Program Verification: Lecture 23

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Deductive Verification of Distributed Systems

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- for each *terminating* sequence of transitions

$$[u_0] \rightarrow_{\mathcal{R}} [u_1] \ldots [u_{n-1}] \rightarrow_{\mathcal{R}} [u_n]$$

the terminating state $[u_n]$ satisfies postcondition B.

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Let us see an example of a parametric Hoare triple involving a slight modification of the `CHOICE` module in Lecture 16.

A Hoare Triple for the CHOICE Module

```
mod CHOICE is
protecting NAT .
sorts MSet State Pred .
subsorts Nat < MSet .
op __ : MSet MSet -> MSet [ctor assoc comm] .
op {_} : MSet -> State .
op tt : -> Pred [ctor] .
op _=C_ : MSet MSet -> Pred [ctor] .  *** MSet containment
vars U V : MSet . var N : Nat .
eq U =C U = tt .
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rl [choice] : {U V} => {U} .
endm
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The Hoare triple: $\{\{M\} \mid \top\}$ CHOICE $\{\{N\} \mid N \subseteq M = tt\}$ is *parametric* on M. It states that for each M every final state reachable from $\{M\}$ is a singleton set $\{N\}$ with N in M.

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Skeirik, Stefanescu and Meseguer at UIUC have in turn made reachability logic *rewrite-theory-independent* by defining it for rewrite theories $\mathcal{R}$.

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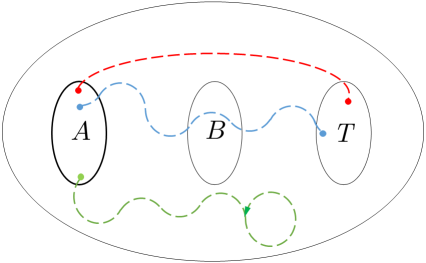
- a generalization of Hoare Logic *partial correctness*, i.e., $A \longrightarrow^{\circledast} B$ generalizes $\{A\}\mathcal{R}\{B\}$
- directly captures *inductive reasoning* in *any* theory $\mathcal{R}$, unlike Hoare Logic, special rules for loops, etc, *unnecessary*

Q: What does the relation $A \longrightarrow^{\circledast} B$ mean?

A: Suppose we have:

(1) a rewrite theory $\mathcal{R}$
(2) pattern fomulas A, B
(3) and terminating states T

Then $A \longrightarrow^{\circledast} B$ means:
for each state $[t] \in [\![A]\!]$
and rewrite path p from $[t]$,
either: (1) p crosses $[\![B]\!]$ or
(2) p is infinite



- - - indicates counterex.
- - - satisfies $A \to^{\circledast} B$
- - - *vacuously* satisfies

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That is, the parameters Y in $A \longrightarrow^{\circledast} B$ are *universally quantified*,

Reachability Logic

Precise Definition

Let $\mathcal{R} = (\Sigma, E \cup B, R)$ be a rewrite theory with good executability conditions, and having a subsignature Ω of constructors and a chosen top sort $State$ of states. Let $\mathcal{C}_{\mathcal{R}}$ denote the canonical reachability model. For a reachability formula $A \longrightarrow^{\circledast} B$ call $Y = vars(A) \cap vars(B)$ its *parameters*.

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We call these two rules *Step+Subsumption* and *Axiom* resp.

The key ideas are: (i) to prove $A \longrightarrow^{\circledast} B$ we may need some *auxiliary lemmas*.

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Reachability Logic

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$$\frac{\displaystyle\bigwedge_j [\{u' \mid \varphi' \longrightarrow^{\circledast} \bigvee_j v'_j \mid \psi'_j\} \cup \mathcal{A},\ \emptyset]\ \vdash_T\ v'_j\alpha \mid \varphi \wedge \psi'_j\alpha \longrightarrow^{\circledast} \bigvee_i v_i \mid \psi_i}{[\{u' \mid \varphi' \longrightarrow^{\circledast} \bigvee_j v'_j \mid \psi'_j\} \cup \mathcal{A},\ \emptyset]\ \vdash_T\ u \mid \varphi \longrightarrow^{\circledast} \bigvee_i v_i \mid \psi_i}$$

The *Step+Subsumption* and *Axiom* Rules

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Next lecture will illustrate the use of the Maude Reachability Logic tool by means of examples.