Predictive Sets

Nishant Chandgotia Benjamin Weiss

Hebrew University of Jerusalem

December, Hebrew University of Jerusalem

Predictive sets

By a process we mean a stationary process with a finite state space unless stated otherwise.

Given a subset $P \subset \mathbb{N}$, a sequence of random variables $X_i; i \in P$ will be denoted by X_P.

A set $P \subset \mathbb{Z}$ is called a predictive set if for all zero-entropy processes $X_{\mathbb{Z}}$, X_0 is measurable with respect to X_P.

Predictive sets

A set $P \subset \mathbb{Z}$ is called a predictive set if for all zero-entropy processes $X_{\mathbb{Z}}$, X_0 is measurable with respect to X_P.

Equivalently, P is a predictive set if for all zero-entropy processes $X_{\mathbb{Z}}$,

$$H(X_0 \mid X_P) = 0.$$

$\mathbb{N}$ is a predictive set.

$k\mathbb{N}$ is predictive

The process $X_{\mathbb{Z}}$ has zero entropy if and only if $X_{k\mathbb{Z}}$ has zero-entropy.

Thus P is a predictive set if and only if kP is also a predictive set.

But $P = k\mathbb{N} + r$ is not predictive (when $r \not\equiv 0 \pmod{k}$): $k = 2, r = 1$

Let us see why this is true for $k = 2$ and $r = 1$.

Take two independent random variables Y_1 taking values -1 or 1 and Y_2 taking values 2 or -2 with equal probability.

Now consider a process $X_\mathbb{Z}$ for which (independent of Y_1 and Y_2)

$$\text{with probability } 1/2,\; X_{2\mathbb{Z}} := Y_1;\; X_{2\mathbb{Z}+1} := Y_2 \quad \text{and}$$
$$\text{with probability } 1/2,\; X_{2\mathbb{Z}} := Y_2;\; X_{2\mathbb{Z}+1} := Y_1.$$

Clearly $X_\mathbb{Z}$ has zero entropy but

$$\mathbb{P}(X_0 > 0 \mid X_{2\mathbb{N}+1}) = 1/2.$$

$P = k\mathbb{N} + r$ is not predictive (when $r \not\equiv 0 \pmod{k}$):
$k = 2, r = 1$

Here, in fact, X_0 is independent of $X_{2\mathbb{N}+1}$. This process is not weak mixing but we can construct one which is weak mixing and has zero entropy.

Some sufficient conditions.

Return-time sets are predictive

Let (X, μ, T) be a probability preserving transformation (ppt). Given a set $U \subset X$ of positive measure, we denote by

$$N(U, U) := \{n \in \mathbb{N} \ : \ \mu(T^n(U) \cap U) > 0\}.$$

A set $A \subset \mathbb{N}$ is called a return-time set if $A = N(U, U)$ for some ppt.

Theorem (Chandgotia, Weiss)

Return-time sets are predictive sets.

Return-time sets are predictive

Theorem (Chandgotia, Weiss)

Return-time sets are predictive sets.

Note that $k\mathbb{N}$ is a return-time set for the transformation $T : \mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ given by $T(i) = i + 1$.

Thus we have generalised our former observation that $k\mathbb{N}$ is predictive.

Return-time sets are predictive

Theorem (Chandgotia, Weiss)

Return-time sets are predictive sets.

This theorem can be formally strengthened for return-time sets coming from zero-entropy ppt. If (X, μ, T) is a zero entropy ppt, $U \subset X$ with $\mu(U) > 0$ and P is a predictive set then $P \cap N(U, U)$ is also a predictive set.

Question

Does every return-time set contain a return-time set of a zero-entropy process?

The intersection of a return-time set of a zero entropy process and a predictive set is predictive

If (X, μ, T) is a zero entropy ppt, $U \subset X$ with $\mu(U) > 0$ and P is a predictive set then $P \cap N(U, U)$ is also a predictive set.

It is easy to see that if $\alpha \in \mathbb{R}/\mathbb{Z}$ and $\epsilon > 0$ then the set

$$\{n \ : \ n\alpha \mod 1 \in (-\epsilon, \epsilon)\}$$

contains a return-time set for $U = (-\epsilon/2, \epsilon/2)$.

Thus if P is predictive then

$$P \cap \{n \ : \ n\alpha \mod 1 \in (-\epsilon, \epsilon)\}$$

is also predictive.

Some necessary conditions

$SIP^\star$ sets

Given a sequence of natural numbers $S = \{s_i\}_{i \in \mathbb{N}} \subset \mathbb{N}$, we write

$$SIP(S) := \left\{ \sum_{i=1}^{\infty} \epsilon_i s_i \ : \ \epsilon_i \in \{-1, 0, 1\} \right\} \cap \mathbb{N}.$$

A set $P \subset \mathbb{N}$ is called $SIP^\star$ if it intersects every SIP set.

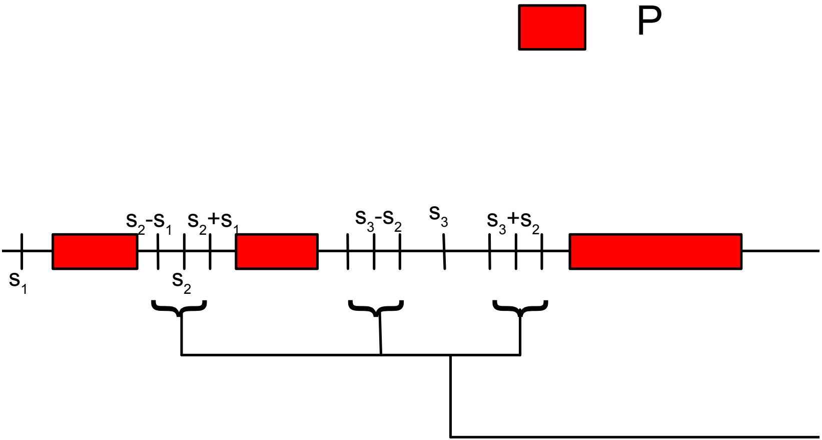
$SIP^\star$ sets

$$SIP(S) := \left\{ \sum_{i=1}^{\infty} \epsilon_i s_i \ : \ \epsilon_i \in \{-1,0,1\} \right\} \cap \mathbb{N}.$$

1. $k\mathbb{N}$ is $SIP^\star$: Given a sequence $S = \{s_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ there exists a subsequence $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ (which are equal modulo k) such that
$$\sum_{t=1}^{k} s_{i_t} \in k\mathbb{N}.$$
Thus $SIP(S) \cap k\mathbb{N} \neq \emptyset$.
2. if $r \not\equiv 0 \pmod{k}$ then $k\mathbb{N} + r$ is not SIP^*: If a sequence $S \subset k\mathbb{N}$ then $SIP(S) \subset k\mathbb{N}$ and $SIP(S) \cap (k\mathbb{N} + r) = \emptyset$.
3. $SIP^\star$ sets have bounded gaps.

$SIP^\star$ sets have bounded gaps.

Suppose P is a set such that it does not have bounded gaps. Then we can fit an SIP set in its complement.



Predictive sets are $SIP^\star$

Theorem (Chandgotia, Weiss)

Predictive sets are $SIP^\star$.

1. If $r \not\equiv 0 \pmod{k}$ then $k\mathbb{N} + r$ is not SIP^*: Thus we have generalised the fact that $k\mathbb{N} + r$ is not predictive.
2. $SIP^\star$ sets have bounded gaps. Thus predictive sets also have bounded gaps.

Sufficient conditions for a set to be predictive:

Theorem (Chandgotia, Weiss)

Return-time sets are predictive sets.

Necessary conditions for a set to be predictive:

Theorem (Chandgotia, Weiss)

Predictive sets are SIP*.

The following question arises naturally.

Question

Are sufficient conditions necessary and necessary conditions sufficient?

Let us give some partial answers.

Are all $SIP^\star$ sets predictive?

If P is a predictive set, $\epsilon > 0$ and $\alpha \in \mathbb{R}/\mathbb{Z}$ then

$$\{n \in \mathbb{N} \; : \; n\alpha \in (-\epsilon, \epsilon)\} \cap P$$

is predictive.

Question

Is the intersection of two predictive sets also predictive? Is the intersection non-empty?

Question

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be irrational and $\epsilon < 1/2$. Is the set

$$\{n \in \mathbb{N} \; : \; n\alpha \in (0, \epsilon)\}$$

predictive?

An uncertain theorem

Question

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be irrational and $\epsilon < 1/2$. Is the set

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$$

predictive?

If the answer is yes then we have two predictive sets

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\} \text{ and } \{n \in \mathbb{N} \ : \ -n\alpha \in (0, \epsilon)\}$$

which do not intersect.

Theorem (Akin and Glasner, 2016)

The set $\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$ is $SIP^\star$.

Thus if the answer is no then we have a $SIP^\star$ set which is not predictive.

So we don't really know if all $SIP^{\star}$ sets are predictive.

There are predictive sets which do not contain return-time sets.

Consider the set
$$Q = \{n^2 \ : \ n \in \mathbb{N}\}.$$

For all $i, k \in \mathbb{N}$ we have that if
$$n^2 = -i + 3i^2 k = i(-1 + 3ik)$$
then since i and $-1 + 3ik$ are prime to each other, they are perfect squares.

But this is impossible because $-1 + 3ik \equiv -1 \pmod 3$. Thus $\mathbb{N} \setminus Q$ contains $-i + 3i^2 k; k \in \mathbb{N}$.

There are predictive sets which do not contain return-time sets.

Hence we have that

$$H(X_{-i} \mid X_{\mathbb{N}\setminus Q}) = 0$$

for all $i \in \mathbb{N}$.

But then for all $i \in \mathbb{Z}$

$$H(X_i \mid X_{\mathbb{N}\setminus Q}) = H(X_i \mid X_{(-\mathbb{N})\cup(\mathbb{N}\setminus Q)}) = 0.$$

But all return-time sets must intersect the set $\{n^2 \ : \ n \in \mathbb{N}\}$ (Sarkozy, Furstenberg). Thus there are predictive sets which are not return-time sets.

Predictive sets

Question

Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence such that $n_{k+1} - n_k$ is also an increasing sequence. Prove that

$$H(X_0 \mid X_{\mathbb{N}\setminus\{n_k \mid k\in\mathbb{N}\}}) = 0.$$

We do not know this even in the case $n_k = k^3$. We will come back to this later if time permits.

Proofs.

Return-time sets are predictive

Let (X, μ, T) be a ppt and $U \subset X$ have positive measure. We will prove that

$$\{n \in \mathbb{N} \ : \ \mu(T^n(U) \cap U) > 0\}$$

is predictive.

We can assume that $X \subset \{0,1\}^{\mathbb{Z}}$ is a closed invariant set, $supp(\mu) = X$ and

$$U = \{x \in X \ : \ x_0 = 1\}.$$

For all $x \in X$ we have that for the set Q_x,

$$Q_x := \{n \in \mathbb{N} \ : \ T^n(x) \in U\}$$

satisfies $(Q_x - Q_x) \cap \mathbb{N} \subset N(U, U)$. By the ergodic theorem we can choose x such that $Q_x \subset \mathbb{N}$ has positive density.

Return-time sets are predictive

Thus return-time sets contain the difference set of a positive density set.

It is sufficient to prove that the difference set of a positive density set is predictive.

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \dots\}$ have density

$$\alpha = \lim_{n\to\infty} \frac{n}{q_n} > 0$$

and $h(X_{\mathbb{Z}}) = 0$.

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \ldots\}$ have density

$$\alpha = \lim_{n\to\infty} \frac{n}{q_n} > 0$$

and $h(X_{\mathbb{Z}}) = 0$.Then

$$\frac{1}{n} H(X_{q_1}, X_{q_2}, \ldots, X_{q_n}) \leq \frac{q_n}{n} \frac{1}{q_n} H(X_1, X_2, \ldots, X_{q_n}) \to \frac{1}{\alpha} h(X_{\mathbb{Z}}) = 0.$$

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \dots\}$ have density

$$\alpha = \lim_{n\to\infty} \frac{n}{q_n} > 0$$

and $h(X_{\mathbb{Z}}) = 0$.Then

$$\frac{1}{n}H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) \leq \frac{q_n}{n}\frac{1}{q_n}H(X_1, X_2, \dots, X_{q_n}) \to \frac{1}{\alpha}h(X_{\mathbb{Z}}) = 0.$$

But

$$\frac{1}{n}H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) \quad = \quad \frac{1}{n}H(X_0 \mid X_{q_2-q_1}, X_{q_3-q_1}, \dots, X_{q_n-q_1})$$

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \dots\}$ have density

$$\alpha = \lim_{n\to\infty} \frac{n}{q_n} > 0$$

and $h(X_{\mathbb{Z}}) = 0$.Then

$$\frac{1}{n} H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) \le \frac{q_n}{n} \frac{1}{q_n} H(X_1, X_2, \dots, X_{q_n}) \to \frac{1}{\alpha} h(X_{\mathbb{Z}}) = 0.$$

But

$$\begin{aligned} \frac{1}{n} H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) &= \frac{1}{n} H(X_0 \mid X_{q_2-q_1}, X_{q_3-q_1}, \dots, X_{q_n-q_1}) \\ &+ \frac{1}{n} H(X_0 \mid X_{q_3-q_2}, X_{q_4-q_2}, \dots, X_{q_n-q_2}) + \dots \end{aligned}$$

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \dots\}$ have density

$$\alpha = \lim_{n\to\infty} \frac{n}{q_n} > 0$$

and $h(X_{\mathbb{Z}}) = 0$.Then

$$\frac{1}{n} H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) \leq \frac{q_n}{n} \frac{1}{q_n} H(X_1, X_2, \dots, X_{q_n}) \to \frac{1}{\alpha} h(X_{\mathbb{Z}}) = 0.$$

But

$$\begin{aligned} \frac{1}{n} H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) &= \frac{1}{n} H(X_0 \mid X_{q_2-q_1}, X_{q_3-q_1}, \dots, X_{q_n-q_1}) \\ &+ \frac{1}{n} H(X_0 \mid X_{q_3-q_2}, X_{q_4-q_2}, \dots, X_{q_n-q_2}) + \dots \\ &+ \frac{1}{n} H(X_0 \mid X_{q_n-q_{n-1}}) \end{aligned}$$

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \dots\}$ have density

$$\alpha = \lim_{n\to\infty} \frac{n}{q_n} > 0$$

and $h(X_{\mathbb{Z}}) = 0$.Then

$$\frac{1}{n} H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) \leq \frac{q_n}{n}\frac{1}{q_n} H(X_1, X_2, \dots, X_{q_n}) \to \frac{1}{\alpha} h(X_{\mathbb{Z}}) = 0.$$

But

$$\begin{aligned}
\frac{1}{n} H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) &= \frac{1}{n} H(X_0 \mid X_{q_2-q_1}, X_{q_3-q_1}, \dots, X_{q_n-q_1}) \\
&+ \frac{1}{n} H(X_0 \mid X_{q_3-q_2}, X_{q_4-q_2}, \dots, X_{q_n-q_2}) + \dots \\
&+ \frac{1}{n} H(X_0 \mid X_{q_n-q_{n-1}}) \\
&\geq H(X_0 \mid X_{(Q-Q)\cap\mathbb{N}})
\end{aligned}$$

Return-time sets are predictive

Thus if Q has positive density then

$$H(X_0 \mid X_{(Q-Q)\cap\mathbb{N}}) = 0$$

and $(Q - Q) \cap \mathbb{N}$ is a predictive set. We showed earlier that every return-time set contains such a set.

Thus return-time sets are predictive.

Predictive sets are $SIP^\star$

In course of the proof we show that for all $SIP(S)$ there exists a weak mixing zero entropy Gaussian process $X_{\mathbb{Z}}$ such that

$$X_0 \text{ is independent of } X_i \text{ for } i \in \mathbb{N} \setminus SIP(S).$$

This shows that $\mathbb{N} \setminus SIP(S)$ is not predictive.

Thus there exists a weak-mixing process in which X_0 can be predicted by $X_{\mathbb{N}}$ but is independent of $X_{2\mathbb{N}+1}$.

Predictive sets are $SIP^\star$: Processes and Spectral measures

From here on we will assume that X_0 is complex-valued, has zero mean and finite variance.

Given any process $X_{\mathbb{Z}}$, the sequence $\mathbb{E}(X_0\overline{X_n}); n \in \mathbb{N}$ is a positive definite sequence.

By Herglotz's theorem, there exists a probability measure μ on $\mathbb{R}/\mathbb{Z}$ such that the Fourier coefficients

$$\hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n}).$$

On the other hand, given any probability measure μ on $\mathbb{R}/\mathbb{Z}$ there exists a Gaussian process $X_{\mathbb{Z}}$ such that

$$\hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n}).$$

Predictive sets are $SIP^{\star}$: Processes and Spectral measures

$X_{\mathbb{Z}}$

Predictive sets are $SIP^\star$: Processes and Spectral measures

$$X_{\mathbb{Z}} \longrightarrow \mathbb{E}(X_0\overline{X_n});\, n \in \mathbb{N}$$

Predictive sets are $SIP^\star$: Processes and Spectral measures

$$X_{\mathbb{Z}} \longrightarrow \mathbb{E}(X_0\overline{X_n}); n \in \mathbb{N} \longrightarrow \mu \text{ on } \mathbb{R}/\mathbb{Z} \text{ such that } \hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n}).$$

Predictive sets are $SIP^{\star}$: Processes and Spectral measures

$X_{\mathbb{Z}} \longrightarrow \mathbb{E}(X_0\overline{X_n}); n \in \mathbb{N} \longrightarrow \mu$ on $\mathbb{R}/\mathbb{Z}$ such that $\hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n})$.

μ on $\mathbb{R}/\mathbb{Z}$

Predictive sets are $SIP^{\star}$: Processes and Spectral measures

$X_{\mathbb{Z}} \longrightarrow \mathbb{E}(X_0\overline{X_n}); n \in \mathbb{N} \longrightarrow \mu$ on $\mathbb{R}/\mathbb{Z}$ such that $\hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n})$.

μ on $\mathbb{R}/\mathbb{Z} \longrightarrow$ Gaussian process $X_{\mathbb{Z}}$ for which $\hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n})$.

Predictive sets are $SIP^\star$: Processes and Spectral measures

$X_{\mathbb{Z}} \longrightarrow \mathbb{E}(X_0\overline{X_n}); n \in \mathbb{N} \longrightarrow \mu$ on $\mathbb{R}/\mathbb{Z}$ such that $\hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n})$.

μ on $\mathbb{R}/\mathbb{Z} \longrightarrow$ Gaussian process $X_{\mathbb{Z}}$ for which $\hat{\mu}(n) = \mathbb{E}(X_0\overline{X_n})$.

If μ is singular then $X_{\mathbb{Z}}$ has zero entropy (Newton and Parry).

For Gaussian processes X_0 and X_n are independent if and only if $\hat{\mu}(n) = 0$.

A Gaussian process $X_{\mathbb{Z}}$ is weak-mixing if and only if μ is continuous.

Predictive sets are $SIP^\star$: Gaussian Processes

μ is singular then $X_{\mathbb{Z}}$ has zero entropy.

For Gaussian processes X_0 and X_n are independent if and only if $\hat{\mu}(n) = 0$.

A Gaussian process $X_{\mathbb{Z}}$ is weak-mixing if and only if μ is continuous.

Predictive sets are $SIP^\star$: Riesz products

Fix a sequence $s_1, s_2, \ldots \subset \mathbb{N}$ such that $s_{i+1} > 3s_i$.

Predictive sets are $SIP^\star$: Riesz products

Fix a sequence $s_1, s_2, \ldots \subset \mathbb{N}$ such that $s_{i+1} > 3s_i$.

The Riesz product is the function $f_r : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ given by

$$
\begin{aligned}
f_r(x) &:= \prod_{k \le r} (1 + \cos(2\pi s_k x)) \\
&= \prod_{k \le r} \left(1 + \frac{\exp(2\pi i s_k x) + \exp(-2\pi i s_k x)}{2} \right).
\end{aligned}
$$

Predictive sets are $SIP^\star$: Riesz products

Fix a sequence $s_1, s_2, \ldots \subset \mathbb{N}$ such that $s_{i+1} > 3s_i$.

The Riesz product is the function $f_r : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ given by

$$\begin{aligned} f_r(x) &:= \prod_{k \le r} (1 + \cos(2\pi s_k x)) \\ &= \prod_{k \le r} \left(1 + \frac{\exp(2\pi i s_k x) + \exp(-2\pi i s_k x)}{2}\right). \end{aligned}$$

As r tends to infinity the limit of $f_r \mu_{Leb}$ is a singular continuous measure μ such that $\hat{\mu}(n) = 0$ for all

$$n \notin SIP(s_1, s_2, \ldots) := \left\{ \sum_{t \in \mathbb{N}} \epsilon_i s_i \ : \ \epsilon_i \in \{-1, 0, 1\} \right\}.$$

Thus $X_{\mathbb{Z}}$ has zero entropy, is weak mixing and $\mathbb{E}(X_0 \overline{X_n}) = 0$ for all

$$n \notin SIP(s_1, s_2, \ldots).$$

Predictive sets are $SIP^\star$

Thus $X_{\mathbb{Z}}$ has zero entropy, is weak mixing and $\mathbb{E}(X_0\overline{X_n}) = 0$ for all

$$n \notin SIP(s_1, s_2, \ldots).$$

If P is predictive then

$$P \cap SIP(s_1, s_2, \ldots) \neq \emptyset.$$

One can use this to prove that predictive sets are SIP^*.

Linear Predictivity

In fact if μ is singular by a theorem of Verblunsky we get the following result:

Theorem

If $X_{\mathbb{Z}}$ is a complex-valued L^2 process for which the spectral measure μ is singular and P is predictive then X_0 is in the closed linear span of $X_i; i \in P$.

I wasn't aware of this even for processes arising from circle rotations and $P = \mathbb{N}$.

On the other hand if the spectral measure has a Lebesgue component but $X_{\mathbb{Z}}$ has zero entropy then we can predict but not linearly predict the process.

Riesz Sets

Using this machinery we can conclude the following result.

Theorem (Chandgotia, Weiss)

If $P \subset \mathbb{N}$ is a set such that $P + i$ is predictive for all $i \in \mathbb{N}$ then for all singular measures μ on $\mathbb{R}/\mathbb{Z}$ there exists $p \in P$ such that the Fourier coefficient

$$\hat{\mu}(p) \neq 0.$$

In other words any measure μ on $\mathbb{R}/\mathbb{Z}$ whose Fourier coefficients are supported on $\mathbb{Z} \setminus P$ must have an absolutely continuous component.

Riesz Sets

Using this machinery we can conclude the following result.

Theorem (Chandgotia, Weiss)

If $P \subset \mathbb{N}$ is a set such that $P + i$ is predictive for all $i \in \mathbb{N}$ then for all singular measures μ on $\mathbb{R}/\mathbb{Z}$ there exists $p \in P$ such that the Fourier coefficient

$$\hat{\mu}(p) \neq 0.$$

In other words any measure μ on $\mathbb{R}/\mathbb{Z}$ whose Fourier coefficients are supported on $\mathbb{Z} \setminus P$ must have an absolutely continuous component.

This is very close to Riesz sets as defined by Yves Meyer in 1968: A set $Q \subset \mathbb{Z}$ is called a Riesz set if all measures on $\mathbb{R}/\mathbb{Z}$ whose Fourier coefficients are supported on Q are absolutely continuous.

Riesz Sets

A set $P \subset \mathbb{N}$ is called totally predictive if $P + i$ is predictive for all $i \in \mathbb{N}$.

Theorem (Chandgotia, Weiss)

If $P \subset \mathbb{N}$ is a totally predictive set which is open in the Bohr topology, then $\mathbb{Z} \setminus P$ is a Riesz set.

Question

If $P \subset \mathbb{N}$ is totally predictive then is $\mathbb{Z} \setminus P$ a Riesz set? If $Q \subset \mathbb{N}$ is a set such that $Q \cup (-\mathbb{N})$ is Riesz then is $\mathbb{N} \setminus Q$ a totally predictive set?

A titillating question

Let $n_{\mathbb{N}}$ be an increasing sequence of natural numbers such that $n_{i+1} - n_i$ is also an increasing sequence. We had asked whether $\mathbb{N} \setminus n_{\mathbb{N}}$ is totally predictive.

It is unknown even for $n_i = i^3$ whether $(-\mathbb{N}) \cup n_{\mathbb{N}}$ is a Riesz set. Wallen (1970) proved that if μ is a measure whose Fourier coefficients are supported on $(-\mathbb{N}) \cup n_{\mathbb{N}}$ then $\mu \star \mu$ is absolutely continuous.

Following an idea by Lindenstrauss, a simple application of Fermat's last theorem and Cauchy Schwarz gives us the following partial result.

Theorem (Chandgotia, Weiss)

If μ is a probability measure whose Fourier coefficients are supported on $\{\pm i^K \ : \ i \in \mathbb{N}\} \cup \{0\}$ for some $k \geq 2$ then μ is not singular.

Summary

Return-time sets are predictive.

The converse is not true.

Predictive sets are $SIP^\star$.

Predictive sets have bounded gaps.

If you were bored. . .

1. Is the intersection of two predictive sets also a predictive set?
2. Are all $SIP^\star$ sets predictive?
3. Is $\{n \ : \ n\alpha \in (0, \epsilon)\}$ a predictive set for irrational α?
4. Let $\{n_k\}_{k \in \mathbb{N}}$ be an increasing sequence such that $n_{k+1} - n_k$ is also an increasing sequence. Prove that

$$H(X_0 \mid X_{\mathbb{N} \setminus \{n_k \ \mid \ k \in \mathbb{N}\}}) = 0.$$

5. What is the relationship between Riesz sets and totally predictive sets?
6. Explore linear prediction.