Nearest Neighbour shifts of finite type

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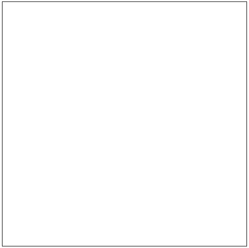
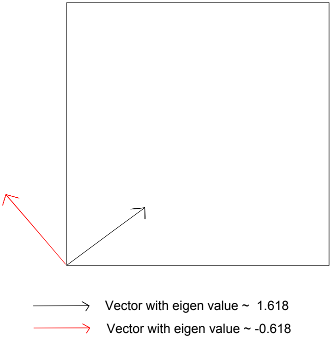
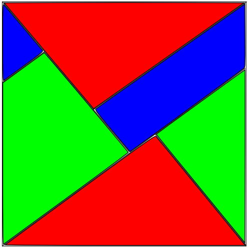
Consider a torus $\mathbb{R}^2/\mathbb{Z}^2$ with the map $\left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]$.



$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has two eigenvalues (~ 1.618 and $-.618$).



We can divide the torus into 3 parts by extending the eigendirections. These are called Markov partitions.



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This phenomenon is much more general: Any automorphism of the torus(with no eigenvalues of modulus 1) can be coded in a similar way.

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A nearest neighbour shift of finite type is a shift space such that $\mathcal{F}$ can be given by patterns on ‘edges’.

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0	1	0	1	0	1	0	1
1	2	1	0	1	2	1	0
0	1	0	1	0	1	0	1
1	0	1	2	1	0	1	2
0	1	0	1	0	1	0	1
1	0	1	2	1	2	1	0
0	1	0	1	0	1	0	1
1	2	1	0	1	2	1	0

Figure: The 3-coloured chessboard in 2 dimensions

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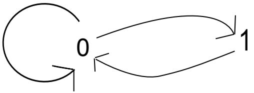

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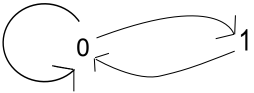

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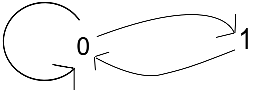
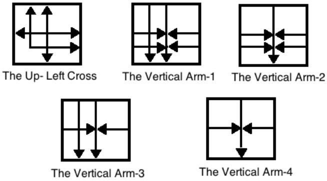


give us the hard square shift space. ($\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \{11\}$). However in higher dimensions given $\mathcal{A}$ and $\mathcal{F}$ there is no algorithm to decide whether the nearest neighbour shift of finite type is non-empty!!!

Let the alphabet be



and their rotations.

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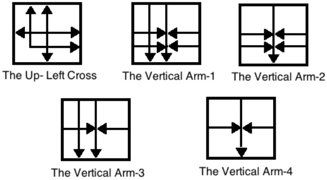

and their rotations. The adjacency rules entail that the arrows should match up.

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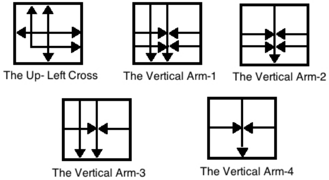
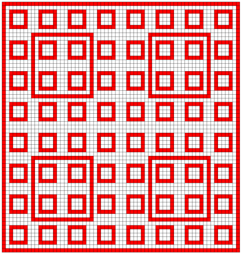


and their rotations. The adjacency rules entail that the arrows should match up. The double arrows form patterns which look like



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These shift spaces are non-empty if and only if the corresponding turing machine does not halt on an empty input.Since the latter is undecidable it is undecidable whether or not given an alphabet and its adjacency rules, the shift space generated is non-empty.

Thank You!

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