FOUR THEOREMS ABOUT ENTROPY IN ERGODIC THEORY

1. General information:

- Classes from 15th of March to 2nd of July.
- Regular classes Wednesday and Friday, 11:00 AM -12:30 PM.
- Further discussions and clearing doubts (by prior appointment): Monday, 11:00 AM-12: 01 PM

2. A general introduction

Suppose you were to take a bi-infinite sequence of independent fair coin tosses. Can you somehow obtain from this (via some bijective coding) a sequence of bi-infinite independent dice throws? Intuitively this seems impossible: There is a lot more 'uncertainty' in the dice throws as compared to the coin tosses. This was a fundamental question in ergodic theory back in the early part of the 20th century [5] which remained unsolved for several decades. The difference between the two processes (dice throws and coin tosses) was eventually realised by Kolmogorov and Sinai in their landmark papers [9, 22] which introduced an invariant called entropy to ergodic theory. Here Kolmogorov was inspired by revolutionary work of Shannon who had introduced entropy rate for stationary processes [21]. Later three classical theorems in ergodic theory solidified the status of entropy in ergodic theory.

(1) *Sinai's factor theorem:* Given a stationary process of entropy h it can be coded into a Bernoulli process of equal or smaller entropy. [23]
(2) *Ornstein's isomorphism theorem:* Two stationary aperiodic Markov chains can be obtained from one another (via some bijective coding) if and only if they have the same entropy. [14]
(3) *Krieger's generator theorem:* If the entropy of an ergodic stationary process (without periodic points) is h, then it can be recoded to a process with a state space of cardinality $\lceil e^h \rceil$. [11]

A natural question one might raise here is whether (for instance) these measurable isomorphisms can be made continuous (up to a set of measure zero). This corresponds to codes between processes such that the coding radius is almost surely finite (or the maps are finitary). This was answered as an affirmative by Keane and Smordinsky [7]. On the other hand, it has only been recently proven by a very nice construction by Uri Gabor that a finitary version of Sinai's factor theorem is not true for processes [4]. In recent times a huge body of work has developed around the existence and non-existence of finitary codings with various properties (like expectation of the coding radius [6, 19]/monotonicity [24, 17]), finitary factors of iid processes for random fields ([28] and many other works by the same author), continuous time processes [25, 26, 10, 27] and for processes on the free group [13]. Yet a lot of this remains poorly understood to date and is still a subject of active interest and we hope to touch up on some of it in this course.

These invariants like entropy aren't limited to just stationary processes. In fact, Kolmogorov introduced it as an invariant for probability preserving transformations and has seen a wide-range of generalisations to actions of amenable groups by Ornstein and Weiss [15], for sofic groups by work of Lewis Bowen [1] and Kerr-Li [8](look also at the introduction by Weiss [30]), for all countable groups by Seward [20].

In a different direction, entropy has turned up at several unexpected places. Here is an example of one such example. Consider the following 'simple' question: Consider the maps $\times 2$ and $\times 3$ on the the torus

$\mathbb{R}/\mathbb{Z}$. It is easy to see that Lebesgue measure is preserved under this maps. One can, with a little bit of work, construct many atomic measures which are also preserved by these maps. Are there others? This was a famous question raised by Furstenberg [3] in 1967 and came to be known as the famous $\times 2 \times 3$ conjecture. Some of the major progress towards resolution of this question was made by Rudolph in [18] where the same was proven under the assumption of positive entropy for the one of the maps (either the $\times 2$ map or (as it turns out equivalently) the $\times 3$ map). This was building upon breakthrough results by Russ Lyons [12] in the same direction. Later these ideas formed a critical kernel of 'an almost' resolution of the Littlewood conjecture by Einsedler, Katok and Lindenstrauss [2].

In this course, we hope to build a foundation for the subject with a view towards these results.

3. Structure of the course

Here is the rough structure that we will abide with. The chronology will perhaps be preserved. About the time-line I am less sure of.

(1) *Week 1:* What is a measure preserving transformation and some common examples.
(2) *Week 2:* Proof of the ergodic theorems.
(3) *Week 3 and 4:* Mixing properties, spectrum of a measure preserving transformation.
(4) *Week 5-8:* In the first couple of weeks we will give an introduction to Shannon's entropy and then see how it can be used to define an invariant for measure preserving transformations. We will compute the entropy for some simple examples like Markov chains and (time permitting) see how to do it for certain automorphisms of the torus.
(5) *Weeks 9-11:* Rudolph's theorem.
(6) *Week 12:* Shannon-McMillan-Breiman theorem.
(7) *Week 13-16:* Krieger's generator theorem, Sinai's factor theorem and Ornstein's isomorphism theorem.

We will follow some unpublished notes of Benjamin Weiss, of Einsedler-Lindenstrauss-Ward, and the standard books of ergodic theory by Peter Walters [29] and Karl Petersen [16]).

4. Prerequisites

(1) Graduate course in probability (familiarity with measure theory, Markov chains and martingales)
(2) Functional analysis
(3) Fourier series
(4) Some familiarity with operator theory. Specifically spectral theory of unitary operators.

Some of these prerequisites can be avoided if you agree to take some facts on faith (specifically in operator theory). Without familiarity with Fourier series, functional analysis and a graduate course in probability, this course will be hard to follow.

References

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