convergent alternating harmonic series (Section 8.6). So the interval of convergence is $-1 \le x < 1$. Here is a subtle point: Although we know the series converges at $x = -1$, Theorem 9.5 guarantees convergence to $\ln(1 - x)$ only at the interior points. So we cannot use Theorem 9.5 to claim that the series converges to $\ln 2$ at $x = -1$. In fact, it does, as shown in Section 9.3.
Related Exercises 33–38

QUICK CHECK 4 Use the result of Example 4 to write a power series representation for $\ln \frac{1}{2} = -\ln 2$.

EXAMPLE 5 Functions to power series Find power series representations centered at 0 for the following functions and give their intervals of convergence.

a. $\tan^{-1} x$ **b.** $\ln\left(\dfrac{1+x}{1-x}\right)$

SOLUTION In both cases, we work with known power series and use differentiation, integration, and other combinations.

a. The key is to recall that

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

and that, by Example 3c,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots, \qquad \text{provided } |x| < 1.$$

We now integrate both sides of this last expression:

$$\int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 - \cdots)\, dx,$$

which implies that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + C.$$

Substituting $x = 0$ and noting that $\tan^{-1} 0 = 0$, the two sides of this equation agree provided we choose $C = 0$. Therefore,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}.$$

By Theorem 9.5, this power series converges for $|x| < 1$. Testing the endpoints separately, we find that it also converges at $x = \pm 1$. Therefore, the interval of convergence is $[-1, 1]$.

➤ Again, Theorem 9.5 does not guarantee that the power series in part (a) converges to $\tan^{-1} x$ at $x = \pm 1$. In fact, it does.

b. We have already seen (Example 4) that

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

Replacing x by $-x$, we have

$$\ln(1 - (-x)) = \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots.$$

➤ Nicolaus Mercator (1620–1687) and Sir Isaac Newton (1642–1727) independently derived the power series for $\ln(1 + x)$, which is called the *Mercator series*.

Subtracting these two power series gives

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) && \text{Properties of logarithms}\\
&= \underbrace{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)}_{\ln(1+x)} - \underbrace{\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right)}_{\ln(1-x)}, \quad \text{for } |x| < 1\\
&= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) && \text{Combine, Theorem 9.4.}\\
&= 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} && \text{Summation notation}
\end{aligned}$$

QUICK CHECK 5 Verify that the power series in Example 5b does not converge at the endpoints $x = \pm 1$.

This power series is the difference of two power series, both of which converge on the interval $|x| < 1$. Therefore, by Theorem 9.4, the new series also converges on $|x| < 1$.
Related Exercises 39–44

If you look carefully, every example in this section is ultimately based on the geometric series. Using this single series, we were able to develop power series for many other functions. Imagine what we could do with a few more basic power series. The following section accomplishes precisely that end. There, we discover basic power series for all the standard functions of calculus.

SECTION 9.2 EXERCISES

Review Questions

1. Write the first four terms of a power series with coefficients c_0, c_1, c_2, c_3 centered at 0.
2. Write the first four terms of a power series with coefficients c_0, c_1, c_2, c_3 centered at 3.
3. What tests are used to determine the radius of convergence of a power series?
4. Explain why a power series is tested for *absolute* convergence.
5. Do the interval and radius of convergence of a power series change when the series is differentiated or integrated? Explain.
6. What is the radius of convergence of the power series $\sum c_k (x/2)^k$ if the radius of convergence of $\sum c_k x^k$ is R?
7. What is the interval of convergence of the power series $\sum (4x)^k$?
8. How are the radii of convergence of the power series $\sum c_k x^k$ and $\sum (-1)^k c_k x^k$ related?

Basic Skills

9–20. Interval and radius of convergence *Determine the radius of convergence of the following power series. Then test the endpoints to determine the interval of convergence.*

9. $\sum \left(\dfrac{x}{3}\right)^k$
10. $\sum (-1)^k \dfrac{x^k}{5^k}$
11. $\sum \dfrac{x^k}{k^k}$
12. $\sum (-1)^k \dfrac{k(x-4)^k}{2^k}$
13. $\sum \dfrac{k^2 x^{2k}}{k!}$
14. $\sum \dfrac{k^k x^k}{(k+1)!}$
15. $\sum \dfrac{x^{2k+1}}{3^{k-1}}$
16. $\sum \left(-\dfrac{x}{10}\right)^{2k}$
17. $\sum \dfrac{(x-1)^k k^k}{(k+1)^k}$
18. $\sum \dfrac{(-2)^k (x+3)^k}{3^{k+1}}$
19. $\sum \dfrac{k^{20} x^k}{(2k+1)!}$
20. $\sum (-1)^k \dfrac{x^{3k}}{27^k}$

21–26. Combining power series *Use the geometric series*

$$f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \qquad \text{for } |x| < 1,$$

to find the power series representation for the following functions (centered at 0). Give the interval of convergence of the new series.

21. $f(3x) = \dfrac{1}{1-3x}$
22. $g(x) = \dfrac{x^3}{1-x}$
23. $h(x) = \dfrac{2x^3}{1-x}$
24. $f(x^3) = \dfrac{1}{1-x^3}$
25. $p(x) = \dfrac{4x^{12}}{1-x}$
26. $f(-4x) = \dfrac{1}{1+4x}$

27–32. Combining power series *Use the power series representation*

$$f(x) = \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}, \qquad \text{for } -1 \le x < 1,$$

to find the power series for the following functions (centered at 0). Give the interval of convergence of the new series.

27. $f(3x) = \ln(1-3x)$
28. $g(x) = x^3 \ln(1-x)$

29. $h(x) = x \ln(1 - x)$

30. $f(x^3) = \ln(1 - x^3)$

31. $p(x) = 2x^6 \ln(1 - x)$

32. $f(-4x) = \ln(1 + 4x)$

33–38. Differentiating and integrating power series *Find the power series representation for g centered at 0 by differentiating or integrating the power series for f (perhaps more than once). Give the interval of convergence for the resulting series.*

33. $g(x) = \dfrac{1}{(1-x)^2}$ using $f(x) = \dfrac{1}{1-x}$

34. $g(x) = \dfrac{1}{(1-x)^3}$ using $f(x) = \dfrac{1}{1-x}$

35. $g(x) = \dfrac{1}{(1-x)^4}$ using $f(x) = \dfrac{1}{1-x}$

36. $g(x) = \dfrac{x}{(1+x^2)^2}$ using $f(x) = \dfrac{1}{1+x^2}$

37. $g(x) = \ln(1 - 3x)$ using $f(x) = \dfrac{1}{1-3x}$

38. $g(x) = \ln(1 + x^2)$ using $f(x) = \dfrac{x}{1+x^2}$

39–44. Functions to power series *Find power series representations centered at 0 for the following functions using known power series. Give the interval of convergence for the resulting series.*

39. $f(x) = \dfrac{1}{1+x^2}$

40. $f(x) = \dfrac{1}{1-x^4}$

41. $f(x) = \dfrac{3}{3+x}$

42. $f(x) = \ln\sqrt{1-x^2}$

43. $f(x) = \ln\sqrt{4-x^2}$

44. $f(x) = \tan^{-1}(4x^2)$

Further Explorations

45. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The interval of convergence of the power series $\sum c_k (x-3)^k$ could be $(-2, 8)$.

b. $\sum(-2x)^k$ converges for $-1/2 < x < 1/2$.

c. If $f(x) = \sum c_k x^k$ on the interval $|x| < 1$, then $f(x^2) = \sum c_k x^{2k}$ on the interval $|x| < 1$.

d. If $f(x) = \sum c_k x^k = 0$ for all x on an interval $(-a, a)$, then $c_k = 0$ for all k.

46–49. Summation notation *Write the following power series in summation (sigma) notation.*

46. $1 + \dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{6} + \cdots$

47. $1 - \dfrac{x}{2} + \dfrac{x^2}{3} - \dfrac{x^3}{4} + \cdots$

48. $x - \dfrac{x^3}{4} + \dfrac{x^5}{9} - \dfrac{x^7}{16} + \cdots$

49. $-\dfrac{x^2}{1!} + \dfrac{x^4}{2!} - \dfrac{x^6}{3!} + \dfrac{x^8}{4!} - \cdots$

50. Scaling power series If the power series $f(x) = \sum c_k x^k$ has an interval of convergence of $|x| < R$, what is the interval of convergence of the power series for $f(ax)$, where $a \neq 0$ is a real number?

51. Shifting power series If the power series $f(x) = \sum c_k x^k$ has an interval of convergence of $|x| < R$, what is the interval of convergence of the power series for $f(x - a)$, where $a \neq 0$ is a real number?

52–57. Series to functions *Find the function represented by the following series and find the interval of convergence of the series.*

52. $\displaystyle\sum_{k=0}^{\infty} (x^2 + 1)^{2k}$

53. $\displaystyle\sum_{k=0}^{\infty} (\sqrt{x} - 2)^k$

54. $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k}}{4k}$

55. $\displaystyle\sum_{k=0}^{\infty} e^{-kx}$

56. $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{3^{2k}}$

57. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{x^2-1}{3}\right)^k$

58. A useful substitution Replace x by $x - 1$ in the series $\ln(1 + x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$ to obtain a power series for $\ln x$ centered at $x = 1$. What is the interval of convergence for the new power series?

59–62. Exponential function *In Section 9.3, we show that the power series for the exponential function centered at 0 is*

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } -\infty < x < \infty.$$

Use the methods of this section to find the power series for the following functions. Give the interval of convergence for the resulting series.

59. $f(x) = e^{-x}$

60. $f(x) = e^{2x}$

61. $f(x) = e^{-3x}$

62. $f(x) = x^2 e^x$

Additional Exercises

63. Powers of x multiplied by a power series Prove that if $f(x) = \displaystyle\sum_{k=0}^{\infty} c_k x^k$ converges on the interval I, then the power series for $x^m f(x)$ also converges on I for positive integers m.

T **64. Remainders** Let

$$f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{and} \quad S_n(x) = \sum_{k=0}^{n-1} x^k.$$

Then, the remainder in truncating the power series after n terms is $R_n = f(x) - S_n(x)$, which now depends on x.

a. Show that $R_n(x) = x^n/(1 - x)$.

b. Graph the remainder function on the interval $|x| < 1$ for $n = 1, 2, 3$. Discuss and interpret the graph. Where on the interval is $|R_n(x)|$ largest? Smallest?

c. For fixed n, minimize $|R_n(x)|$ with respect to x. Does the result agree with the observations in part (b)?

d. Let $N(x)$ be the number of terms required to reduce $|R_n(x)|$ to less than 10^{-6}. Graph the function $N(x)$ on the interval $|x| < 1$. Discuss and interpret the graph.

65. Product of power series Let

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} d_k x^k.$$

a. Multiply the power series together as if they were polynomials, collecting all terms that are multiples of 1, x, and x^2. Write the first three terms of the product $f(x)g(x)$.

b. Find a general expression for the coefficient of x^n in the product series, for $n = 0, 1, 2, \ldots$.

66. Inverse sine Given the power series

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots$$

for $-1 < x < 1$, find the power series for $f(x) = \sin^{-1} x$ centered at 0.

T **67. Computing with power series** Consider the following function and its power series:

$$f(x) = \frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}, \quad \text{for } -1 < x < 1.$$

a. Let $S_n(x)$ be the first n terms of the series. With $n = 5$ and $n = 10$, graph $f(x)$ and $S_n(x)$ at the sample points $x = -0.9, -0.8, \ldots, -0.1, 0, 0.1, \ldots, 0.8, 0.9$ (two graphs). Where is the difference in the graphs the greatest?

b. What value of n is needed to guarantee that $|f(x) - S_n(x)| < 0.01$ at all of the sample points?

QUICK CHECK ANSWERS

1. $g(0) = 0$ **2.** For any value of x with $x > 6$ or $x < -2$, the series diverges by the Divergence Test. The Root or Ratio Test gives the same result. **3.** $|x| < 1$, $R = 1$ **4.** Substituting $x = 1/2$, $\ln(1/2) = -\ln 2 = -\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k k}$.

9.3 Taylor Series

In the preceding section we saw that a power series represents a function on its interval of convergence. This section explores the opposite question: Given a function, what is its power series representation? We have already made significant progress in answering this question because we know how Taylor polynomials are used to approximate functions. We now extend Taylor polynomials to produce power series—called *Taylor series*—that provide series representations of functions.

Taylor Series for a Function

Suppose a function f has derivatives $f^{(k)}(a)$ of *all* orders at the point a. If we write the Taylor polynomial of degree n for f centered at a and allow n to increase indefinitely, a power series is obtained. The power series consists of a Taylor polynomial of order n plus terms of higher degree called the *remainder*:

$$\underbrace{c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n}_{\text{Taylor polynomial of order } n} + \underbrace{c_{n+1}(x-a)^{n+1} + \cdots}_{\text{remainder}}$$

$$= \sum_{k=0}^{\infty} c_k (x-a)^k$$

The coefficients of the Taylor polynomial are given by

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad \text{for } k = 0, 1, 2, \ldots.$$

These coefficients are also the coefficients of the power series. Furthermore, this power series has the same matching properties as the Taylor polynomials; that is, the function f and the power series agree in *all* of their derivatives at a. This power series is called the **Taylor series for f centered at a**. It is the natural extension of the set of Taylor polynomials for f at a. The special case of a Taylor series centered at 0 is called a **Maclaurin series**.

➤ Maclaurin series are named after the Scottish mathematician Colin Maclaurin (1698–1746), who described them (with credit to Taylor) in a textbook in 1742.