SOLUTION To show that the power series converges to f, we must show that $\lim_{n\to\infty} |R_n(x)| = 0$ for $-\infty < x < \infty$. According to Taylor's Theorem with $a = 0$,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1},$$

where c is between 0 and x. Notice that $f^{(n+1)}(c) = \pm\sin c$ or $f^{(n+1)}(c) = \pm\cos c$. In all cases, $|f^{(n+1)}(c)| \le 1$. Therefore, the absolute value of the remainder term is bounded as

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Holding x fixed and using $\lim_{n\to\infty} x^n/n! = 0$, we see that $\lim_{n\to\infty} R_n(x) = 0$ for all x. Therefore, the given power series converges to $f(x) = \cos x$ for all x; that is, $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$. The convergence of the Taylor series to $\cos x$ is illustrated in Figure 9.18.

Related Exercises 47–50 ◀

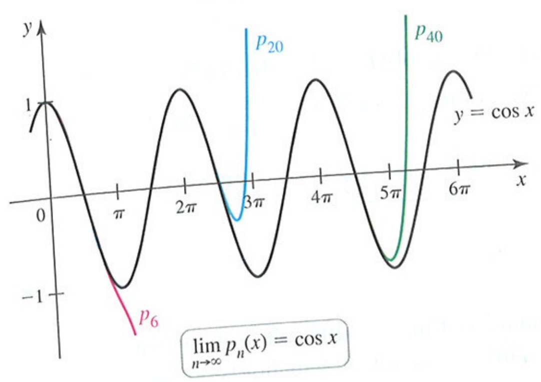

FIGURE 9.18

The procedure used in Examples 5 and 6 can be carried out for all of the Taylor series we have worked with so far (with varying degrees of difficulty). In each case, the Taylor series converges to the function it represents on the interval of convergence. Table 9.5 summarizes commonly used Taylor series centered at 0 and the functions to which they converge.

➤ Table 9.5 asserts, without proof, that in several cases the Taylor series for f converges to f at the endpoints of the interval of convergence. Proving convergence at the endpoints generally requires advanced techniques. It may also be done using the following theorem:

Suppose the Taylor series for f centered at 0 converges to f on the interval $(-R, R)$. If the series converges at $x = R$, then it converges to $\lim_{x\to R^-} f(x)$. If the series converges at $x = -R$, then it converges to $\lim_{x\to -R^+} f(x)$.

For example, this theorem would allow us to conclude that the series for $\ln(1 + x)$ converges to $\ln 2$ at $x = 1$.

Table 9.5

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \quad \text{for } |x| < 1 \quad \text{and} \quad \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \binom{p}{0} = 1$$

SECTION 9.3 EXERCISES

Review Questions

1. How are the Taylor polynomials for a function f centered at a related to the Taylor series of the function f centered at a?
2. What conditions must be satisfied by a function f to have a Taylor series centered at a?
3. How do you find the coefficients of the Taylor series for f centered at a?
4. How do you find the interval of convergence of a Taylor series?
5. Suppose you know the Maclaurin series for f and it converges for $|x| < 1$. How do you find the Maclaurin series for $f(x^2)$ and where does it converge?
6. For what values of p does the Taylor series for $f(x) = (1 + x)^p$ centered at 0 terminate?
7. In terms of the remainder, what does it mean for a Taylor series for a function f to converge to f?
8. Write the Maclaurin series for e^{2x}.

Basic Skills

9–16. Maclaurin series

a. *Find the first four nonzero terms of the Maclaurin series for the given function.*
b. *Write the power series using summation notation.*
c. *Determine the interval of convergence of the series.*

9. $f(x) = e^{-x}$
10. $f(x) = \cos 2x$
11. $f(x) = (1 + x^2)^{-1}$
12. $f(x) = \ln(1 + x)$
13. $f(x) = e^{2x}$
14. $f(x) = (1 + 2x)^{-1}$
15. $f(x) = \tan^{-1} x$
16. $f(x) = \sin 3x$

17–22. Taylor series centered at $a \ne 0$

a. *Find the first four nonzero terms of the Taylor series for the given function centered at a.*
b. *Write the power series using summation notation.*

17. $f(x) = \sin x,\ a = \pi/2$
18. $f(x) = \cos x,\ a = \pi$
19. $f(x) = 1/x,\ a = 1$
20. $f(x) = 1/x,\ a = 2$
21. $f(x) = \ln x,\ a = 3$
22. $f(x) = e^x,\ a = \ln 2$

23–28. Manipulating Taylor series *Use the Taylor series in Table 9.5 to find the first four nonzero terms of the Taylor series for the following functions centered at 0.*

23. $\ln(1 + x^2)$
24. $\sin x^2$
25. $\frac{e^x - 1}{x}$
26. $\cos \sqrt{x}$
27. $(1 + x^4)^{-1}$
28. $x \tan^{-1} x^2$

29–34. Binomial series

a. *Find the first four nonzero terms of the Taylor series centered at 0 for the given function.*
b. *Use the first four terms of the series to approximate the given quantity.*

29. $f(x) = (1 + x)^{-2}$; approximate $1/1.21 = 1/1.1^2$.
30. $f(x) = \sqrt{1 + x}$; approximate $\sqrt{1.06}$.
31. $f(x) = \sqrt[4]{1 + x}$; approximate $\sqrt[4]{1.12}$.
32. $f(x) = (1 + x)^{-3}$; approximate $1/1.331 = 1/1.1^3$.
33. $f(x) = (1 + x)^{-2/3}$; approximate $1.18^{-2/3}$.
34. $f(x) = (1 + x)^{2/3}$; approximate $1.02^{2/3}$.

35–40. Working with binomial series *Use properties of power series, substitution, and factoring to find the first four nonzero terms of the Taylor series centered at 0 for the following functions. Give the interval of convergence for the new series. Use the Taylor series*

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots, \quad \text{for } -1 < x \le 1.$$

35. $\sqrt{1 + x^2}$
36. $\sqrt{4 + x}$
37. $\sqrt{9 - 9x}$
38. $\sqrt{1 - 4x}$
39. $\sqrt{a^2 + x^2},\ a > 0$
40. $\sqrt{4 - 16x^2}$

41–46. Working with binomial series *Use properties of power series, substitution, and factoring of constants to find the first four nonzero terms of the Taylor series centered at 0 for the following functions. Use the Taylor series*

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

41. $(1 + 4x)^{-2}$
42. $\frac{1}{(1 - 4x)^2}$
43. $\frac{1}{(4 + x^2)^2}$
44. $(x^2 - 4x + 5)^{-2}$
45. $\frac{1}{(3 + 4x)^2}$
46. $\frac{1}{(1 + 4x^2)^2}$

47–50. Remainder terms *Find the remainder in the Taylor series centered at the point a for the following functions. Then show that $\lim_{n\to\infty} R_n(x) = 0$ for all x in the interval of convergence.*

47. $f(x) = \sin x,\ a = 0$
48. $f(x) = \cos 2x,\ a = 0$
49. $f(x) = e^{-x},\ a = 0$
50. $f(x) = \cos x,\ a = \pi/2$

Further Explorations

51. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. The function $f(x) = \sqrt{x}$ has a Taylor series centered at 0.
   b. The function $f(x) = \csc x$ has a Taylor series centered at $\pi/2$.
   c. If f has a Taylor series that converges only on $(-2, 2)$, then $f(x^2)$ has a Taylor series that also converges only on $(-2, 2)$.
   d. If p is the Taylor series for f centered at 0, then $p(x - 1)$ is the Taylor series for f centered at 1.
   e. The Taylor series for an even function about 0 has only even powers of x.

52–59. Any method

a. *Use any analytical method to find the first four nonzero terms of the Taylor series centered at 0 for the following functions. In most cases you do not need to use the definition of the Taylor series coefficients.*

b. *If possible, determine the radius of convergence of the series.*

52. $f(x) = \cos(2x) + 2\sin x$ **53.** $f(x) = \dfrac{e^x + e^{-x}}{2}$

54. $f(x) = \sec x$ **55.** $f(x) = (1 + x^2)^{-2/3}$

56. $f(x) = \tan x$ **57.** $f(x) = \sqrt{1 - x^2}$

58. $f(x) = b^x$ for $b > 0$ **59.** $f(x) = \dfrac{1}{x^4 + 2x^2 + 1}$

T 60–63. Alternative approach *Compute the coefficients for the Taylor series for the following functions about the given point a and then use the first four terms of the series to approximate the given number.*

60. $f(x) = \sqrt{x}$ with $a = 36$; approximate $\sqrt{39}$.

61. $f(x) = \sqrt[3]{x}$ with $a = 64$; approximate $\sqrt[3]{60}$.

62. $f(x) = 1/\sqrt{x}$ with $a = 4$; approximate $1/\sqrt{3}$.

63. $f(x) = \sqrt[4]{x}$ with $a = 16$; approximate $\sqrt[4]{13}$.

64. Geometric/binomial series Recall that the Taylor series for $f(x) = 1/(1 - x)$ about 0 is the geometric series $\sum_{k=0}^{\infty} x^k$. Show that this series can also be found as a case of the binomial series.

65. Integer coefficients Show that the coefficients in the Taylor series (binomial series) for $f(x) = \sqrt{1 + 4x}$ about 0 are integers.

66. Choosing a good center Suppose you want to approximate $\sqrt{72}$ using four terms of a Taylor series. Compare the accuracy of the approximations obtained using the Taylor series for $\sqrt{x}$ centered at 64 and 81.

67. Alternative means By comparing the first four terms, show that the Maclaurin series for $\sin^2 x$ can be found (a) by squaring the Maclaurin series for $\sin x$ or (b) by using the identity $\sin^2 x = (1 - \cos 2x)/2$.

68. Alternative means By comparing the first four terms, show that the Maclaurin series for $\cos^2 x$ can be found (a) by squaring the Maclaurin series for $\cos x$ or (b) by using the identity $\cos^2 x = (1 + \cos 2x)/2$.

69. Designer series Find a power series that has $(2, 6)$ as an interval of convergence.

70–71. Patterns in coefficients *Find the next two terms of the following Taylor series.*

70. $\sqrt{1 + x}$: $1 + \dfrac{1}{2}x - \dfrac{1}{2 \cdot 4}x^2 + \dfrac{1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \cdots$.

71. $\dfrac{1}{\sqrt{1 + x}}$: $1 - \dfrac{1}{2}x + \dfrac{1 \cdot 3}{2 \cdot 4}x^2 - \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots$.

72. Composition of series Use composition of Taylor series to find the first three terms of the Maclaurin series for the following functions.

a. $e^{\sin x}$ **b.** $e^{\tan x}$ **c.** $\sqrt{1 + \sin^2 x}$

Applications

T 73–76. Approximations *Choose a Taylor series and a center point a to approximate the following quantities with an accuracy of at least 10^{-4}.*

73. $\cos 40°$ **74.** $\sin(0.98\pi)$

75. $\sqrt[3]{83}$ **76.** $1/\sqrt[4]{17}$

77. Different approximation strategies Suppose you want to approximate $\sqrt[3]{128}$ to within 10^{-4} of the exact value.

a. Use a Taylor polynomial centered at 0.

b. Use a Taylor polynomial centered at 125.

c. Compare the two approaches. Are they equivalent?

Additional Exercises

78. Mean Value Theorem Explain why the Mean Value Theorem (Theorem 4.9 of Section 4.6) is a special case of Taylor's Theorem.

79. Version of the Second Derivative Test Assume that f has at least two continuous derivatives on an interval containing a with $f'(a) = 0$. Use Taylor's Theorem to prove the following version of the Second Derivative Test:

a. If $f''(x) > 0$ on some interval containing a, then f has a local minimum at a.

b. If $f''(x) < 0$ on some interval containing a, then f has a local maximum at a.

80. Nonconvergence to f Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

a. Use the definition of the derivative to show that $f'(0) = 0$.

b. Assume the fact that $f^{(k)}(0) = 0$, for $k = 1, 2, 3, \ldots$. (You can write a proof using the definition of the derivative.) Write the Taylor series for f centered at 0.

c. Explain why the Taylor series for f does not converge to f for $x \neq 0$.

QUICK CHECK ANSWERS

1. When evaluated at $x = a$, all terms of the series are zero except for the first term, which is $f(a)$. Therefore the series equals $f(a)$ at this point. **2.** $1 - x + x^2 - x^3 + x^4 - \cdots$ **3.** $2x + 2x^2 + x^3$; $1 - x + x^2/2$ **4.** 6, 1/16 **5.** 1.05, 1.04875

9.4 Working with Taylor Series

We now know the Taylor series for many familiar functions and we have tools for working with power series. The goal of this final section is to illustrate additional techniques associated with power series. As you will see, power series cover the entire landscape of calculus from limits and derivatives to integrals and approximation.

Limits by Taylor Series

An important use of Taylor series is evaluating limits. A couple of examples illustrate the essential ideas.

EXAMPLE 1 **A limit by Taylor series** Evaluate $\displaystyle\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{3x^4}$.

> L'Hôpital's Rule may be impractical when it must be used more than once on the same limit or when derivatives are difficult to compute.

SOLUTION Because the limit has the indeterminate form 0/0, l'Hôpital's Rule can be used, which requires four applications of the rule. Alternatively, because the limit involves values of x near 0, we substitute the Maclaurin series for $\cos x$. Recalling that

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots, \quad \text{Table 9.5, page 620}$$

we have

$$\begin{aligned}
\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{3x^4} &= \lim_{x \to 0} \frac{x^2 + 2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) - 2}{3x^4} && \text{Substitute for } \cos x. \\
&= \lim_{x \to 0} \frac{x^2 + \left(2 - x^2 + \frac{x^4}{12} - \frac{x^6}{360} + \cdots\right) - 2}{3x^4} && \text{Simplify.} \\
&= \lim_{x \to 0} \frac{\frac{x^4}{12} - \frac{x^6}{360} + \cdots}{3x^4} && \text{Simplify.} \\
&= \lim_{x \to 0} \left(\frac{1}{36} - \frac{x^2}{1080} + \cdots\right) = \frac{1}{36}. && \text{Simplify, evaluate limit.}
\end{aligned}$$

> In using a series approach to evaluating limits, it is often not obvious how many terms of the Taylor series to use. When in doubt, include extra (higher-power) terms. The dots in the calculation stand for powers of x greater than the last power that appears.

Related Exercises 7–20

QUICK CHECK 1 Use the Taylor series $\sin x = x - x^3/6 + \cdots$ to verify that $\lim_{x \to 0} (\sin x)/x = 1$.

EXAMPLE 2 **A limit by Taylor series** Evaluate

$$\lim_{x \to \infty} \left[6x^5 \sin\left(\frac{1}{x}\right) - 6x^4 + x^2\right].$$

SOLUTION A Taylor series may be centered at any finite point in the domain of the function, but we don't have the tools needed to expand a function about $x = \infty$. Using a technique introduced earlier, we replace x by $1/t$ and note that as $x \to \infty$, $t \to 0^+$. The new limit becomes

$$\begin{aligned}
\lim_{x \to \infty} \left[6x^5 \sin\left(\frac{1}{x}\right) - 6x^4 + x^2\right] &= \lim_{t \to 0^+} \left(\frac{6\sin t}{t^5} - \frac{6}{t^4} + \frac{1}{t^2}\right) && \text{Replace } x \text{ by } 1/t. \\
&= \lim_{t \to 0^+} \left(\frac{6\sin t - 6t + t^3}{t^5}\right). && \text{Common denominator}
\end{aligned}$$

This limit has the indeterminate form 0/0. We now expand $\sin t$ in a Taylor series centered at $t = 0$. Because

$$\sin t = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \cdots, \quad \text{Table 9.5, page 620}$$