Four-Cycle Free Graphs and Entropy Minimality

Nishant Chandgotia

University of British Columbia

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Outline

- Entropy Minimality and Hom Shifts
- Mixing Conditions and Entropy Minimality
- Rigidity and Flexibility in the Space of 3-Colourings.

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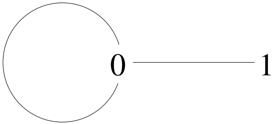
Examples:(Hard Square model)



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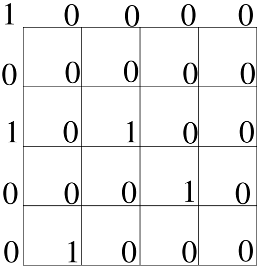

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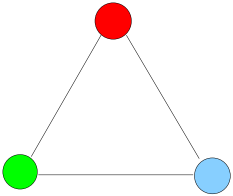

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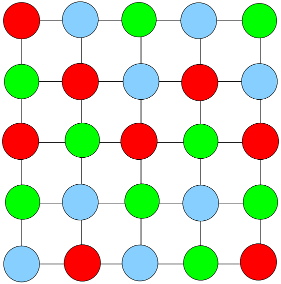

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$$h_{top}(X) := \lim_{n \longrightarrow \infty} \frac{\log |\mathcal{B}(X) \cap \mathcal{A}^{\{1,2,\dots,n\}^d}|}{n^d}.$$

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(Quas and Trow '00) Every shift space X contains an entropy minimal shift space $Y \subset X$ such that $h_{top}(X) = h_{top}(Y)$.

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Remark: We will concentrate on X_{C_3}, the space of all 3-colourings.

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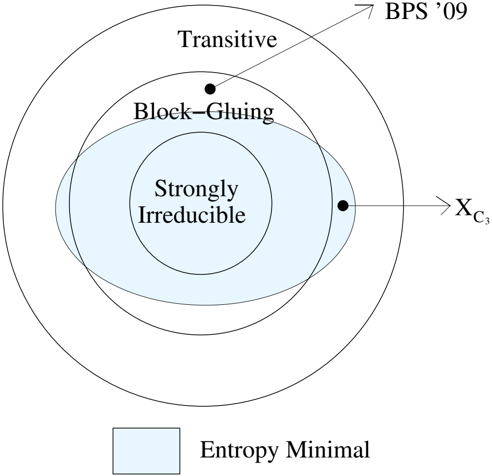
(Lightwood and Schraudner '12) A shift of finite type is entropy minimal if and only if the set of all 'non-universal' boundary patterns is 'poor'.

Mixing Conditions and Entropy Minimality



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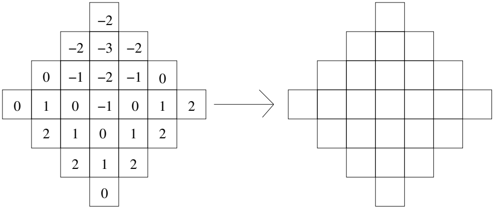

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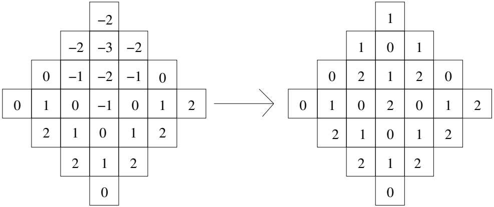

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Pattern in X_{C_3}

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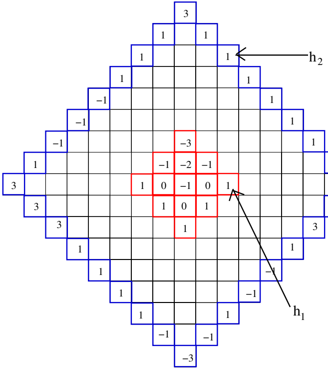

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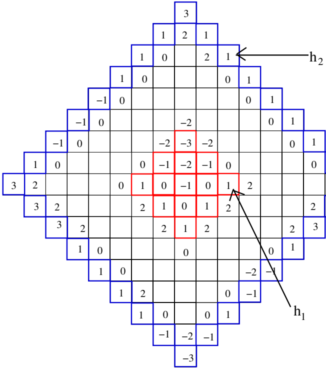

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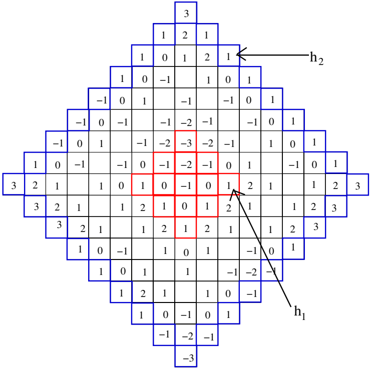

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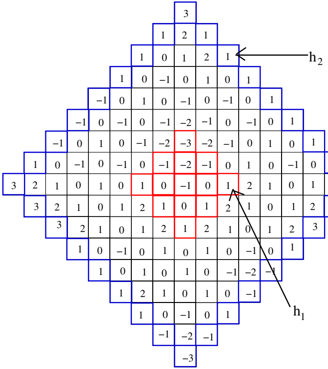

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$\mathbb{Z}$ is replaced by the universal cover of $\mathcal{H}$.

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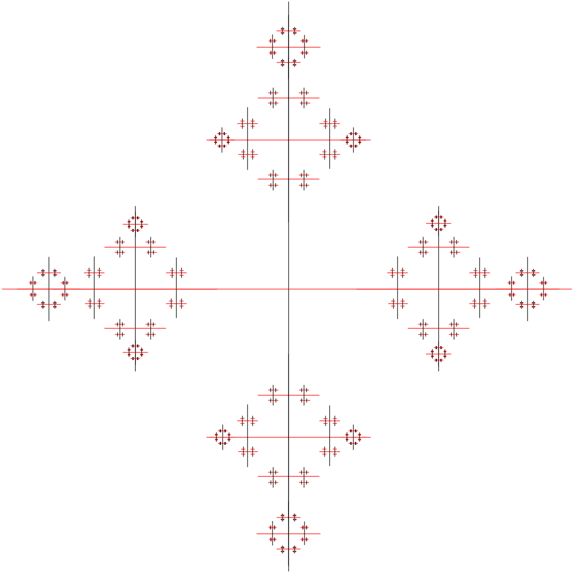
Question: What shift spaces are conjugate to $X_{\mathcal{H}}$ for some graph $\mathcal{H}$?



Thank You!