$
71. $\sum_{k=0}^{\infty} x^k$
72. $\sum_{k=1}^{\infty} \frac{x^k}{k}$
73. $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$
74. $\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2}$
75. $\sum_{k=1}^{\infty} \frac{x^k}{2^k}$

76. **Infinite products** An infinite product $P = a_1 a_2 a_3 \ldots$, which is denoted $\prod_{k=1}^{\infty} a_k$, is the limit of the *sequence of partial products* $\{a_1, a_1 a_2, a_1 a_2 a_3, \ldots\}$.
    a. Show that the infinite product converges (which means its sequence of partial products converges) provided the series $\sum_{k=1}^{\infty} \ln a_k$ converges.
    b. Consider the infinite product
    $$P = \prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right) = \frac{3}{4}\cdot\frac{8}{9}\cdot\frac{15}{16}\cdot\frac{24}{25}\cdots.$$
    Write out the first few terms of the sequence of partial products,
    $$P_n = \prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)$$
    (for example, $P_2 = \frac{3}{4}$, $P_3 = \frac{2}{3}$). Write out enough terms to determine the value of the product, which is $\lim_{n\to\infty} P_n$.
    c. Use the results of parts (a) and (b) to evaluate the series
    $$\sum_{k=2}^{\infty}\ln\left(1-\frac{1}{k^2}\right).$$

77. **Infinite products** Use the ideas of Exercise 76 to evaluate the following infinite products.
    a. $\prod_{k=0}^{\infty} e^{1/2^k} = 1\cdot e^{1/2}\cdot e^{1/4}\cdot e^{1/8}\cdots$
    b. $\prod_{k=2}^{\infty}\left(1-\frac{1}{k}\right) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdots$

78. **An early limit** Working in the early 1600s, the mathematicians Wallis, Pascal, and Fermat were attempting to determine the area of the region under the curve $y = x^p$ between $x = 0$ and $x = 1$, where p is a positive integer. Using arguments that predated the Fundamental Theorem of Calculus, they were able to prove that
$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^p = \frac{1}{p+1}.$$
Use what you know about Riemann sums and integrals to verify this limit.

QUICK CHECK ANSWERS

1. 10; $(k+2)(k+1)$; $1/(k+1)$ **2.** To use the Comparison Test, we would need to show that $1/(k+1) > 1/k$, which is not true. **3.** If $\lim_{k\to\infty}\frac{a_k}{b_k} = L$ for $0 < L < \infty$, then $\lim_{k\to\infty}\frac{b_k}{a_k} = \frac{1}{L}$ where $0 < 1/L < \infty$.

8.6 Alternating Series

Our previous discussion focused on infinite series with positive terms, which is certainly an important part of the entire subject. But there are many interesting series with terms of mixed sign. For example, the series

$$1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\cdots$$

has the pattern that two positive terms are followed by two negative terms and vice versa. Clearly, infinite series could have a variety of sign patterns, so we need to restrict our attention.

Fortunately, the simplest sign pattern is also the most important. We consider **alternating series** in which the signs strictly alternate, as in the series

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots.$$

The factor $(-1)^{k+1}$ (or $(-1)^k$) has the pattern $\{\ldots, 1, -1, 1, -1, \ldots\}$ and provides the alternating signs.