Predictive Sets

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Can you guess?

Suppose I give you a sequence: 1,

Can you guess?

Suppose I give you a sequence: 1,1,

Can you guess?

Suppose I give you a sequence: 1,1, 1,

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Suppose I give you a sequence: 1,1, 1, 1, 1, 1,

Can you guess?

Suppose I give you a sequence: 1,1, 1, 1, 1, 1,

What comes next?

Can you guess?

Suppose I give you a sequence: 1,1, 1, 1, 1, 1,

What comes next?

It is probably going to be 1.

Can you guess?

What about this one: 1,

Can you guess?

What about this one: 1,2,

Can you guess?

What about this one: 1,2, 3,

Can you guess?

What about this one: 1,2, 3, 1,

Can you guess?

What about this one: 1,2, 3, 1, 2,

Can you guess?

What about this one: 1,2, 3, 1, 2, 3,

Can you guess?

What about this one: 1,2, 3, 1, 2, 3,

It is probably going to be 1 again.

Can you guess?

What about this one: 1,2, 3, 1, 2, 3,

It is probably going to be 1 again.

But it could very well have been part of

$$\ldots, 4, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 1, 2, 3, 4, \ldots$$

in which case it should have been 4.

Be careful with your guesses.

As mathematicians, we know that without enough information about how the sequence comes about there is not much point in guessing.

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But what if instead I give you an infinite sequence and tell you before hand that the sequence is periodic. Then we can always predict precisely.

Coin flips

But knowing the past of the sequence does not always help even if we know how the sequence is generated.

Coin flips

But knowing the past of the sequence does not always help even if we know how the sequence is generated.

For instance, consider the sequence $\ldots, H, T, H, H, T$ obtained from independent coin flips. Since the coin flips are independent, we get no information about what is about to happen next.

Rotations

Till now we have seen that we can predict periodic sequences. Let us look at almost periodic ones.

Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ denote the circle. Given $x, \alpha \in \mathbb{T}$ we consider the rotation $x, x+\alpha, x+2\alpha, \ldots$.

We split the circle into two parts $[0, 1/2)$ and $[1/2, 1)$. If the point falls on the first part we record a 0 and if it falls on the second half we record a 1.

Thus starting with a point x we get a sequence in 0 and 1.

Rotations

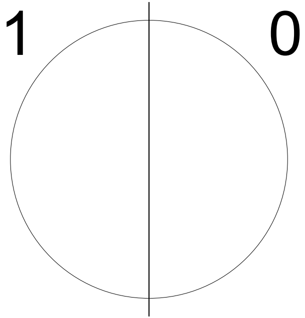

Figure : Recording a circle rotation by 0 and 1:

Rotations

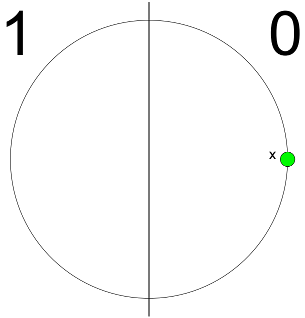

Figure : Recording a circle rotation by 0 and 1: 0

Rotations

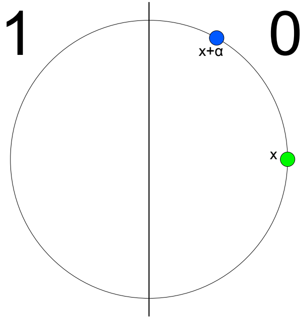

Figure : Recording a circle rotation by 0 and 1: $0, 0$

Rotations

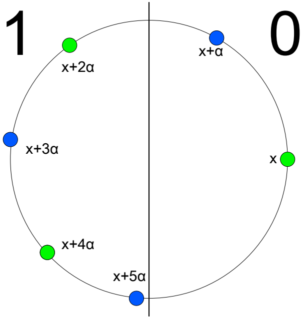

Figure : Recording a circle rotation by 0 and 1: 0, 0, 1, 1, 1, 1

Rotations

Now suppose I give you an irrational rotation number α and the point being rotated is x. If I give you the past of the sequence $x_n = 1_{x+n\alpha \in [1/2,1)}$, that is, $\ldots, x_{-2}, x_{-1}$, can you guess x_0?

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We can do more. We can obtain x. If $x_n = 0$ then $x + n\alpha \in [0, 1/2)$ and otherwise it is in $[1/2, 1)$.

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We can do more. We can obtain x. If $x_n = 0$ then $x + n\alpha \in [0, 1/2)$ and otherwise it is in $[1/2, 1)$.

Thus

$$x \in \cap_{n\in(\mathbb{N});x_{-n}=0}(n\alpha + [0, 1/2)) \cap_{n\in(\mathbb{N});x_{-n}=1} (n\alpha + [1/2, 1)).$$

Rotations

We had

$$x \in \bigcap_{t \in (\mathbb{N}); x_{-t}=0} (t\alpha + [0, 1/2)) \bigcap_{t \in (\mathbb{N}); x_{-t}=1} (t\alpha + [1/2, 1)).$$

Let

$$I_n := \bigcap_{t \in [1,n]; x_{-t}=0} (t\alpha + [0, 1/2)) \bigcap_{t \in [1,n]; x_{-t}=1} (t\alpha + [1/2, 1)).$$

This is a decreasing sequence of intervals and we have that

$$x \in \cap I_n.$$

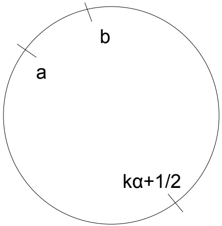
Rotations

$$I_n := \bigcap_{t\in[1,n];x_{-t}=0} (t\alpha + [0,1/2)) \bigcap_{t\in[1,n];x_{-t}=1} (t\alpha + [1/2,1)).$$

and

$$x \in \cap I_n.$$

In fact if $I_n = [a, b]$ then there is $k > n$ such that $k\alpha + 1/2$ is close to $(a + b)/2$.



Rotations

But then

$$\begin{aligned} I_k &\subset [a,b] \cap \left(k\alpha + 1/2 + [0,1/2]\right) \text{ or} \\ I_k &\subset [a,b] \cap \left(k\alpha + 1/2 + [-1/2,1)\right). \end{aligned}$$

Rotations

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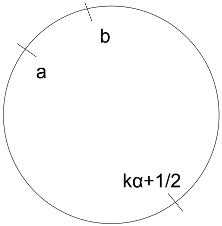

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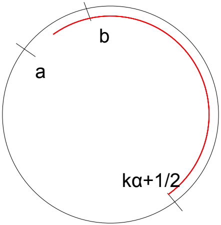

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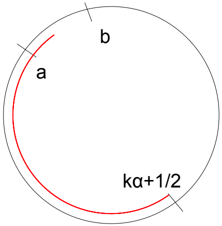

Rotations

Either way $|I_k|$ is close to $1/2|I_n|$. Thus $\cap I_n$ is a single point. Therefore $x \in \cap I_n$ is completely determined.

So given the past of the sequence we can predict its future.

Let us make everything more formal.

Stationary Processes

A stationary process is a sequence of random variables $\ldots, X_{-2}, X_{-1}, X_0, \ldots$ such that the distribution of

$$X_0, X_1, \ldots, X_n$$

is the same as that of

$$X_k, X_{k+1}, \ldots, X_{k+n}.$$

Examples of Stationary Processes

The results of independent coin flips gives us a stationary process.

$$\ldots, H, T, H, T, T, T, H, \ldots,$$

Non-examples of Stationary Processes

The sequence $X_n := 1_{n\alpha \in [0,1/2)}$ is not stationary. If we shift the sequence then we will get $X_n := 1_{(n+1)\alpha \in [0,1/2)}$ which is not the same sequence.

Examples of Stationary Processes

However if x has been chosen according to Lebesgue measure on the circle $\mathbb{R}/\mathbb{Z}$ then the sequence $X_n := 1_{x+n\alpha\in[0,1/2)}$ is stationary.

We shall always assume that our stationary processes are finite-valued (except when explicitly stated).

The process $\ldots, X_{-2}, X_{-1}, X_0, \ldots$ will be denoted by $X_{\mathbb{Z}}$.

Given $P \subset \mathbb{Z}$, X_P is $(X_p)_{p \in \mathbb{Z}}$.

Deterministic Processes

Let us assume that X_0 takes value in a finite set A. By stationarity we have that $X_k \in A$ for all $k \in \mathbb{Z}$ with probability one.

We say that a stationary process $X_{\mathbb{Z}}$ is called deterministic if there is a measurable function $\Phi : A^{-\mathbb{N}} \to A$ such that

$$\Phi(X_{-\mathbb{N}}) = X_0$$

with probability one.

Deterministic Processes: Examples

A stationary process $X_{\mathbb{Z}}$ is called deterministic if there is a measurable function $\Phi : A^{-\mathbb{N}} \to A$ such that

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with probability one.

The process given by

$$X_n := 1_{x+n\alpha \in [0,1/2)}; n \in \mathbb{Z}$$

and x chosen according to the Lebesgue measure on $\mathbb{R}/Z$ is a deterministic process.

Deterministic Processes: Non-examples

A stationary process $X_{\mathbb{Z}}$ is called deterministic if there is a measurable function $\Phi : A^{-\mathbb{N}} \to A$ such that

$$\Phi(X_{-\mathbb{N}}) = X_0$$

with probability one.

The process given by

$$X_n := \text{ result of } n^{\text{th}} \text{ independent coin flips}$$

is not a deterministic process.

Predictive Sets

A set $P \subset \mathbb{N}$ is called predictive if for all deterministic processes $X_{\mathbb{Z}}$ there exists a function $\Phi : A^{-P} \to A$ such that

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Predictive Sets: Examples

A set $P \subset \mathbb{N}$ is called predictive if for all deterministic processes $X_{\mathbb{Z}}$ there exists a function $\Phi : A^{-P} \to A$ such that

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Thus by definition $\mathbb{N}$ is a predictive set.

Predictive Sets: Examples

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Suppose $\mathbb{N} + k \subset P$ then P is predictive. Because then by stationarity, X_{-P} can predict X_{-k} and hence X_{-k+1} and subsequently X_0.

Predictive Sets: Non-examples

A set $P \subset \mathbb{N}$ is called predictive if for all deterministic processes $X_{\mathbb{Z}}$ there exists a function $\Phi : A^{-P} \to A$ such that

$$\Phi(X_{-P}) = X_0.$$

The set, P, of odd numbers is not predictive. Let $X_{\mathbb{Z}} =$

$$\dots, 1, 2, 1, 2, 1, 2, \dots$$

$$\dots, 1, 3, 1, 3, 1, 3, \dots$$

$$\dots, 2, 1, 2, 1, 2, 1, \dots$$

$$\dots, 3, 1, 3, 1, 3, 1, \dots$$

with probability $1/4$ each. Clearly $X_{\mathbb{Z}}$ is a deterministic process. However with probability $1/2$, $X_P = 1$ in which case we will have no clue what X_0 ought to be.

In fact, there are weak-mixing zero-entropy processes $X_{\mathbb{Z}}$ where X_0 is independent of X_P for the set of odd numbers P.

The proof goes via Riesz products and Gaussian processes.

Even numbers

Are the even numbers predictive?

Even numbers

Are the even numbers predictive?

For this we need to introduce some entropy theory.

Entropy of random variables

Shannon revolutionised information theory in 1948 by bringing in a host of new ideas and technology.

At the centre of this revolution was entropy.

For a random variable X taking values in the set A, the entropy of X is given by

$$H(X) := -\sum_{a \in A} \mathbb{P}(X = a) \log(\mathbb{P}(X = a))$$

assuming $0 \log 0 = 0$.

Shannon Entropy

$$H(X) := -\sum_{a \in A} \mathbb{P}(X = a) \log(\mathbb{P}(X = a)).$$

Clearly $\mathbb{P}(X = a) \log(\mathbb{P}(X = a)) \geq 0$. It is equal to 0 if and only if $\mathbb{P}(X = a) = 1$ for some $a \in A$.

Thus $H(X) = 0$ if and only if it is completely determined.

Shannon Entropy

$$H(X) := -\sum_{a \in A} \mathbb{P}(X = a) \log(\mathbb{P}(X = a)).$$

On the other hand $\theta \to -\log \theta$ is a convex function. By Jensen's inequality we have

$$\begin{aligned} H(X) &:= -\sum_{a \in A} \mathbb{P}(X = a) \log(\mathbb{P}(X = a)) \\ &\leq \log(\sum_{a \in A} \frac{\mathbb{P}(X = a)}{\mathbb{P}(X = a)}) \\ &= \log(|A|) \end{aligned}$$

with equality if and only if $\mathbb{P}(X = a) = \frac{1}{|A|}$ for all $a \in A$.

Shannon Entropy

Thus $H(X) = 0$ if and only if X is deterministic and

$H(X) \leq \log(|A|)$ with equality if and only if X is uniformly distributed.

Shannon Entropy

Further $H(X|Y) := H(X,Y) - H(Y)$ and similarly one can prove

$H(X|Y) = 0$ if and only if X is a function of Y.

Kolmogorov Sinai Entropy

For a process $X_{\mathbb{Z}}$, Kolmogorov Sinai entropy is defined by the limit

$$
\begin{aligned}
h(X_{\mathbb{Z}}) &:= \lim_{n\to\infty} \frac{1}{n} H(X_0, X_1, \ldots, X_{n-1}) \\
&= \lim_{n\to\infty} \frac{1}{n}(H(X_0) + H(X_1|X_0) + \ldots, H(X_{n-1}|X_{n-2}, \ldots, X_0))) \\
&= \lim_{n\to\infty} \frac{1}{n}(H(X_0) + H(X_0|X_{-1}) + \ldots, H(X_0|X_{-1}, \ldots, X_{-n+1})) \\
&= H(X_0|X_{-1}, X_{-2}, \ldots).
\end{aligned}
$$

Thus $h(X_{\mathbb{Z}}) = 0$ if and only if $H(X_0|X_{-1}, X_{-2}, \ldots) = 0$ if and only if X_0 is a function of $X_{-\mathbb{N}}$.

Back to the evens

$X_{\mathbb{Z}}$ is deterministic if and only if $h(X_{\mathbb{Z}}) = 0$.
It is easy to see that

$$\begin{aligned} h(X_{\mathbb{Z}}) &:= \lim_{n\to\infty} \frac{1}{n} H(X_0, X_1, \ldots, X_{n-1}) \\ &\geq \lim_{n\to\infty} \frac{1}{n} H(X_0, X_k, X_{2k} \ldots,) \\ &\geq \frac{1}{k} h(X_{k\mathbb{Z}}). \end{aligned}$$

Thus if $X_{\mathbb{Z}}$ is deterministic then $X_{k\mathbb{Z}}$ is also deterministic.

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\end{aligned}
$$

Thus if $X_{\mathbb{Z}}$ is deterministic then $X_{k\mathbb{Z}}$ is also deterministic.

Caution: This is not true unless the state space is finite.

Back to the evens

If $X_{\mathbb{Z}}$ is deterministic then $X_{k\mathbb{Z}}$ is also deterministic.

We know that $\mathbb{N}$ is predictive. Thus Y_0 is a function of $Y_{-\mathbb{N}}$ for all deterministic processes $Y_{\mathbb{Z}}$. Thus we have that $k\mathbb{N}$ is also a predictive set.

Results: Return-time sets

Let X be a Polish space with a probability measure μ and $T : (X, \mu) \to (X, \mu)$ be a measurable isomorphism. The triple (X, μ, T) is called a probability preserving transformation (or a ppt).

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Given a set $U \subset X$ of positive measure we define

$$N(U, U) := \{n \in \mathbb{N} \ : \ \mu(T^n(U) \cap U) > 0\}.$$

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A return-time set is a set of the above type.

Evens are return-times sets for $X := \{-1, 1\}$, $T(x) = -x$, uniform probability measure μ and $U := \{1\}$.

Results: Return-time sets

Theorem

Return-time sets are predictive.

Thus, one can prove that for any $\alpha \in \mathbb{R}/\mathbb{Z}$ and $\epsilon > 0$, the set

$$\{n \ : \ n\alpha \in (-\epsilon, \epsilon)\}$$

is predictive. In fact, if P is a predictive set then

$$P \cap \{n \ : \ n\alpha \in (-\epsilon, \epsilon)\}$$

is also predictive.

Results: SIP^*

Given a sequence $s_1, s_2, \ldots$ we write

$$SIP(s_1, s_2, \ldots) := \{\sum_{i=1}^{\infty} \epsilon_i s_i \ : \ \epsilon_i \in \{-1, 0, 1\}\} \cap \mathbb{N}.$$

A set $P \subset \mathbb{N}$ is called $SIP^\star$ if it intersects every SIP set.

Clearly one of the numbers of the type $\sum_{i=1}^{\infty} \epsilon_i s_i$ is even for all sequences $s_1, s_2, \ldots$ (either s_1, s_2 or $s_1 + s_2$ has to be even).

Thus the even numbers are $SIP^\star$ and similarly $k\mathbb{N}$ for all $k \in \mathbb{N}$.

The odd numbers are not $SIP^\star$. They do not intersect the SIP generated by even numbers.

Results: SIP^*

Theorem

Predictive sets are SIP^*.

Given an SIP Q, we construct a weak-mixing process X_Z for which X_0 is independent of $X_{\mathbb{N}\setminus Q}$.

An easy consequence is that predictive sets P have bounded gaps meaning $\mathbb{N} \setminus P$ cannot contain intervals of unbounded length.

Let us see why.

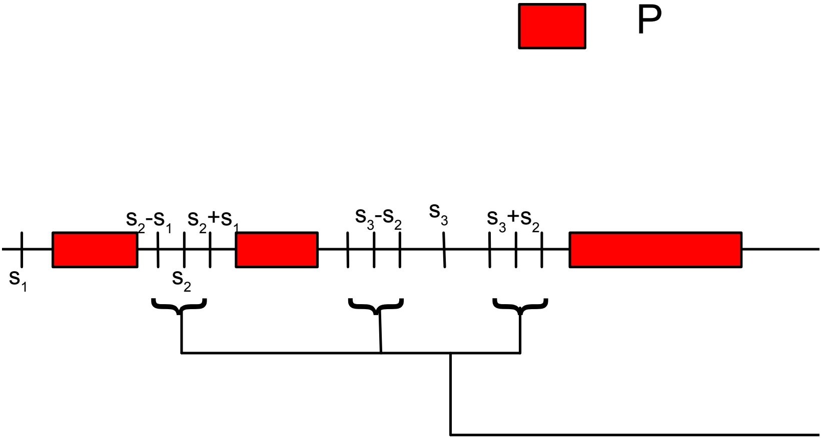
$$SIP(s_1, s_2, \ldots) := \{\sum_{i=1}^{\infty} \epsilon_i s_i \ : \ \epsilon_i \in \{-1, 0, 1\}\} \cap \mathbb{N}.$$

Predictive sets intersect every SIP set.

Suppose P is a predictive set such that $\mathbb{N} \setminus P$ contains intervals of unbounded length.

Then we can fit an SIP set in $\mathbb{N} \setminus P$.

Fitting SIP sets in $\mathbb{N} \setminus P$ if P has does not have bounded gaps



Sufficient conditions for a set to be predictive:

Theorem (Chandgotia, Weiss)

Return-time sets are predictive sets.

Necessary conditions for a set to be predictive:

Theorem (Chandgotia, Weiss)

Predictive sets are SIP*.

The following question arises naturally.

Question

Are sufficient conditions necessary and necessary conditions sufficient?

Let us give some partial answers.

Are all $SIP^\star$ sets predictive?

If P is a predictive set, $\epsilon > 0$ and $\alpha \in \mathbb{R}/\mathbb{Z}$ then

$$\{n \in \mathbb{N} \ : \ n\alpha \in (-\epsilon, \epsilon)\} \cap P$$

is predictive.

Question

Is the intersection of two predictive sets also predictive? Is the intersection non-empty?

Are all $SIP^\star$ sets predictive?

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Question

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ *be irrational and* $\epsilon < 1/2$. *Is the set*

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$$

predictive?

An uncertain theorem

Question

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be irrational and $\epsilon < 1/2$. Is the set

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$$

predictive?

If the answer is yes then we have two predictive sets

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\} \text{ and } \{n \in \mathbb{N} \ : \ -n\alpha \in (0, \epsilon)\}$$

which do not intersect.

Theorem (Akin and Glasner, 2016)

The set $\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$ is $SIP^\star$.

Thus if the answer is no then we have a $SIP^\star$ set which is not predictive.

There are predictive sets which do not contain return-time sets.

Consider the set

$$Q = \{n^2 \ : \ n \in \mathbb{N}\}.$$

For all $i, k \in \mathbb{N}$ we have that if

$$n^2 = -i + 3i^2 k = i(-1 + 3ik)$$

then since i and $-1 + 3ik$ are prime to each other, they are perfect squares.

But this is impossible because $-1 + 3ik \equiv -1 \pmod 3$. Thus $\mathbb{N} \setminus Q$ contains $-i + 3i^2 k; k \in \mathbb{N}$.

There are predictive sets which do not contain return-time sets.

Hence we have that

$$H(X_{-i} \mid X_{\mathbb{N}\setminus Q}) = 0$$

for all $i \in \mathbb{N}$.

But then for all $i \in \mathbb{Z}$

$$H(X_i \mid X_{\mathbb{N}\setminus Q}) = H(X_i \mid X_{(-\mathbb{N})\cup(\mathbb{N}\setminus Q)}) = 0$$

for all $i \in \mathbb{Z}$.

It is well known that any return-time set must intersect the set $\{n^2 \ : \ n \in \mathbb{N}\}$. Thus there are predictive sets which are not return-time sets.

Predictive sets

Question

Let $\{n_k\}_{k\in\mathbb{N}}$ *be an increasing sequence such that* $n_{k+1} - n_k$ *is also an increasing sequence. Prove that*

$$H(X_0 \mid X_{\mathbb{N}\setminus\{n_k \mid k\in\mathbb{N}\}}) = 0.$$

We do not know this even in the case $n_k = k^3$.

Predictive sets

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We do not know this even in the case $n_k = k^3$.

Some partial results use the Fermat's last theorem.

Connections with harmonic analysis

Suppose μ is a probability measure on the circle $\mathbb{T}$. Then the relation

$$\mathbb{E}(X_0\overline{X_n}) = \hat{\mu}(n)$$

defines a unique Gaussian process with mean zero.

A Gaussian process $X_{\mathbb{Z}}$ has zero entropy if and only if μ is singular.

But then X_0 can be predicted by $X_{-\mathbb{N}}$. By properties of Gaussian processes this implies that 1 is in the linear span of $e^{2\pi inx}; n \in \mathbb{N}$.

Theorem

In general if P is a predictive set then for any singular probability measure μ there exists $p \in P$ such that $\hat{\mu}(p) \neq 0$.

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In general if P is a predictive set then for any singular probability measure μ there exists $p \in P$ such that $\hat{\mu}(p) \neq 0$.

This relates predictive sets with Riesz sets defined by Yves Myer (1975) and points to many very interesting directions.

A set Q is a Riesz set if for any singular complex-valued measure μ, there is a $q \in Q$ such that $\hat{\mu}(p) \neq 0$.

There is a lot of parallel between Riesz sets and totally predicitive sets, set $P \subset \mathbb{N}$ which can predict the entire process.

Summary

Return-time sets are predictive.

The converse is not true.

Predictive sets are $SIP^{\star}$.

Predictive sets have bounded gaps.

Questions

1. Is the intersection of two predictive sets also a predictive set?
2. Are all $SIP^\star$ sets predictive?
3. Is $\{n \ : \ n\alpha \in (0,\epsilon)\}$ a predictive set for irrational α?
4. Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence such that $n_{k+1} - n_k$ is also an increasing sequence. Prove that

$$H(X_0 \mid X_{\mathbb{N}\setminus\{n_k \ \mid \ k\in\mathbb{N}\}}) = 0.$$

5. What is the relationship between Riesz sets and totally predictive sets?

Thank you!